SCATTERING FROM A THIN RANDOM FLUID LAYER

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ABSTRACT

The problem of a plane wave obliquely incident from a homogeneous ideal fluid upon a random fluid layer is considered. Expressions for the diffusely backscattered intensity is derived. The thin layer is taken to have properties which vary only in the two in-plane directions. A first Born approximation is used and the average intensity of the incoherent part of the backscattered wave is found to be proportional to the two-dimensional spatial Fourier transform of the auto- and cross-covariance functions of the layer. Within the confines of the validity of the first Born approximation, therefore, the inverse scattering problem may be straightforward, provided the necessary experimental data of the incoherent wave can be obtained. Also given in this study is a sufficient condition to estimate the range in which the second order term in the Born series can be neglected.

Introduction

Wave propagation in randomly inhomogeneous media has been investigated extensively and continuously for over three decades. The involved problems include electromagnetic, acoustic and elastic waves in such random media as fog, clouds, rain, biological tissues, slurries, composites and polycrystalline materials. Notable references include work by Bergmann,[1] Waterman and Truell,[2] Twersky,[3] Ishimaru,[4] Frisch,[5] Varadan, *et al*,[6] Sobczyk,[7] etc., in which the major interest has been the prediction of the effective propagation constants for the coherent, or ensemble average, waves. This literature has concluded that the effective propagation constants are, in the dilute limit, dependent upon the underlying microstructure predominantly through the forward scattering amplitude of the individual scatterers. In the non-dilute case, the effective propagation constants have been found to depend upon the microstructure in a very complicated way. In neither case are general microstructural features simply manifested; the inverse problem is not easy.

A notable exception to the general complexity of the inverse scattering problems is provided by the case in which the first Born approximation is valid. If the scattering is weak enough that most rays scatter no more than once (single scattering approximation), the scattered field may be found [7] to be proportional to the spatial Fourier transforms of the scatterer's properties.

The scalar wave equation for a thick random one-dimensional slab, with its refraction index changing only in the thickness direction, has been studied extensively by many authors.[8–16] In these investigations, the plane incident wave is usually assumed to be normally incident onto the slab, and a stationary Markov process is also commonly used to describe the weak random variation of the refraction index of the slab, whose mean value is the same as that of the background fluid. Analytical and/or numerical results are given for the mean (coherent) reflection and transmission coefficients,[9, 11, 13, 15, 16] and for the mean power (or equivalently, intensity) reflection and transmission coefficients.[8, 10–13, 15, 16]

Problems of scattering from a two-dimensional or three-dimensional random medium are quite different from those of 1-D. Solutions for scattering from a 3-D random medium occupying a small finite volume (with properties having mean values equal to those of the background homogeneous medium) or a whole space can be found in the literature (see, for example, Frisch,[5] and Sobczyk[7]). But no solutions are available in the literature, to the best knowledge of the authors, for the case of a semi-infinite random medium, a half-space or an infinite layer with transverse heterogeneity.

In this paper we seek the field scattered from a thin ideal fluid layer whose material density and modulus vary randomly in the in-plane directions. This problem is conceived as a simpler version of problems entailing thin solid layers and solid half-spaces, which are usually of greater practical interest in such field as materials characterization. We study this simpler problem first before attempting the generally more complicated solid media cases. Coherent reflections and transmissions through layers have been discussed before, by Achenbach and Li,[17] for the case of an array of cracks, and by Twersky,[18] for screens of discrete scatterers. Here we focus on the incoherent part of the scattered field from a continuously varying layer of inhomogeneous refraction index. We find in section I and II that, within the confines of the validity of the first Born approximation, the scattered field is indeed proportional to the Fourier transforms of the layer properties. The proportionality is, furthermore, found to vary only slowly with such parameters as the incident and listening (detecting) directions, leading one to hypothesize the existence of a relatively simple inverse scattering algorithm for laboratory data. We also give a sufficient condition, in section III, for the estimation of the parameter range in which the first Born approximation may be valid. The last section of this paper is concerned with the more general case in which both the refraction index and the density of the layer are varying randomly in the in-plane directions.

I. Fluid Layer With Spatially Varying Refraction Index

Consider an ideal, homogeneous fluid of density ρ_0 and propagation constant $k_0 = \omega / c_0$, where c_0 is the sound speed in the fluid full space and ω is the angular frequency of the acoustic wave. A thin ideal fluid layer of thickness h, density $\rho_L(1+\varepsilon\nu)$ and propagation parameter $k_L\sqrt{1+\varepsilon\mu}$, where $k_L = \omega / c_L$ and $\varepsilon \ll 1$, lies between $z = 0$ and $z = h$ (see Fig.1). ρ_L and k_L^2 are the ensemble average values of the density and the square of the propagation parameter. $\nu = \nu(x, y, z)$ and $\mu = \mu(x, y, z)$ are Gaussian random variations with zero mean. We assume in this section that $\nu = 0$, i.e. that the density in the layer is homogeneous.

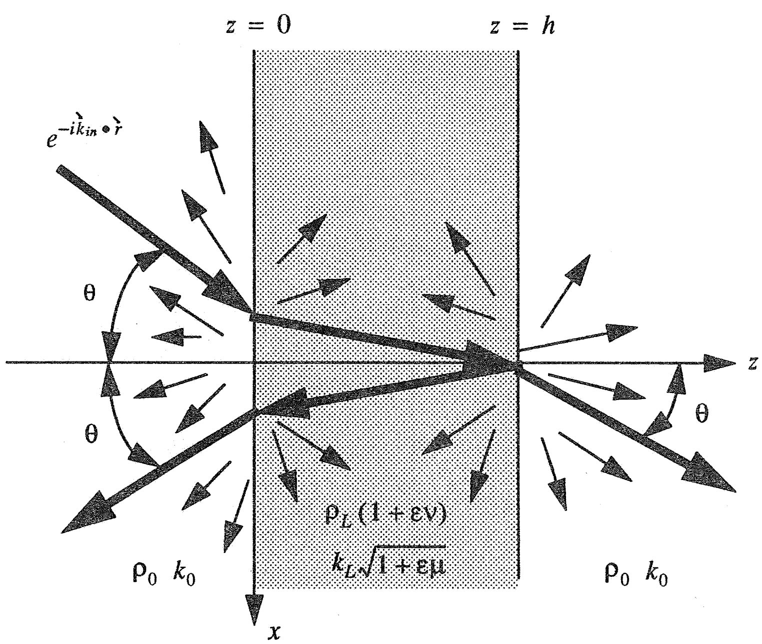

Fig.1: Graphic illustration of the scattering problem from a thin fluid layer

The correlation distance in the random layer is assumed to be of the same order as the acoustic wavelengths. It will be seen in section III that the Born approximation may be valid for layers whose thickness to wavelength ratios are small. We are therefore led, upon speculating that the correlations are approximately isotropic, to neglect variations in layer properties in the direction perpendicular to the plane of the layer: $\nu = \nu(x, y)$ and $\mu = \mu(x, y)$ are independent of z.

The time harmonic wave field (acoustic pressure P in the fluids) satisfies the reduced wave equations (we suppress the factor $e^{i\omega t}$ in all the following)

$$\begin{cases} \nabla^2 P + k_0^2 P = 0, & z<0 \ \ or \ \ h<z, \\ \nabla^2 P + k_L^2 P = -\varepsilon k_L^2 \mu(x, y) P, & 0<z<h. \end{cases} \tag{1}$$

The continuity conditions at the interfaces, $z = 0$ and $z = h$, are

$$\begin{cases} [P]_{z=0} = [P]_{z=h} = 0, \\ \left[\frac{1}{\rho\omega^2}\frac{\partial}{\partial z}P\right]_{z=0} = \left[\frac{1}{\rho\omega^2}\frac{\partial}{\partial z}P\right]_{z=h} = 0, \end{cases} \tag{2}$$

where the square bracket notation signifies a discontinuity: $[f]_{z=z_0} = f|_{z=z_0^+} - f|_{z=z_0^-}$.

The solution to Eqs. (1) and (2) may be expanded in a Neumann, or Born, series:

$P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + O(\varepsilon^3)$,

where P_0, P_1 and P_2 are solutions of the differential equations

$$\begin{cases} \nabla^2 P_0 + k_0^2 P_0 = 0, & z<0 \ \ or \ \ h<z, \\ \nabla^2 P_0 + k_L^2 P_0 = 0, & 0<z<h, \end{cases} \tag{3}$$

$$\begin{cases} \nabla^2 P_1 + k_0^2 P_1 = 0, & z<0 \ \ or \ \ h<z, \\ \nabla^2 P_1 + k_L^2 P_1 = -k_L^2 \mu P_0, & 0<z<h, \end{cases} \tag{4}$$

$$\begin{cases} \nabla^2 P_2 + k_0^2 P_2 = 0, & z<0 \ \ or \ \ h<z, \\ \nabla^2 P_2 + k_L^2 P_2 = -k_L^2 \mu P_1, & 0<z<h. \end{cases} \tag{5}$$

The continuity conditions of Eq. (2) are imposed at each order, and appropriate radiation boundary conditions are also imposed at $z \to \pm\infty$. A unit plane wave is incident from the homogeneous fluid half-space $z<0$ onto the layer. Without loss of generality, the incident wave may be assumed to be propagating in the x-z plane with an angle of incidence θ between its propagation direction $\vec{k}_{in} = \hat{x}k_0\sin\theta + \hat{z}k_0\cos\theta$ and z-axis (Fig.1). P_0, which includes the incident plane wave, is given by

$$P_0(x,y,z) = \begin{cases} e^{-i(k_0 x\sin\theta + k_0 z\cos\theta)} + A_0 e^{-i(k_0 x\sin\theta - k_0 z\cos\theta)}, & z<0, \\ [B_0 e^{-ik_L z\cos\phi} + C_0 e^{ik_L z\cos\phi}]\, e^{-ik_L x\sin\phi}, & 0<z<h, \\ D_0 e^{-i(k_0 x\sin\theta + k_0 z\cos\theta)}, & h<z, \end{cases} \tag{6}$$

where $k_0\sin\theta = k_L\sin\phi$ follows from Snell's Law. The coefficients can be determined from the continuity conditions at $z = 0$ and $z = h$ (Eq. (2) with P replaced by P_0) and found to be

$$\begin{Bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{Bmatrix} = \frac{1}{\Delta_0} \begin{Bmatrix} (1-\alpha_0^2)\,(e^{-ik_L h\cos\phi} - e^{ik_L h\cos\phi}) \\ -2\,(1+\alpha_0)\,e^{ik_L h\cos\phi} \\ 2\,(1-\alpha_0)\,e^{-ik_L h\cos\phi} \\ -4\alpha_0 e^{ik_0 h\cos\theta} \end{Bmatrix}, \tag{7}$$

where we have defined $\alpha_0 = (\rho_0 k_L\cos\phi)/(\rho_L k_0\cos\theta)$ and $\Delta_0 = (1-\alpha_0)^2 e^{-ik_L h\cos\phi} - (1+\alpha_0)^2 e^{ik_L h\cos\phi}$. This concludes the analysis of the zeroth order solution P_0.

Upon defining the two-dimensional spatial Fourier transform pair as

$$\mathcal{FT}(f) = \tilde{f}(k_x,k_y) = \int_{-\infty}^{\infty} dxdy\; f(x,y)\,e^{i(k_x x + k_y y)}, \qquad f(x,y) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_x dk_y\; \tilde{f}(k_x,k_y)\,e^{-i(k_x x + k_y y)}, \tag{8}$$

and taking the Fourier transform of Eq. (4), we obtain The ordinary differential equations in z for the first order Born approximation term P_1:

$$\begin{cases} \dfrac{d^2}{dz^2}\tilde{P}_1 + \eta_0^2\tilde{P}_1 = 0, & z<0 \quad or \quad z>h, \\ \dfrac{d^2}{dz^2}\tilde{P}_1 + \eta_L^2\tilde{P}_1 = -k_L^2\,\mathcal{FT}(\mu P_0), & 0<z<h, \end{cases} \tag{9}$$

where $\eta_0^2 = k_0^2 - k_x^2 - k_y^2$ and $\eta_L^2 = k_L^2 - k_x^2 - k_y^2$. We now define

$$\tilde{G}_L(k_x,k_y;x_l,y_l) = (1/2i\eta_L)\,e^{i(k_x x_l + k_y y_l) - i\eta_L|z-z_l|}, \tag{10}$$

which is the spatially Fourier transformed Green's function for the equation

$$\nabla^2 G_L + k_L^2 G_L = -\delta(x-x_l)\,\delta(y-y_l)\,\delta(z-z_l), \qquad 0<(z,z_l)<h. \tag{11}$$

Note that the function G_L so chosen is only a particular solution of Eq. (11) inside the layer. It has not been made to satisfy continuity or boundary conditions at the two interfaces. It is not necessary to use a Green's function which satisfies the continuity Eq. (2). It suffices to use a Green's function which satisfies the PDE, and to impose boundary and continuity conditions on the total pressure field P_1. A particular solution of Eq. (4) (for $0<z<h$) may then be written as:

$$J_1(x, y, z) = k_L^2 \int_0^h dz_l [G_L \otimes (\mu P_0)], \tag{12}$$

where " $\otimes$ " represents a spatial convolution defined by

$$f(x, y) \otimes g(x, y) = \int_{-\infty}^{\infty} dx_l dy_l\ f(x_l, y_l)\, g(x-x_l, y-y_l) = g(x, y) \otimes f(x, y).$$

The full solution is then of the form

$$\tilde{P}_1(k_x, k_y; z) = \begin{cases} A_1 e^{i\eta_0 z}, & z<0, \\ B_1 e^{-i\eta_L z} + C_1 e^{i\eta_L z} + J_{1\mu}(k_x, k_y; z)\, \tilde{\mu}(k_x - k_0 \sin\theta, k_y), & 0<z<h, \\ D_1 e^{-i\eta_0 z}, & h<z, \end{cases} \tag{13}$$

where $J_{1\mu}$ is defined by $J_{1\mu}(k_x, k_y; z)\, \tilde{\mu}(k_x - k_0 \sin\theta, k_y) = \mathcal{FT}[J_1(x, y, z)]$ (see Appendix A).

The terms with factors A_1, B_1, C_1 and D_1 form the homogeneous part of the total solution. The constants are determined by invocation of the continuity conditions at $z = 0$ and $z = h$ (the Fourier transform of Eq. (2) with P replaced by P_1). The coefficient A_1 determines the field scattered back into the half-space $z<0$ from which the incident wave comes and is therefore of particular interest here. Appendix A gives the details of the solution for A_1. The result is

$$\{A_1, B_1, C_1, D_1\} = \{A_{1\mu}, B_{1\mu}, C_{1\mu}, D_{1\mu}\}\ \tilde{\mu}(k_x - k_0 \sin\theta, k_y), \tag{14}$$

where

$$\begin{cases} A_{1\mu}(k_x, k_y; k_0, k_L; \phi, h) = -[2\alpha/\Delta]\ \{(1+\alpha)\, e^{i\eta_L h}\Gamma_1 - (1-\alpha)\,\Gamma_2\}, \\ B_{1\mu}(k_x, k_y; k_0, k_L; \phi, h) = [(1-\alpha)/\Delta]\ \{(1+\alpha)\, e^{i\eta_L h}\Gamma_1 - (1-\alpha)\,\Gamma_2\}, \\ C_{1\mu}(k_x, k_y; k_0, k_L; \phi, h) = -[(1-\alpha)/\Delta]\ \{(1-\alpha)\, e^{-i\eta_L h}\Gamma_1 - (1+\alpha)\,\Gamma_2\}, \end{cases} \tag{14a}$$

and

$$\begin{cases} \Gamma_1 = \dfrac{k_L^2}{2\eta_L\gamma_1\gamma_2}\{B_0\gamma_2(e^{-i\gamma_1 h}-1)+C_0\gamma_1(e^{-i\gamma_2 h}-1)\}, \\ \Gamma_2 = \dfrac{k_L^2}{2\eta_L\gamma_1\gamma_2}\{B_0\gamma_1(e^{-i\eta_L h}-e^{-ik_L h\cos\phi})+C_0\gamma_2(e^{-i\eta_L h}-e^{ik_L h\cos\phi})\}, \\ \alpha=(\rho_0\eta_L)/(\rho_L\eta_0), \qquad \Delta=(1-\alpha)^2 e^{-i\eta_L h}-(1+\alpha)^2 e^{i\eta_L h}, \qquad \gamma_{1,2}=\eta_L\pm k_L\cos\phi. \end{cases} \tag{14b}$$

$D_{1\mu}$ may be obtained similarly and the first order field beyond the layer constructed from it.

The first order backscattered field in real space is given by an inverse transform of $\tilde{P}_1$ as

$$P_1(x,y,z) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_x dk_y A_1(k_x,k_y)\, e^{-i\vec{k}\bullet\vec{r}}, \tag{15}$$

where $\vec{k} = k_x\hat{x}+k_y\hat{y}+(-\eta_0)\hat{z}$. Clearly, A_1 determines the amplitude of the first order backscattered (propagating towards negative z-direction) acoustic pressure field in direction $\vec{k}$. The total first order incoherently scattered acoustic pressure $P_1(x,y,z<0)$ is thus a superposition of plane waves $\tilde{P}_1$ in all possible directions $\vec{k}$.

II. Properties of the 1st Order Solution and Application to Inverse Problem

Consider a receiver with a narrow beam pattern placed in the far field at $z<0$, with its beam oriented towards the scatterer in a "listening direction" $\hat{n}$. We denote by T the receiver's sensitivity which is a function of the angle between the direction $\hat{k} = \vec{k}/k_0$ of the plane wave component $\tilde{P}_1(\vec{k})$ and the receiver beam axis $\hat{n}$, i.e., a function of $\hat{n}\bullet\hat{k}$. The received signal is then

$$V(\hat{n}) = \int_{-\infty}^{\infty} dk'_x\, dk'_y\, \tilde{P}(k'_x, k'_y)\, T(\hat{n}\bullet\hat{k}'). \tag{16}$$

The ensemble averaged "intensity" in the signal is then given by

$$<|V(\hat{n})|^2> = \int_{-\infty}^{\infty} dk'_x\, dk'_y \int_{-\infty}^{\infty} dk''_x\, dk''_y\, <\tilde{P}(k'_x,k'_y)\tilde{P}^*(k''_x,k''_y)>T(\hat{n}\bullet\hat{k}')\,T^*(\hat{n}\bullet\hat{k}'') \;. \tag{17}$$

Because $<\mu> = 0$, all terms of order ε vanish. Also because the ensemble average of any odd order moments of a centered Gaussian process is zero, we have $<|\tilde{P}|^2> = |\tilde{P}_0|^2 + \varepsilon^2(<|\tilde{P}_1|^2> + \tilde{P}_0<\tilde{P}_2^*> + <\tilde{P}_2>\tilde{P}_0^*) + O(\varepsilon^4)$. We show later in the next section that the two terms containing $<\tilde{P}_2>$ vanish unless $\hat{n}$ is in the specular reflection direction (the coherent direction). Thus the leading order part of the incoherent intensity is due to the $<\tilde{P}_1\tilde{P}_1^*>$ term.

Next, we define R_μ as the autocovariance function of the random variation μ, and use the fact that the spectral variations are delta correlated as[19]

$$\begin{cases} <\mu(x_1, y_1)\,\mu(x_2, y_2)> = R_\mu(x_1 - x_2, y_1 - y_2), \\ <\tilde{\mu}(k_x, k_y)\,\tilde{\mu}^*(k'_x, k'_y)> = (2\pi)^2\tilde{R}_\mu(k_x, k_y)\,\delta(k_x - k'_x)\,\delta(k_y - k'_y), \end{cases} \tag{18}$$

where " * " represents the complex conjugate. This allows us to express the ensemble average incoherent signal intensity as

$$<|V(\hat{n})|^2> = (2\pi\varepsilon)^2\int_{-\infty}^{\infty} dk'_x\, dk'_y\, |A_{1\mu}(k'_x, k'_y)|^2 |T(\hat{n}\bullet\hat{k}')|^2 \tilde{R}_\mu(k'_x - k_0\sin\theta, k'_y)\, e^{-2(Im\eta_0)z}. \tag{19}$$

$T(\hat{n}\bullet\hat{k}')$ is related to the directivity of the transducer (usually approximately analyzed as a vibrating piston), and is sharply peaked in the direction $\hat{n}$. Thus one may approximate

$$|T(\hat{n}\bullet\hat{k}')|^2 \approx T_0^2\,\delta(\hat{n}\bullet\hat{k}' - 1),$$

and then substitute this into Eq. (19) to obtain (see Appendix B for details)

$$<|V(\hat{n})|^2> \approx (2\pi)^3 (k_0 T_0 \varepsilon)^2 |A_{1\mu}(k_0 n_x, k_0 n_y)|^2 \tilde{R}_\mu(k_0 n_x - k_0\sin\theta, k_0 n_y)\, n_z, \tag{20}$$

or simply

$$<|V(\hat{n})|^2> \propto [|A_{1\mu}(k_0 n_x, k_0 n_y)|^2 n_z]\,[\varepsilon^2\tilde{R}_\mu(k_0 n_x - k_0\sin\theta, k_0 n_y)]. \tag{21}$$

Equation (21) is the chief result of this section. It shows the relationship between the Fourier transform of the autocovariance function R_μ (usually called the spatial spectral density of μ, which characterizes the second order statistics of the material property of the layer) and the received signal intensity. The simple proportionality is not unanticipated. Similar relations have been observed, and the first Born approximations responsible for this validated, by numerous authors. Here we have derived, for the first time, this relationship for the case of a thin layer. Owing to the simple algebraic relation, the inverse scattering problem for the characterization of the heterogeneity of the material properties is relatively simple, if the experimental data on $\langle |V(\hat{n})|^2_{inc} \rangle$ can be measured in the laboratory. We also notice that, in order to use expression (20) or (21) to determine the spectral density $\tilde{R}_\mu$, one would want the coefficient of proportion, $|A_{1\mu}|$, to be a smooth function of incident and listening directions, so that when those parameters change, the changes in the signal intensity will chiefly reflect the changes in the spectral density $\tilde{R}_\mu$. It is with this issue that the remainder of this section is concerned.

It is difficult to analyze $|A_{1\mu}|$ directly from Eqs. (14a,b). We therefore plot it in Figures 2-5 and enumerate the properties below. In the plots, we use $\theta_{in} = \theta$ as the incident angle to z-axis, and $\theta_{out} = \theta_n$ as the angle between the listening direction $\hat{n}$ and z-axis.

1) We first note that, Γ_1, Γ_2 and $A_{1\mu}$ are functions of $k_x^2 + k_y^2 = k_0^2 \sin^2\theta_{out}$ and do not depend on k_x or k_y separately. This means that any variation in the incoherent acoustic intensity in directions along the generators of a conic, whose axis is in z direction, will be directly related to the spatial Fourier transform of the autocovariance function, $\tilde{R}_\mu$.

2) The quantity $|A_{1\mu}|\sqrt{\cos\theta_{out}}$, which is proportional to $|V(\hat{n})|$ (see Eq. (21)), is almost independent of the incident angle θ_{in} (see Fig.2 and 3), except for some cases where $k_L/k_0 = c_0/c_L > 1$ and $\rho_L c_L/\rho_0 c_0 \le 1.0$ (see Fig.4).

3) $|A_{1\mu}|\sqrt{\cos\theta_{out}}$ changes very little and very smoothly with listening directions $\hat{n}$ for the cases where the average acoustic impedance ratio $\rho_L c_L/\rho_0 c_0 \ge 1$, except near $\theta_{out} = 90^o$, i.e., listening

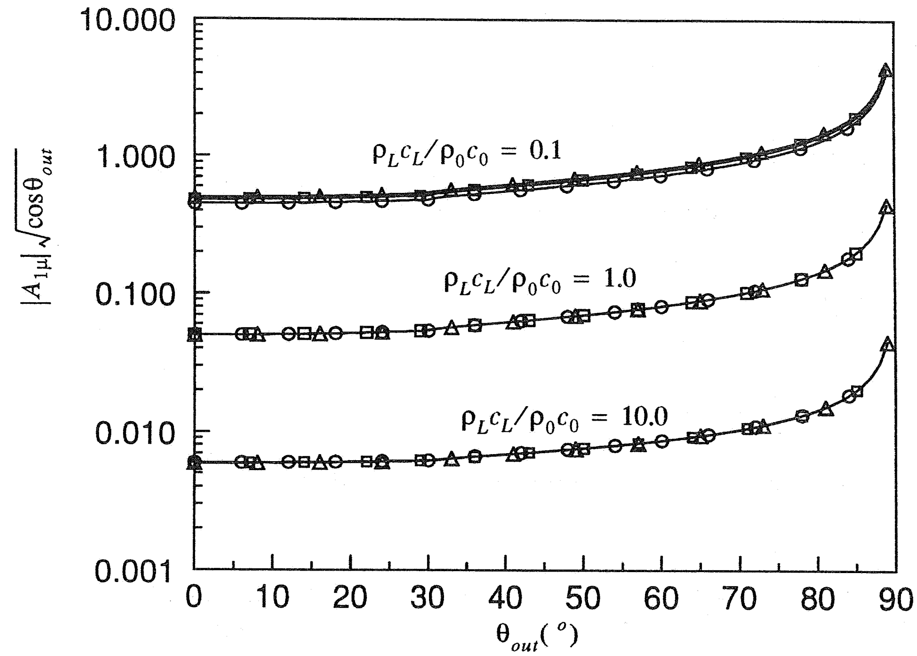

Fig.2: $|A_{1\mu}|\sqrt{\cos\theta_{out}}$ as a function of the listening angle. $k_L h = 0.1$, $k_L/k_0 = 0.5$. Normal incidence ($\theta_{in} = 0^o$, circle) and oblique incidence ($\theta_{in} = 20^o$, square; $\theta_{in} = 30^o$, triangle).

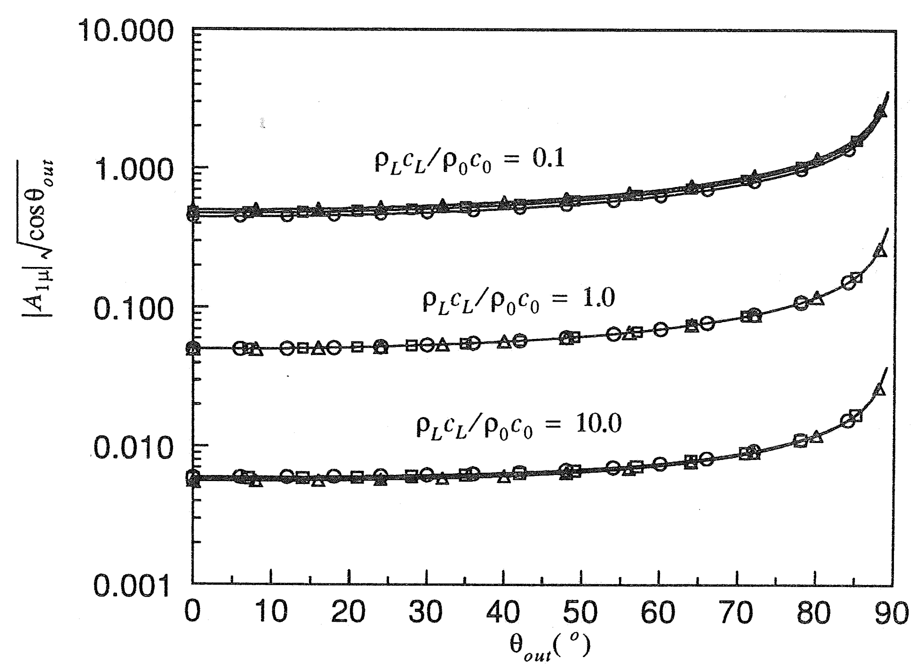

Fig.3: $|A_{1\mu}|\sqrt{\cos\theta_{out}}$ as a function of the listening angle. $k_L h = 0.1$, $k_L/k_0 = 1.0$. Normal incidence ($\theta_{in} = 0^o$, circle) and oblique incidence ($\theta_{in} = 50^o$, square; $\theta_{in} = 89^o$, triangle).form of) the autocovariance function $\tilde{R}_\mu$

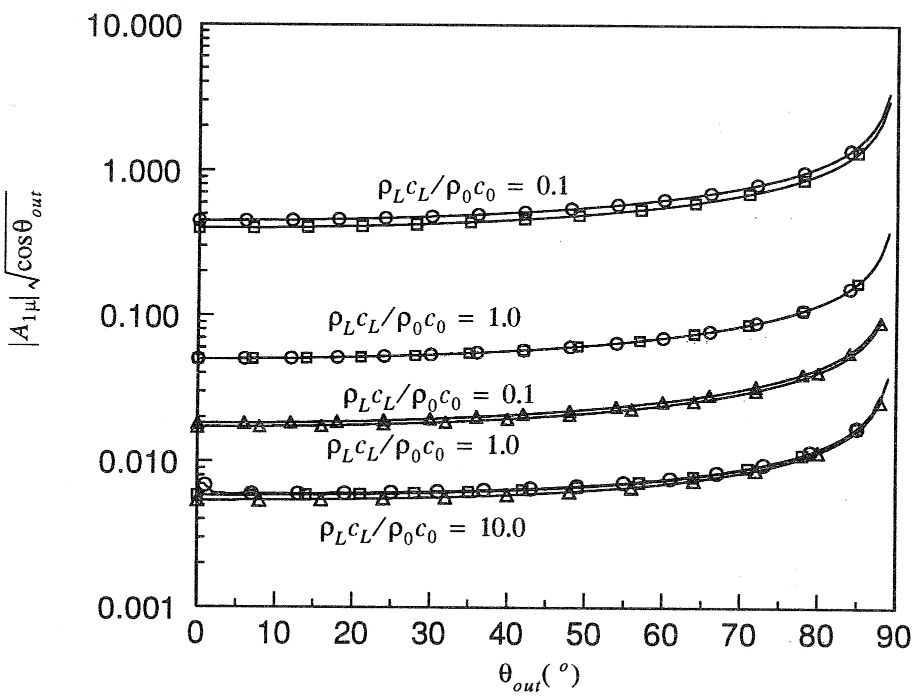

Fig.4: $|A_{1\mu}|\sqrt{\cos\theta_{out}}$ as a function of the listening angle. $k_L h = 0.1$, $k_L/k_0 = 5.0$. Normal incidence ($\theta_{in} = 0^o$, circle) and oblique incidence ($\theta_{in} = 50^o$, square; $\theta_{in} = 89^o$, triangle).

in a direction parallel to the interface. Even for the cases where $\rho_L c_L/\rho_0 c_0 < 1$, $|A_{1\mu}|\sqrt{\cos\theta_{out}}$ changes fairly smoothly.

4) In figure 5, we can see that although the magnitude of $A_{1\mu}$ as a function of the acoustic impedance ratio $\rho_L c_L/\rho_0 c_0$, is very complicated, we can approximate it as $|A_{1\mu}| = a(\rho_L c_L/\rho_0 c_0)^{-b}$, where $a > 0$ depends on $k_L h, \theta_{in}, \theta_{out}, k_L/k_0 = c_0/c_L$ and $b > 0$ depends on $k_L h, \theta_{in}$ only. All the slopes b are about $b \approx 0.05$ for $k_L h = 0.1$, $\theta_{in} = 20^0$ and different θ_{out}.

The above properties of $A_{1\mu}$ guarantee that, by changing the incident and/or listening directions within a certain proper range, the changes in the received signal intensity will reflect mainly the corresponding changes in the spectral density $\tilde{R}_\mu$ of the layer medium.

In the above calculations, we only considered the range of $k^2 = k_x^2 + k_y^2 \le k_0^2$. Outside this range, η_0 is purely imaginary and $\tilde{P}_1$ evanesces for $z < 0$ or $z > h$. There is thus no contribution in the far field.

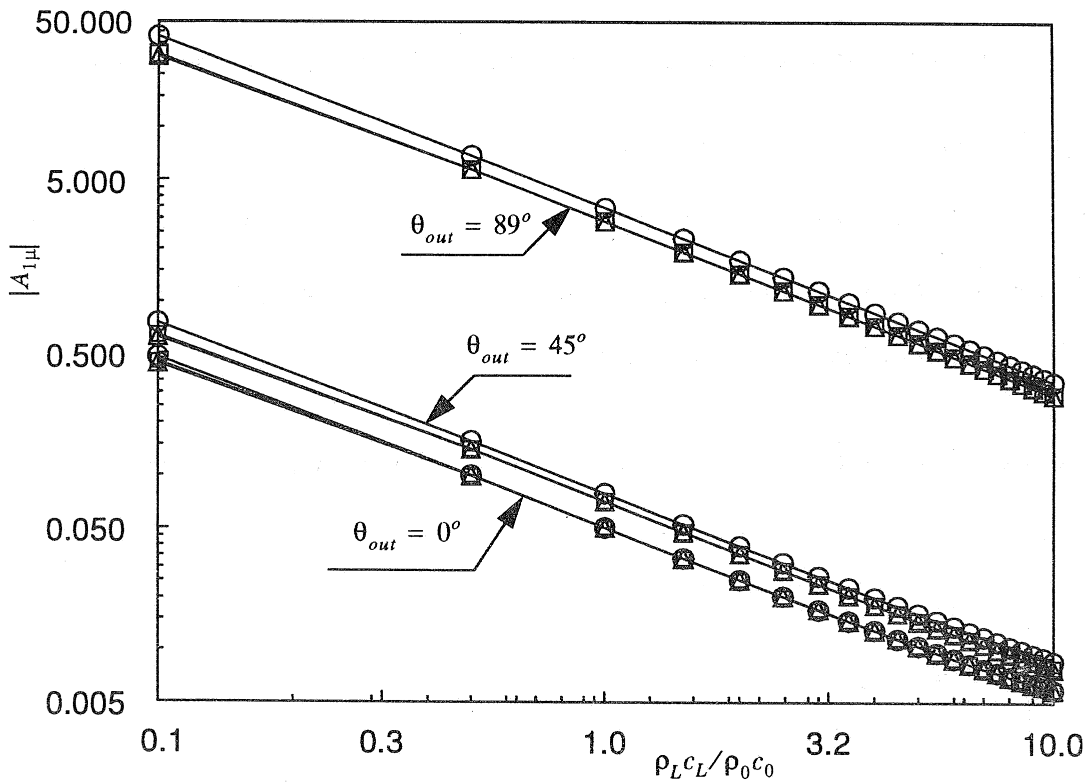

Fig.5: $|A_{1\mu}|$ as a function of the acoustic impedance ratio with the sound speed ratio fixed (log-log plot). $k_L h = 0.1$, $\theta_{in} = 20^o$. And for $c_0/c_L = k_L/k_0 = 0.5$ (circle); $c_0/c_L = 1.0$ (square); $c_0/c_L = 5.0$ (triangle).

III. Higher Order Consideration

The utility of Eq. (21) depends upon the validity of the first Born (single scattering) approximation. Here we ask under what conditions the higher order terms in the Neumann series may be neglected. To do this, we analyze the averaged acoustic intensity to higher order in ε

$$<|P|^2> = |P_0|^2 + \varepsilon^2 (P_0<P_2^*> + <P_2>P_0^* + <|P_1|^2>) + \varepsilon^4 (P_0<P_4^*> + <P_4>P_0^* + <P_1P_3^*> + <P_3P_1^*> + <|P_2|^2>) + O(\varepsilon^6), \tag{22}$$

and compare the $O(\varepsilon^4)$ order terms with the $O(\varepsilon^2)$ order terms. Since, for a centered Gaussian random process μ, the odd order statistics such as $<\mu>$, $<\mu\mu\mu>$ etc. are zero, all the terms of odd order ε vanish in the above equation. The averaged field $<P>$ vanishes at all orders if the listening direction $\hat{n}$ is other than the directions for coherent transmission and specular reflection. Thus,

$$<|P|^2> = \varepsilon^2<|P_1|^2> + \varepsilon^4 (<P_1P_3^*> + <P_3P_1^*> + <|P_2|^2>) + O(\varepsilon^6), \tag{23}$$

if we restrict our attention to the incoherent directions. Furthermore, if our attention is confined to the limiting case where $k_L h \ll 1$, then the $P_1P_3^*$ terms are found to be negligible compared to $|P_2|^2$ (see Appendix E). Thus one needs to evaluate only P_2.

The solution of Eq. (5) for the second order scattered acoustic pressure is similar to that for $\tilde{P}_1$ and given by

$$\tilde{P}_2(k_x, k_y; z) = \begin{cases} A_2 e^{i\eta_0 z}, & z < 0, \\ B_2 e^{-i\eta_L z} + C_2 e^{i\eta_L z} + \tilde{J}_2, & 0 < z < h, \\ D_2 e^{-i\eta_0 z}, & h < z, \end{cases} \tag{24}$$

where

$$\tilde{J}_2 = k_L^2 \int_0^h dz_l \mathcal{FT}\{G_L \otimes (\mu P_1)\}. \tag{25}$$

As in section I, we find that (see Appendix C)

$$\begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk'_x\, dk'_y\, \tilde{\mu}(k'_x - k_0 \sin\theta, k'_y)\, \tilde{\mu}(k_x - k'_x, k_y - k'_y) \begin{Bmatrix} A_{2\mu}(k_x, k_y; k'_x, k'_y) \\ B_{2\mu}(k_x, k_y; k'_x, k'_y) \\ C_{2\mu}(k_x, k_y; k'_x, k'_y) \\ D_{2\mu}(k_x, k_y; k'_x, k'_y) \end{Bmatrix}, \tag{26}$$

with

$$\begin{cases} A_{2\mu}(k_x, k_y; k'_x, k'_y) = -\dfrac{2\alpha}{\Delta}[(1+\alpha)e^{i\eta_L h}\Gamma_3 - (1-\alpha)\Gamma_4], \\ B_{2\mu}(k_x, k_y; k'_x, k'_y) = \dfrac{1-\alpha}{\Delta}[(1+\alpha)e^{i\eta_L h}\Gamma_3 - (1-\alpha)\Gamma_4], \\ C_{2\mu}(k_x, k_y; k'_x, k'_y) = -\dfrac{1-\alpha}{\Delta}[(1-\alpha)e^{-i\eta_L h}\Gamma_3 - (1+\alpha)\Gamma_4], \end{cases} \tag{26a}$$

and

$$\begin{cases} \Gamma_3 = \dfrac{k_L^4}{4\eta_L\eta'_L\gamma'_1\gamma'_2}\{\dfrac{[e^{-i(\eta_L+\eta'_L)h}-1]}{\eta_L+\eta'_L}(B_0\gamma'_1+C_0\gamma'_2) + \dfrac{[e^{-i(\eta_L-\eta'_L)h}-1]}{\eta_L-\eta'_L}(B_0\gamma'_2 e^{-i\gamma'_1 h} + C_0\gamma'_1 e^{-i\gamma'_2 h})\} \\ \quad -\dfrac{k_L^2}{\gamma'_1\gamma'_2}\Gamma_1 + \dfrac{k_L^2}{2\eta_L}\left\{\dfrac{B'_{1\mu}[1-e^{-i(\eta_L+\eta'_L)h}]}{\eta_L+\eta'_L} + \dfrac{C'_{1\mu}[1-e^{-i(\eta_L-\eta'_L)h}]}{\eta_L-\eta'_L}\right\}, \\ \Gamma_4 = \dfrac{-k_L^4}{4\eta_L\eta'_L\gamma'_1\gamma'_2}\{\dfrac{[e^{-i\eta'_L h}-e^{-i\eta_L h}]}{\eta_L-\eta'_L}(B_0\gamma'_1+C_0\gamma'_2) + \dfrac{[e^{i\eta'_L h}-e^{-i\eta_L h}]}{\eta_L+\eta'_L}(B_0\gamma'_2 e^{-i\gamma'_1 h} + C_0\gamma'_1 e^{-i\gamma'_2 h})\} \\ \quad -\dfrac{k_L^2}{\gamma'_1\gamma'_2}\Gamma_2 + \dfrac{k_L^2}{2\eta_L}\left\{\dfrac{B'_{1\mu}[e^{-i\eta'_L h}-e^{-i\eta_L h}]}{\eta_L-\eta'_L} + \dfrac{C'_{1\mu}[e^{i\eta'_L h}-e^{-i\eta_L h}]}{\eta_L+\eta'_L}\right\}, \end{cases} \tag{26b}$$

where the prime " ' " signifies evaluation of (k_x, k_y) at (k'_x, k'_y). The ε^4 term from Eq. (23) is then

$$\langle \tilde{P}_2(k_{1x}, k_{1y})\tilde{P}_2^*(k_{2x}, k_{2y})\rangle = (\frac{1}{2\pi})^4 \int_{-\infty}^{\infty} dk'_{1x}\, dk'_{1y} \int_{-\infty}^{\infty} dk'_{2x}\, dk'_{2y} A_{2\mu}(\vec{k}_1, \vec{k}'_1) A_{2\mu}^*(\vec{k}_2, \vec{k}'_2)$$
$$\times \langle \tilde{\mu}(k'_{1x} - k_0\sin\theta, k'_{1y})\tilde{\mu}(k_{1x}-k'_{1x}, k_{1y}-k'_{1y})\tilde{\mu}^*(k'_{2x}-k_0\sin\theta, k'_{2y})\tilde{\mu}^*(k_{2x}-k'_{2x}, k_{2y}-k'_{2y})\rangle. \tag{27}$$

As shown in Appendix D, the right hand side of Eq. (27) vanishes in any incoherent direction unless $\vec{k}_1 = \vec{k}_2$. For this case, with $\vec{k}_1 = \vec{k}_2 = \vec{k}$, one obtains (see Appendix D)

$$\left.\frac{\langle \varepsilon^2\tilde{P}_2\varepsilon^2\tilde{P}_2^*\rangle}{\langle \varepsilon\tilde{P}_1\varepsilon\tilde{P}_1^*\rangle}\right|_{\hat{n}} = (\frac{1}{2\pi})^2 \int_{-\infty}^{\infty} dk'_x\, dk'_y \frac{\varepsilon^4\tilde{R}_\mu(k'_{1x}-k_0\sin\theta, k'_{1y})\tilde{R}_\mu(k_x-k'_{1x}, k_y-k'_{1y})}{\varepsilon^2\tilde{R}_\mu(k_x-k_0\sin\theta, k_y)}$$
$$\times \frac{A_{2\mu}(k_x, k_y; k'_x, k'_y)A_{2\mu}^*(k_x, k_y; k'_x, k'_y) + A_{2\mu}(k_x, k_y; k'_x, k'_y)A_{2\mu}^*(k_{1x}, k_{1y}; k_{1x}-k'_{1x}+k_0\sin\theta, k_y-k'_{1y})}{A_{1\mu}(k_x, k_y)A_{1\mu}^*(k_x, k_y)}. \tag{28}$$

The term $|A_{2\mu}/A_{1\mu}|$ is plotted in figure 6 as a function of acoustic impedance ratio $\rho_L c_L/\rho_0 c_0$. We can see that $|A_{2\mu}/A_{1\mu}|$ is of order unity for different values of (k'_x, k'_y), when $\rho_L c_L/\rho_0 c_0$ is of order 1. For order analysis, it is useful to assume that, for any (k'_x, k'_y) in the interesting region, $A_{2\mu}(k_x, k_y; k'_x, k'_y) \approx A_{2\mu}(k_x, k_y; k_x, k_y)$. One can then take $A_{2\mu}$ out of the integral over k' in Eq. (28).

If $A_{1\mu}$ and $A_{2\mu}$ are expanded in powers of the small parameter $k_L h$, we find an approximate expression, valid for small layer thickness,

$$\left.\frac{A_{2\mu}}{A_{1\mu}}\right|_{\grave{k}'=\grave{k}} \approx i\frac{(\alpha-2)k_L h}{2(\eta_L/k_L)} + O((k_L h)^2) = i\frac{k_L h}{2(\eta_L/k_L)}\left[\frac{\rho_0 c_0}{\rho_L c_L}\frac{\eta_L/k_L}{\eta_0/k_0} - 2\right] + O((k_L h)^2). \tag{29}$$

Fig.7 shows that the ratio $|A_{2\mu}/A_{1\mu}|$ is approximately linear to the nondimensional layer thickness $k_L h$, for small $k_L h$, as predicted by the expression (29). Fig.8 shows that the expression (29) matches the exact form for the ratio if the acoustic impedance ratios are of order 1. Figures (6-8) therefore indicate that the approximation (29) may be used for order analysis if the impedance mismatch is not large. If this approximation is substituted into (28) and notice is taken that $\eta_0/k_0 \sim O(1)$ and $\eta_L/k_L \sim O(1)$ unless $k^2 \approx k_0^2$, one obtains

$$\left|\frac{\langle \varepsilon^2 \tilde{P}_2 (\varepsilon^2 \tilde{P}_2)^* \rangle}{\langle \varepsilon \tilde{P}_1 \varepsilon \tilde{P}_1^* \rangle}\right| \sim (k_L h)^2 \left[\frac{\rho_0 c_0}{\rho_L c_L} - 2\right]^2 \left|\frac{\varepsilon^4 \mathcal{FT}(R_\mu e^{ik_0 x \sin\theta} R_\mu)}{\varepsilon^2 \mathcal{FT}(R_\mu e^{ik_0 x \sin\theta})}\right| \sim (k_L h)^2 \left[\frac{\rho_0 c_0}{\rho_L c_L} - 2\right]^2 |\varepsilon^2 \tilde{R}_\mu| \tag{30}$$

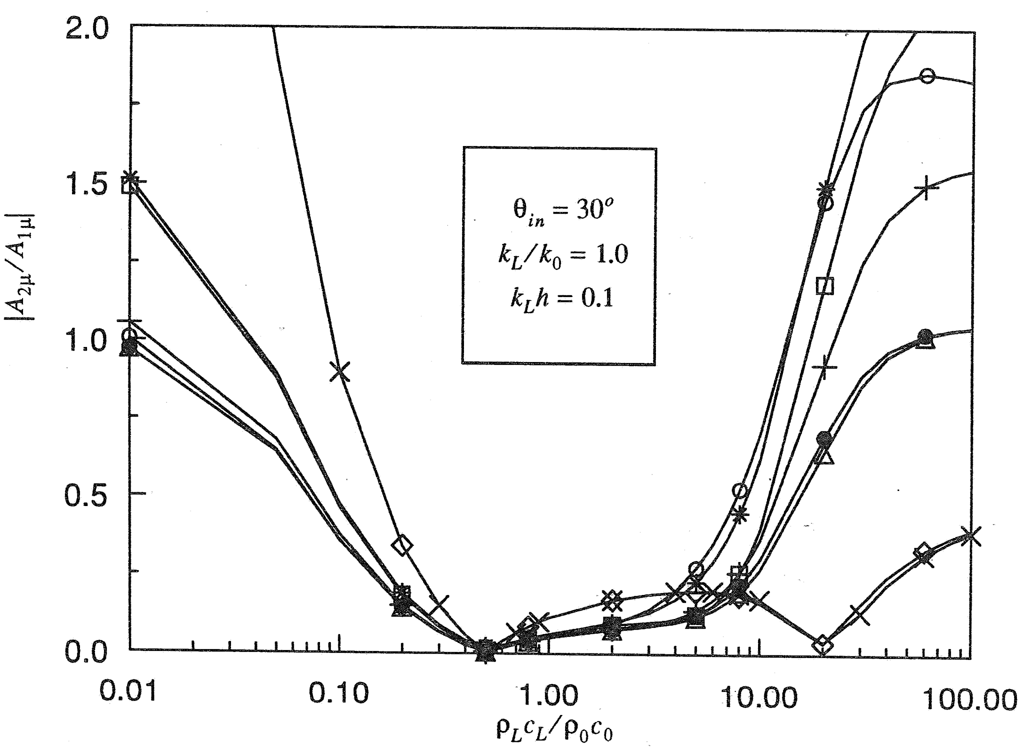

Fig.6: $|A_{2\mu}/A_{1\mu}|$ as a function of the acoustic impedance ratio. For $k/k_0 = 0.3$: $k'/k_0 = 0.1$ (dot); 0.3 (circle); 0.6 (square) and 0.9 (diamond). For $k/k_0 = 0.6$: $k'/k_0 = 0.1$ (triangle); 0.3 (plus); 0.6 (star) and 0.9 (cross).

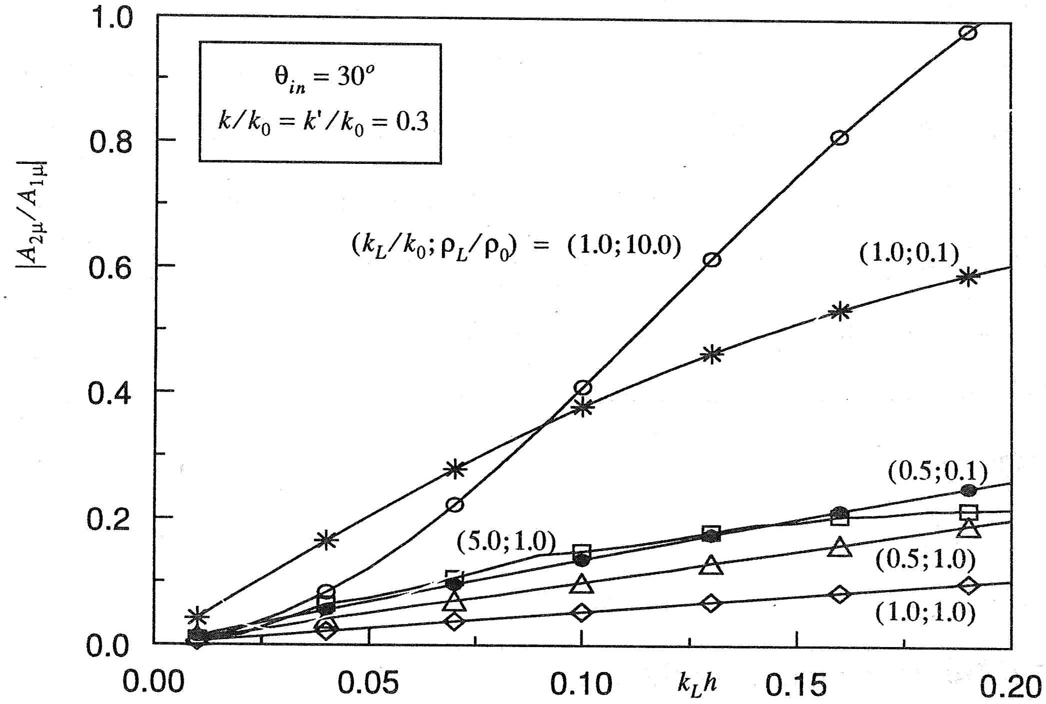

Fig.7: $|A_{2\mu}/A_{1\mu}|$ as a function of the (nondimensional) layer thickness.

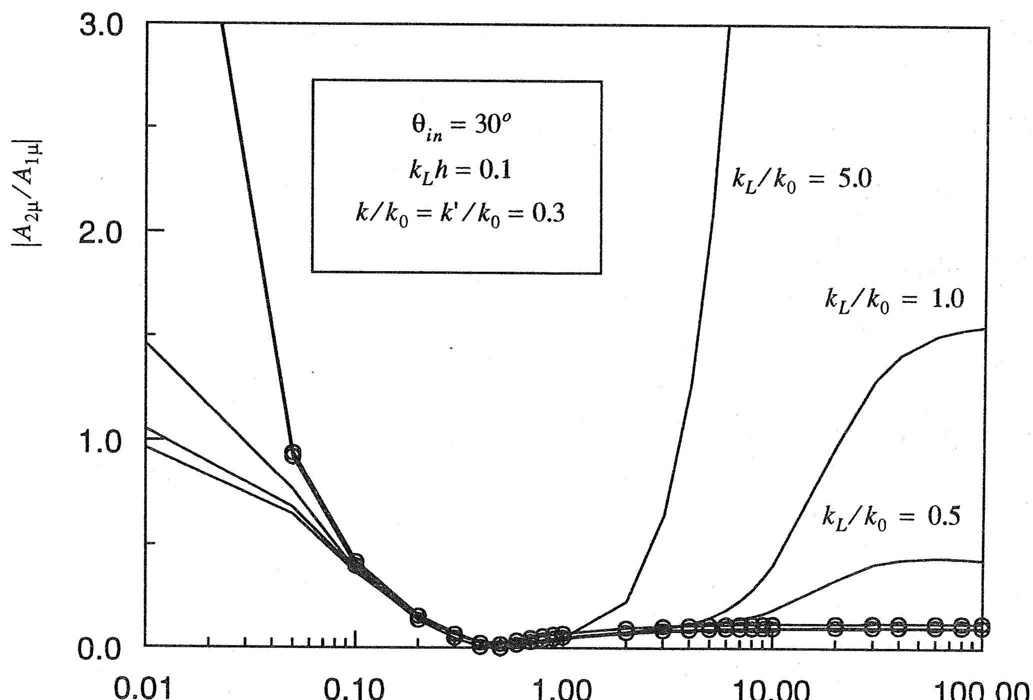

Fig.8: Comparison of the exact and the approximate (lines with circles) expressions of $|A_{2\mu}/A_{1\mu}|$.

Equation (30) provides an estimate of the relative importance of the higher order terms in the Neumann series. As expected, the first Born approximation is good in the limit of a very thin layer. Physically, one might have expected that high impedance mismatch would enhance the importance of the multiply scattered fields, due to strong internal reverberation. Our calculation – the exact numerical results of $|A_{2\mu}/A_{1\mu}|$ as in figures 6 and 8, conforms to this expectation. It is also of interest to notice that, although Eq. (30) is valid only for moderate values of the impedance mismatch, it does qualitatively show more severe constraints on ε and $k_L h$ for very small values of the layer to background impedance ratios.

IV. General Inhomogeneous Fluid Layer

Now we consider a more general case of Fig.1 in which both the sound speed and the ambient density of the fluid layer are inhomogeneous. Let

$$(\omega/c)^2 = k_L^2(1+\varepsilon\mu), \qquad \rho = \rho_L(1+\varepsilon\nu), \tag{31}$$

where $\mu = \mu(x, y), \nu = \nu(x, y)$ and $\varepsilon \ll 1$. The linear acoustic equation is now[10]

$$\nabla^2 P + k_L^2 P = -\varepsilon k_L^2 \mu P + \varepsilon \nabla\nu \bullet \nabla P, \qquad 0<z<h. \tag{32}$$

The 2-D spatial Fourier transform of Eq. (32) is

$$\frac{d^2}{dz^2}\tilde{P} + \eta_L^2 \tilde{P} = -\varepsilon k_L^2 \mathcal{FT}(\mu P) - \varepsilon (k_L^2 - \eta_L^2)\, \mathcal{FT}(\nu P). \tag{33}$$

The zero order solution P_0 is given, as before, in section I, Eq. (6), and, the first Born approximation term P_1, now satisfies

$$\begin{cases} \dfrac{d^2}{dz^2}\tilde{P}_1 + \eta_0^2 \tilde{P}_1 = 0, & z<0 \quad or \quad h<z, \\ \dfrac{d^2}{dz^2}\tilde{P}_1 + \eta_L^2 \tilde{P}_1 = -k_L^2 \mathcal{FT}(\mu P_0) - (k_L^2 - \eta_L^2)\, \mathcal{FT}(\nu P_0), & 0<z<h. \end{cases} \tag{34}$$

The solution of this equation can be expressed as

$$\tilde{P}_1 = \begin{cases} A_1 e^{i\eta_0 z}, & z<0, \\ B_1 e^{-i\eta_L z} + C_1 e^{i\eta_L z} + \tilde{W}, & 0<z<h, \\ D_1 e^{-i\eta_0 z}, & h<z, \end{cases} \quad (35)$$

where $\tilde{W} = J_{1\mu}\tilde{\mu}(k_x - k_0 \sin\theta, k_y) + J_{1\nu}\tilde{\nu}(k_x - k_0\sin\theta, k_y)$ is the Fourier transform of

$$W = k_L^2 \int_0^h dz_l G_L \otimes (\mu P_0) + (k_L^2 - \eta_L^2) \int_0^h dz_l G_L \otimes (\nu P_0), \quad (36)$$

and $\tilde{J}_{1\mu}$ is given by Eq. (A5) in Appendix A, and $\tilde{J}_{1\nu}$ is just $\tilde{J}_{1\mu}$ when k_L^2 is replaced by $(k_L^2 - \eta_L^2)$ and μ replaced by ν. The continuity conditions which couple $\tilde{P}_1$ and $\tilde{P}_0$ are

$$\begin{cases} \tilde{P}_1\big|_{z=0^-, h^+} = \tilde{P}_1\big|_{z=0^+, h^-}, \\ \dfrac{1}{\omega^2\rho_0}\dfrac{d}{dz}\tilde{P}_1\Big|_{z=0^-, h^+} = \dfrac{1}{\omega^2\rho_L}\dfrac{d}{dz}\tilde{P}_1\Big|_{z=0^+, h^-} - \dfrac{1}{\omega^2\rho_L}\dfrac{d}{dz}(\nu\tilde{P}_0)\Big|_{z=0^+, h^-}, \end{cases} \quad (37)$$

so that $\tilde{P}_1$ has sources within the layer and at the interfaces.

In terms of V defined by $V = (d/dz)(W - \nu P_0)$, and in terms of α, Δ defined in section I, one can solve Eqs. (36) and (37) to obtain

$$A_1 = -(\alpha/\Delta)\,\{\Lambda_1 \tilde{W}|_{z=0} - \Lambda_2 \tilde{V}|_{z=0}/(i\eta_L) - 2\,[\tilde{W}|_{z=h} + \alpha\tilde{V}|_{z=h}/(i\eta_L)]\}, \quad (38)$$

where $\Lambda_{1,2} = (1-\alpha)\,e^{i\eta_L h} \pm (1+\alpha)\,e^{-i\eta_L h}$. From the expression (6) for P_0, we may write

$$\frac{d}{dz}\mathcal{FT}(\nu P_0) = ik_L\cos\phi\,(B_0 e^{-ik_L z\cos\phi} - C_0 e^{ik_L z\cos\phi})\,\tilde{\nu}(k_x - k_0\sin\theta, k_y),$$

so that

$$A_1 = A_{1\mu}\tilde{\mu}(k_x - k_0\sin\theta, k_y) + A_{1\nu}\tilde{\nu}(k_x - k_0\sin\theta, k_y), \quad (39)$$

where $A_{1\mu}$ is given by Eq. (15), and

$$\begin{cases} A_{1\nu} = (1 - \eta_L^2/k_L^2)\,A_{1\mu} + \Gamma_5/\Delta, \\ \Gamma_5 = \dfrac{\alpha k_L\cos\phi}{\eta_L}\,[\,(B_0 - C_0)\,\Lambda_2 + 2\alpha\,(B_0 e^{-ik_0 h\cos\phi} - C_0 e^{ik_0 h\cos\phi})\,]. \end{cases} \quad (40)$$

The ensemble average intensity will be written in terms of the auto- and cross-covariances

$$\begin{cases} <\nu(x_1, y_1)\,\nu(x_2, y_2)> = R_\nu(x_1 - x_2, y_1 - y_2)\,, \\ <\mu(x_1, y_1)\,\nu(x_2, y_2)> = R_{\mu\nu}(x_1 - x_2, y_1 - y_2)\,. \end{cases} \tag{41}$$

The result of the received signal intensity is, for any incoherent direction $\hat{n}$,

$$<|V(\hat{n})|^2> \propto \{|A_{1\mu}|^2\tilde{R}_\mu(k_x - k_0\sin\theta, k_y) + |A_{1\nu}|^2\tilde{R}_\nu(k_x - k_0\sin\theta, k_y) + 2Re\,[A_{1\mu}A^*_{1\nu}\tilde{R}_{\mu\nu}(k_x - k_0\sin\theta, k_y)]\}\,n_z. \tag{42}$$

Equation (42) may, like Eq. (21), also be useful in solving inverse scattering problems.

V. Summary and Conclusions

A first Born (or single scattering) approximation has been used to obtain the incoherent intensity in the pressure field backscattered from a randomly heterogeneous thin layer. The intensity is found to be proportional to the two-dimensional spatial Fourier transform of the auto- and cross-covariance functions R_μ, R_ν and $R_{\mu\nu}$ of the heterogeneities. These quantities, R, characterize the statistics of the mechanical properties of the layer. The coefficient of proportion, $|A_1|^2 n_z$, is generally found to vary smoothly as a function of angle. These features make the results potentially useful for inverse scattering problems in which material heterogeneities are to be characterized, provided the scattered incoherent intensity field can be measured.

An order estimation of the parameters region, in which the higher order Born terms can be ignored, is also given. The criterion states that, for larger mismatch of the average impedances for the layer and background fluid, a thinner layer is needed to assure that the single scattering solution is a good approximation.

These results will inform the work, now in progress, on the problem of scattering from thin solid layers immersed in fluids and deposited on half-spaces. We anticipate, in restricted parameters regions, to obtain similar simple proportionality between the incoherently backscattered intensity and the auto- and cross-covariance functions of material heterogeneities.

ACKNOWLEDGMENT

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Appendix A

We have, from the zero order solution (6) and the definition of the spatial convolution,

$$G_L \otimes (\mu P_0) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dx_l dy_l \int_{-\infty}^{\infty} dk_x dk_y \frac{1}{2i\eta_L} e^{-i\eta_L |z - z_l| + i[k_x(x_l - x) + k_y(y_l - y)]} \times \mu(x_l, y_l) [B_0 e^{-ik_L z_l \cos\phi} + C_0 e^{ik_L z_l \cos\phi}] e^{-ik_L x_l \sin\phi}. \tag{A1}$$

By changing the order of integration, one obtains

$$G_L \otimes (\mu P_0) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_x dk_y \tilde{\mu}(k_x - k_L \sin\phi, k_y) \tilde{\Pi}(k_x, k_y; z, z_l) e^{-i(k_x x + k_y y)}, \tag{A2}$$

where $\tilde{\Pi}$ is given by

$$\tilde{\Pi}(k_x, k_y; z, z_l) = \frac{1}{2i\eta_L} e^{-i\eta_L |z - z_l|} [B_0 e^{-ik_L z_l \cos\phi} + C_0 e^{ik_L z_l \cos\phi}]. \tag{A3}$$

Combine Eqs. (12), (13) and (A2), and use the definition of the Fourier transform pair (8), one has

$$\tilde{J}_{1\mu}(k_x, k_y; z) = k_L^2 \int_0^h dz_l \tilde{\Pi}(k_x, k_y; z, z_l). \tag{A4}$$

After carrying out the integration over z_l, one can obtain

$$\tilde{J}_{1\mu}(k_x, k_y; z) = \frac{k_L^2}{2\eta_L \gamma_1 \gamma_2} \{ [B_0 \gamma_1 + C_0 \gamma_2] e^{-i\eta_L z} + [B_0 \gamma_2 e^{-i\gamma_1 h} + C_0 \gamma_1 e^{-i\gamma_2 h}] e^{i\eta_L z} - 2\eta_L (B_0 e^{-ik_L z \cos\phi} + C_0 e^{ik_L z \cos\phi}) \}. \tag{A5}$$

If we denote

$$\tilde{J}_{1\mu}|_{z=0} = \Gamma_1 = (1/i\eta_L)(\partial \tilde{J}_{1\mu}/\partial z)|_{z=0}, \qquad \tilde{J}_{1\mu}|_{z=h} = \Gamma_2 = -(1/i\eta_L)(\partial \tilde{J}_{1\mu}/\partial z)|_{z=h},$$

as given by Eq. (14b), then from the continuity conditions similar to Eq. (6) and use Snell's law, one can arrive at the result as given by Eq. (14a).

Appendix B

We first note that Eq. (19) will be cut off for all $Im\eta'_0 > 0$, i.e., all the components with $\hat{k}_x'^2 + \hat{k}_y'^2 > 1$ have no contribution to the far field. Thus Eq. (19) becomes

$$<|V(\hat{n})|^2> = (2\pi\varepsilon)^2 \int_{\hat{k}_x'^2 + \hat{k}_y'^2 \le 1} dk'_x\, dk'_y\, |A_{1\mu}(k'_x, k'_y)|^2 |T(\hat{n} \bullet \hat{k}')|^2 \tilde{R}_\mu(k'_x - k_0 \sin\theta, k'_y), \tag{B1}$$

for signals in the far field and since $Im\eta'_0 = 0$ for $\hat{k}_x'^2 + \hat{k}_y'^2 \le 1$. Assume that

$$|T(\hat{n} \bullet \hat{k}')|^2 \approx T_0^2 \delta(\hat{n} \bullet \hat{k}' - 1), \tag{B2}$$

and substitute into (B1), one has

$$<|V(\hat{n})|^2> \approx (2\pi)^2 T_0^2 \varepsilon^2 k_0^2 \int_{\hat{k}_x'^2 + \hat{k}_y'^2 \le 1} d\hat{k}'_x\, d\hat{k}'_y\, |A_{1\mu}(k'_x, k'_y)|^2 \delta(\hat{n} \bullet \hat{k}' - 1) \tilde{R}_\mu(k'_x - k_0 \sin\theta, k'_y). \tag{B3}$$

In spherical coordinates, $\hat{k}' = \sin\theta' \cos\phi' \hat{x} + \sin\theta' \sin\phi' \hat{y} + \cos\theta' \hat{z}$, the integration in (B3) in now over all the solid angles of the upper semi-sphere and

$$<|V(\hat{n})|^2> \approx (2\pi)^3 T_0^2 \varepsilon^2 k_0^2 \int_0^{2\pi} d\phi' \int_0^{\pi/2} d\theta' \sin\theta' \cos\theta' \delta(\hat{n} \bullet \hat{k}' - 1) F(\theta', \phi'), \tag{B4}$$

with $F = |A_{1\mu}|^2 \tilde{R}_\mu$. Now we rotate the coordinates in such a way that the new z''-axis is coincide with $\hat{n}$. Then $(\theta', \phi') \Rightarrow (\theta'', \phi'')$ and $\hat{n} \bullet \hat{k}' = \cos\theta''$. Also notice that $\sin\theta' d\phi' d\theta' = \sin\theta'' d\phi'' d\theta''$ is the area element on the sphere, then

$$<|V(\hat{n})|^2> \approx (2\pi)^2 T_0^2 \varepsilon^2 k_0^2 \int d\phi'' \int d\theta'' \sin\theta'' \cos\theta' \delta(\cos\theta'' - 1) F(\theta'', \phi''). \tag{B5}$$

Since F is a smooth function, the main contribution to the integral can then be evaluated at $F(\theta'', \phi'')|_{\theta'' \to 0^+}$ which is independent of (θ'', ϕ'') since $\hat{k}' \to \hat{n}$ as $\theta'' \to 0$. Thus (B5) becomes

$$\begin{aligned} <|V(\hat{n})|^2> &\approx (2\pi)^2 T_0^2 \varepsilon^2 k_0^2 F(\hat{n}) \int d\phi'' \int d\theta'' \sin\theta'' \cos\theta' \delta(\cos\theta'' - 1) \\ &\approx (2\pi)^3 T_0^2 \varepsilon^2 k_0^2 F(\hat{n}) \lim_{\theta'' \to 0^+} \frac{\sin\theta'' \cos\theta'}{|\sin\theta''|} \\ &= (2\pi)^3 T_0^2 \varepsilon^2 k_0^2 |A_{1\mu}(k_0 n_x, k_0 n_y)|^2 \tilde{R}_\mu(k_0 n_x - k_0 \sin\theta, k_0 n_y)\, n_z, \end{aligned} \tag{B6}$$

since $\cos\theta' \to \cos\theta_n = n_z$ as $\theta'' \to 0$. Eq. (B6) is the result given in Eq. (20).

Appendix C

From (10) and the solution for P_1 in (13), one can carry out the calculation of (25) and write

$$\tilde{J}_2(k_x, k_y; z) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk'_x\, dk'_y\, \tilde{\mu}(k'_x - k_0 \sin\theta, k'_y)\, \tilde{\mu}(k_x - k'_x, k_y - k'_y)\, J_{2\mu}(k_x, k_y; k'_x, k'_y; z), \tag{C1}$$

with

$$\begin{cases} J_{2\mu}(k_x, k_y; k'_x, k'_y; z) = \Phi_0(k_x, k_y; k'_x, k'_y; z) + \Phi_1(k_x, k_y; k'_x, k'_y; z), \\ \Phi_0 = \dfrac{-k_L^4}{4\eta_L \eta'_L \gamma'_1 \gamma'_2} \Big\{ \dfrac{B_0\gamma'_1 + C_0\gamma'_2}{\eta_L^2 - \eta_L'^2} [2\eta_L e^{-i\eta'_L z} - (\eta_L + \eta'_L) e^{-i\eta_L z} - (\eta_L - \eta'_L) e^{i\eta_L z} e^{-i(\eta_L + \eta'_L)h}] \\ \quad + \dfrac{B_0\gamma'_2 e^{-i\gamma'_1 h} + C_0\gamma'_1 e^{-i\gamma'_2 h}}{\eta_L^2 - \eta_L'^2} [2\eta_L e^{i\eta'_L z} - (\eta_L - \eta'_L) e^{-i\eta_L z} - (\eta_L + \eta'_L) e^{i\eta_L z} e^{-i(\eta_L - \eta'_L)h}] \\ \quad - \dfrac{2\eta'_L}{\gamma_1\gamma_2} [B_0 (2\eta_L e^{-ik_L z\cos\phi} - \gamma_1 e^{-i\eta_L z} - \gamma_2 e^{i\eta_L z} e^{-i\gamma_1 h}) + C_0 (2\eta_L e^{ik_L z\cos\phi} - \gamma_2 e^{-i\eta_L z} - \gamma_1 e^{i\eta_L z} e^{-i\gamma_2 h})] \Big\}, \\ \Phi_1 = -\dfrac{k_L^2 B'_{1\mu}}{2\eta_L(\eta_L^2 - \eta_L'^2)} \{ (\eta_L + \eta'_L) e^{-i\eta_L z} - 2\eta_L e^{-i\eta'_L z} + (\eta_L - \eta'_L)(e^{i\eta_L z} e^{-i(\eta_L + \eta'_L)h}) \} \\ \quad - \dfrac{k_L^2 C'_{1\mu}}{2\eta_L(\eta_L^2 - \eta_L'^2)} \{ (\eta_L - \eta'_L) e^{-i\eta_L z} - 2\eta_L e^{i\eta'_L z} + (\eta_L + \eta'_L)(e^{i\eta_L z} e^{-i(\eta_L - \eta'_L)h}) \}, \end{cases} \tag{C2}$$

where the prime " ′ " means to evaluate at $(k_x, k_y) = (k'_x, k'_y)$. Similar as in section I, one can use the boundary conditions at $z = 0$ and $z = h$ for (25), and also use (C2), to obtain Eqs. (26a,b).

Appendix D

If only the two-point correlation functions are non-zero,[5] we have

$$\begin{aligned} &\langle \tilde{\mu}(k'_{1x} - k_0\sin\theta, k'_{1y})\, \tilde{\mu}(k_{1x} - k'_{1x}, k_{1y} - k'_{1y})\, \tilde{\mu}^*(k'_{2x} - k_0\sin\theta, k'_{2y})\, \tilde{\mu}^*(k_{2x} - k'_{2x}, k_{2y} - k'_{2y}) \rangle \\ &= \langle \tilde{\mu}(k'_{1x} - k_0\sin\theta, k'_{1y})\, \tilde{\mu}(k_{1x} - k'_{1x}, k_{1y} - k'_{1y}) \rangle \langle \tilde{\mu}^*(k'_{2x} - k_0\sin\theta, k'_{2y})\, \tilde{\mu}^*(k_{2x} - k'_{2x}, k_{2y} - k'_{2y}) \rangle \\ &+ \langle \tilde{\mu}(k'_{1x} - k_0\sin\theta, k'_{1y})\, \tilde{\mu}^*(k'_{2x} - k_0\sin\theta, k'_{2y}) \rangle \langle \tilde{\mu}(k_{1x} - k'_{1x}, k_{1y} - k'_{1y})\, \tilde{\mu}^*(k_{2x} - k'_{2x}, k_{2y} - k'_{2y}) \rangle \\ &+ \langle \tilde{\mu}(k'_{1x} - k_0\sin\theta, k'_{1y})\, \tilde{\mu}^*(k_{2x} - k'_{2x}, k_{2y} - k'_{2y}) \rangle \langle \tilde{\mu}(k_{1x} - k'_{1x}, k_{1y} - k'_{1y})\, \tilde{\mu}^*(k'_{2x} - k_0\sin\theta, k'_{2y}) \rangle. \end{aligned} \tag{D1}$$

Using (18), one can obtain from (D1) and (27) that

$$<\tilde{P}_2(k_{1x}, k_{1y})\tilde{P}_2^*(k_{2x}, k_{2y})> \; = \; Int1 + Int2 + Int3, \tag{D2}$$

where

$$\begin{cases} \text{Int}\,1 = \delta(k_{1x}-k_0\sin\theta)\,\delta(k_{1y})\,\delta(k_{2x}-k_0\sin\theta)\,\delta(k_{2y}) \int_{-\infty}^{\infty} dk'_{1x}\, dk'_{1y} \int_{-\infty}^{\infty} dk'_{2x}\, dk'_{2y}\, A_{2\mu}(\vec{k}_1, \vec{k}'_1)\, A^*_{2\mu}(\vec{k}_2, \vec{k}'_2) \\ \qquad \times \tilde{R}_\mu(k'_{1x}-k_0\sin\theta, k'_{1y})\,\tilde{R}_\mu(k'_{2x}-k_0\sin\theta, k'_{2y}), \\ \text{Int}\,2 = \delta(k_{1x}-k_{2x})\,\delta(k_{1y}-k_{2y}) \int_{-\infty}^{\infty} dk'_{1x}\, dk'_{1y}\, A_{2\mu}(\vec{k}_1, \vec{k}'_1)\, A^*_{2\mu}(\vec{k}_2, \vec{k}'_1) \\ \qquad \times \tilde{R}_\mu(k'_{1x}-k_0\sin\theta, k'_{1y})\,\tilde{R}_\mu(k_{1x}-k'_{1x}, k_{1y}-k'_{1y}), \\ \text{Int}\,3 = \delta(k_{1x}-k_{2x})\,\delta(k_{1y}-k_{2y}) \int_{-\infty}^{\infty} dk'_{1x}\, dk'_{1y}\, A_{2\mu}(\vec{k}_1, \vec{k}'_1)\, A^*_{2\mu}(k_{1x}, k_{1y}; k_{1x}-k'_{1x}+k_0\sin\theta, k_y-k'_{1y}) \\ \qquad \times \tilde{R}_\mu(k'_{1x}-k_0\sin\theta, k'_{1y})\,\tilde{R}_\mu(k_{1x}-k'_{1x}, k_{1y}-k'_{1y}). \end{cases} \tag{D3}$$

Here we have used the property that the auto-covariance function is real and symmetric and then its Fourier transform is also real. We can see from (D3) that $Int1 = 0$ for incoherent waves. Substitute (D2) into (D3) and use

$$<\tilde{P}_1(k_{1x}, k_{1y})\tilde{P}_1^*(k_{2x}, k_{2y})> \; = \; (2\pi)^2\delta(k_{1x}-k_{2x})\,\delta(k_{1y}-k_{2y})\,A_{1\mu}(k_{1x}, k_{1y})\,A^*_{1\mu}(k_{2x}, k_{2y})\,\tilde{R}_\mu(k_{1x}-k_0\sin\theta, k_{2y}),$$

one can obtain Eq. (28).

Appendix E

Similar as before, $\tilde{P}_3$ is given by

$$\tilde{P}_3(k_x, k_y; z) \; = \; \begin{cases} A_3 e^{i\eta_0 z}, & z < 0, \\ B_3 e^{-i\eta_L z} + C_3 e^{i\eta_L z} + \tilde{J}_3, & 0 < z < h, \\ D_3 e^{-i\eta_0 z}, & h < z, \end{cases} \tag{E1}$$

where

$$J_3(x, y, z) \; = \; k_L^2 \int_0^h dz_l \int_{-\infty}^{\infty} dx_l dy_l G_L(x, y, z; x_l, y_l, z_l)\, \mu(x_l, y_l)\, P_2(x_l, y_l, z_l). \tag{E2}$$

After using the expressions for $\tilde{G}_L$ and $\tilde{P}_2$, one has

$$J_3(x,y,z) = \left(\frac{1}{2\pi}\right)^6 k_L^2 \int_0^h dz_l \int_{-\infty}^{\infty} dk_x dk_y \int_{-\infty}^{\infty} dk'_x\, dk'_y \int_{-\infty}^{\infty} dk''_x\, dk''_y\; e^{-i(k_x x + k_y y)} \frac{e^{-i\eta_L |z-z_l|}}{2i\eta_L}$$
$$\times \tilde{\mu}(k_x - k'_x, k_y - k'_y)\, \tilde{\mu}(k'_x - k''_x, k'_y - k''_y)\, \tilde{\mu}(k''_x - k_0 \sin\theta, k''_y)$$
$$\times \{ B_{2\mu}(k'_x, k'_y; k''_x, k''_y)\, e^{-i\eta'_L z_l} + C_{2\mu}(k'_x, k'_y; k''_x, k''_y)\, e^{i\eta'_L z_l} + J_{2\mu}(k'_x, k'_y; k''_x, k''_y; z_l) \}. \tag{E3}$$

Then one can write

$$\tilde{J}_3(k_x, k_y; z) = \left(\frac{1}{2\pi}\right)^4 \int_{-\infty}^{\infty} dk'_x\, dk'_y \int_{-\infty}^{\infty} dk''_x\, dk''_y\; J_{3\mu}(k'_x, k'_y; k''_x, k''_y; z)$$
$$\times \tilde{\mu}(k_x - k'_x, k_y - k'_y)\, \tilde{\mu}(k'_x - k''_x, k'_y - k''_y)\, \tilde{\mu}(k''_x - k_0 \sin\theta, k''_y), \tag{E4}$$

and

$$J_{3\mu}(k'_x, k'_y; k''_x, k''_y; z) = \Psi_2(k'_x, k'_y; k''_x, k''_y; z) + \Psi_1(k'_x, k'_y; k''_x, k''_y; z) + \Psi_0(k'_x, k'_y; k''_x, k''_y; z), \tag{E5}$$

with

$$\Psi_2(k'_x, k'_y; k''_x, k''_y; z) = -\frac{k_L^2}{2\eta_L(\eta_L^2 - \eta''^2_L)}$$
$$\times \{ B_{2\mu}(k'_x, k'_y; k''_x, k''_y)\, [2\eta_L e^{-i\eta''_L z} - (\eta_L + \eta''_L) e^{-i\eta_L z} - (\eta_L - \eta''_L) e^{i\eta_L z} e^{-i(\eta_L + \eta''_L)h}]$$
$$+ C_{2\mu}(k'_x, k'_y; k''_x, k''_y)\, [2\eta_L e^{i\eta''_L z} - (\eta_L - \eta''_L) e^{-i\eta_L z} - (\eta_L + \eta''_L) e^{i\eta_L z} e^{-i(\eta_L + \eta''_L)h}] \}, \tag{E6}$$

$$\Psi_1(k'_x, k'_y; k''_x, k''_y; z) = \frac{k_L^4}{4\eta_L \eta'_L (\eta'^2_L - \eta''^2_L)}$$
$$\times \{ B_{1\mu}(k''_x, k''_y)\, [\frac{\eta'_L + \eta''_L}{\eta_L^2 - \eta'^2_L} (2\eta_L e^{-i\eta'_L z} - (\eta_L + \eta'_L) e^{-i\eta_L z} - (\eta_L - \eta'_L) e^{i\eta_L z} e^{-i(\eta_L + \eta'_L)h})$$
$$- \frac{2\eta'_L}{\eta_L^2 - \eta''^2_L} (2\eta_L e^{-i\eta''_L z} - (\eta_L + \eta''_L) e^{-i\eta_L z} - (\eta_L - \eta''_L) e^{i\eta_L z} e^{-i(\eta_L + \eta''_L)h})$$
$$+ \frac{\eta'_L - \eta''_L}{\eta_L^2 - \eta'^2_L} (2\eta_L e^{i\eta'_L z} - (\eta_L - \eta'_L) e^{-i\eta_L z} - (\eta_L + \eta'_L) e^{i\eta_L z} e^{-i(\eta_L - \eta'_L)h})\, e^{-i(\eta'_L + \eta''_L)h}]$$
$$+ C_{1\mu}(k''_x, k''_y)\, [\frac{\eta'_L - \eta''_L}{\eta_L^2 - \eta'^2_L} (2\eta_L e^{-i\eta'_L z} - (\eta_L + \eta'_L) e^{-i\eta_L z} - (\eta_L - \eta'_L) e^{i\eta_L z} e^{-i(\eta_L + \eta'_L)h})$$
$$- \frac{2\eta'_L}{\eta_L^2 - \eta''^2_L} (2\eta_L e^{i\eta''_L z} - (\eta_L - \eta''_L) e^{-i\eta_L z} - (\eta_L + \eta''_L) e^{i\eta_L z} e^{-i(\eta_L + \eta''_L)h})$$
$$+ \frac{\eta'_L + \eta''_L}{\eta_L^2 - \eta'^2_L} (2\eta_L e^{i\eta'_L z} - (\eta_L - \eta'_L) e^{-i\eta_L z} - (\eta_L + \eta'_L) e^{i\eta_L z} e^{-i(\eta_L - \eta'_L)h})\, e^{-i(\eta'_L - \eta''_L)h}] \}, \tag{E7}$$

and

$$\Psi_0(k'_x, k'_y\,;\,k''_x, k''_y\,;z) = \frac{k_L^6}{8\eta_L\eta'_L\eta''_L\;\gamma''_1\gamma''_2}$$

$$\times\Big\{\frac{B_0\gamma''_1 + C_0\gamma''_2}{\eta'^2_L - \eta''^2_L}\Big[\frac{2\eta'_L}{\eta^2_L - \eta''^2_L}(2\eta_L e^{-i\eta''_L z} - (\eta_L + \eta''_L)e^{-i\eta_L z} - (\eta_L - \eta''_L)e^{i\eta_L z}e^{-i(\eta_L + \eta''_L)h})$$

$$-\frac{\eta'_L + \eta''_L}{\eta^2_L - \eta''^2_L}(2\eta_L e^{-i\eta'_L z} - (\eta_L + \eta'_L)e^{-i\eta_L z} - (\eta_L - \eta'_L)e^{i\eta_L z}e^{-i(\eta_L + \eta'_L)h})$$

$$-\frac{\eta'_L - \eta''_L}{\eta^2_L - \eta''^2_L}(2\eta_L e^{i\eta'_L z} - (\eta_L - \eta'_L)e^{-i\eta_L z} - (\eta_L + \eta'_L)e^{i\eta_L z}e^{-i(\eta_L - \eta'_L)h})e^{-i(\eta'_L + \eta''_L)h}\Big]$$

$$+\frac{B_0\gamma''_2\, e^{-i\gamma''_1 h} + C_0\gamma''_1\, e^{-i\gamma''_2 h}}{\eta'^2_L - \eta''^2_L}\Big[\frac{2\eta'_L}{\eta^2_L - \eta''^2_L}(2\eta_L e^{i\eta''_L z} - (\eta_L - \eta''_L)e^{-i\eta_L z} - (\eta_L + \eta''_L)e^{i\eta_L z}e^{-i(\eta_L - \eta''_L)h})$$

$$-\frac{\eta'_L - \eta''_L}{\eta^2_L - \eta''^2_L}(2\eta_L e^{-i\eta'_L z} - (\eta_L + \eta'_L)e^{-i\eta_L z} - (\eta_L - \eta'_L)e^{i\eta_L z}e^{-i(\eta_L + \eta'_L)h})$$

$$-\frac{\eta'_L + \eta''_L}{\eta^2_L - \eta''^2_L}(2\eta_L e^{i\eta'_L z} - (\eta_L - \eta'_L)e^{-i\eta_L z} - (\eta_L + \eta'_L)e^{i\eta_L z}e^{-i(\eta_L - \eta'_L)h})e^{-i(\eta'_L - \eta''_L)h}\Big] \tag{E8}$$

$$-\frac{2\eta''_L}{\gamma'_1\gamma'_2}B_0\Big[\frac{2\eta'_L}{\gamma_1\gamma_2}(2\eta_L e^{-ik_L z\cos\phi} - \gamma_1 e^{-i\eta_L z} - \gamma_2 e^{i\eta_L z}e^{-i\gamma_1 h})$$

$$-\frac{\gamma'_1}{\eta^2_L - \eta'^2_L}(2\eta_L e^{-i\eta'_L z} - (\eta_L + \eta'_L)e^{-i\eta_L z} - (\eta_L - \eta'_L)e^{i\eta_L z}e^{-i(\eta_L + \eta'_L)h})$$

$$-\frac{\gamma'_2}{\eta^2_L - \eta'^2_L}(2\eta_L e^{i\eta'_L z} - (\eta_L - \eta'_L)e^{-i\eta_L z} - (\eta_L + \eta'_L)e^{i\eta_L z}e^{-i(\eta_L - \eta'_L)h})e^{-i\gamma'_1 h}\Big]$$

$$-\frac{2\eta''_L}{\gamma'_1\gamma'_2}C_0\Big[\frac{2\eta'_L}{\gamma_1\gamma_2}(2\eta_L e^{ik_L z\cos\phi} - \gamma_2 e^{-i\eta_L z} - \gamma_1 e^{i\eta_L z}e^{-i\gamma_2 h})$$

$$-\frac{\gamma'_2}{\eta^2_L - \eta'^2_L}(2\eta_L e^{-i\eta'_L z} - (\eta_L + \eta'_L)e^{-i\eta_L z} - (\eta_L - \eta'_L)e^{i\eta_L z}e^{-i(\eta_L + \eta'_L)h})$$

$$-\frac{\gamma'_1}{\eta^2_L - \eta'^2_L}(2\eta_L e^{i\eta'_L z} - (\eta_L - \eta'_L)e^{-i\eta_L z} - (\eta_L + \eta'_L)e^{i\eta_L z}e^{-i(\eta_L - \eta'_L)h})e^{-i\gamma'_2 h}\Big]\Big\},$$

Now if we expand (E6-8) as Tayler series of the small layer thickness $k_L h$, and substitute into (E5), we can find that $J_{3\mu} \sim O((k_L h)^4)$. So $\tilde{P}_1\tilde{P}_3^* \sim O((k_L h)^5)$ is a higher order term compared with $\tilde{P}_2\tilde{P}_2^* \sim O((k_L h)^4)$.

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[19] Notice that, although for any spatially stationary process, $\mu(x, y)$ varies in the whole x-y plane in such a manner that the condition, on which the classical theory of Fourier analysis relies:

$$\int_{-\infty}^{\infty} dxdy \ |\mu(x, y)| < \infty$$

is not satisfied, its autocovariance function R_{μ} does have Fourier transform even in the classical sense.[20] Furthermore, we usually do not need to consider, in practical sense, the properties of the medium at any far away position, we can simply assume that $\mu(x, y) = 0$ outside a large circle $x^2 + y^2 = R^2$ with R very large. Under this assumption, we can take (spatial) Fourier transforms of $\mu(x, y)$ without any problem. Still remember that what we really need and use is the (spatial) Fourier transform of the autocovariance function or the (spatial) spectral density function.

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