Attenuated Leaky Rayleigh Waves

by Quan Qi
Dept. of Theoretical and Applied Mechanics
University of Illinois at Urbana-Champaign
Urbana, IL 61801
Abstract

The attenuation of leaky Rayleigh waves due to viscous damping and heat conduction in thin boundary layers is studied by matched asymptotic method. Viscosity of the fluid is considered unimportant except in a thin viscous boundary layer at the interface. By keeping the leading order effect, shown by $Re^{-1/2} = (\omega v)^{1/2} / c_s$, where $Re$ is the Reynolds number, $\omega$ is the frequency, $v$ is the kinematic viscosity of the fluid, and $c_s$ is the shear velocity of the solid substrate, a new characteristic equation is obtained. One of the numerically obtained solutions gives the leaky Rayleigh wave speed and the attenuation coefficient. It is shown that, together with radiation, viscosity and heat conduction in the boundary layer also contribute to the attenuation of the leaky Rayleigh waves. Furthermore, it is shown that, due to the effect of the viscous boundary layer, the attenuated leaky Rayleigh wave speed may be smaller than the conventional Rayleigh wave speed at the interface of a solid half space and a vacuum. Moreover, the correction to wave speeds due to viscosity and heat conduction is shown to be insignificant for a fluid layer on a solid substrate and can be considered unimportant in most cases. Finally, a new wave mode sustained by the viscous boundary layer alone is discovered in the limit of small fluid - solid density ratio. This mode exists for appropriate frequency-layer thickness combinations. For air the corresponding propagation speed is shown to be higher than the sound speed and the corresponding attenuation is significant. These results may be used to improve our interpretation of experimental results of acoustic signature of materials.
1. Introduction

Leaky Rayleigh waves are frequently used in non-destructive testing (NDE) of materials and structures. Here, we present a study of the effect of viscosity and heat conduction on the propagation of the leaky Rayleigh waves at the interface of a fluid layer (or half-space) and a solid half-space substrate. The name attenuated leaky Rayleigh wave is used here to refer to the leaky Rayleigh waves under the influence of viscous boundary layers.

Comprehensive studies of Rayleigh wave and leaky Rayleigh wave can be found in Ewing, Jardetzky and Press (1957) and Viktorov (1967), especially the latter where the leaky Rayleigh wave at the interfaces of a solid half space with both a fluid half space and a fluid layer have been investigated in great detail. Becker and Richardson (1970) were the first to include in their investigation the effect of attenuation. However, their studies are limited to the viscoelastic materials and do not include the attenuation due to viscosity and heat conduction in fluid. In an important contribution, Mott (1971) stressed the importance of including viscous loss in the media. However, unlike Victorov, no quantitative information was presented about how the Rayleigh wave speed and attenuation coefficient might be affected by the viscous loss. Comprehensive review and more recent development on surface waves can be found in Uberall (1972), Ash and Paige (1985), and Parker and Maugin (1988).

The effect of viscosity and heat conduction on wave speeds and attenuations in the boundary layer at the interface between fluid and solid is investigated here by matched asymptotic method. By enforcing non-slip interfacial condition and stress continuities, a new characteristic equation is derived. Different known characteristic equations may be viewed as special cases of this more general characteristic equation. Complex roots are obtained by solving the characteristic equation numerically. The real part of one of the roots gives the attenuated leaky Rayleigh wave speed and the imaginary part is related to the attenuation.

In what follows, the solution are presented respectively for solid substrate, fluid layer and boundary layer, with special attention paid to boundary layer solutions. Tedious but straightforward enforcement of the velocity matching and stress continuities is given in Appendix and it leads to a general characteristic equation. Numerical solution is subsequently obtained and compared with the known results in the literature. It is shown that the attenuated leaky Rayleigh waves at the interface of a solid half space and a fluid half space may propagate at speeds lower than the conventional Rayleigh wave speed in a vacuum, and attenuation per Rayleigh wave length increases due to the presence of viscous boundary layer. For a fluid layer on a solid substrate, the effect of
viscous boundary layer is shown to be insignificant. Finally, a unique wave mode associated with viscous boundary layer alone is shown to exist in the limit of small fluid-solid density ratio for proper combinations of parameters. For air this mode propagates at a speed higher than the sound speed and is independent of solid substrate.

2. Formulation and solutions

For simplicity, only the viscosity in the thin Stokes boundary layer at the interface will be considered. Fluid away from the interface will be assumed inviscid and solid is assumed to be elastic. Consequently, the wave propagations are governed by the well-known wave equation in solid and bulk fluid. In the boundary layer at the interface, appropriate boundary layer equations must be used. The characteristic equation for the leaky Rayleigh wave speed and attenuation coefficient is obtained by requiring the continuities of stresses at the solid-fluid interface and the matching of velocity field.

Since the treatment of a fluid layer is more general compared with that of a fluid half-space, we consider the formulation for a fluid layer and results for a fluid half-space will be derived accordingly.

Consider the geometry shown in Fig. 1. Solid half-space is given by \( z > 0 \), fluid layer is given by \( 0 > z > -h \), where \( h \) is the thickness of the fluid layer, and \( x \) is the direction of propagation for the leaky Rayleigh wave. In the case of fluid half space, \( h \rightarrow \infty \).

For solid substrate and fluid layer away from the interface, solutions may be obtained in the form of displacement potentials and velocity potentials. In the viscous boundary layer, we may consider, in most general cases, the effects of heat conduction and compressibility and it is convenient to consider the solutions in terms of velocities. The principal contribution of this paper is to derive a new characteristic equation that incorporates the effect of the thermal and viscous boundary layers on the propagation of leaky Rayleigh waves.

2.1 Solutions in solid half-space

Since we are concerned with a two dimensional problem, two displacement potentials are required in solid substrate, say, \( \phi \) and \( \psi \) such that displacements in the \( x \) and \( z \) directions can be found from (Victorov 1967),

\[
U = \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} \quad \text{and} \quad W = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} .
\]

The normal stress in the \( z \) direction and the shear stress can be related to these potentials by,
where $\lambda, \mu$ are the two Lame constants of the solid materials.

Solutions (to the wave equation) that behave properly as $z \to \infty$ are:

$$
\varphi = A e^{-qz} e^{i(kx - \omega t)}, \quad \psi = B e^{-sz} e^{i(kx - \omega t)},
$$

where $\omega$ is the frequency, $t$ is time, $k$ is the unknown wave number to be found, $A, B$ are two unknown integration constants and $q, s$ are defined by,

$$
q = \sqrt{k^2 - k_i^2}, \quad s = \sqrt{k^2 - k_t^2}.
$$

Wave numbers $k_p, k_i$ are related to longitudinal and shear waves speeds $c_p, c_t$ through,

$$
k_p = \frac{\omega}{c_p}, \quad k_i = \frac{\omega}{c_t}.
$$

Therefore, the displacements are,

$$
U = ikA e^{-qz} e^{i(kx - \omega t)} + sBe^{-sz} e^{i(kx - \omega t)}
$$

$$
W = -qA e^{-qz} e^{i(kx - \omega t)} + ikBe^{-sz} e^{i(kx - \omega t)}
$$

and stresses at the interface are

$$
\sigma_{zz} \bigg|_{z=0} = \{ \lambda (q^2 - k^2) A + 2\mu (Aq^2 - ikkB) \} e^{i(kx - \omega t)}
$$

$$
\sigma_{xz} \bigg|_{z=0} = \{ -\mu (2ikqA + k^2 B + s^2 B) \} e^{i(kx - \omega t)}.
$$

2.2 Solutions in the fluid layer away from the interface

The velocity potential $\phi$ in fluid layer can be shown to have the following form:

$$
\phi = i\omega (D \sin rz + E \cos rz) e^{i(kx - \omega t)},
$$

where $D, E$ are two additional integration constants and the pressure in fluid is given by
\[
p = -\rho_0 \frac{\partial \phi}{\partial t} = \rho_0 \omega^2 (D \sin rz + E \cos rz) e^{i(kx - \omega t)},
\]

where \( \rho_0 \) is the fluid density and

\[
r = \sqrt{k_f^2 - k^2}.
\]

Wave number in fluid is related to sound speed in fluid \( c_f \) by

\[
k_f = \frac{\omega}{c_f}.
\]

At the free surface \( z = -h \), pressure perturbation vanishes and we have,

\[
E = D \tan rh.
\]

Finally, velocities can be found from \( \mathbf{u} = \nabla \phi = (v_x, v_z) \) to be

\[
v_x = -k \omega D \frac{\sin [r(z+h)]}{\cos rh} e^{i(kx - \omega t)}
\]

\[
v_z = i \rho_0 \omega D \frac{\cos [r(z+h)]}{\cos rh} e^{i(kx - \omega t)}.
\]

Using Eq. (12), we can re-write pressure in fluid layer as,

\[
p = \rho_0 \omega^2 D \frac{\sin [r(z+h)]}{\cos rh} e^{i(kx - \omega t)}.
\]

Accompanying the pressure variation in a compressible fluid, there is also a temperature variation. In this case, it is given by,

\[
T_f = \left( \frac{\beta T_0}{C_p} \right) \omega^2 D \frac{\sin [r(z+h)]}{\cos rh} e^{i(kx - \omega t)},
\]

where \( T_0, \beta \) and \( C_p \) are ambient temperature, coefficient of thermal expansion, and specific heat at constant pressure, respectively.
2.3 Solutions inside the boundary layer at the interface

Because of the neglect of the viscosity in fluid layer, the non-slip boundary condition cannot be enforced between solutions in solid half-space and in fluid layer. For fluids with low viscosity and/or low frequency acoustic waves, this approximation may be a acceptable. But for more viscous fluids or high frequencies, as we shall show, the neglect of non-slip condition at the interface may lead to inadequate results.

Here, we introduce a set of boundary layer equations that capture the effects of heat conduction and compressibility of the fluid in the boundary layer. The solution to these equations can be applied to bridge the “gap” between solutions in the solid half-space and in the fluid layer so that the continuities of all the physical quantities across the interface are ensured.

It can be shown that the leading order boundary layer equations for a compressible and heat conducting fluid are given by,

$$\frac{\partial \rho^{(i)}}{\partial t} + \rho_0^{(i)} \frac{\partial u^{(i)}}{\partial x} + \rho_0^{(i)} \frac{\partial v^{(i)}}{\partial z} = 0,$$

$$\frac{\partial u^{(i)}}{\partial t} + \frac{1}{\rho_0^{(i)}} \frac{\partial p^{(i)}}{\partial x} = v \frac{\partial^2 u^{(i)}}{\partial z^2},$$

$$\frac{\partial p^{(i)}}{\partial z} = 0,$$

$$\rho_0 C_p \frac{\partial T^{(i)}}{\partial t} = \beta T_0 \frac{\partial p^{(i)}}{\partial t} + \kappa \frac{\partial^2 T^{(i)}}{\partial z^2},$$

$$p^{(i)} = \frac{c_f^2}{\gamma} [p^{(i)} + \beta \rho_0 T^{(i)}],$$

where $u^{(i)}, v^{(i)}, \rho^{(i)}, p^{(i)}$ and $T^{(i)}$ are velocities, density, pressure and temperature perturbations inside the boundary layer, respectively. Superscript $i$ is used to denote the solutions inside the boundary layer, or inner solutions. Symbols $\rho_0, \gamma, c_f$ and $\kappa$ are ambient density, the kinematic viscosity, specific heat ratio, sound speed and thermal conductivity in fluid medium, respectively. For Rayleigh wave propagations it suffices to retain only the leading order linearized equations. Equation (16) is the mass conservation, Eq. (17) is the momentum conservation in the $x$ direction, momentum conservation in the $z$ direction (Eq. (18)) states that pressure is continuous across the boundary layer because of its thinness, Eq. (19) is the energy equation and finally, Eq. (20) is the linearized state equation.

Solution to these equations proceed as follows. First, from Eq. (18) we have,
\[ p^{(i)} = p_{z=0}^{1} = \rho_0 \omega^2 D \tan rh e^{i(kx - \omega t)} = \rho_0 \omega^2 D \tan rh e^{i\theta}, \quad (21) \]

where \( \theta = kx - \omega t \) is introduced for later convenience. Equation (17) is then solved to find the velocity in the \( x \) direction,

\[ u_{z=0}^{(i)} = \omega D k \tan rh e^{i\theta} + A_1 e^{\sqrt{2}v} e^{i\theta}, \quad (22) \]

where \( A_1 \) is another integration constant to be determined. By requiring that velocities be continuous across the interface, i.e., \( u^{(i)} \bigg|_{z=0} = \frac{\partial U}{\partial t} \bigg|_{z=0} \), we find from Eqs. (6) and (22),

\[ A_1 = \omega k A - i \omega B - \omega k D \tan rh. \quad (23) \]

Similarly, temperature variation inside the boundary layer can be found from Eq. (19),

\[ T^{(i)} = \left( \frac{\beta T_0}{C_p} \right) \left[ \omega^2 D \tan rh e^{i\theta} + A_2 e^{\sqrt{2}v} e^{i\theta} \right], \quad (24) \]

where \( A_2 \) is an additional integration constant and \( Pr = \frac{\rho_0 C_p \nu}{\kappa} \) is the Prandtl number of the fluid. For simplicity, we shall assumed that the heat capacity of the solid is much larger than that of the fluid. Consequently, the temperature at the solid surface is taken to be ambient and \( T^{(i)} \bigg|_{z=0} = 0 \) giving,

\[ A_2 = -\omega^2 D \tan rh. \quad (25) \]

Correspondingly,

\[ T^{(i)} = \left( \frac{\beta T_0}{C_p} \right) \left( \omega^2 D \tan rh e^{i\theta} \right) \left[ 1 - e^{\sqrt{2}v} \right]. \quad (26) \]

The density variation is found from Eq. (20) to be

\[ \rho^{(i)} = \frac{\gamma}{\gamma - 1} P^{(i)} - (\beta \rho_0) T^{(i)} \]

\[ = \left[ \frac{\rho_0 \gamma}{2c_f^2} - \frac{\beta^2 T_0}{C_p} \omega^2 \left[ 1 - e^{i \sqrt{2}v (1 - i) z} \right] \right] D \tan rh e^{i\theta}. \quad (27) \]
Finally, the velocity component in the $z$ direction inside the boundary layer can be found from Eq. (16) to be,

$$v^{(i)} = v_0^{(i)}(x, t) + \left[-ik^2D\omega \tan rh - \frac{v}{2\omega}i(1 + i)kA_1\left(e^{\frac{1-i}{2}\sqrt{\omega z}} - 1\right)\right]$$

$$+ \frac{i\gamma\omega^3D}{c_f^2} \tan rh - \omega^3i\frac{\beta^2T_0}{C_p}D\tan rh$$

$$+ \omega^3i\frac{\beta^2T_0}{C_p}D\tan rh \left[\frac{1-i}{\omega Pr} e^{\frac{1}{2\omega}Pr} - 1\right].$$

(28)

Similarly, the integration constant $v_0^{(i)}$ can be determined from requiring that the vertical velocity be continuous at the interface, namely, $v^{(i)}|_{z=0} = \frac{\partial W}{\partial t}|_{z=0}$. This leads to,

$$v_0^{(i)} = -i\omega (-qA + ikB) e^{i\theta}$$

(29)

and

$$v^{(i)} = \left[iqA\omega + kB\omega - \frac{v}{2\omega}i(1 + i)kA_1\left(e^{\frac{1-i}{2}\sqrt{\omega z}} - 1\right)\right]$$

$$+ \left[i\omega^3\frac{D}{c_f^2} - ik^2D\omega\right] \tan rh$$

$$+ \omega^3i\frac{\beta^2T_0}{C_p}D\tan rh \left[\frac{1-i}{\omega Pr} e^{\frac{1}{2\omega}Pr} - 1\right] e^{i\theta}$$

(30)

where the identity $(\beta^2T_0c_f^2)/C_p = \gamma - 1$ has been used for simplification.

### 2.4 Characteristic equations

Now that all the solutions inside the boundary layer become available, we can match the vertical velocity with its counterpart in the fluid layer and the shear and normal stress at the solid-fluid interface. This process, though tedious, is straightforward. The details are given in the Appendix and the resulting characteristic equation for attenuated leaky Rayleigh wave is:
where \( k = \omega / c \) is the unknown wave number to be found and
\[
R = \frac{c_i^2}{\omega v}
\] is the Reynolds number based on the shear wave speed in solid \( c_i \), the kinematic viscosity \( \nu \), and the frequency \( \omega \). For common materials and normal driving frequencies, this Reynolds number is large. However, for high frequency and/or viscous fluids and solid substrate with low shear viscosity, the contribution due to the viscous boundary layer may be important.

To find the attenuated leaky Rayleigh wave speed and the attenuation coefficient, Eq. (31) must be solved numerically. Noting that both \( 1/R^{1/2} \) and \( 1/R \) are present in Eq. (31), we may simplify the equation significantly if terms associated with \( 1/R \) can be regarded as negligible. This approximation is reasonable, because \( R \gg R^{1/2} \approx 1 \) generally holds. The essential feature of the attenuated leaky Rayleigh wave is retained after this simplification.

It can be shown that this process results in the following characteristic equation,
Clearly, in the limit $R \to \infty$, the effect of the viscous boundary layer vanishes and characteristic equation for an ideal fluid layer is recovered:

\[
(k^2 + s^2)^2 - 4k^2 qs = \frac{\rho_0 q}{\rho_s r} \frac{(\omega c_t^2)^2}{(1-i)} \tanrh.
\]  

Equation (34) is exactly the same equation given in Viktorov (1968 p. 54). Furthermore, when fluid layer is so thick that it may be treated as a fluid half space, we have in the limit $h \to \infty$,

\[
(k^2 + s^2)^2 - 4k^2 qs - \frac{\rho_0 q}{\rho_s r} \frac{(\omega c_t^2)^2}{(1-i)} = \frac{(-i)}{rR^{1/2}} \frac{4\sqrt{2}k^2 qsc_t}{\omega (1-i)} \left( k^2 + k_f^2 \gamma - \frac{1}{\sqrt{Pr}} \right)
\]

\[
- 2 \frac{\rho_0 s \omega k^2}{\rho_s \sqrt{2} c_t} (1+i) - \left( \frac{\rho_0}{\rho_s} \right)^2 \frac{\omega^3 k^2}{c_t^3} (1+i)
\]

\[
+ 2k^2 \frac{\rho_0 q \omega}{\rho_s c_t} \frac{\sqrt{2}}{1-i} + \frac{\rho_0 \omega s}{\rho_s c_t} \frac{(1-i)}{\sqrt{2}} (k^2 + s^2)
\]

\[
+ \left( \frac{\rho_0}{\rho_s} \right)^2 \frac{\omega^3}{c_t^3} s q \left( \frac{1-i}{\sqrt{2}} \right) - \frac{\rho_0}{\rho_s} (k^2 + s^2) \frac{\omega^2}{c_t^2} \frac{\sqrt{2}}{1-i}
\]

\[
- (k^2 + s^2)^2 \frac{\sqrt{2} c_t}{(1-i) \omega} \left( k^2 + k_f^2 \gamma - \frac{1}{\sqrt{Pr}} \right) - \frac{\rho_0}{\rho_s} \frac{k^2 \omega}{c_t \sqrt{2}} [2sq - (s^2 + k^2)]
\]

which is the characteristic equation for attenuated leaky Rayleigh waves at the interface of a solid half space and a fluid half space. Apparently, terms on the right hand side of the above equation
are the contribution due to the thin boundary layers at the interface. In the other extreme limit of vanishing fluid layer thickness, \( \tan rh \to 0 \) and the original equation for Rayleigh wave is recovered:

\[
(k^2 + s^2)^2 - 4k^2qs = 0. \tag{36}
\]

In this case, fluid layer is replaced by a vacuum and boundary layer disappears accordingly. Finally, when the fluid density is negligibly small compared with that of the solid, \( \rho_0 / \rho_s \to 0 \) and Eq. (33) leads to,

\[
\left[ (k^2 + s^2)^2 - 4k^2qs \right] \frac{\tan rh}{rR^{1/2}} \left( \frac{2c_t}{\sqrt{\gamma \rho}} \left( k^2 + k_f^2 \frac{\gamma - 1}{\sqrt{Pr}} \right) - 1 \right) = 0. \tag{37}
\]

Consequently in this limit, in addition to the well known Rayleigh wave at the interface of a solid substrate and a vacuum (Eq. (36)), there is yet another possible mode determined by,

\[
\left[ \frac{\tan rh}{r} \left( \frac{2\nu}{\sqrt{\omega(1 - i)}} \right) \left( k^2 + k_f^2 \frac{\gamma - 1}{\sqrt{Pr}} \right) - 1 \right] = 0, \tag{38}
\]

where we note that \( R^{1/2} = c_t / \sqrt{\nu \omega} \) has been substituted. Clearly, this mode, due to the presence of thin viscous boundary layers alone, is independent of solid substrate and has no counterpart in inviscid flows. No solution exists in the limit \( \nu \to 0 \). Later, we shall examine this mode in more detail. The discussion presented so far confirms that Eq. (33) is indeed a generalized characteristic equation for the attenuated leaky Rayleigh wave propagation.

3. Wave speed and attenuation coefficient

Complex root-finding is accomplished with Muller’s method and we used IMSL Math/LIBRARY’s ZANLY (1991) subroutine for this purpose. In what follows, results for attenuated leaky Rayleigh wave speed and attenuation coefficient are presented for a solid-fluid half space and a solid half space and a fluid layer, separately.
3.1 A solid half space and a fluid half space

In this case, Eq. (35) must be solved numerically. Here, the following parameters can be varied: the Reynolds number $R$, density ratio $\rho_0/\rho_s$, ratio of the sound speed in fluid to the shear wave speed in solid $c_f/c_s$, ratio of the shear wave speed to longitudinal speed in solid $c_t/c_l$ (or the Poisson ratio), and the Prandtl number of the fluid $Pr$.

For ease of comparison with Viktorov (1967), the relative difference of the Rayleigh wave speed $c/c_R - 1$, where $c_R$ is the Rayleigh wave speed when the fluid is replaced by a vacuum, is plotted as a function of the fluid-solid density ratio for different values of the ratio of the shear wave speed to the sound speed in fluid. Furthermore, the attenuation per Rayleigh wave length is shown for comparison with Viktorov, i.e.,

$$k_2\lambda_R = 2\pi k_2 c_R^2 \omega,$$

where $k_2$ is the imaginary part of the wave number $k = \omega/c = k_1 + ik_2$.

Figure 2 is for the inviscid limit and results agree well with those given in Viktorov (1967, p. 54). Figure 2 (a) shows that the inviscid leaky Rayleigh wave speed increases monotonically with the fluid-solid density ratio. The lower the ratio of shear wave speed to sound speed $r$, the sharper the increase. Attenuation per Rayleigh wavelength is given in Figure 2 (b). It is seen that smaller $r$ corresponds to larger attenuation per wave length. In the calculation, a Poisson ratio of 0.3 is used and Viktorov has shown that the inviscid leaky Rayleigh wave speed decreases monotonically with the increase of Poisson ratio.

A Reynolds number of 900 is used in obtaining Fig. 3. It is noted from Fig. 3 (a) that for relatively smaller values of $r$ attenuated leaky Rayleigh wave speed still increases with the fluid-solid density ratio, although this increase is smaller than that in the corresponding inviscid limit. For relatively larger values of $r$, however, this trend is reversed: attenuated Rayleigh wave speed decreases with the increase of the density ratio. Note that for $r = 2.0$, there exists a minimum attenuated leaky Rayleigh wave speed at about $\rho_0/\rho_s = 0.3$. These results suggest that, due to the effect of viscous boundary layer, the propagation speed of the leaky Rayleigh waves can be smaller than its counterpart at the interface of a vacuum and a solid substrate. An increase in attenuation per Rayleigh wavelength is observed from Fig. 3 (b) when the boundary layer effect is present. The larger the fluid-solid density ratios ($\rho_0/\rho_s = 0.9$) and the smaller the ratio of shear wave speed to sound speed ($r = 1.5$), the more significant the increase.
Results for a Reynolds number of 100 are given in Fig. 4. It is now seen that more reduction in attenuated leaky Rayleigh wave speeds occurs for large ratio of shear wave speed to the sound speed. The larger the ratio of the density \( r \), the smaller the attenuated leaky Rayleigh wave speed. For relative small \( r \), the trend of increasing attenuation continues with the decrease of Reynolds number, as can be seen from Fig. 4 (b).

To appreciate the variation with Reynolds number, a fixed value of \( \rho_0/\rho_s = 0.9 \), corresponding to the most visible effect of boundary layer, is used to generate Fig. 5. Note from Fig. 5 (a) that a sharp increase with the square root of Reynolds number \( Re^{1/2} \) in the attenuated leaky Rayleigh wave speed takes place for relatively small values of \( Re^{1/2} \). Curves flatten out quickly as \( Re^{1/2} \) becomes large. The dependence of the attenuation per wavelength on \( Re^{1/2} \) is interesting (Fig. 5 (b)): for relatively small values of \( r \), it decreases with the increases of \( Re^{1/2} \); for relatively large values of \( r \), it actually increases with \( Re^{1/2} \). These results appear to suggest that there exist a threshold Reynolds number \( Rec \) beyond which viscous boundary layer stops influencing the leaky Rayleigh wave propagation. A rough estimate according to Fig. 5 gives: \( Rec = 2500 \).

### 3.2 A solid half space and a fluid layer

In this case, the attenuated leaky Rayleigh wave speed and attenuation coefficient may be obtained from solving Eq. (33) numerically.

In the following limits,

\[
\frac{h}{\lambda_f} \rightarrow n\left[\frac{2}{\sqrt{1 - k^2/k_f^2}}\right], \quad \text{where} \quad n = 0, 1, 2, \ldots, \\
\tan rh \rightarrow 0 \quad \text{and the effect of fluid layer vanishes identically according to Eq. (33). In this case, the leaky Rayleigh waves take the same values of the conventional Rayleigh wave speed for a solid substrate and a vacuum. In the other limits,}
\]

\[
\frac{h}{\lambda_f} \rightarrow (1 + 2n)\sqrt{\frac{4}{\sqrt{1 - k^2/k_f^2}}}, \quad \text{where} \quad n = 0, 1, 2, \ldots,
\]

\( \tan rh \rightarrow \infty \) and there are no solutions to Eq. (33). Namely, no leaky Rayleigh waves can propagate at these discrete values of the ratio of the fluid layer thickness to the wave length in fluid. However, since \( k \) is unknown, these values have to be determined numerically.
The data for a layer of Gaster oil on steel have been used to produce Fig. 6. Namely, we have $c_f/c_i = 0.553$, $c_f/c_f = 2.132$, $\rho/\rho_s = 0.1234$. Furthermore, $Pr = 100$ and $\gamma = 1.004$ are used in the calculation. Corresponding to $h/\lambda_f = 0$ and $h/\lambda_f = 0.58$, the fluid layer does not influence the Rayleigh wave speeds. These correspond to the first two roots in Eq. (40) for $n = 0, 1$. On the other hand, near $h/\lambda_f = 0.3$, we see a discontinuity corresponding to the first root of Eq. (41) for $n = 0$. On the right side of the discontinuity, according to Victorov (1967), there are two branches of solutions. The lower branch approaches the limit: $c_f/c_f - 1$ (Victorov 1967) and the upper branch crosses horizontal axis to repeat the curves on the left of the discontinuity. Only the upper branch is shown in Fig. 6, because these results are sufficient in explaining the effect of viscous boundary layer on the leaky Rayleigh wave propagations.

Three curves shown correspond to three different Reynolds numbers: $Re = 100, 900, \infty$. The curve corresponding to $Re = \infty$ gives the inviscid limit and it agrees well with what is presented in Victorov. It appears that decreasing Reynolds number shifts the curve towards the right slightly. Except near the discontinuity the effect of viscous boundary layer on the attenuated leaky Rayleigh wave speeds is so small that it may be considered unimportant. Close to the discontinuity, different Reynolds number may lead to different waves speeds and the precise value can be determined once $Re$ and $h/\lambda_f$ are known.

### 3.3 Solution of Equation (38)

Waves governed by Eq. (38) is in liquid layer alone because no dependence on the properties of the solid substrate is present. First, we may consider the limit $h \to \infty$ and Eq. (38) can be rearranged giving,

$$\left[ \frac{k_f(1 - i)}{2 \sqrt{2}} \frac{\sqrt{\nu}}{1 - \zeta^2} \left( \frac{\zeta^2 + \gamma - 1}{Pr} \right) - 1 \right] = 0, \text{ where } \zeta = k/k_f = c_f/c_i. \tag{42}$$

In this simple limit, the following explicit solutions can be found:

$$\zeta = \pm \frac{1 - \gamma \omega - \omega k_f}{\sqrt{2}Pr} i \pm \frac{\omega k_f}{\sqrt{2}Pr} \left( \frac{1 - \gamma \omega}{2k_f^2} \right)^{1/2}, \tag{43}$$

where the correct sign should be taken to ensure that propagating waves are attenuated in the $x$ direction. As an example, we consider waves sustained by the viscous boundary layer in air at frequency $10^{10}$ Hz. The corresponding root is found from Eq. (43) to be,
\[ \xi = 0.712 + 0.266 \, i. \]  

Consequently, the corresponding wave speed is given by,

\[ c = \frac{c_f \text{Re}(\xi)}{|\xi|^2} = \frac{331 \times 0.712}{0.5777} = 407.95 \, \text{m/s} \]  

and the corresponding attenuation per acoustic wavelength is,

\[ 2\pi \text{Im}(\xi) = 1.6713. \]

For a light fluid layer on a (heavy) solid substrate, Eq. (38) have to be solved numerically. Results for air is presented in Fig. 7 as a function of the product of layer thickness and wavenumber. Two set of data correspond to two different frequencies. Note that solutions only exist for certain frequency-fluid layer thickness combinations: no solution is found for \( 0.8 > (h\omega)/c_f > 0.5 \) at \( 10^8 \, \text{Hz} \) and for \( 0.65 > (h\omega)/c_f \) at \( 10^{10} \, \text{Hz} \). It is seen from Figure 7 (a) that wave speeds decrease with the increase of \( (h\omega)/c_f \) and are faster than the sound speed in air within the region examined. Figure 7 (b) shows that these waves are attenuated significantly, although the attenuation decreases with the increase of \( (h\omega)/c_f \). For given \( (h\omega)/c_f \) lower frequencies give lower wave speed and higher attenuation. Clearly, these waves are dispersive.

To sum up, a new set of waves is found in the limit \( \rho_0/\rho_s \rightarrow 0 \), where \( \rho_0 \) is the fluid density and \( \rho_s \) is the density of the solid substrate. Solutions in general exist when the frequency is sufficiently high. The corresponding wave speeds in air are shown to be higher than the sound speed and attenuation is significant. These waves propagate in fluid layer alone and their existence depends on the presence of the viscous boundary layer.

4. Concluding remarks

The effect of viscous boundary layer on the propagation of the leaky Rayleigh waves is investigated in this paper. It is found that attenuated leaky Rayleigh waves at the interface of a solid half space and fluid half space can in fact be smaller than the Rayleigh wave speed in the vacuum. This should be compared with the results for an idea fluid where the leaky Rayleigh wave speed is always larger than that in the vacuum. Furthermore, it is shown that, together with radiation, viscous boundary layer also contributes to the attenuation and the most significant contribution takes place when the fluid-solid density ratio becomes large. For a fluid layer on a solid substrate, the effect of viscous boundary layer is shown to be much less pronounced. For most cases, this effect
may be considered unimportant. Finally, a new wave mode is found in the limit of very light fluids or very heavy solids. This mode is entirely due to the presence of the viscous boundary layer and is independent of the solid substrate. The corresponding wave speeds in air are shown to be larger than the sound speed at high frequency and these waves are attenuated.

<table>
<thead>
<tr>
<th>Table 1: Reynolds number at increasing frequencies</th>
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<tr>
<td>$Re$</td>
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<tr>
<td>A-G1</td>
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<tr>
<td>A-G2</td>
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</table>

In the numerical solutions, we have chosen $Re = 100, 900, \infty$ to show the effect of viscous boundary layer on the propagation of the leaky Rayleigh waves. In Table 1, the corresponding Reynolds numbers for aluminum-Gaster oil (A-G1) and aluminum-glycerin (A-G2) are listed for different frequencies. Note that these are also the materials Victorov (1967) considered in his numeric calculations. Clearly, the corresponding Reynolds numbers at high frequencies are indeed small and the contribution of the viscous boundary layer effect will be important. The effect of viscous boundary layer will be more significant for other materials with lower shear velocity.

ACKNOWLEDGEMENTS

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APPENDIX

In this appendix, we derive the characteristic equation for the attenuated leaky Rayleigh wave. The following three conditions are enforced to determine the three unknown constants of integrations: $A$, $B$ and $D$.

$$
\lim_{\nu \to 0} v^{(i)}(z) = \lim_{z \to 0} v^{(i)}(z),
\quad (1.a)
$$

$$
\lim_{z \to 0^-} \sigma_{zz} = \lim_{z \to 0^+} \left\{ -p + 2\rho_0 v \frac{\partial}{\partial z} v^{(i)} - \frac{2\rho_0 v}{3} \left( \frac{\partial u^{(i)}}{\partial x} + \frac{\partial v^{(i)}}{\partial z} \right) \right\},
\quad (2.a)
$$

and

$$
\lim_{z \to 0^-} \sigma_{xz} = \lim_{z \to 0^+} \left\{ \rho_0 v \left( \frac{\partial}{\partial x} v^{(i)} + \frac{\partial}{\partial z} u^{(i)} \right) \right\}.
\quad (3.a)
$$

Equation (1.a) is the matching condition for velocities in the $z$ direction in the boundary layer and in the fluid layer away from the interface. Equations (2.a) and (3.a) are stress continuity conditions at the interface. Using Eqs. (7), (10), (14), (21), (22), (23), and (30), we find,

$$
A \left( q + \sqrt{\frac{2\nu k^2}{\omega}} \right) - B \left( ik + \sqrt{\frac{2\nu k^2}{\omega}} \right) - D \left[ r + \sqrt{\frac{2\nu k^2}{\omega}} \tan rh + \frac{2\nu k^2}{\omega} \frac{\tan rh}{(1-i)C_p} \right] = 0,
\quad (4.a)
$$

$$
A \left( \tilde{\lambda} \left( q^2 - k^2 \right) - 2\tilde{\mu} q^2 + \rho_0 v i k^2 \omega \right) + B \left( \rho_0 v k s \omega - 2\tilde{\mu} ik s \right)
+ D \tan rh \left( \rho_0 \omega^2 - 2\rho_0 v i \frac{\gamma}{c_f^2} \omega^3 \right) = 0,
\quad (5.a)
$$

$$
A \left[ \rho_0 \omega k (1-i) \sqrt{\frac{\omega v}{2}} - \rho_0 v k q \omega + 2\tilde{\mu} ik q \right] + B \left[ \tilde{\mu} \left( k^2 + s^2 \right) - \rho_0 \omega s (1+i) \sqrt{\frac{\omega v}{2}} - ik \omega \right]
- D \rho_0 \omega k (1-i) \sqrt{\frac{\omega v}{2}} = 0.
\quad (6.a)
$$

The following well-known relations may be used to simplify the above equations,

$$
\tilde{\lambda} + 2\tilde{\mu} = \rho_s c_i^2, \quad \tilde{\lambda} = \rho_s c_l^2 - 2\rho_s c_i^2,
\quad (7.a)
$$

where $\rho_s$ is the density of the solid, $c_i$ is the shear wave speed and $c_l$ is the longitudinal wave speed in solid. Furthermore, a Reynolds number based on the shear speed of the solid is introduced as follows,

$$
R = \frac{c_i^2}{\omega v}.
\quad (8.a)
$$

These lead to,
Clearly, the condition for non-trivial solution is satisfied by setting the determinant to zero and this gives us a characteristic equation for the determination of attenuated leaky Rayleigh waves speed and the attenuation coefficient, i.e., Eq. (31).
Figure captions:

Figure 1. Geometry for the propagation of leaky Rayleigh wave at the interface between a fluid layer and a solid substrate. The solid substrate is assumed to be semi-infinite and the thickness of the fluid layer equals $h$. Results for a fluid half-space is obtained by taking $h \to \infty$.

Figure 2. Leaky Rayleigh wave speeds (a) and attenuation per wavelength (b) as a function of the fluid-solid density ratio. Each curve corresponds to a specific ratio of shear wave speed in solid to sound speed in fluid. In the inviscid limit, $Re = \infty$.

Figure 3. Attenuated leaky Rayleigh wave speeds (a) and attenuation per wavelength (b) for $Re = 900$ as a function of the fluid-solid density ratio.

Figure 4. Attenuated leaky Rayleigh wave speeds (a) and attenuation per wavelength (b) for $Re = 100$ as a function of the fluid-solid density ratio.

Figure 5. Attenuated leaky Rayleigh wave speeds (a) and attenuation per wavelength (b) as a function of the Reynolds number. Each curve corresponds to a specific ratio of shear wave speed in solid to sound speed in fluid.

Figure 6. Attenuated leaky Rayleigh wave for fluid layers on a solid substrate as a function of the product of the layer thickness and wavenumber. Each curve corresponds to a different Reynolds number. The effect of Reynolds number is insignificant away from the discontinuity.

Figure 7. Real and imaginary part of the ratio of sound speed to wave speed $c_f/c$. The wave speed is seen to be faster than the sound speed and attenuation is significant. Solutions are available only for appropriate combinations of layer thickness and frequencies.
Figure 1.
Figure 2.
Figure 3.
Figure 4.
Figure 5.
\[ \frac{c}{c_R} - 1 \]

\[ \frac{c_t}{c_R} - 1 \]

\[ Re = \infty, 900 \]
\[ Re = 100 \]

Figure 6.
Figure 7.
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<th>Title</th>
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