

# Does Fully-Developed Turbulence Exist? Reynolds Number Independence versus Asymptotic Covariance

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## ABSTRACT

By analogy with recent arguments concerning the mean velocity profile of wall-bounded turbulent shear flows, we suggest that there may exist corrections to the  $2/3$  law of Kolmogorov, which are proportional to  $(\ln \text{Re})^{-1}$  at large  $\text{Re}$ . Such corrections to K41 are the only ones permitted if one insists that the functional form of statistical averages at large  $\text{Re}$  be invariant under a natural redefinition of  $\text{Re}$ . The family of curves of the observed longitudinal structure function  $D_{LL}(r, \text{Re})$  for different values of  $\text{Re}$  is bounded by an envelope. In one generic scenario, close to the envelope,  $D_{LL}(r, \text{Re})$  is of the form assumed by Kolmogorov, with corrections of  $O((\ln \text{Re})^{-2})$ . In an alternative generic scenario, both the Kolmogorov constant  $C_K$  and corrections to Kolmogorov's linear relation for the third order structure function  $D_{LLL}(r)$  are proportional to  $(\ln \text{Re})^{-1}$ . Recent experimental data of Praskovsky and Oncley appear to show a definite dependence of  $C_K$  on  $\text{Re}$ , which if confirmed, would be consistent with the arguments given here.

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# 1. Introduction

The term *fully-developed turbulence* traditionally refers to a unique state of turbulent behaviour believed to occur for sufficiently large but finite Reynolds number ( $Re$ ). This state is characterised by local isotropy and homogeneity and associated universal behaviour of statistical properties, such as moments of the longitudinal velocity difference  $v_r \equiv (\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})) \cdot \mathbf{r}/|\mathbf{r}|$ . The first theoretical description along these lines was given by Kolmogorov and Obukhov in 1941 (referred to as K41).<sup>1</sup> The assumption that such a limiting state exists, and may be found at large but finite  $Re$ , is non-trivial, and in our view, has not properly been established experimentally. The purpose of this note is to investigate how a breakdown of this assumption would be manifested.

The mathematical expression of the assumption of a limiting state of fully-developed turbulence is that statistical averages of the flow exhibit complete similarity<sup>2</sup> with respect to  $Re$ . To explain this statement, let us consider the second order longitudinal structure function  $D_{LL}(r) \equiv \langle (v_r)^2 \rangle$ . In K41,  $D_{LL}(r)$  has the form

$$D_{LL}(r) = C_K(\bar{\epsilon}r)^{2/3}, \quad (1.1)$$

where  $r$  lies in the inertial range, and  $\bar{\epsilon}$  is the mean rate of energy dissipation per unit mass. Kolmogorov's form for  $D_{LL}(r)$  is based upon both dimensional considerations, and assumptions about limiting behaviour. Dimensional analysis shows that the form of  $D_{LL}(r)$  must be given by

$$D_{LL}(r) = (\bar{\epsilon}r)^{2/3} F(Re, r/L), \quad (1.2)$$

where  $F(x, y)$  is a universal function to be determined,  $L$  is the external or integral scale and  $r$  is always considered to lie in the inertial range. Kolmogorov assumed that in the limits  $x \rightarrow \infty$  and  $y \rightarrow 0$ , the function  $F(x, y)$  simply takes the constant value  $C_K$ . In other words, there is *complete similarity* with respect to the variables  $Re$  and  $r/L$ .

The existence of the limit of  $F(x, y)$  as  $y \rightarrow 0$  has been questioned<sup>3-5</sup> due to intermittency — fluctuations of the energy dissipation rate about its mean value  $\bar{\epsilon}$ . *Incomplete similarity* in the variable  $r/L$  would require the non-existence of a finite and non-zero limit of  $F(x, y)$  as  $y \rightarrow 0$ , and leads in the simplest case to the form

$$D_{LL}(r) = C_K(\bar{\epsilon}r)^{2/3} \left( \frac{r}{L} \right)^\alpha, \quad (1.3)$$

where  $\alpha$  is the so-called intermittency exponent, believed to be small and non-negative.

In the present note, we argue that there may be an alternative way, which we term *asymptotic covariance*, in which a lack of complete similarity can occur. We try to use physical arguments to constrain the mathematical form that this might take. Asymptotic covariance in  $Re$  would imply that there is no unique limiting state of fully-developed turbulence. Instead, the manner in which statistical averages evolve with  $Re$  for large  $Re$  is governed by a functional form that in the simplest case is universal.

Precisely the same set of arguments can be made for a seemingly different, but related problem: the mean velocity profile in a wall-bounded turbulent shear flow. There is a well-known (and probably superficial) analogy<sup>6</sup> between the boundary layers and universal scaling regimes of both the spatial structure of wall-bounded turbulent shear flows and the local structure of turbulence. For example, consider turbulent flow in a pipe. The viscous wall layer is analogous to the dissipative range in fully-developed turbulence; the velocity profile (conventionally described by the universal von Kármán-Prandtl logarithmic law) outside the viscous wall layer, but on scales much smaller than the pipe radius is analogous to the inertial range (conventionally described by K41); and the non-universal finite-size effects on the flow associated with scales of order the pipe radius are analogous to the non-universal behaviour of fully-developed turbulence at the integral scale. This problem was already considered in detail,<sup>7</sup> and it was shown that the existing data do not exclude the possibility there of asymptotic covariance in  $Re$ .

This paper is organised as follows. In section 2, we review the analysis of the wall-bounded shear flow. In particular, we propose a principle of *asymptotic covariance*, in which we insist that the functional form of statistical averages at large  $Re$  be invariant under redefinition of  $Re$ . This implies that a form of incomplete similarity must occur in terms of the variable  $\ln Re$ . In section 3, we consider the analogous arguments for the local structure of fully-developed turbulence. We conclude in section 4 with a brief discussion of experimental data and some final comments.

## 2. Wall-bounded turbulent shear flows

We consider a wall-bounded shear flow which is statistically steady and homogeneous in the longitudinal direction. Its properties vary only in the lateral direction, perpendicular to the wall. A classic example, which we shall always have in mind, is the flow in a pipe far from the entrance and outlet.

### 2.1. MEAN VELOCITY PROFILE

Von Kármán<sup>8</sup> and Prandtl<sup>9</sup> obtained the law for the variation of the mean longitudinal velocity  $\bar{u}$  in an intermediate region of the turbulent shear flow, outside a small “viscous” sub-layer near the wall. Within this sub-layer, the stress due to molecular momentum transfer is comparable in magnitude with that due to turbulent momentum transfer by vortices. The universal, Reynolds number independent von Kármán-Prandtl logarithmic law has the form

$$\phi = \frac{1}{\kappa} \ln \eta + \tilde{C}, \quad (2.1)$$

where

$$\phi \equiv \bar{u}/u_*, \quad \eta \equiv u_* y/\nu, \quad u_* \equiv (\tau/\rho)^{1/2}, \quad (2.2)$$

$y$  is the distance from the wall, and  $\rho$  and  $\nu$  are the fluid density and kinematic viscosity respectively. The constants  $\kappa$  (the von Kármán constant) and  $\tilde{C}$  are universal according to the logic of the derivation. The logarithmic law follows from a strong assumption of complete similarity, namely that in the intermediate region the contribution of the molecular viscosity and the external length scale (*e.g.*, the diameter of the pipe) could be completely neglected.

It was shown in ref. 7 that this assumption is questionable, and an alternative relationship was proposed, corresponding to incomplete similarity:

$$\phi = C\eta^\alpha, \quad (2.3)$$

where the coefficients  $C$  and  $\alpha$  may be expressed in terms of the small parameter

$$\epsilon \equiv \frac{1}{\ln \text{Re}} \quad (2.4)$$

by the expansions

$$\alpha = \frac{3}{2}\epsilon + O(\epsilon^2), \quad C = \frac{1}{\sqrt{3}\epsilon} + \frac{5}{2} + O(\epsilon). \quad (2.5)$$

It is important that the power  $\alpha$  in (2.3) depends upon the Reynolds number for two reasons. First, it is well-known that a Re-independent power law form is inconsistent with the data.<sup>10</sup>

Second, Eyink<sup>11</sup> has argued that a Re-independent power law is inconsistent with the rigorous bound on the energy dissipation rate given by Doering and Constantin,<sup>12</sup> (making the plausible assumption that their bound can be carried over to the pipe geometry), but the form given in (2.5) is consistent with the bound. Eyink's argument is as follows: the average dissipation per unit mass  $\bar{\epsilon}$  is

$$\bar{\epsilon} = \frac{1}{L} \int_0^L \epsilon'(y) dy, \quad \epsilon'(y) = u_*^2 \frac{\partial \bar{u}}{\partial y}, \quad (2.6)$$

which for the von Kármán-Prandtl law (2.1) can be estimated to be

$$\bar{\epsilon} = \frac{U^3}{L} \frac{1}{\kappa (\ln \text{Re})^2}, \quad \text{Re} \rightarrow \infty. \quad (2.7)$$

For the velocity profile (2.3), however, it can be shown that

$$\epsilon'(y) \sim \left(\frac{L}{y}\right)^{1-\alpha} [2^\alpha \alpha (\alpha + 1) (\alpha + 2)]^{(3+\alpha)/(1+\alpha)} \text{Re}^{-2\alpha/(1+\alpha)}. \quad (2.8)$$

If  $\alpha$  were Re-independent, the dissipation in the intermediate region would vanish sharply as  $\text{Re} \rightarrow \infty$ . However, a more physically plausible alternative is that there is only a weak dependence of the dissipation on Re for  $\text{Re} \rightarrow \infty$ :

$$\text{Re}^\alpha \sim \text{constant} \quad \text{i.e., } \alpha \sim \frac{\text{constant}}{\ln \text{Re}}. \quad (2.9)$$

The corresponding result for the dissipation is that

$$\bar{\epsilon} \sim \frac{U^3}{L} \frac{1}{(\ln \text{Re})^2}, \quad \text{Re} \rightarrow \infty, \quad (2.10)$$

which is very similar to the estimate for the von Kármán-Prandtl law.

It seems that the scaling law (2.3) manifests a lack of universality, by virtue of its dependence on Re. However, this is not so, in the following sense. Instead of the traditional universal straight line in the  $\phi$ - $\ln \eta$  plane corresponding to (2.1), there is a one-parameter family of curves (2.3) occupying a certain region of the plane. This region is nevertheless universal, in the sense that it is bounded by the envelope of the family, which is a universal curve. The equation for the envelope is obtained by eliminating  $\ln \text{Re}$  from (2.3), written in the form  $\phi = F(\ln \eta, \ln \text{Re})$ ,

and the tangency condition  $\partial F/\partial \ln \text{Re} = 0$ , and is found to have the universal form

$$\phi = 5 \left( \sigma^{-1} + \frac{1}{2} \right) \eta^{\sigma\sqrt{3}/10}, \quad \sigma \equiv \left( 1 + \frac{20}{\sqrt{3} \ln \eta} \right)^{1/2} - 1. \quad (2.11)$$

This envelope is in fact close to (2.1) even for moderate values of  $\ln \eta$  and  $\text{Re}$ , if  $\kappa \sim 0.4$  and  $\tilde{C} = 5.1$ . For presently unachievably large values of  $\text{Re}$  and  $\ln \eta$ , the universal envelope (2.11) assumes the form

$$\phi = \frac{\sqrt{3}e}{2} \ln \eta + \frac{5e}{2}, \quad (2.12)$$

where the coefficient  $2/\sqrt{3}e \approx 0.424$  is close to the generally accepted value of the von Kármán constant  $\kappa$ , but  $5e/2 \approx 6.79$  is larger than the generally accepted value of  $\tilde{C} \approx 5.1$ – $5.5$ .

Thus, even when there is a lack of similarity, it seems that for  $\ln \text{Re} \gg 1$  in the intermediate region of the shear flow, the state is described by the relation  $\phi = F(\ln \eta, \ln \text{Re})$ , universal in the sense that the same function  $F$  applies to all turbulent shear flows.

## 2.2. ASYMPTOTIC COVARIANCE

It is instructive that the universal relation (2.3) – (2.5) contains  $\text{Re}$  only through the dependence on  $\ln \text{Re}$ . In fact, this is inevitable, and a consequence of what we propose to call *asymptotic covariance*. Asymptotic covariance provides a general constraint on the way in which lack of complete similarity in  $\text{Re}$  may be exhibited in a turbulent flow. Let us assume that we consider a simple turbulent flow, such as that in a pipe, and that a putative state of isotropic, homogeneous turbulence is present on some scale  $r \ll L$ , where  $L$  is taken to be the diameter of the pipe. The *nature* of the turbulent state should be insensitive to small changes in the cross-sectional average input and output flow rate  $U$  or the diameter  $L$ . Thus, we require that at sufficiently large  $\text{Re}$ , the *functional form* of statistical averages, such as  $D_{LL}(r)$  or  $\phi$ , should not be influenced by a redefinition of  $\text{Re}$  such that

$$\text{Re} \rightarrow \text{Re}' \equiv Z\text{Re} = \text{Re} + \delta\text{Re}, \quad \delta\text{Re}/\text{Re} \rightarrow 0. \quad (2.13)$$

For example, at sufficiently large  $\text{Re}$ , we could use for  $L$  the radius of the pipe instead of the diameter, or for  $U$  the maximum velocity instead of the average one *etc.*, without changing the functional form of  $D_{LL}(r)$  or  $\phi$ . Let us suppose that we consider such a statistical average, whose dependence on  $\text{Re}$  (and  $\eta$  in the case of the shear flow considered in this section) is through its dependence

on a function  $F(\ln \eta, \psi(\text{Re}))$  having for definiteness a uniformly bounded first derivative, and with  $\psi$  an unboundedly growing function of its argument, to be determined by the following considerations.

For any  $Z > 0$

$$\begin{aligned}\psi(\text{Re}) &= \psi(\text{Re}_0) + \int_{\text{Re}_0}^{\text{Re}} \frac{d\psi(\text{Re}')}{d\text{Re}'} d\text{Re}' \\ \psi(Z\text{Re}) &= \psi(Z\text{Re}_0) + \int_{\text{Re}_0}^{\text{Re}} \frac{d\psi(Z\text{Re}')}{d\text{Re}'} d\text{Re}'.\end{aligned}\tag{2.14}$$

Here,  $\text{Re}_0$  is some reference value of  $\text{Re}$ . The first term on the right hand sides of (2.14) at large  $\text{Re}$  is small in comparison with the second term, because  $\psi$  is unbounded. However,  $d\psi(Z\text{Re})/d\text{Re} = Z\psi'(Z\text{Re})$  where  $'$  denotes differentiation with respect to the argument. Asymptotic covariance is equivalent to the statement that

$$\psi'(\text{Re}) = \psi'(Z\text{Re}).\tag{2.15}$$

Thus, the right hand side of (2.15) is simply  $Z\psi'(Z\text{Re})$ , and

$$\text{Re} \psi'(\text{Re}) = Z \text{Re} \psi'(Z\text{Re})\tag{2.16}$$

giving  $\psi(\text{Re}) \propto \ln \text{Re}$ .

Let us now consider the question of whether one could use the Taylor microscale Reynolds number  $\text{Re}_\lambda$ , which is often assumed to vary approximately as  $\sqrt{\text{Re}}$ . From the point of view of advanced similarity methods and dimensional analysis, the important point about  $\text{Re}$  is that it represents a characterization of the system which can be made *a priori*. That is, it is not an emergent property of the flow (*i.e.*, depending upon the *solution* of the equation of motion), but a property of the *constraints* or *boundary conditions* placed upon the flow. On the other hand,  $\text{Re}_\lambda$  represents the response of the flow, and in general, for arbitrary  $\text{Re}$ , will not necessarily have a unique, universal dependence on  $\text{Re}$ .<sup>13</sup> However, for an asymptotically covariant theory, at very large  $\text{Re}$ , the variation of  $\ln \text{Re}_\lambda$  with  $\ln \text{Re}$  may be written in the form

$$\frac{d \ln \text{Re}_\lambda}{d \ln \text{Re}} = a_0 + \frac{a_1}{\ln \text{Re}} + O(1/(\ln \text{Re})^2)\tag{2.17}$$

where  $a_0, a_1$  etc. are constants, so that

$$\text{Re}_\lambda \sim \text{Re}^{a_0} (\ln \text{Re})^{a_1} \quad (2.18)$$

showing that there is a unique power law relationship between  $\text{Re}_\lambda$  and  $\text{Re}$ .

### 3. Local structure of fully-developed turbulence

As explained in the Introduction, Kolmogorov's form (1.1) for

$$D_{LL}(r) = (\bar{\epsilon}r)^{2/3} F(\text{Re}, r/L) \quad (3.1)$$

is based upon both dimensional considerations, and assumptions about limiting behaviour. Let us now examine the question of the lack of complete similarity of (1.2), based upon considerations analogous to those used above for wall-bounded turbulent shear flows.

We will assume that the spatial fluctuations in the energy flux diminish at higher  $\text{Re}$ , because the flow configurations or processes which correspond to these fluctuations become increasingly dense throughout the flow. In this picture, the scaling assumed in K41 becomes more and more accurate at very high  $\text{Re}$ . We will refer to this assumption as *asymptotic Kolmogorov scaling*.<sup>14</sup>

The behaviour of  $F(x, y)$  as  $x \rightarrow \infty$  is the central issue on which we focus. The manner in which  $F$  could fail to attain a finite limit is constrained by asymptotic covariance, which implies that

$$D_{LL}(r) = (\bar{\epsilon}r)^{2/3} G(\ln \text{Re}, r/L), \quad (3.2)$$

for some function  $G$  to be determined, and intermittency, *i.e.* incomplete similarity of  $G(x, y)$  in the limit  $y \rightarrow 0$ . Conventionally, it is assumed that the manner of violation of K41 is that  $G(x, y) \sim y^\alpha$  as  $y \rightarrow 0$ , where  $\alpha$  is an intermittency correction, assumed to be  $\text{Re}$ -independent and estimated experimentally to be positive. In contrast, we wish to investigate the consequences of assuming that as  $x \rightarrow \infty$  and  $y \rightarrow 0$

$$D_{LL}(r) = (\bar{\epsilon}r)^{2/3} A(\ln \text{Re}) \left( \frac{r}{L} \right)^{\alpha(\ln \text{Re})} \quad (3.3)$$

where the prefactor  $A$  and intermittency exponent  $\alpha$  depend on  $\ln \text{Re}$ . In particular, the assumption of asymptotic covariance implies that we can write

$$\alpha = \alpha_0 + \frac{\alpha_1}{\ln \text{Re}} + O((\ln \text{Re})^{-2}); \quad (3.4)$$

further assuming asymptotic Kolmogorov scaling implies that  $\alpha_0 = 0$ . Such a form is consistent with the experimental results of Castaing *et al.*<sup>15</sup> As before,



instead of a universal straight line in the  $D_{LL}(r)/(\bar{\epsilon}r)^{2/3} - (r/L)$  plane, we obtain a one parameter family of curves, occupying a certain portion of the plane. The boundary of this family is a universal, Re-independent curve, which satisfies both (3.3) and the condition  $\partial D_{LL}(r)/\partial \text{Re} = 0$ , *i.e.*

$$\frac{dA}{d(\ln \text{Re})} = A(\ln \text{Re}) \frac{\alpha_1 \ln(r/L)}{(\ln \text{Re})^2}. \quad (3.5)$$

Empirically it is found that the intermittency correction  $\alpha$  is positive, so that  $\alpha_1 > 0$ . However, in the inertial range,  $r \ll L$ , so that  $dA/d(\ln \text{Re}) < 0$ . Thus, we may expand  $A$  in the small parameter  $\epsilon \equiv 1/\ln \text{Re}$

$$A(\ln \text{Re}) = A_0 + A_1 \epsilon + O(\epsilon^{-2}), \quad (3.6)$$

where  $A_0, A_1$  are non-negative constants. There are two cases to consider: (a)  $A_0 \neq 0$ , and (b)  $A_0 = 0$ . We will briefly discuss the experimental estimates<sup>16</sup> of  $C_K$  in the following section, but for now, we remark that both the scatter and the recent report of a systematic dependence on  $\text{Re}$ <sup>17</sup> encourage us to consider not only the conventional case (a) but also case (b).

In case (a), both intermittency corrections to the exponent  $\alpha$  and to the Kolmogorov constant  $A_0$  vanish logarithmically as  $\ln \text{Re} \rightarrow \infty$ . More precisely, the relation (3.5) gives

$$\frac{\alpha_1 \ln(r/L)}{\ln \text{Re}} = -\frac{A_1}{\ln \text{Re}} \frac{1}{[A_0 + \frac{A_1}{\ln \text{Re}} + O((\ln \text{Re})^{-2})]} \quad (3.7)$$

so that the envelope is given by

$$D_{LL}^{\text{env}}(r) = (\bar{\epsilon}r)^{2/3} A_0 (1 + O(\ln(r/L))). \quad (3.8)$$

Note the correction term in this formula: the cancellations that occur to first order in  $\epsilon$  make the deviations from K41 very difficult to detect. Indeed, near the envelope, individual plots of  $D_{LL}(r)$  at fixed but large  $\text{Re}$  will exhibit corrections to K41 only of order  $O(\epsilon^2)$ .

In case (b), the results are more dramatic. The envelope condition (3.5) becomes

$$\frac{dA}{d(\ln \text{Re})} = -\frac{A}{\ln \text{Re}} \quad (3.9)$$

so that we obtain the condition:

$$1 = -\frac{\alpha_1 \ln(r/L)}{\ln \text{Re}}. \quad (3.10)$$

The universal boundary in the  $D_{LL}(r)/(\bar{\epsilon}r)^{2/3} - (r/L)$  plane is then represented

by the curve

$$\frac{D_{LL}(r)}{(\bar{\epsilon}r)^{2/3}} = -\frac{A_1}{e\alpha_1\ln(r/L)}. \quad (3.11)$$

The equation of a member of the family of curves at fixed  $\text{Re}$ , near the envelope is

$$D_{LL}(r) = (\bar{\epsilon}r)^{2/3} \frac{A_1}{e \ln \text{Re}}, \quad (3.12)$$

which is K41, but with a Kolmogorov constant  $C_K = A_1/e \ln \text{Re}_\lambda$ .

The phenomenological considerations given above also have implications for higher moments. For example, consider the third moment  $D_{LLL}(r) \equiv \langle (v_r)^3 \rangle$ , which is related to  $D_{LL}(r)$  by Kolmogorov's relation

$$D_{LLL}(r) = -\frac{4}{5}\bar{\epsilon}r + 6\nu \frac{dD_{LL}(r)}{dr}. \quad (3.13)$$

Using the form (3.3) for  $D_{LL}(r)$ , and assuming case (b) above, we find that for large  $\ln \text{Re}$

$$D_{LLL}(r) = -\frac{4}{5}\bar{\epsilon}r \left( 1 - \frac{5A_1}{\ln \text{Re}} \left( \frac{\eta}{r} \right)^{4/3} e^{-3\alpha_1/4} + O((\ln \text{Re})^{-2}) \right). \quad (3.14)$$

The correction to Kolmogorov's "4/5 law" is Reynolds number dependent, in contrast to the situation in K41.

## 4. Discussion

Sreenivasan<sup>16</sup> has recently surveyed the available data on  $C_K$ , for different flows and  $\text{Re}$ , and came to the conclusion that there is no Reynolds number dependence for the data taken as a whole. However, there is considerable scatter in the data, and as noted by Sreenivasan, a controlled series of measurements in a *single* flow geometry for a wide range of values of  $\text{Re}$  has yet to be performed.

To our knowledge, the only measurements which explore the systematics of the possible Reynolds number dependence of  $C_K$  are those of Praskovsky and Oncley,<sup>17</sup> in which a weak dependence of  $C_K$  on  $\text{Re}$  is reported. These authors attempted to fit this dependence with a power law, but, as shown in figure 1, their data are consistent with the logarithmic forms proposed here. Figure 1 shows the data, together with two best fits to the form given by (3.6), for the two cases  $A_0 \neq 0$  and  $A_0 = 0$ . The former case, with a non-zero limit of  $C_K$  as  $\text{Re} \rightarrow \infty$ , may be a slightly better fit than the latter. One measure of this

is shown in the insets of figure 1: Inset (A) shows the scatter when  $C_K \ln \text{Re}_\lambda$  is plotted against  $\text{Re}_\lambda$ , whilst inset (B) shows the scatter when  $\Delta C_K \ln \text{Re}_\lambda$  is plotted against  $\text{Re}_\lambda$ , where  $\Delta C_K \equiv C_K - 0.45$ . A slight upwards trend with increasing  $\text{Re}_\lambda$  may be discernable in (A), whereas the scatter in (B) seems to be more uniform. Clearly, these data (with only eight points) are not sufficient to draw any strong conclusions; nevertheless, they are not inconsistent with either of the two possibilities  $A_0 \neq 0$  or  $A_0 = 0$  in (3.6).

Therefore, it would be of considerable fundamental value to obtain a confirmation (or otherwise) of the results of Praskovsky and Oncley, with greater precision. In particular, evidence that  $A_0 = 0$  would indicate that fully-developed turbulence is not a unique, Reynolds number independent state, approached at large enough Reynolds number, but instead is a state with universal Reynolds number and external scale dependence, identical for different flows, but with no attainable limit.

In conclusion, we have made two main points in this paper. First, we have suggested that there may not be complete similarity in  $\text{Re}$  — in particular, we have suggested that the lack of complete similarity is exhibited as asymptotic covariance — and we have detailed some of the consequences of this. In particular, a universal scaling should be approached at large  $\text{Re}$ . Second, the universal scaling may be that of K41, indicating the existence of a unique fully-developed turbulent state, or may have no asymptotic limit, indicating the non-existence of such a simple state.

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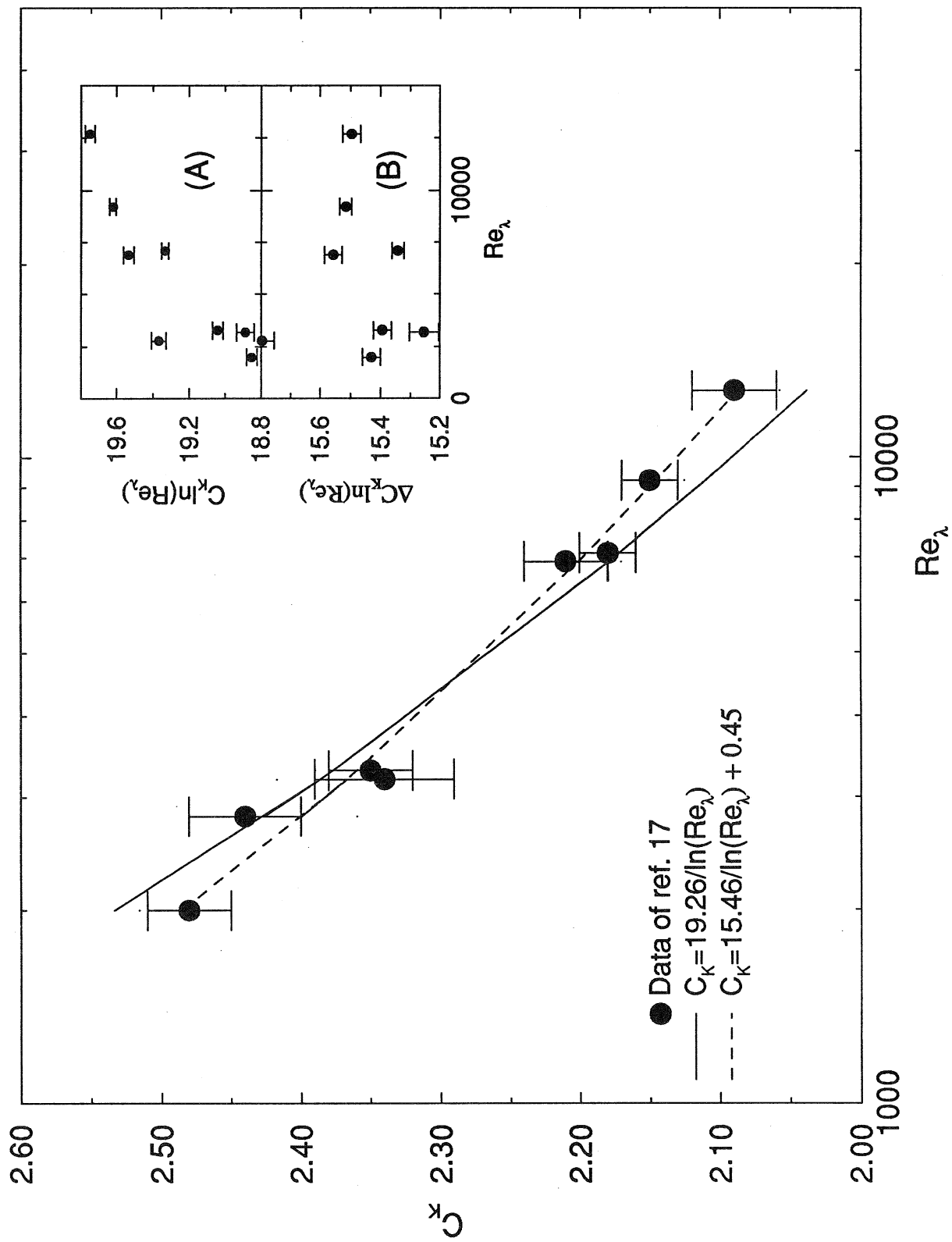
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13. An analogous situation arises in condensed matter systems subject to an external magnetic field  $\mathbf{H}$ ; in such cases, it is usually not appropriate to work with the magnetic induction  $\mathbf{B}$ , which includes also magnetization  $\mathbf{M}$ , *i.e.*, the response of the system to  $\mathbf{H}$ .
14. Note that asymptotic Kolmogorov scaling occurs in a recent diagrammatic analysis by V.S. L'vov and V.V. Lebedev, *JETP Lett.* **59**, 577 (1994); V.S. L'vov and I. Procaccia, *Phys. Rev. Lett.* **74**, 2690 (1995).
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## FIGURE CAPTIONS

Fig. 1. The Kolmogorov constant, as measured by Praskovsky and Oncley,<sup>17</sup> plotted on a logarithmic scale against Taylor microscale Reynolds number. Also shown are best fits to the functional form given by (3.6), for the two cases  $A_0 \neq 0$  and  $A_0 = 0$ . Inset (A) shows the scatter when  $C_K \ln \text{Re}_\lambda$  is plotted against  $\text{Re}_\lambda$ . Inset (B) shows the scatter when  $\Delta C_K \ln \text{Re}_\lambda$  is plotted against  $\text{Re}_\lambda$ , where  $\Delta C_K \equiv C_K - 0.45$ .



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