The influence of inertia on configurational forces in a deformable solid

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We propose a modification of Gurtin's (1994, 1995) approach to the theory of configurational forces, so as to account properly for inertia. Writing the configurational balance for an evolving part of the body in the form of a momentum balance, we arrive at a representation of the configurational stress that coincides with the dynamic energy-momentum tensor of Eshelby (1971) and also obtain a representation for the configurational momentum that coincides with the pseudomomentum of Maugin, Epstein & Trimarco (1992). An advantage of our formulation is that it leads to an expression for the internal configurational force that, even when inertia is taken into account, includes no inertial contribution. Rather, that force consists solely of terms that reflect changes in material structure such as those that may accompany processes such as defect propagation, phase transitions, and fracture.

1. Introduction

At the continuum-level, the description of any phenomenon involving a kinematic process in addition to that associated with the motion of material particles requires the introduction of supplemental ingredients, ingredients that lead ultimately to an equation that complements the equations arising from the standard balance laws and governs the evolution of the additional kinematic variable.

Such theories have arisen in connection with the modeling of defects (Peach & Koehler 1950; Eshelby 1951; Kosevich 1979), directed media such as liquid crystals (Ericksen 1960, 1961), plasticity (Kratochvil & Dillon 1969; Lubliner 1969; Rice 1971), phase transitions (Truskinovsky 1986; Abeyaratne & Knowles 1989; Gurtin & Struthers 1990), and fracture (Freund 1990; Gurtin & Podio-Guidugli 1996).

In early works on defect and crack theory, a central role was played by the notion of force acing on a defect, measuring the rate of change of the total energy under a change in the structure or position of the defect. It was Eshelby (1951, 1970) discovery that, at least in crack theory and the elastic inclusion problem, the expressions for these configurational forces could be unified by introducing a new tensor field, a field that he called the "energy-momentum tensor," whose divergence embodies the inhomogeneity of underlying medium. In the framework of elastostatics as a field theory, this tensor can, indeed, be associated with a conservation law of divergence-form.

Even so, for years this conservation law lacked the status of basic balance in continuum mechanics:† its condition as a derived, rather than a primitive, relation restricted its use as an heuristic tool in constructing theories of complex phenomena where, a-priori, little information concerning the configurational change process is available.

Recently, an innovative approach to this problem has been developed by Gurtin (1994, 1995), who, to describe situations where the material structure of a body may undergo configurational changes, incorporated configurational forces and an associated balance in the framework of modern continuum mechanics and thermodynamics. This approach leads to a further unification, encompassing a broader spectrum of known theories, and proves also to be a powerful heuristic tool in dealing with nonstandard phenomena.

The configurational forces introduced by Gurtin (1994, 1995) include a stress tensor and an internal body force. Given a part of a body, the configurational stress tensor generates a traction acting on the boundary of that part. That traction represents the resultant force exerted by the remainder of the body on that part, a force that may be thought of as driving possible configurational transformations within the part. The internal configurational force is indeterminate in the absence of configurational changes but otherwise is determined by a constitutive equation that governs the kinetics underlying such changes.

The configurational balance, which merely asserts that the total configurational force be balanced over each part the body, is a supplemental law to be imposed in addition to the standard balances present in a classical theory, with the understanding that it degenerate to an identity (in fact equivalent to the linear momentum balance) in the absence of configurational changes of the body. Upon localization, the configurational balance yields an equation with the classical divergence structure of a static balance.

In this context, inertial fields may be treated by retaining the static form for the configurational balance, but including the rates of change of momentum and kinetic energy in the deformational balances and dissipation imbalance. Nevertheless, doing so yields a representation for the internal force as the sum consisting of a contribution that responds to configurational changes and an additional term that reflects inertial effects.

Here, we take the view that the inclusion of the inertial terms in the internal force is inconsistent with the notion that the internal configurational force is a field that acts in response to configurational changes. We also believe that, when inertia is taken into account, a configurational momentum should exist and that the configurational balance should have the structure of a momentum balance.

This paper is organized as follows: first, in § 2, we review the formulation of the theory as proposed by Gurtin (1994, 1995); then, in § 3, we state our results—giving representations for configurational stress, configurational momentum, and internal configurational force. In § 4, we show how these representations may be

† In support of this assertion, we recall Eshelby's (1975) remark: "...the energy-momentum tensor and kindred concepts... are of interest for they own sake, but they have received scarcely any attention from applied mathematicians, even during the extensive re-examination and extension of continuum mechanics which has been under way for the last couple of decades."

derived using Gurtin's (1994, 1995) approach based on evolving parts of the body. Finally, in §§ 5 and 6, we address issues related to invariance under spatial and material changes of observer.

Throughout our discussion we ignore external body forces.

2. Preliminary considerations

Here, we give a brief overview of the theory of configurational forces due to Gurtin (1994, 1995).

Consider a body identified with the region $\mathcal{B}$, of three-dimensional space $\mathbb{R}^3$, that it occupies in a fixed reference configuration with mass density ϱ. Let $\boldsymbol{y} : \mathcal{B} \times I \to \mathbb{R}^3$ denote a smooth motion, over a time interval I, of $\mathcal{B}$ and write $\boldsymbol{F}$ and $\dot{\boldsymbol{y}}$ for the deformation gradient and particle velocity field. Further, let ψ and $\boldsymbol{S}$ denote, respectively, the free-energy density and first Piola-Kirchhoff stress.

Suppose, until further notice, that inertial effects are of negligible importance. Then, the standard statements imposing linear momentum balance and dissipation imbalance for a *fixed* part $\mathcal{P}$ of $\mathcal{B}$ are

$$\int_{\partial\mathcal{P}} \boldsymbol{S}\boldsymbol{m}\, da = \boldsymbol{0} \qquad \text{and} \qquad \overline{\int_{\mathcal{P}} \psi\, dv}^{\,\boldsymbol{\cdot}} \le \int_{\partial\mathcal{P}} \boldsymbol{S}\boldsymbol{m}\cdot\dot{\boldsymbol{y}}\, da, \tag{2.1}$$

with $\boldsymbol{m}$ the outward unit normal to $\partial\mathcal{P}$.

Consider now a part $\mathcal{R}$ of $\mathcal{B}$ whose boundary $\partial\mathcal{R}$ evolves in the direction of its outward unit normal $\boldsymbol{m}$ with scalar normal velocity $\mathcal{V}$. Then, letting $\boldsymbol{v}$ denote a generic velocity for $\partial\mathcal{R}$, we must have

$$\boldsymbol{v}\cdot\boldsymbol{m} = \mathcal{V}. \tag{2.2}$$

At each instant t of I, the image $\boldsymbol{y}(\partial\mathcal{R}(t), t)$ of $\partial\mathcal{R}(t)$ under the motion is in turn an evolving surface with velocity field

$$\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{v}. \tag{2.3}$$

Of fundamental importance in Gurtin's (1994, 1995) approach to configurational forces is the view that the motion of $\partial\mathcal{R}$ in the reference configuration $\mathcal{B}$ represents a kinematic process independent of the motion of material particles in the image $\boldsymbol{y}(\mathcal{R}, \cdot)$ of $\mathcal{R}$. As such, that process requires the introduction of an additional field, the *configurational stress* $\boldsymbol{C}$, that acts on the surface $\partial\mathcal{R}$ and governs its evolution. Intuitively, should $\mathcal{R}$ contain structural defects, the configurational traction associated with $\boldsymbol{C}$ represents the resultant of the total force exerted on those defects by that portion of $\mathcal{B}$ external to $\mathcal{R}$.

By definition, the configurational stress expends power over the velocity $\boldsymbol{v}$, so that its working on $\partial\mathcal{R}$ is

$$\int_{\partial\mathcal{R}} \boldsymbol{C}\boldsymbol{m}\cdot\boldsymbol{v}\, da. \tag{2.4}$$

A further premise of Gurtin's (1994, 1995) approach is that, for an evolving part $\mathcal{R}$ of $\mathcal{B}$, the deformational stress should act over the velocity $\mathring{\boldsymbol{y}}$. Hence, the working on $\partial\mathcal{R}$ of the deformational stress has the form

$$\int_{\partial\mathcal{R}} \boldsymbol{Sm}\cdot\mathring{\boldsymbol{y}}\, da. \tag{2.5}$$

Now, in light of (2.4) and (2.5), the dissipation imbalance for $\mathcal{R}$ takes the form

$$\overline{\int_{\mathcal{R}} \psi\, dv} \le \int_{\partial\mathcal{R}} \boldsymbol{Cm}\cdot\boldsymbol{v}\, da + \int_{\partial\mathcal{R}} \boldsymbol{Sm}\cdot\mathring{\boldsymbol{y}}\, da, \tag{2.6}$$

and invariance of the working with respect to the choice of the velocity field $\boldsymbol{v}$ for $\partial\mathcal{R}$ together with the arbitrary nature of $\mathcal{R}$ yields a representation,

$$\boldsymbol{C} = \psi\boldsymbol{1} - \boldsymbol{F}^T\boldsymbol{S}, \tag{2.7}$$

identifying the configurational stress tensor with the Eshelby (1971, equation (13)) tensor.

To study situations involving configurational changes, a configurational balance of the form

$$\int_{\partial\mathcal{R}} \boldsymbol{Cm}\, da + \int_{\mathcal{R}} \boldsymbol{f}\, dv = \boldsymbol{0} \tag{2.8}$$

is postulated, where $\boldsymbol{f}$, the *internal configurational force*, is an object that acts in response to configurational changes. Notice that a power expenditure associated with $\boldsymbol{f}$ is not included in the dissipation imbalance (2.6), which accounts only for the power expended by those forces acting *external* to $\mathcal{R}$.

Taking the view (cf. Gurtin 1994, 1995), that the internal configurational force $\boldsymbol{f}$ is determined by the balance (2.8), localizing that balance about a point where $\boldsymbol{C}$ is differentiable, and invoking the representation (2.7) for $\boldsymbol{C}$ and to the local version of the linear momentum balance $(2.1)_1$, we obtain an expression

$$\boldsymbol{f} = -\nabla\psi + (\nabla(\boldsymbol{F}^T))\boldsymbol{S} \tag{2.9}$$

for that force. Here $(\nabla(\boldsymbol{F}^T))\boldsymbol{S}$ is the vector field defined by the requirement that

$$[(\nabla(\boldsymbol{F}^T))\boldsymbol{S}]\cdot\boldsymbol{g} = (\nabla(\boldsymbol{Fg}))\cdot\boldsymbol{S} - (\boldsymbol{F}^T\boldsymbol{S})\cdot(\nabla\boldsymbol{g}) \tag{2.10}$$

hold for any vector field $\boldsymbol{g}$.

Considering, for instance, an inhomogeneous elastic material, so that ϱ depends on position $\boldsymbol{x}$,

$$\psi(\boldsymbol{x}, t) = \hat{\psi}(\boldsymbol{F}(\boldsymbol{x}, t), \boldsymbol{x}), \qquad \text{and} \qquad \boldsymbol{S} = \frac{\partial}{\partial\boldsymbol{F}}\hat{\psi}(\boldsymbol{F}, \cdot), \tag{2.11}$$

(2.9) reduces to

$$\boldsymbol{f}(\boldsymbol{x}, t) = -\nabla\hat{\psi}(\boldsymbol{A}, \boldsymbol{x})|_{\boldsymbol{A}=\boldsymbol{F}(\boldsymbol{x},t)}. \tag{2.12}$$

To illustrate the importance of the configurational balance (2.8) in situations involving configurational changes, consider a theory for coherent solid-solid phase transitions in which the bulk phases are modeled as elastic, and phase interfaces are treated as sharp nonmaterial surfaces across which $\boldsymbol{y}$ is continuous but $\boldsymbol{F}$ and $\dot{\boldsymbol{y}}$ may suffer finite jump discontinuities. Let $\mathfrak{S}$ be such a surface and let $\boldsymbol{n}$ and $V_{\mathfrak{S}}$ denote its orientation and scalar normal velocity. To account for transition kinetics, the framework must be extended to include an *interfacial internal configurational force* $\mathscr{f}$ (which may be viewed as the singular part of a distributional internal configurational force), whereby the configurational balance becomes

$$\int_{\partial\mathcal{P}} \boldsymbol{C}\boldsymbol{m}\,da + \int_{\mathcal{P}} \boldsymbol{f}\,dv + \int_{\mathfrak{S}\cap\mathcal{P}} \mathscr{f}\,da = \boldsymbol{0}. \tag{2.13}$$

In this setting, the expressions (2.7) and (2.9) for $\boldsymbol{C}$ and $\boldsymbol{f}$ remain valid, while $\mathscr{f}$ must have the form

$$\mathscr{f} = f\mathbb{n}. \tag{2.14}$$

Localizing the balances $(2.1)_1$ and (2.13) and the imbalance $(2.1)_2$ at a point on $\mathfrak{S}$, we obtain

$$[\![\boldsymbol{S}\mathbb{n}]\!] = \boldsymbol{0}, \qquad [\![\boldsymbol{C}\mathbb{n}]\!] + \mathscr{f} = \boldsymbol{0}, \qquad \text{and} \qquad fV_{\mathfrak{S}} \geq 0. \tag{2.15}$$

Granted that f is given by a constitutive response function depending on $\mathbb{n}$, $V_{\mathfrak{S}}$, and the tangential deformation gradient $\mathbb{F}$, $(2.15)_3$ implies (cf. Gurtin 1994, equation (3.7)) that

$$f = -bV_{\mathfrak{S}}, \tag{2.16}$$

with $b = \hat{b}(\mathbb{F}, \mathbb{n}, V_{\mathfrak{S}}) \geq 0$ the *kinetic coefficient.* Thus the normal component of configurational balance $(2.15)_2$ yields the evolution equation

$$\mathbb{n}\cdot[\![(\psi\boldsymbol{1} - \boldsymbol{F}^T\boldsymbol{S})\mathbb{n}]\!] = bV_{\mathfrak{S}} \tag{2.17}$$

for the interface $\mathfrak{S}$.

The above construction can be modified to account for inertia. In doing so, Gurtin (1994) suggests that, for an evolving part $\mathcal{R}$ of $\mathcal{B}$, the momentum balance and dissipation imbalance should be written as

$$\left.\begin{aligned} \int_{\partial\mathcal{R}} (\boldsymbol{S}\boldsymbol{m} + \varrho\mathcal{V}\dot{\boldsymbol{y}})\,da &= \overline{\int_{\mathcal{R}} \varrho\dot{\boldsymbol{y}}\,dv}^{\,\boldsymbol{\cdot}}, \\ \overline{\int_{\mathcal{R}} (\psi + \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)\,dv}^{\,\boldsymbol{\cdot}} - \int_{\partial\mathcal{R}} \tfrac{1}{2}\varrho\mathcal{V}|\dot{\boldsymbol{y}}|^2\,da &\leq \int_{\partial\mathcal{R}} (\boldsymbol{C}\boldsymbol{m}\cdot\boldsymbol{v} + \boldsymbol{S}\boldsymbol{m}\cdot\mathring{\boldsymbol{y}})\,da. \end{aligned}\right\} \tag{2.18}$$

Having done so, the representation for the configurational stress remains as given in (2.7), while the expression for the internal configurational force is generally altered. To illustrate this consider the particular case of coherent solid-solid phase transitions, in which case (2.18) yield the modifications

$$\boldsymbol{f} = -\nabla\psi + \tfrac{1}{2}|\dot{\boldsymbol{y}}|^2\nabla\varrho + (\nabla(\boldsymbol{F}^T))\boldsymbol{S} + \varrho\boldsymbol{F}^T\ddot{\boldsymbol{y}} \tag{2.19}$$

and

$$f = -bV_{\mathfrak{S}} + \tfrac{1}{2}\varrho V_{\mathfrak{S}}^2[\![|\boldsymbol{F}\mathbb{n}|^2]\!] \tag{2.20}$$

of (2.9) and (2.16), modifications wherein the terms $\varrho\boldsymbol{F}^T\ddot{\boldsymbol{y}}$ and $\frac{1}{2}\varrho V_{\mathfrak{S}}^2[\![|\boldsymbol{F}\mathbb{n}|^2]\!]$ account for inertial effects.†

† Observe that, by virtue of the identity $[\![\dot{\boldsymbol{y}}]\!] = -V_{\Sigma}[\![\boldsymbol{F}\mathbb{n}]\!]$, the quantities $\varrho\boldsymbol{F}^T\ddot{\boldsymbol{y}}$ and $\frac{1}{2}\varrho V_{\Sigma}^2[\![|\boldsymbol{F}\mathbb{n}|^2]\!]$ may be regarded as the regular and singular part, respectively, of a distributional field.

3. Representations. Configurational balance

The role of the internal configurational force is to act in response to alterations of material structure. To serve in that capacity, the internal configurational force should be constitutively determined and free of inertial terms—which may contribute to the internal force in the absence of configurational changes.‡

In response to this difficulty we argue that, so long as inertia (and, hence, kinetic energy) is taken into account:

(*i*) the bulk configurational stress should have the form

$$\boldsymbol{C} = (\psi - \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)\boldsymbol{1} - \boldsymbol{F}^T\boldsymbol{S}, \tag{3.1}$$

which reflects the influence of kinetic energy and coincides with the *dynamic energy-momentum tensor* of Eshelby (1971, equation (53));

(*ii*) the internal configurational force should have the form

$$\boldsymbol{f} = -\nabla\psi + \tfrac{1}{2}|\dot{\boldsymbol{y}}|^2\nabla\varrho + (\nabla(\boldsymbol{F}^T))\boldsymbol{S}, \tag{3.2}$$

which accounts for spatial variations in the free energy density, mass density, and deformation gradient but includes no inertial effects;

(*iii*) we should account for configurational momentum $\boldsymbol{q}$, which admits the representation

$$\boldsymbol{q} = -\varrho\boldsymbol{F}^T\dot{\boldsymbol{y}} \tag{3.3}$$

and corresponds to the *pseudomomentum* of Maugin, Epstein & Trimarco (1992, equation (16));

(*iv*) the configurational balance for a part $\mathcal{P}$ of $\mathcal{B}$ should be a *momentum* balance, having the form

$$\int_{\partial\mathcal{P}} \boldsymbol{Cm}\,da + \int_{\mathcal{P}} \boldsymbol{f}\,dv = \overline{\int_{\mathcal{P}} \boldsymbol{q}\,dv}^{\,\cdot}. \tag{3.4}$$

Bearing in mind the expressions (3.1), (3.2), and (3.3) for $\boldsymbol{C}$, $\boldsymbol{f}$, and $\boldsymbol{q}$ and the identity

$$\dot{\boldsymbol{q}} + \nabla(\tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2) = \tfrac{1}{2}|\dot{\boldsymbol{y}}|^2\nabla\varrho - \varrho\boldsymbol{F}^T\ddot{\boldsymbol{y}}, \tag{3.5}$$

the balance (3.4) is equivalent to the balance

$$\int_{\partial\mathcal{P}} \boldsymbol{C}'\boldsymbol{m}\,da + \int_{\mathcal{P}} \boldsymbol{f}'\,dv = \boldsymbol{0} \tag{3.6}$$

involving $\boldsymbol{C}' = \psi\boldsymbol{1} - \boldsymbol{F}^T\boldsymbol{S}$ and $\boldsymbol{f}' = -\nabla\psi + (\nabla(\boldsymbol{F}^T))\boldsymbol{S} + \varrho\boldsymbol{F}^T\ddot{\boldsymbol{y}}$.

Consider, now, a coherent solid-solid phase transition involving an interface $\mathfrak{S}$ with orientation $\mathbb{m}$ and scalar normal velocity $V_{\mathfrak{S}}$. Suppose that $\mathfrak{S}$ possesses an *interfacial energy density* $\psi_{\mathfrak{S}}$, an *interfacial deformational stress* $\mathbb{S}$, an *interfacial configurational stress* $\mathbb{C}$, and an *interfacial internal configurational force* $\mathbb{f}$, but is massless. Then, the configurational balance (3.4) should be modified to read

$$\int_{\partial\mathcal{P}} \boldsymbol{Cm}\,da + \int_{\mathcal{P}} \boldsymbol{f}\,dv + \int_{\partial(\mathfrak{S}\cap\mathcal{P})} \mathbb{C}\mathbb{m}\,dl + \int_{\mathfrak{S}\cap\mathcal{P}} \mathbb{f}\,da = \overline{\int_{\mathcal{P}} \boldsymbol{q}\,dv}^{\,\cdot}, \tag{3.7}$$

‡ This is indeed true of the expressions for $\boldsymbol{f}$ and f obtained by Gurtin (1995) ignoring inertia (cf. (2.9) and (2.16)).

with $\mathbb{m}$ the outward unit normal to $\partial(\mathfrak{S}\cap\mathcal{P})$. Under these circumstances, (3.1), (3.2), and (3.3) remain valid and we find that

$$\mathbb{C} = \psi_{\mathfrak{S}}\mathbb{P} - \mathbb{F}^T\mathbb{S} + \mathbb{n}\otimes\mathbb{c} \tag{3.8}$$

and

$$\mathbb{f} = f\mathbb{n} - \nabla_{\mathfrak{S}}\psi_{\mathfrak{S}} + (\nabla_{\mathfrak{S}}(\mathbb{F}^T))\mathbb{S} + \mathbb{L}\mathbb{c}, \tag{3.9}$$

with $\mathbb{P} = \mathbf{1} - \mathbb{n}\otimes\mathbb{n}$ the *interfacial projector*, $\mathbb{F} = \nabla_{\mathfrak{S}}(\boldsymbol{y}|_{\mathfrak{S}}) = \boldsymbol{F}\mathbb{P}$ the *interfacial deformation gradient*, $\mathbb{L} = -\nabla_{\mathfrak{S}}\mathbb{n}$ the *interfacial curvature tensor*, and $\mathbb{c} = \mathbb{C}^T\mathbb{n}$ the *interfacial configurational shear.*

4. Derivation of configurational fields and the associated balance

Following Gurtin (1994, 1995), we consider an evolving part $\mathcal{R}$ and use $\boldsymbol{m}$ and $\boldsymbol{v}$ to denote the unit outward normal field to $\partial\mathcal{R}$ and a velocity field for $\partial\mathcal{R}$. We also write $\mathcal{V} = \boldsymbol{v}\cdot\boldsymbol{m}$ for the normal velocity of $\partial\mathcal{R}$. For simplicity, we restrict attention to the situation discussed by Gurtin (1994, § 3) (see also Gurtin 1995, § 3), where neither phase interfaces nor defects are present. Extension of our approach to more general situations is straightforward.

(*a*) *Linear momentum balance for an evolving part*

For an evolving part $\mathcal{R}$ of $\mathcal{B}$, Gurtin (1994, equation $(3.2)_1$) writes the linear momentum balance as

$$\int_{\partial\mathcal{R}} (\boldsymbol{Sm} + \mathcal{V}\boldsymbol{p})\,da = \overline{\int_{\mathcal{R}} \boldsymbol{p}\,dv}^{\,\boldsymbol{\cdot}}, \tag{4.1}$$

with $\boldsymbol{p} = \varrho\dot{\boldsymbol{y}}$ the linear momentum. Introducing

$$\boldsymbol{S}^{\text{eff}} = \boldsymbol{S} + \mathcal{V}\boldsymbol{p}\otimes\boldsymbol{m}, \tag{4.2}$$

the *effective deformational stress* on $\partial\mathcal{R}$, we may write (4.1) in the form

$$\int_{\partial\mathcal{R}} \boldsymbol{S}^{\text{eff}}\boldsymbol{m}\,da = \overline{\int_{\mathcal{R}} \boldsymbol{p}\,dv}^{\,\boldsymbol{\cdot}}. \tag{4.3}$$

(*b*) *Configurational momentum balance for an evolving part*

To account for configurational changes, we introduce a *configurational stress* $\boldsymbol{C}$, an *internal configurational force* $\boldsymbol{f}$, and a *configurational momentum* $\boldsymbol{q}$ and require that $\boldsymbol{C}$, $\boldsymbol{f}$, and $\boldsymbol{q}$ satisfy, for each fixed subregion $\mathcal{P}$ of $\mathcal{B}$, the *configurational balance*

$$\int_{\partial\mathcal{P}} \boldsymbol{Cm}\,da + \int_{\mathcal{P}} \boldsymbol{f}\,dv = \overline{\int_{\mathcal{P}} \boldsymbol{q}\,dv}^{\,\boldsymbol{\cdot}}, \tag{4.4}$$

or that the equivalent local balance

$$\text{div}\boldsymbol{C} + \boldsymbol{f} = \dot{\boldsymbol{q}} \tag{4.5}$$

hold on $\mathcal{B}$.

For an evolving part $\mathcal{R}$, we write (4.4) in a form analogous to (4.1)

$$\int_{\partial\mathcal{R}} (\boldsymbol{C}\boldsymbol{m} + \mathcal{V}\boldsymbol{q})\,da + \int_{\mathcal{R}} \boldsymbol{f}\,dv = \overline{\int_{\mathcal{R}} \boldsymbol{q}\,dv}^{\boldsymbol{\cdot}}, \tag{4.6}$$

which suggests that we define the *effective configurational stress* on $\partial\mathcal{R}$ to be

$$\boldsymbol{C}^{\text{eff}} = \boldsymbol{C} + \mathcal{V}\boldsymbol{q}\otimes\boldsymbol{m}. \tag{4.7}$$

(c) *Total power expended on an evolving part. Dissipation imbalance*

In reckoning the working of the deformational traction acting on $\partial\mathcal{R}$, we posit that it is the *effective deformational traction* $\boldsymbol{S}^{\text{eff}}\boldsymbol{m}$ that is active. Further, following Gurtin (1994, 1995), we require that $\boldsymbol{S}^{\text{eff}}\boldsymbol{m}$ expend power over the rate $\mathring{\boldsymbol{y}} = \dot{\boldsymbol{y}} + \boldsymbol{F}\boldsymbol{v}$ at which $\boldsymbol{y}$ changes following the motion of $\partial\mathcal{R}$ as defined by $\boldsymbol{v}$.

In the same spirit, we suppose that it is the *effective configurational stress* $\boldsymbol{C}^{\text{eff}}$ that must be accounted for when writing the configurational contribution to the working. Hence the net power expended on $\mathcal{R}$ by deformational and configurational forces is

$$W(\mathcal{R}) = \int_{\partial\mathcal{R}} (\boldsymbol{C}^{\text{eff}}\boldsymbol{m}\cdot\boldsymbol{v} + \boldsymbol{S}^{\text{eff}}\boldsymbol{m}\cdot\mathring{\boldsymbol{y}})\,da. \tag{4.8}$$

We suppose that the dissipation imbalance for $\mathcal{R}$ reads, simply,

$$\overline{\int_{\mathcal{R}} (\psi + \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)\,dv}^{\boldsymbol{\cdot}} \le W(\mathcal{R}). \tag{4.9}$$

Our treatment of the free and kinetic energy terms appearing in the imbalance (4.9) is symmetric, unlike that of Gurtin (1994) (cf. $(2.18)_2$), who includes an efflux term involving kinetic, but not free, energy. Here no such efflux appears.

(d) *Invariance under reparametrization*

We begin by using the definitions (2.3), (4.2), and (4.7) of $\mathring{\boldsymbol{y}}$, $\boldsymbol{S}^{\text{eff}}$, and $\boldsymbol{C}^{\text{eff}}$ to rewrite (4.8) in the form

$$W(\mathcal{R}) = \int_{\partial\mathcal{R}} \boldsymbol{S}\boldsymbol{m}\cdot\dot{\boldsymbol{y}}\,da + \int_{\partial\mathcal{R}} \mathcal{V}\boldsymbol{p}\cdot\dot{\boldsymbol{y}}\,da + \int_{\partial\mathcal{R}} (\boldsymbol{G}\boldsymbol{m} + \mathcal{V}\boldsymbol{g})\cdot\boldsymbol{v}\,da, \tag{4.10}$$

with $\boldsymbol{G}$ and $\boldsymbol{g}$ defined by

$$\boldsymbol{G} = \boldsymbol{C} + \boldsymbol{F}^{T}\boldsymbol{S} \qquad \text{and} \qquad \boldsymbol{g} = \boldsymbol{q} + \boldsymbol{F}^{T}\boldsymbol{p}. \tag{4.11}$$

Following Gurtin (1994, 1995), we stipulate that the working $W(\mathcal{R})$ be invariant under changes of parametrization of $\partial\mathcal{R}$. Since $\boldsymbol{v}\cdot\boldsymbol{m} = \mathcal{V}$ is intrinsic to the motion of $\partial\mathcal{R}$, such changes affect the tangential component of $\boldsymbol{v}$, but leave the normal component of $\boldsymbol{v}$ unaltered. Hence, the requirement that $W(\mathcal{R})$ be invariant under reparametrization of $\partial\mathcal{R}$ is equivalent to the requirement that $(\boldsymbol{G}\boldsymbol{m} + \mathcal{V}\boldsymbol{g})\cdot\boldsymbol{l} = 0$ for any tangential vector field $\boldsymbol{l}$ on $\partial\mathcal{R}$; hence, there exists a scalar field Φ such that

$$\boldsymbol{G}\boldsymbol{m} + \mathcal{V}\boldsymbol{g} = \Phi\boldsymbol{m}, \tag{4.12}$$

and $W(\mathcal{R})$ can be written in the intrinsic form

$$W(\mathcal{R}) = \int_{\partial\mathcal{R}} \boldsymbol{Sm}\cdot\dot{\boldsymbol{y}}\, da + \int_{\partial\mathcal{R}} (\Phi + \boldsymbol{p}\cdot\dot{\boldsymbol{y}})\mathcal{V}\, da. \tag{4.13}$$

Now, the identity (4.12) must hold for any evolving part $\partial\mathcal{R}$, or equivalently for any $\mathcal{V}$ and $\boldsymbol{m}$. Choosing $\mathcal{V}$ arbitrarily this implies that $\boldsymbol{g}$ is proportional to $\boldsymbol{m}$ for any $\boldsymbol{m}$, so that

$$\boldsymbol{g} = \boldsymbol{0}, \tag{4.14}$$

which in turn implies that $\boldsymbol{Gm} = \Phi\boldsymbol{m}$, for any $\boldsymbol{m}$, i.e.

$$\boldsymbol{G} = \Phi\boldsymbol{1}. \tag{4.15}$$

Next, drawing on the identity

$$\overline{\int_{\mathcal{R}} g\, dv}^{\,\cdot} = \int_{\mathcal{R}} \dot{g}\, dv + \int_{\partial\mathcal{R}} g\mathcal{V}\, da, \tag{4.16}$$

with $g = \psi$, and recalling that $\boldsymbol{p} = \varrho\dot{\boldsymbol{y}}$, we may use (4.13) to write the dissipation imbalance (4.9) as

$$\int_{\mathcal{R}} \overline{(\psi + \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)}^{\,\cdot}\, dv \le \int_{\partial\mathcal{R}} \boldsymbol{Sm}\cdot\dot{\boldsymbol{y}}\, da + \int_{\partial\mathcal{R}} (\Phi - (\psi - \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2))\mathcal{V}\, da. \tag{4.17}$$

This inequality must hold for any evolving part $\mathcal{R}$ of $\mathcal{B}$ and thus, as it is always possible to find a second evolving part $\mathcal{R}'$ such that, at a given instant t, $\mathcal{R}'(t) = \mathcal{R}(t)$ but for which $\mathcal{V}'(\cdot, t)$ and $\mathcal{V}(\cdot, t)$ are not necessarily coincident, (4.17) implies that

$$\Phi = \psi - \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2. \tag{4.18}$$

Thus, in concert with the results (4.15) and (4.14), the definitions (4.11) of $\boldsymbol{G}$ and $\boldsymbol{g}$ yield representations

$$\boldsymbol{C} = (\psi - \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)\boldsymbol{1} - \boldsymbol{F}^T\boldsymbol{S} \qquad \text{and} \qquad \boldsymbol{q} = -\boldsymbol{F}^T\boldsymbol{p} \tag{4.19}$$

for the configurational stress $\boldsymbol{C}$ and the configurational momentum $\boldsymbol{q}$.

(*e*) *Internal configurational force*

As we are supposing that no structural change is present, the internal configurational force $\boldsymbol{f}$ is not determined by a constitutive relation, so that we view the configurational momentum balance (4.5) as *defining* $\boldsymbol{f}$. Hence, on inserting the representations (4.19) for $\boldsymbol{C}$ and $\boldsymbol{q}$ into that balance and appealing to the local statement of linear momentum balance, we obtain

$$\boldsymbol{f} = -\nabla\psi + \tfrac{1}{2}|\dot{\boldsymbol{y}}|^2\nabla\varrho + (\nabla(\boldsymbol{F}^T))\boldsymbol{S}. \tag{4.20}$$

Observe that, in terms of the free energy per unit mass ϕ, the expression (4.20) for $\boldsymbol{f}$ takes the form

$$\boldsymbol{f} = -\varrho\nabla\phi - (\phi - \tfrac{1}{2}|\dot{\boldsymbol{y}}|^2)\nabla\varrho + (\nabla(\boldsymbol{F}^T))\boldsymbol{S}. \tag{4.21}$$

If, in particular, the material is elastic but *inhomogeneous*, (4.20) implies that

$$\boldsymbol{f}(\boldsymbol{x}, t) = -\varrho(\boldsymbol{x})\nabla\hat{\phi}(\boldsymbol{A}, \boldsymbol{x})|_{\boldsymbol{A}=\boldsymbol{F}(\boldsymbol{x},t)} - (\hat{\phi}(\boldsymbol{F}, \boldsymbol{x}) - \tfrac{1}{2}|\dot{\boldsymbol{y}}(\boldsymbol{x}, t)|^2)\nabla\varrho(\boldsymbol{x}). \tag{4.22}$$

As a result of (4.22), the bulk internal configurational force must vanish in a homogeneous and elastic material—regardless of inertial effects.

5. Consequences of invariance under spatial observer changes

Consider a Galilean spatial change of observer

$$\boldsymbol{y} \mapsto \boldsymbol{y}^* = \boldsymbol{Q}(\boldsymbol{y} + t\boldsymbol{s}), \tag{5.1}$$

with $\boldsymbol{s}$ a constant vector and $\boldsymbol{Q}$ a fixed rotation. Given a field g, let g^* denote the associated field as viewed under the change of observer. Assume, as is standard, that the mass density ϱ, the energy density ψ, and the stress $\boldsymbol{S}$ transform according to

$$\varrho^* = \varrho, \qquad \psi^* = \psi, \qquad \boldsymbol{S}^* = \boldsymbol{Q}\boldsymbol{S}. \tag{5.2}$$

Now, the simple consequences

$$|\dot{\boldsymbol{y}}^*|^2 = |\dot{\boldsymbol{y}}|^2 + 2\dot{\boldsymbol{y}}\cdot\boldsymbol{s} + |\boldsymbol{s}|^2 \qquad \text{and} \qquad \boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F} \tag{5.3}$$

of (5.1) and the expressions $(4.19)_1$, (4.20), and $(4.19)_2$ determining $\boldsymbol{C}$, $\boldsymbol{f}$, and $\boldsymbol{q}$ imply that that, under the change of observer, the configurational fields transform according to

$$\left.\begin{aligned} \boldsymbol{C}^* &= \boldsymbol{C} - \varrho(\dot{\boldsymbol{y}}\cdot\boldsymbol{s} + \tfrac{1}{2}|\boldsymbol{s}|^2)\boldsymbol{1}, \\ \boldsymbol{f}^* &= \boldsymbol{f} + (\dot{\boldsymbol{y}}\cdot\boldsymbol{s} + \tfrac{1}{2}|\boldsymbol{s}|^2)\nabla\varrho, \\ \boldsymbol{q}^* &= \boldsymbol{q} - \varrho\boldsymbol{F}^T\boldsymbol{s}; \end{aligned}\right\} \tag{5.4}$$

whereby a direct calculation yields

$$\int_{\partial\mathcal{R}} (\boldsymbol{C}^{\text{eff}})^*\boldsymbol{m}\,da + \int_{\mathcal{R}} \boldsymbol{f}^*\,dv - \overline{\int_{\mathcal{R}} \boldsymbol{q}^*\,dv}^{\,\cdot} = \int_{\partial\mathcal{R}} \boldsymbol{C}^{\text{eff}}\boldsymbol{m}\,da + \int_{\mathcal{R}} \boldsymbol{f}\,dv - \overline{\int_{\mathcal{R}} \boldsymbol{q}\,dv}^{\,\cdot}, \tag{5.5}$$

for any evolving part $\mathcal{R}$ of $\mathcal{B}$, showing that, granted the expression (4.20) for the internal force, the configurational momentum balance is invariant under Galilean spatial observer changes.

In standard theories where configurational forces are superfluous, the foregoing requirement is a fundamental axiom, an axiom typically imposed tacitly in the formulation of the linear momentum balance (see, for example, Noll 1963). Here, we require that it hold also in the presence of configurational changes, where (4.20) is no longer a trivial identity, the internal force being determined by a constitutive relation.

As a further application of these ideas, we show that the total energy dissipated

$$\Delta(\mathcal{R}) = W(\mathcal{R}) - \overline{\int_{\mathcal{R}} (\psi + \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)\,dv}^{\,\cdot} \tag{5.6}$$

within an evolving part $\mathcal{R}$ is invariant under Galilean spatial observer changes if and only if mass and linear momentum balance hold on $\mathcal{R}$. Toward this, we first compute directly that

$$\left.\begin{aligned} (\boldsymbol{C}^{\text{eff}})^*\boldsymbol{m}\cdot\boldsymbol{v} &= \boldsymbol{C}^{\text{eff}}\boldsymbol{m}\cdot\boldsymbol{v} - \varrho\mathcal{V}(\mathring{\boldsymbol{y}}\cdot\boldsymbol{s} + \tfrac{1}{2}|\boldsymbol{s}|^2), \\ (\boldsymbol{S}^{\text{eff}})^*\boldsymbol{m}\cdot\mathring{\boldsymbol{y}}^* &= \boldsymbol{S}^{\text{eff}}\boldsymbol{m}\cdot\mathring{\boldsymbol{y}} + \boldsymbol{S}^{\text{eff}}\boldsymbol{m}\cdot\boldsymbol{s} + \varrho\mathcal{V}(\mathring{\boldsymbol{y}}\cdot\boldsymbol{s} + |\boldsymbol{s}|^2), \end{aligned}\right\} \tag{5.7}$$

whereby the variation of the dissipation reads

$$\Delta^*(\mathcal{R}) - \Delta(\mathcal{R}) = \tfrac{1}{2}\Big(\int_{\partial\mathcal{R}} \varrho\mathcal{V}\,da - \overline{\int_{\mathcal{R}} \varrho\,dv}^{\,\cdot}\Big)|\boldsymbol{s}|^2 + \Big(\int_{\partial\mathcal{R}} \boldsymbol{S}^{\text{eff}}\boldsymbol{m}\,da - \overline{\int_{\mathcal{R}} \boldsymbol{p}\,dv}^{\,\cdot}\Big)\cdot\boldsymbol{s}. \quad (5.8)$$

As a polynomial in $\boldsymbol{s}$, that variation vanishes identically if and only if the corresponding coefficients do, which yields, for a part $\mathcal{R}$ evolving with scalar normal velocity $\mathcal{V}$, the mass balance

$$\int_{\partial\mathcal{R}} \varrho\mathcal{V}\,da = \overline{\int_{\mathcal{R}} \varrho\,dv}^{\,\cdot} \quad (5.9)$$

and the linear momentum balance (4.3).

6. Consequences of invariance under material observer changes

In analogy with the treatment of Gurtin & Struthers (1990), we may prove that the configurational balance holds if and only if the total dissipation is invariant under material observer changes. The basic role of such observer changes in a mathematical theory of continuous media is to detect inhomogeneities, defects and, in general, phenomena related to configurational changes. If $\boldsymbol{x} \in \mathbb{R}^3$, a *material observer change* is a time-dependent translation of the form

$$\boldsymbol{x} \mapsto \boldsymbol{x}^* = \boldsymbol{x} + t\boldsymbol{c}, \quad (6.1)$$

with $\boldsymbol{c}$ a constant vector. Let $\boldsymbol{y}^*$ be the motion relative to moving material observer, so that $\boldsymbol{y}^*(\boldsymbol{x}^*, t) = \boldsymbol{y}(\boldsymbol{x}, t)$. Then, in terms of $\boldsymbol{y}^*$, the velocity at time t of a material particle $\boldsymbol{x}$ in $\mathcal{B}$ is determined by

$$\dot{\boldsymbol{y}}^*(\boldsymbol{x}^*, t) = \frac{\partial}{\partial t}\boldsymbol{y}^*(\boldsymbol{x} + t\boldsymbol{c}, t) = \frac{\partial}{\partial t}\boldsymbol{y}^*(\boldsymbol{z}, t)|_{\boldsymbol{z}=\boldsymbol{x}^*} + \boldsymbol{F}^*(\boldsymbol{x}^*, t)\boldsymbol{c}. \quad (6.2)$$

Given a moving part $\mathcal{R}$, with velocity field $\boldsymbol{v}$ for $\partial\mathcal{R}$, we thus have

$$\dot{\boldsymbol{y}}^* = \dot{\boldsymbol{y}}, \qquad \boldsymbol{v}^* = \boldsymbol{v} + \boldsymbol{c}, \qquad \text{and} \qquad \mathring{\boldsymbol{y}}^* = \mathring{\boldsymbol{y}}, \quad (6.3)$$

where, as before, a superposed asterisk indicates that the corresponding field is measured by the moving observer. Notice that the velocity following reference material points is invariant under changes of material observers.

We assume that the linear and configurational momentum and the internal force, being quantities naturally associated with material particles, are invariant under changes of material observer (thus these fields depend only on the velocity of material particles),

$$\boldsymbol{p}^* = \varrho^*\dot{\boldsymbol{y}}^* = \boldsymbol{p}, \qquad \boldsymbol{q}^* = -(\boldsymbol{F}^*)^T\boldsymbol{p}^* = \boldsymbol{q}, \qquad \boldsymbol{f}^* = \boldsymbol{f}, \quad (6.4)$$

and the same understanding holds for the kinetic energy $\frac{1}{2}\varrho^*|\dot{\boldsymbol{y}}^*|^2$ as measured by the moving observer. Moreover, we require that the effective stresses be invariant, writing

$$(\boldsymbol{S}^{\text{eff}})^* = \boldsymbol{S}^* + \mathcal{U}\boldsymbol{p}^*\otimes\boldsymbol{m}, \qquad (\boldsymbol{C}^{\text{eff}})^* = \boldsymbol{C}^* + \mathcal{U}\boldsymbol{q}^*\otimes\boldsymbol{m}, \quad (6.5)$$

where $\mathcal{U}$ is the normal velocity of $\partial\mathcal{R}$ following material points, that is, $\mathcal{U} = \mathcal{V}^* - \boldsymbol{c}\cdot\boldsymbol{m} = \mathcal{V}$, with $\mathcal{V}^* = \boldsymbol{v}^*\cdot\boldsymbol{m}$ and $\boldsymbol{m}$ the outward unit normal to $\mathcal{R}$.

The foregoing invariance requirement may be justified by noting that the terms $\mathcal{U}\boldsymbol{p}^*\otimes\boldsymbol{m}$ and $\mathcal{U}\boldsymbol{q}^*\otimes\boldsymbol{m}$ represent the effluxes of deformational and configurational momenta, fields that are associated to material particles, and should thus be proportional to the normal velocity of $\partial\mathcal{R}$ with respect to said particles. We notice finally that the free energy and the mass density are invariant under a material observer change: $\varrho^* = \varrho$ and $\psi^* = \psi$.

To discuss its invariance, we write the dissipation imbalance for a moving part $\mathcal{R}$ in a different form, which displays explicitly the analogy between the deformational and configurational fields: the idea is to replace the *Hamiltonian density* $(\psi + \frac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)$ by the *Lagrangian density* $(\psi - \frac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)$, so that (4.9) becomes

$$\overline{\int_{\mathcal{R}} (\psi - \tfrac{1}{2}\varrho|\dot{\boldsymbol{y}}|^2)\, dv}^{\boldsymbol{\cdot}} \le \int_{\partial\mathcal{R}} \boldsymbol{S}\boldsymbol{m}\cdot\mathring{\boldsymbol{y}}\, da + \int_{\partial\mathcal{R}} \mathcal{V}\boldsymbol{p}\cdot\mathring{\boldsymbol{y}}\, da - \overline{\int_{\mathcal{R}} \boldsymbol{p}\cdot\dot{\boldsymbol{y}}\, dv}^{\boldsymbol{\cdot}}$$
$$+ \int_{\partial\mathcal{R}} \boldsymbol{C}\boldsymbol{m}\cdot\boldsymbol{v}\, da + \int_{\partial\mathcal{R}} \mathcal{V}\boldsymbol{q}\cdot\boldsymbol{v}\, da + \int_{\mathcal{R}} \boldsymbol{f}\cdot\boldsymbol{0}\, dv - \overline{\int_{\mathcal{R}} \boldsymbol{q}\cdot\boldsymbol{0}\, dv}^{\boldsymbol{\cdot}}. \tag{6.6}$$

This form of the dissipation imbalance conveys our view that the linear and configurational momentum should play symmetric roles in the theory. The last two terms on the right-hand side of (6.6) vanish because the velocity of material particles in the reference configuration vanishes with respect to a fixed observer.

Notice that (6.6) could have been used at the outset, in § 4(*d*), to obtain the representations (4.19) for $\boldsymbol{C}$ and $\boldsymbol{q}$. Moreover, in the absence of configurational fields its use is standard to obtain the identity $\boldsymbol{p} = \varrho\dot{\boldsymbol{y}}$ as a constitutive relation for the linear momentum.

Computing now the difference of the dissipation as measured by a fixed and a moving material observer, we notice that the velocities $\dot{\boldsymbol{y}}$ and $\mathcal{V}$ in (6.6) are to be interpreted as $\dot{\boldsymbol{y}}^*$ and $\mathcal{U}$, the velocities following material points. In particular, we find that

$$\Delta^*(\mathcal{R}) - \Delta(\mathcal{R}) = \Big(\int_{\partial\mathcal{R}} \boldsymbol{C}^{\text{eff}}\boldsymbol{m}\, da + \int_{\mathcal{R}} \boldsymbol{f}\, dv - \overline{\int_{\mathcal{R}} \boldsymbol{q}\, dv}^{\boldsymbol{\cdot}}\Big)\cdot\boldsymbol{c}, \tag{6.7}$$

whereby the total dissipation is invariant under material observer changes if and only if the configurational momentum balance (3.4) holds.

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826	Riahi, D. N.	Effects of roughness on nonlinear stationary vortices in rotating disk flows—*Mathematical and Computer Modeling,* in press (1996)	June 1996
827	Riahi, D. N.	Nonlinear instabilities of shear flows over rough walls	June 1996
828	Weaver, R. L.	Multiple scattering theory for a plate with sprung masses: Mean and mean-square responses	July 1996
829	Moser, R. D., M. M. Rogers, and D. W. Ewing	Self-similarity of time-evolving plane wakes	July 1996
830	Lufrano, J. M., and P. Sofronis	Enhanced hydrogen concentrations ahead of rounded notches and cracks—Competition between plastic strain and hydrostatic constraint	July 1996
831	Riahi, D. N.	Effects of surface corrugation on primary instability modes in wall-bounded shear flows	Aug. 1996
832	Bechel, V. T., and N. R. Sottos	Measuring debond length in the fiber pushout test	Aug. 1996
833	Riahi, D. N.	Effect of centrifugal and Coriolis forces on chimney convection during alloy solidification	Sept. 1996
834	Cermelli, P., and E. Fried	The influence of inertia on configurational forces in a deformable solid	Oct. 1996