A transformation of the point vortex equations

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A transformation of the point vortex equations is introduced and studied. The transformation substitutes two point vortices for each vortex used in the calculation of induced velocity at a point, in particular in the calculation of the velocity of another vortex in the system. For certain symmetric configurations, such as the vortex polygons, this substitution simplifies the calculation of velocity very considerably. We consider the case of three vortices where the circumcircle of the vortex triangle may be used for the transformation. The transformation suggests a new differential relation for the vortex velocities in a many-vortex system and we give applications of this result.

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1. THE TRANSFORMATION

We assume the equations of motion for a system of point vortices on the unbounded plane to hold in their usual form[1,2]:

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \tag{1}$$

Equation (1) gives the instantaneous velocity of vortex α, in a system of N interacting point vortices, as a sum over contributions from the other N–1 vortices. The prime on the summation sign means $\beta \neq \alpha$. The asterisk on the left hand side signifies complex conjugation.

Choose the origin of coordinates such that both z_α and a given z_β are non-zero. Then the contribution from vortex β to the velocity of vortex α may be written:

$$\frac{1}{2\pi i} \frac{\Gamma_\beta}{z_\alpha - z_\beta} = \frac{1}{2\pi i} \left(\frac{-\Gamma_\beta}{z_\alpha - \dfrac{z_\alpha^2}{z_\beta}} + \frac{\Gamma_\beta}{z_\alpha} \right). \tag{2}$$

This can now be interpreted as the velocity induced by two new vortices, one of circulation Γ_β at the origin, the other of circulation $-\Gamma_\beta$ at z_α^2/z_β.

The point z_α^2/z_β has a well known geometrical significance. If we draw the circle centred at the origin through vortex α, and if we invert the position of vortex β in this circle[3], we obtain the point $|z_\alpha|^2/z_\beta^*$. On the other hand, if we reflect the position of vortex β in the line through the origin and vortex α, we obtain the point $z_\alpha z_\beta^*/z_\alpha^*$. Hence, if we first invert and then reflect (or first reflect and then invert – it makes no difference), we obtain the desired point z_α^2/z_β.

Since z_α^2/z_β arises by inverting and reflecting β using vortex α for both the inverting circle radius and for the reflecting line, as just described, we shall call the procedure of determining z_α^2/z_β from z_α and z_β the *invection* of β with respect to α, and z_β the *invected point* with respect to z_α.

The choice of origin plays no role in the resulting value of the mutually induced velocities of the vortices according to (1). However, in (2), or at least in the geometrical interpretation of this equation, the origin would appear to play a role, both in the construction of the invected point and as the location of the point at which to place the vortex responsible for the second term in (2). This freedom to choose the location of the origin is something that can be exploited in applying the transformation, as we shall see below.

In summary, when calculating the induced velocity of a point vortex (which is not at the origin) in a system of such vortices, the contribution from any other vortex may be thought of as arising from two vortices: one of the same strength placed at the origin, the other of opposite strength placed at the invected point.

2. A PHYSICAL EXPLANATION

Although the algebraic manipulations in (1) and (2) are clear enough, it may be of interest to give a more 'physical' explanation of why the transformation works, i.e., of why, when calculating the velocity of vortex α, the two vortices introduced via (2) are indeed equivalent to the original vortex β. To this end consider Figure 1, where we show a vortex of strength Γ at point Q and a 'field point' P at which the we desire the velocity produced by the vortex at Q. Draw an arbitrary circle through P centred at the origin, O. First, assume, as in Fig.1, that the origin is chosen such that Q is outside this circle. If we think of the circle as a stationary rigid boundary in the usual sense of ideal hydrodynamics (i.e., a surface or curve along which the fluid velocity must be tangential), it is well known that the velocity field everywhere outside the circle can equivalently be given by the vortex at Q and two 'image vortices', one located at the inverse point, Q′, with respect to the circle, the other at the circle center. The image vortex at Q′ has circulation –Γ. The image at the center has circulation Γ. (This may be seen, for example, by applying Milne-Thomson's circle theorem, mentioned below, or in several other ways.) The velocity at P with the rigid cylinder present equals the sum of the direct velocity from the vortex at Q and the two velocity contributions from the images. Consider also (Fig.2) a vortex of strength Γ at the point Q* which is the reflection of Q in OP. It has an image system which is the reflection of the images of Q in OP. The transformation says that the velocity at P due to Q equals the velocity at P due to the images of a similar vortex placed at Q*.

That this is correct can be seen from the 'circle theorem' of Milne-Thomson[4]. According to that theorem the modification to a flow with complex potential $f(z)$, and with no singularities inside the circle $|z| \leq R$, due to the placement of a rigid circle/cylinder $|z| = R$ in the flow can be stated solely in terms of f itself. Indeed, the complex potential $w(z)$ with the cylinder inserted is given by

$$w(z) = f(z) + f_*(z), \tag{3a}$$

where $f^*(z)$ is the analytic function defined by

$$f_*(z) = \left[f\left(\frac{R^2}{z^*}\right)\right]^*. \tag{3b}$$

As before, the asterisks on the right hand side indicate complex conjugation. In (3a) $f(z)$, of course, is the 'original' flow and $f_*(z)$ represents the modifying influence of the inserted circle/cylinder.

It is a corollary of Milne-Thomson's theorem that on the circle itself the normal velocity due to $f(z)$ is opposite to the normal velocity due to $f_*(z)$. This must clearly be the case in order for the total normal velocity to vanish. What is not usually noted is that the *tangential* velocity due to $f(z)$ equals the tangential velocity due to $f_*(z)$. We leave the calculation verifying this result to the reader.

It is now clear why the transformation in (2) works: The normal velocity due to the images of Q is opposite to the normal velocity due to Q itself. However, by symmetry the normal velocity due to a vortex at Q* is also opposite to the normal velocity due to the vortex at Q. Hence, the normal velocity due to the images of the vortex at Q* equals the normal velocity of the vortex at Q. Again by symmetry the tangential velocity of a vortex at Q* equals the tangential velocity due to Q. By the circle theorem it also equals the tangential velocity of the images of Q*. Thus, by substituting the images of the vortex at the reflected point Q*, both the normal and the tangential induced velocities due to Q are obtained with the correct signs at the field point P.

If the vortex is inside the circle at the point we called Q′ above, say, then its invected point is the point we called Q*. The transformation says that the velocity, $\mathbf{v}_{Q'}$, induced at P by a vortex of unit circulation at Q′ equals the sum of the velocity induced by a unit vortex at the origin, $\mathbf{v}_O$, and the velocity of a vortex of circulation −1 at Q*, the invected point of Q′. With the velocity of a vortex of circulation +1 placed at Q* designated as $\mathbf{v}_{Q*}$, we wish to verify the formula

$$\mathbf{v}_{Q'} = \mathbf{v}_O - \mathbf{v}_{Q*}.$$

But this is simply the statement that

$$\mathbf{v}_{Q*} = \mathbf{v}_O - \mathbf{v}_{Q'},$$

which is just the already verified result of the transformation for a unit vortex placed at the point Q* outside the circle. Hence, regardless of the relative situation of Q, P and O (except that Q and P cannot coincide with O) the transformation has been verified by a somewhat different argument.

3. APPLICATION TO STEADILY ROTATING CONFIGURATIONS

Let us proceed immediately to some examples of how this transformation may be applied. At first sight one might not feel that substituting two vortices for one, as happens in the transformation, could ever be useful. The objective is, clearly, to identify situations where as many vortices as possible are situated on the same circle. The examples chosen have this property. Rather than writing things out algebraically, we shall proceed geometrically, which seems most in the spirit of the transformation itself.

Consider the configuration in Fig.3 consisting of an odd number of identical vortices, all of circulation Γ, arranged at the vertices of a regular polygon. We have used $N = 5$, but the argument we give is easily seen to work for arbitrary, odd N (see below). We wish to calculate the velocity of vortex '1'. As our circle we use the circle through the five vortices. According to the transformation just described we can replace vortex '2' by a vortex of opposite circulation, $-\Gamma$, at the invected point, simultaneously adding a similar vortex, of circulation Γ, at the origin. However, the invected point of '2' coincides with vortex '5', i.e., the invection cancels '5' by placing an opposite vortex on top of it. Thus, so far as the velocity of '1' is concerned we may remove '2' and '5' and place one vortex of circulation Γ at the center. In effect, the transformation has in this case replaced two vortices, '2' and '5', by one at the center. Similarly, '3' and '4' may be replaced by a vortex of circulation Γ at the center. From the vantage point of vortex '1', the configuration in Fig.3a is successively replaced by that in Fig.3b and then by that in Fig.3c. We have removed the 'ghosts' of the removed and cancelled vortices to produce Fig.3d. Calculating the velocity of '1' is now trivial. If the circle passing through the vortices has radius R, the velocity induced at '1' from 2Γ at the center will be tangential and of magnitude $2\Gamma/2\pi R = \Gamma/\pi R$ directed counter-clockwise for positive Γ. Our choice of 'vortex 1' was quite arbitrary, so we see that all vortices will have the same tangential velocity, i.e., the entire configuration rotates about its center with angular velocity $\Gamma/\pi R^2$.

The argument is easily extended to general, odd $N = 2n + 1$. As one goes around the polygon vortices annihilate their opposites and n vortices of circulation Γ pile up at the center. The resulting velocity of the vortex in question (and, by symmetry, of any vortex in the polygon) is then $n\Gamma/2\pi R$ or $(N - 1)\Gamma/4\pi R$.

For even N, illustrated in Fig.4 for the case $N = 6$, the construction is only slightly more complicated. Now, as we go around the polygon, we remove/cancel all vortices except one, viz the vortex diagonally across from vortex '1' in the original polygon. The vortices at the center are at a distance R from vortex '1'. The diametrically opposite vortex 'left over' is at a distance 2R from it. With $N = 2n$, the induced velocity now becomes $(n - 1)\Gamma/2\pi R + \Gamma/2\pi(2R)$ or, once again, $(N - 1)\Gamma/4\pi R$. The result is, of course, well known.[5]

Adding a vortex at the center of the polygon is easy. For example, in Fig.5 we have an equilateral triangle of vortices of circulation Γ with a vortex of circulation $-\Gamma$ at the center.

Invection of one of the other vortices making up the triangle cancels the last corner vortex. Furthermore, when we place a vortex of circulation Γ at the origin, we cancel the opposite vortex that is there! There is then nothing left to advect the vortex whose velocity we wish to obtain and, indeed, every vortex is stationary in this configuration. The transformation allows this result to be obtained essentially without calculation.

4. APPLICATION TO THE THREE-VORTEX PROBLEM

Unless they are collinear (in which case the calculation of induced velocities is very simple), three vortices are always on a circle, the circumcircle of the vortex triangle. Figure 6a shows a generic example. Our objective is to use the transformation from §1 to calculate the velocity of any of the vortices, such as vortex '1' in Fig.6a.

At first sight the transformation seems useless since applying it to vortices '2' and '3' in turn simply replaces the original vortex triangle by a reflected but otherwise similar triangle with opposite vortices at the corners and with an added vortex at the center (Fig. 6b). However, we can also compute the velocity of '1' by superposition of Fig.6a and Fig.6b taking the contribution from each multiplied by 1/2. This gives Fig.7. We now see that the velocity of vortex '1' may be thought of as arising from a vortex at the center and from two, coaxial vortex pairs. Clearly, the center vortex contributes the entire tangential velocity, which is easily seen to be

$$\frac{\Gamma_2+\Gamma_3}{4\pi R}, \qquad (4)$$

where R is the radius of the circumcircle.

Each of the two vortex pairs contributes a purely radial velocity. We need a bit of geometry to calculate it. We shall label the vertices of the vortex triangle A, B, C and use the same letters to designate the angles in the triangle at these vertices. It will be quite clear when we mean the point and when we mean the angle. We use the symbols a, b, c to designate the lengths of the sides opposite to A, B, C, respectively. The arc clockwise from A to B is then 2C. Hence, the angle between the diameter through A and the side AB is $\pi/2-C$. Since the induced velocity of '1' from '2' is at right angles to the side AB, the angle between this velocity and the diameter must then be just C. Projecting through this angle to get the radial component, and recalling that there are two equal contributions from the two vortices making up the pair, we obtain

$$-\frac{\Gamma_2}{2\pi c}\cos C \qquad (5a)$$

as the radial component of the velocity of vortex '1' from the $\pm\Gamma_2/2$ pair. Similarly, the $\pm\Gamma_3/2$ pair contributes a radial velocity

$$\frac{\Gamma_3}{2\pi b}\cos B. \tag{5b}$$

The net radial velocity of vortex '1' is the sum of (5a) and (5b). From plane geometry we have the 'sine relation' for the vortex triangle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}. \tag{6}$$

Thus, we get the radial velocity of vortex '1' in the form

$$\frac{\Gamma_3 \cot B - \Gamma_2 \cot C}{4\pi R}. \tag{7}$$

The expressions (4) and (7) give the induced velocity of vortex '1' due to vortices '2' and '3', decomposed into a tangential and a radial component with respect to the circumcircle of the triangle. Similar expressions, obtained by permutation of indices, give the velocity components of vortices '2' and '3'. We have summarized these expressions in Table I. Note that the formulae assume that vortices '1', '2' and '3' appear in clockwise order.

Table I: *Radial and tangential velocity components with respect to the circumcircle of three '1', '2' and '3' vortices appearing in clockwise order.*

Vortex	Radial velocity	Tangential velocity
Vortex '1'	$\frac{\Gamma_3 \cot B - \Gamma_2 \cot C}{4\pi R}$	$\frac{\Gamma_2 + \Gamma_3}{4\pi R}$
Vortex '2'	$\frac{\Gamma_1 \cot C - \Gamma_3 \cot A}{4\pi R}$	$\frac{\Gamma_3 + \Gamma_1}{4\pi R}$
Vortex '3'	$\frac{\Gamma_2 \cot A - \Gamma_1 \cot B}{4\pi R}$	$\frac{\Gamma_1 + \Gamma_2}{4\pi R}$

These results can, of course, also be obtained algebraically working from Eq.(1) with $N = 3$: In order to calculate the right hand side in (1) one selects the origin to be at the center of the circumcircle, and then sets $z_1 = Re^{i\varphi_1}$, $z_2 = Re^{i\varphi_2}$, $z_3 = Re^{i\varphi_3}$, where R is the radius of the circumcircle at the instant under consideration. The velocity contributions now decompose into radial and tangential components as in Table I. The relations $\varphi_2 - \varphi_3 = 2A$, $\varphi_3 - \varphi_1 = 2B$, $\varphi_1 - \varphi_2 = 2C$, which depend on vortices '1', '2', and '3' appearing clockwise around the circumcircle, are required.

An intriguing corollary of these formulae arises for a system of two identical vortices and one of opposite circulation. Let the numbering be chosen such that $\Gamma_1 = \Gamma_2 = -\Gamma_3 = \Gamma$. According to (4) the tangential velocity of both vortices '1' and '2' will vanish. From (7) the radial velocity of '1' is $-\Gamma[\cot B + \cot C]/4\pi R$. But $\cot B + \cot C = \sin(B + C)/\sin B \sin C = \sin A/\sin B \sin C = 2aR/bc$, where we have used (6) and that the sum of angles in a triangle is π. The radial velocity of '1', then, equals $-\Gamma a/2\pi bc$.

From the dynamics of this system we know that the Hamiltonian is an integral of the motion. Because of the values of the circulations the Hamiltonian is, in essence, the logarithm of $c/ab = h$, so this quantity is constant during the motion. Hence, the velocity of vortex '1' is $-\Gamma a/2\pi bc = -\Gamma/2\pi hb^2$. Similarly, the velocity of vortex '2' is $-\Gamma/2\pi ha^2$. These simple results were obtained by Gröbli[6] through rather lengthy algebraic manipulations of the equations of motion. (The results on the velocity magnitudes actually follow readily from Eqs.(1) in this case using the arguments we have just given.) The transformation provides the added information that the velocities of both '1' and '2' are directed radially to the circumcircle.

We can also calculate the rate of change of the distance between vortices '2' and '3' due to vortex '1'. Consider

$$\frac{da^2}{dt} = 2(\mathbf{x}_2 - \mathbf{x}_3)\bullet(\mathbf{v}_2 - \mathbf{v}_3). \tag{8}$$

Here $\mathbf{x}_2$, $\mathbf{x}_3$ are the position vectors for vortices '2' and '3', and $\mathbf{v}_2$, $\mathbf{v}_3$ are the velocity vectors of these vortices. We calculate the scalar product using the radial and tangential components of velocity just found.

It is easy to see that the angle between $\mathbf{x}_2 - \mathbf{x}_3$ and the radial component of either $\mathbf{v}_2$ or $\mathbf{v}_3$ is $\pi/2 - A$. Similarly, the angle between between $\mathbf{x}_2 - \mathbf{x}_3$ and the tangential component of either $\mathbf{v}_2$ or $\mathbf{v}_3$ is A. Thus, using Table I

$$\frac{da^2}{dt} = \frac{a}{2\pi R} \times [(\Gamma_3 - \Gamma_2)\cos A + (\Gamma_2 - \Gamma_3)\cot A \sin A + \Gamma_1 (\cot C - \cot B)\sin A].$$

The terms proportional to Γ_2 and Γ_3 in the expression just written cancel, which means that the mutually induced velocities of vortices '2' and '3' make no net contribution to the change in length of the segment '23'. This is actually well known from the solution of the two-vortex problem: two-vortex interaction leaves the separation between the two vortices invariant. The terms proportional to Γ_1 in the tangential velocity also cancel since they effect the same rotation of the endpoints of the line segment '23'. This is a new bit of insight resulting from the particular decomposition of the velocities obtained here.

The above expression now simplifies considerably:

$$\frac{da^2}{dt} = \Gamma_1 \frac{a}{2\pi R} (\cot C - \cot B) \sin A, \tag{9}$$

which could have been written down directly if the two cancellations that we have just verified had been introduced from the outset.

The right hand side of (9) is usually written in terms of the sides and the area, Δ, of the vortex triangle. To do so note that since $A + B + C = \pi$, $\sin A = \sin(B + C) = \sin B \cos C + \cos B \sin C = \sin B \sin C (\cot C + \cot B)$. Thus, (9) becomes

$$\frac{da^2}{dt} = \Gamma_1 \frac{a}{2\pi R} (\cot^2 C - \cot^2 B) \sin B \sin C = \Gamma_1 \frac{a}{2\pi R} [\frac{1}{\sin^2 C} - \frac{1}{\sin^2 B}] \sin B \sin C.$$

We now use (6) again to write this as

$$\frac{da^2}{dt} = \frac{2}{\pi} \Gamma_1 [\frac{1}{c^2} - \frac{1}{b^2}] \frac{abc}{4R}.$$

Finally, we use the geometric relation,

$$\Delta = \frac{1}{2} bc \sin A = \frac{abc}{4R}, \tag{10}$$

to obtain

$$\frac{da^2}{dt} = \frac{2}{\pi} \Gamma_1 \Delta [\frac{1}{c^2} - \frac{1}{b^2}], \tag{11}$$

which still assumes that the numbering is chosen so that vortices '1', '2', and '3' appear in clockwise order. Equation (11), also due originally to Gröbli[6] and then rediscovered by Novikov[7] and Aref[8] a century later, is key to solving the three-vortex problem. The derivation given here is probably the simplest one known at present.

5. A DIFFERENTIAL RELATION FOR VORTEX VELOCITIES

Substituting (2) on the right hand side of (1), and assuming no vortex is at the origin, gives

$$z_\alpha^2 \frac{dz_\alpha^*}{dt} = \frac{-1}{2\pi i} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{\frac{1}{z_\alpha} - \frac{1}{z_\beta}} + \frac{z_\alpha}{2\pi i} \sum_{\beta \neq \alpha}^{N} \Gamma_\beta,$$

or, multiplying by Γ_α and summing over α:

$$\sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha^2 \frac{dz_\alpha^*}{dt} = -\frac{1}{2\pi i} \sum_{\alpha,\beta=1}^{N}{}' \frac{\Gamma_\alpha \Gamma_\beta}{\frac{1}{z_\alpha} - \frac{1}{z_\beta}} + \frac{1}{2\pi i} \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha (S - \Gamma_\alpha),$$

where $S = \sum_{\alpha=1}^{N} \Gamma_\alpha$. The first term on the right vanishes by antisymmetry and we are left with

$$2\pi i \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha^2 \frac{dz_\alpha^*}{dt} = \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha \sum_{\beta=1}^{N} \Gamma_\beta - \sum_{\alpha=1}^{N} \Gamma_\alpha^2 z_\alpha. \tag{12}$$

The first term on the right hand side of (12) vanishes if the sum of the vortex strengths is zero. If the sum of the vortex strengths is not zero, we can always shift the origin to the center of vorticity, i.e., we can always assume that the first term on the right hand side is zero. In all cases, then, the first term on the right hand side of the preceding equation can be assumed to vanish. We then have

$$2\pi i \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha^2 \frac{dz_\alpha^*}{dt} = -\sum_{\alpha=1}^{N} \Gamma_\alpha^2 z_\alpha. \tag{13}$$

This relation can, of course, also be derived from (1) but we are not aware of (13) appearing previously in the literature on point vortex dynamics.

If the vortices are identical, and we have already arranged for $\sum_{\alpha=1}^{N} z_\alpha$ to vanish, then we see that the right hand side of (13) will also vanish and we obtain the simpler result

$$\sum_{\alpha=1}^{N} z_\alpha^2 \frac{dz_\alpha^*}{dt} = 0. \tag{14}$$

As an example of the application of (14) consider N identical vortices in a steadily rotating configuration, for which $\frac{dz_\alpha^*}{dt} = -i\omega z_\alpha^*$. Then we must have

$$\sum_{\alpha=1}^{N} z_\alpha |z_\alpha|^2 = 0 \tag{15a}$$

or, in terms of the x- and y-coordinates of the vortices

$$\sum_{\alpha=1}^{N} (x_\alpha^2 + y_\alpha^2)^2 x_\alpha = \sum_{\alpha=1}^{N} (x_\alpha^2 + y_\alpha^2)^2 y_\alpha = 0. \tag{15b}$$

This constraint on third order 'moments' of the coordinates in rotating patterns of identical vortices appears not to be generally known. There is, in fact, a hierarchy of moment relations that may be written for such configurations of which (15b) is one of the simplest.

As a second application of (13) we give this example: There are configurations of $n(n+1)/2$ vortices of circulation $+\Gamma$ and $n(n-1)/2$ vortices of circulation $-\Gamma$ that are completely stationary. Three vortices at the corners of an equilateral triangle with an opposite vortex at the center – the configuration mentioned previously in conjunction with Fig.5 – arises for $n = 2$. Let the vortices with circulation $+\Gamma$ be at z_α, $\alpha = 1, \ldots, n(n+1)/2$, those with circulation $-\Gamma$ be at ζ_β, $\beta = 1, \ldots, n(n-1)/2$. Assume the origin of coordinates has been chosen such that

$$\sum_{\alpha=1}^{n(n+1)/2} z_\alpha - \sum_{\beta=1}^{n(n-1)/2} \zeta_\beta = 0,$$

which is possible since the total circulation of the system is non-zero. Then (13) tells us that

$$\sum_{\alpha=1}^{n(n+1)/2} z_\alpha + \sum_{\beta=1}^{n(n-1)/2} \zeta_\beta = 0,$$

since all vortex velocities vanish in these configurations. Thus, we conclude that

$$\sum_{\alpha=1}^{n(n+1)/2} z_\alpha = \sum_{\beta=1}^{n(n-1)/2} \zeta_\beta = 0. \tag{16}$$

In particular, for n = 2 the single negative vortex can only be placed at the centroid of the three positive vortices as in the configuration in Fig.5. (The three positive vortices must also form an equilateral triangle, but this requires further considerations.)

A third application is the following: There are configurations of n vortices of circulation $+\Gamma$ and n vortices of circulation $-\Gamma$ that translate uniformly without any change in the relative positions of the vortices. The vortex pair consisting of two point vortices with opposite circulations is the simplest example for n = 1 but more complicated examples exist. According to a results of Bartman[9], n is restricted to being a 'triangular number, $n = m(m + 1)/2$, m = 1, 2, ... In particular, there are no solutions of this kind for four vortices (n = 2), two of either sign of the circulation.

As before, let the vortices with circulation $+\Gamma$ be at z_α, $\alpha = 1, ..., n$, those with circulation $-\Gamma$ be at ζ_β, $\beta = 1, ..., n$. Let the common velocity of translation of all vortices be V. Set

$$Z_+ = \sum_{\alpha=1}^{n} z_\alpha \quad ; \quad Z_- = \sum_{\alpha=1}^{n} \zeta_\alpha .$$

The linear impulse of the vortex system, which is a constant of the motion, is

$$Q + iP = \Gamma \sum_{\alpha=1}^{n} (z_\alpha - \zeta_\alpha) = \Gamma(Z_+ - Z_-). \tag{17}$$

Multiplying (1) by $\Gamma_\alpha z_\alpha$ and summing gives

$$(Q + iP)\, V^* = \frac{1}{4\pi i} \sum_{\alpha,\beta=1}^{N}{}' \Gamma_\alpha \Gamma_\beta, \tag{18}$$

where V is the common velocity of all the vortices. For the current, special selection of circulations this relation becomes

$$\Gamma(Z_+ - Z_-)\, V^* = -\frac{n}{2\pi i}\,\Gamma^2. \qquad (19)$$

Equation (13) gives

$$2\pi i V^* \sum_{\alpha=1}^{n} (z_\alpha^2 - \zeta_\alpha^2) = -\Gamma(Z_+ + Z_-),$$

or, multiplying by $Z_+ - Z_-$ on both sides,

$$n \sum_{\alpha=1}^{n} (z_\alpha^2 - \zeta_\alpha^2) = \left(\sum_{\alpha=1}^{n} z_\alpha\right)^2 - \left(\sum_{\alpha=1}^{n} \zeta_\alpha\right)^2$$

or

$$\left(\frac{1}{n}\sum_{\alpha=1}^{n} z_\alpha\right)^2 - \frac{1}{n}\sum_{\alpha=1}^{n} z_\alpha^2 = \left(\frac{1}{n}\sum_{\alpha=1}^{n} \zeta_\alpha\right)^2 - \frac{1}{n}\sum_{\alpha=1}^{n} \zeta_\alpha^2. \qquad (20)$$

The 'variances' of the complex coordinates of the positive and the negative vortices must be equal.

The result (20) is sufficient to rule out the existence of translating configurations of the type under consideration here for $n = 2$: Indeed, for that case, (20) tells us that

$$\tfrac{1}{4}(z_1 + z_2)^2 - \tfrac{1}{2}(z_1^2 + z_2^2) = \tfrac{1}{4}(\zeta_1 + \zeta_2)^2 - \tfrac{1}{2}(\zeta_1^2 + \zeta_2^2),$$

or

$$(z_1 - z_2)^2 = (\zeta_1 - \zeta_2)^2$$

or, finally,

$$z_1 - z_2 = \pm(\zeta_1 - \zeta_2). \qquad (21)$$

Equation (21) with the + sign together with (19) for $n = 2$, gives

$$V^*(z_1 - \zeta_1) = V^*(z_2 - \zeta_2) = -\frac{\Gamma}{2\pi i}. \qquad (22)$$

Now, the equation of motion (1) for z_1 becomes

$$V^* = \frac{\Gamma}{2\pi i}\left(\frac{1}{z_1 - z_2} - \frac{1}{z_1 - \zeta_1} - \frac{1}{z_1 - \zeta_2}\right) = V^* + \frac{\Gamma}{2\pi i}\left(\frac{1}{z_1 - z_2} - \frac{1}{z_1 - \zeta_2}\right),$$

implying $z_2 = \zeta_2$, which is unacceptable. Equation (21) with the – sign runs into a similar contradiction. In summary, we have a direct proof that there cannot exist translating states of the type assumed for n = 2.

6. SUMMARY

A simple transformation of the point vortex equations has been introduced and studied. The transformation may be thought of either as a simple algebraic transcription of the equations or in geometrical terms – and it is usually best to keep both interpretations in mind. When calculating the velocity of any one vortex in a system of interacting vortices, it substitutes two vortices for each of the other vortices in the system. This will only lead to a simplification of the end result if the configuration has certain symmetries so that the substitutions dictated by the transformation, somehow, result in an overall reduction in the complexity of the vortex system. We identified several examples, admittedly rather special configurations, where the transformation does, indeed, lead to a simplification. There may be further examples, e.g., the special solutions for four interacting vortices studied by Rott[10] in which the four vortices are always situated on a circle. However, we have not at present found a way to substantially simplify the treatment of this problem using the transformation.

The transformation provides new formulae for the velocities of three interacting vortices based on using the circumcircle of the vortex triangle. These relations appear to be quite useful in re-deriving several known results for systems of this kind.

The algebraic nature of the transformation suggests a new differential relation between vortex velocities and coordinates given as Eq.(13). Several applications of this result were shown. Again, further applications are likely to exist.

The transformation suggests that one seek out other point transformations that may simplify the equations of point vortex motion. In particular, the transformation reported here is only valid instantaneously, i.e., it does not 'map' a given vortex interaction problem onto another in a way that continues to be valid and useful as the motion evolves. For this reason the transformation may be of limited use for general dynamical considerations.

The work reported here is based on paper ED2 presented at the 54th Annual Meeting of the American Physical Society, Division of Fluid Dynamics, held in San Diego in November, 2001. The author should like to thank several members of the audience for thoughtful comments following the presentation.

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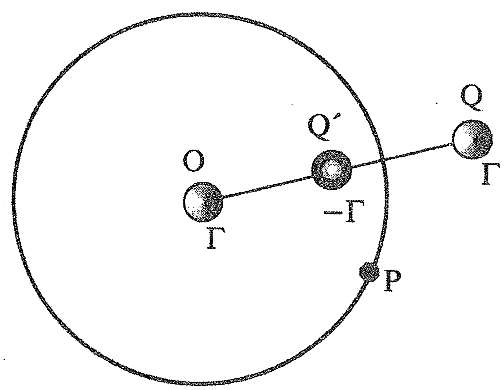

Figure 1: Image system for a vortex at Q in the presence of a solid cylinder (circle) through P.

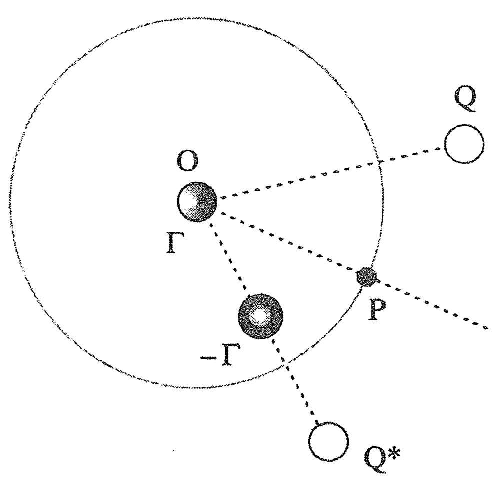

Figure 2: The velocity at P from a vortex at Q is equal to the velocity from the images of a similar vortex at the reflected point Q*.

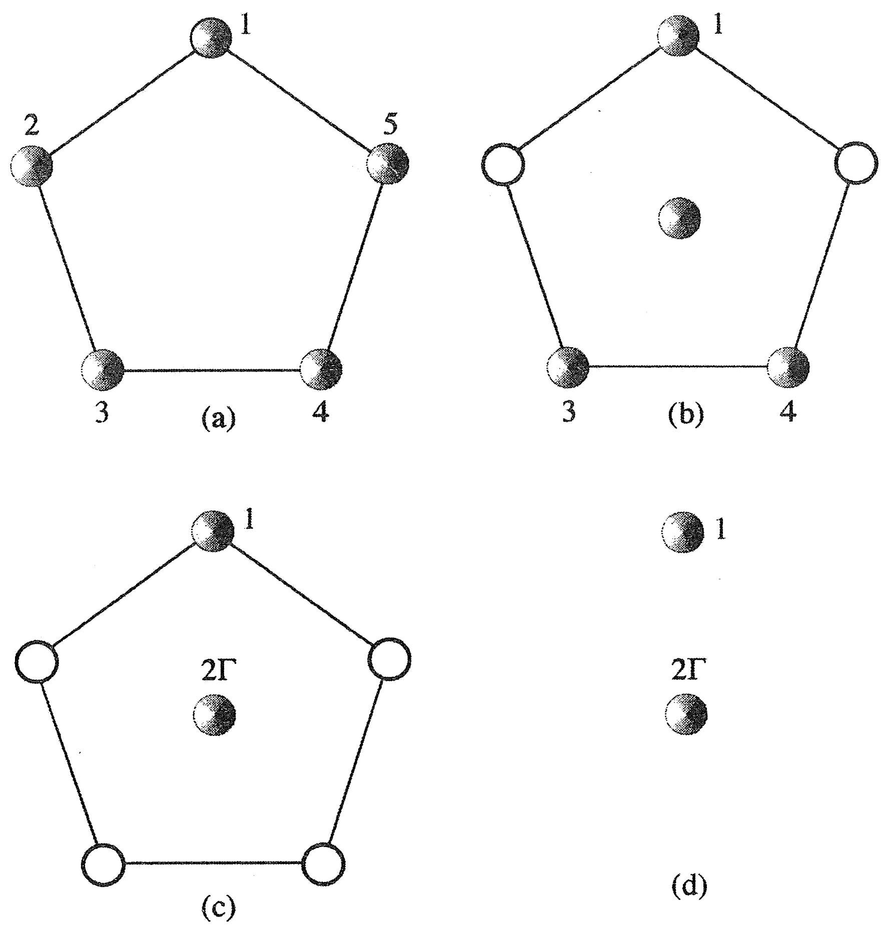

Figure 3: Velocity of vortex '1' from remaining vortices in a regular pentagon, (a), upon successive applications of the transformation. (b) Vortex '2' cancels '5' and puts a vortex at the center. (c) Vortex '3' cancels '4' and puts a second vortex at the center. (d) In the resulting system velocity calculation is trivial.

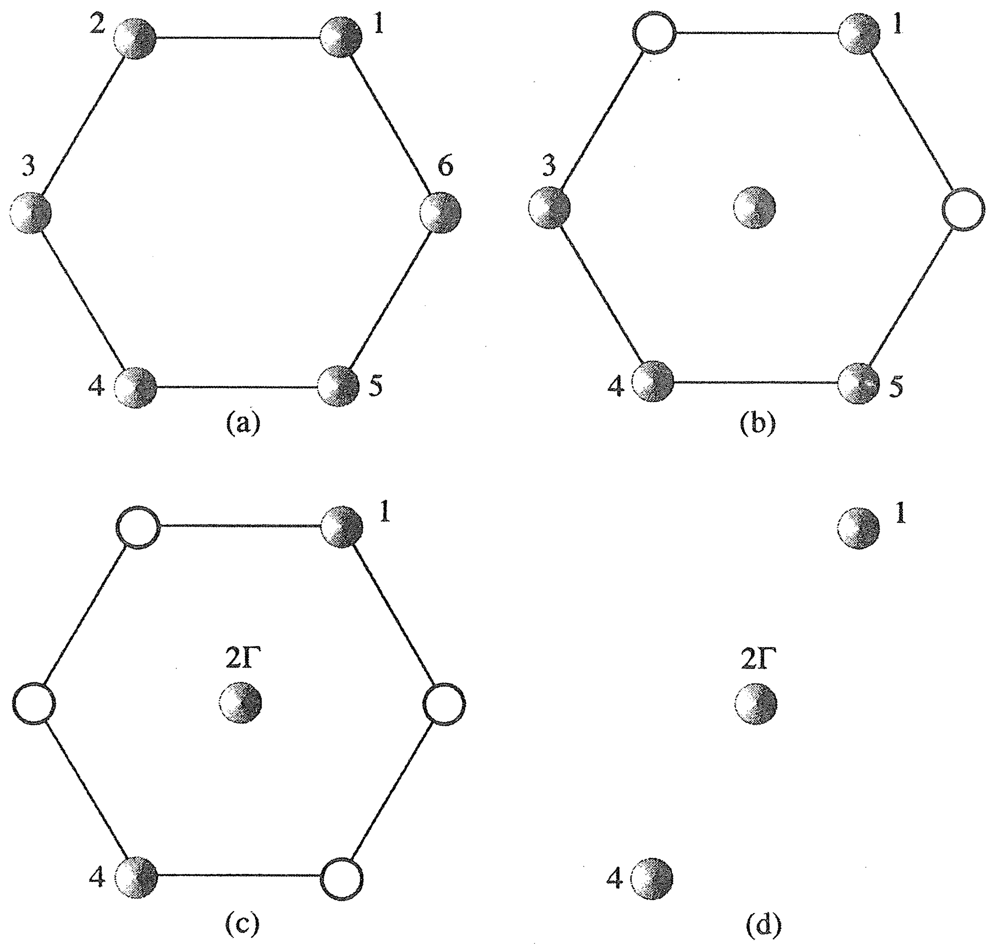

Figure 4: Velocity of vortex '1' from remaining vortices in a regular hexagon, (a), upon successive applications of the transformation. (b) Vortex '2' cancels '6' and puts a vortex at the center. (c) Vortex '3' cancels '5' and puts a second vortex at the center. (d) In the resulting system velocity calculation is trivial.

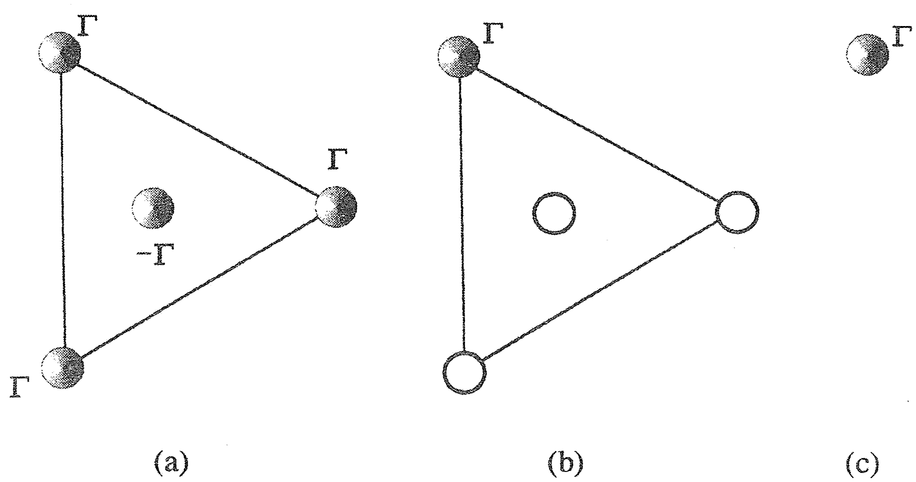

Figure 5: Three identical vortices in an equilateral triangle, with an opposite vortex at the center, is a completely stationary configuration. The transformation shows why this is so.

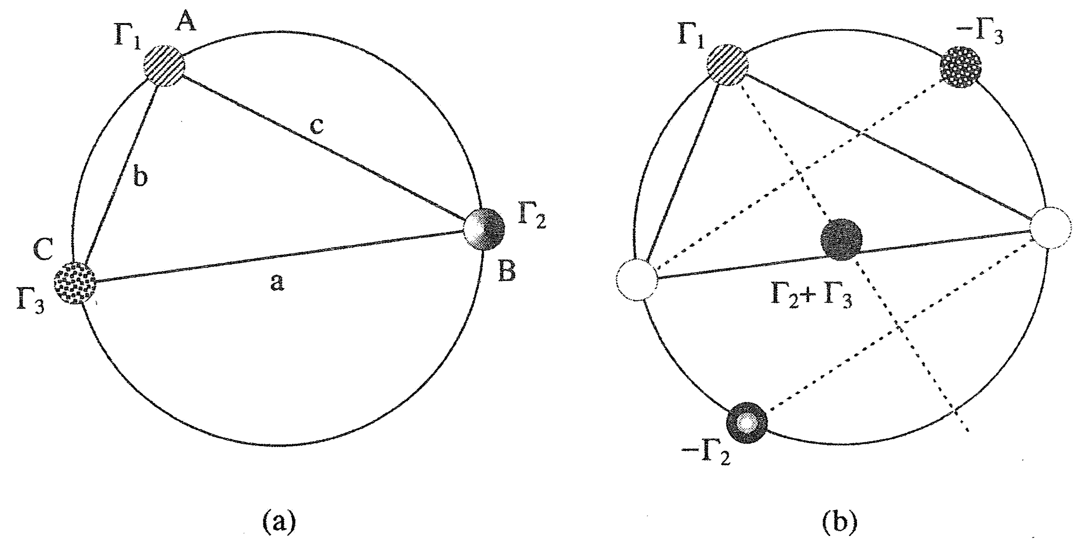

Figure 6: (a) Three vortices and their circumcircle. (b) The transformed system for calculating the velocity of vortex '1'.

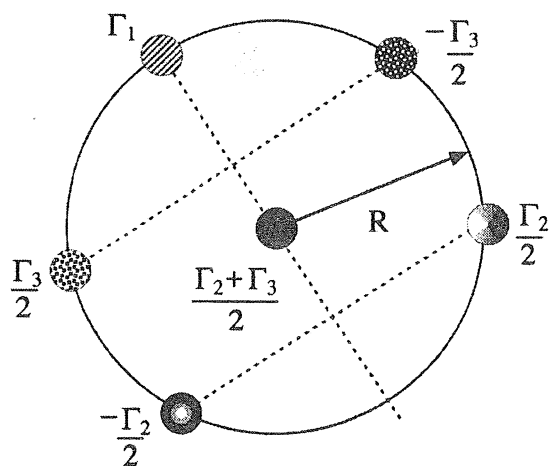

Figure 7: Combining Figs. 6(a) and 6(b) to calculate the velocity of vortex '1'; R is the radius of the circumcircle.

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