

CONVOLUTION METHOD IN ELASTODYNAMICS

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DISSERTATION

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Abstract

Theoretical solid mechanics is mainly based on the equations of motion in terms of displacement fields. Using the constitutive and kinematic equations, the equations of motion that initially involve the stress fields are written in terms of spatial and time derivatives of the displacement field. These equations are at the center of analytical and computational attempts in solid mechanics. However, this approach contains seemingly unresolvable challenges. As an example, in nonlinear elastodynamics, it is well-understood that the Newmark average acceleration method (the trapezoidal rule) fails to conserve the balance of energy and the balance of angular momentum. While several algorithms in the literature attempted to address this challenge, all of them are based on a rather intricate formulation that still falls short in some practical problems. In this dissertation, by contrast, wave phenomena in elastic materials are investigated through the so-called alternative field equations where the convolution product appears in the place of time derivative.

In the first step of the study, the stress equations of motion for electromagneto-elastic materials are derived. As the name suggests, in the new framework, the equations of motion are written in terms of the stress field. These equations provide a new foundation to establish numerical methods in terms of stress rather than displacement, which is an appropriate strategy for addressing problems with Neumann boundary conditions. In the next step, based on the alternative field equations, convolutional variational principles for electromagneto-elastic materials are established.

Upon convolutional variational principles, a new time-domain finite element method, namely, Convolution Finite Element Method (CFEM), is then introduced for linear elastodynamics problems. It is shown that this new dynamic finite element approach inherits surprising characteristics that cannot be found in the classical methods. Finally, by using the CFEM, a new solution procedure for nonlinear elastodynamics is established. The new algorithm, which is based on the CFEM and Newton-Raphson method, leads to a simple solution procedure while conserving energy and angular momentum.

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Chapter 1

Introduction

There are various approaches to analyze and design solid materials ranging from microscale to macroscale methods. In this regard, the most successful macroscale method is the continuum approach. The deformable solid body is considered as a continuous medium, and the discontinuity existing in atomic scales is disregarded. The concept of strain is introduced to measure the deformations, while the notion of stress is employed to compute internal forces. The relation between stress and strain is defined through the so-called constitutive equations. These equations, along with equations of motion and boundary and initial conditions, form an initial boundary value problem that governs the motion of the deformable body.

The classical approach to finding the solution of the initial boundary value problem is to write the governing equations in terms of displacement fields. This form can be achieved by connecting the displacement field to strain and stress through compatibility and constitutive equations. For example, the displacement governing partial differential equations (i.e., Navier equations of elastodynamics) for isotropic and homogenous materials can be straightforwardly written:

$$\mu \nabla^2 \mathbf{u}(\mathbf{x}, t) + (\mu + \lambda) \nabla \nabla \cdot \mathbf{u}(\mathbf{x}, t) + \rho \mathbf{F}(\mathbf{x}, t) = \rho \ddot{\mathbf{u}}(\mathbf{x}, t), \quad (1.1)$$

where μ and λ are Lamé constants, ρ is the mass density of the material, $\mathbf{u}(\mathbf{x}, t)$ is displacement field, and $\mathbf{F}(\mathbf{x}, t)$ is the body force per unit mass per volume. These equations, along with boundary and initial conditions, are then solved either analytically or numerically. Once the displacement has been found, the strain and stress can be obtained through compatibility and constitutive equations. Alternatively, one can write the governing equations in terms of stresses. For example, in elastostatic problems, the counterpart of displacement governing equations, in the absence of body force, is the

Beltrami-Michell equations:

$$\sigma_{ij,kk} + \frac{1}{1+\nu} \sigma_{kk,ij} - \frac{\nu}{1+\nu} \sigma_{mm,kk} \delta_{ij} = 0, \quad (1.2)$$

where ν is Poisson's ratio, σ_{ij} are components of the stress tensor, and δ_{ij} is Kronecker delta. These equations have found many applications for planar problems and in the mechanics of composites. For instance, the Airy stress functions are defined such that the above compatibility equations reduce to a biharmonic equation. For the elastodynamics problems, on the other hand, there exists another set of equations, the so-called Ignaczak stress equations of motion [43, 44] (here given for an isotropic medium):

$$(\sigma_{(ik,k),j})(\mathbf{x},t) - \frac{\rho}{2\mu} (\ddot{\sigma}_{ij}(\mathbf{x},t) - \frac{\lambda}{2\mu + 3\lambda} \ddot{\sigma}_{kk}(\mathbf{x},t) \delta_{ij}) = 0, \quad (1.3)$$

where $A_{(i,j)}$ denotes the symmetric part of the second-order tensor A_{ij} . It has been shown that the stress tensor can be uniquely obtained from the above governing equations and traction boundary conditions by assuming vanishing initial stress. Once the stress field has been found, one can obtain the displacement from:

$$u_i(\mathbf{x},t) = t * \sigma_{ij,j}, \quad (1.4)$$

where $*$ denotes the convolution product. (1.4) later became the motivation to develop a new type of field equations, so-called alternative field equations, and establish convolutional variational principles.

The purpose of this dissertation is to investigate the potentials of the alternative formulation in analyzing materials. In this regard, we first develop the formulation for electromagneto-elastic materials. To this end, we briefly explain the fundamentals of continuum electromagnetic theory in chapter 2. Then, in chapter 3, a stress-based approach to the analysis of linear electromagneto-elastic materials is proposed. Firstly, field equations for linear electromagneto-elastic solids are given in detail. Next, as a counterpart of coupled governing equations in terms of the displacement field, generalized stress equations of motion for analyzing three-dimensional (3D) problems are obtained.

Then, a sufficient set of conditions for the corresponding solution of generalized stress equations of motion to be unique are detailed in a uniqueness theorem. A numerical passage to obtain the solution of equations is then given by generalizing a reciprocity theorem in terms of stress for such materials. Finally, as particular cases of the general 3D form, the stress equations of motion for planar problems (plane strain and Generalized plane stress) for transversely isotropic media are formulated.

In chapter 4, we establish the convolutional variational framework for electromagneto-elastic materials. First, a general variational form, imposing no restriction on the fields involved, is given based on coupled constitutive equations and the alternative field equations. Subsequently, the result is derived when satisfaction of the strain-displacement equation is presumed as a restriction. Next, the variational form for kinematically admissible processes and, in turn, for kinematically admissible displacement-potential processes are found. Finally, the principles characterizing the stress field instead of the displacement field are formulated.

We propose a new finite element method based on alternative field equations and convolutional variational principles in chapter 5. First, a weak formulation—corresponding to convolutional variational principles—is defined, which paves the way for introducing a particular type of time-wise shape functions. Next, some mathematical characteristics of the method are investigated, and upon determining them, a new solution procedure for the elastodynamics problem is proposed. Subsequently, several numerical examples are considered, including a single degree of freedom mass-spring-damper system as the prototype of structural dynamics along with 1d and 2d elastodynamics problems for the case of the wave motion in elastic solids.

Eventually, based on the proposed finite element method, a new algorithm for the non-linear wave propagation in elastic media is suggested in chapter 6. The satisfaction in the weak sense of the alternative equations for the non-linear equations of motion is analogous to the linear case. The Newton-Raphson procedure and the consistent tangential matrix are applied to the weak formulation where the convolution finite element method is used as the linear solver in each iteration. It is shown that the method successfully and adaptively conserves the constants of motion, i.e., the conservation of linear and angular momenta and energy. More importantly, in contrast to the energy-momentum

conserving time integration methods, the proposed algorithm obtains accurate results for larger time steps. Several examples are considered to illustrate the capabilities of the method. In particular, free vibration of bilinear stiffening and softening mass-spring systems are used to show that the new method conserves energy and yields an accurate solution when an adequate number of time-wise terms is assumed. Also, a particular form of a 2d mass-spring system is considered as a prototype multi-degree of freedom systems. Finally, a non-linear elastic bar problem is analyzed as the prototype of elastodynamics.

Chapter 2

Continuum Mechanics of Electromagneto-elastic Materials¹

This Section summarizes the foundations of continuum electromagnetic theory, and, in this regard, we follow [18] both notionally and conceptually.

To begin with, consider a deformable body whose volume denoted by V , a subset of three-dimensional Euclidean space which is simply connected, open and bounded, with boundary ∂V . Furthermore, let denote $\bar{V} = V \cup \partial V$. Two different coordinate systems, i.e., the material and spatial coordinate systems, are used in this section. To be distinguishable, similar to [18], bold letter indices and small ones stand for the material and spatial coordinates, respectively. Also, in the remainder of the chapter, standard index notation for Cartesian coordinates is occasionally used.

First, Maxwell's equations in a continuous matter, in a fixed frame, are [18]:

$$\begin{aligned}\nabla \cdot \mathbf{B} &= \mathbf{0}, \\ \nabla \cdot \mathbf{D} &= q_e, \\ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= \mathbf{0}, \\ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} &= \frac{1}{c} \mathbf{J},\end{aligned}\tag{2.1}$$

where

$$\begin{aligned}\mathbf{H} &= \mathbf{B} - \mathbf{M}, \\ \mathbf{D} &= \mathbf{E} + \mathbf{P},\end{aligned}\tag{2.2}$$

in which $\mathbf{E}$, $\mathbf{P}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{H}$, $\mathbf{J}$, and q_e are the electric field, volume electric polarization, electric displacement field, magnetic induction, volume magnetic polarisation, the magnetic field strength, electric current and charge density, respectively. It is worthwhile to be mentioned that above

¹ Adapted with permission from [57]

governing equation can be obtained through corresponding microscopic scales by means of statistical averaging (See [18] for details).

To generalize those equations in moving frames, which is the case for the balance laws of continuum mechanics in general, one needs to ensure that those equations are form-invariant under a Galilean transformation [18]:

$$\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{V}t, \quad \tilde{t} = t. \quad (2.3)$$

Here $\mathbf{V}$ is the constant velocity of the moving frame with respect to the fixed one. The necessary and sufficient conditions when the velocity of the motion of the continuum is much less than c , the speed of light in vacuum, can be written as: [18]:

$$\begin{aligned} \tilde{q}_e &= q_e, \\ \tilde{\mathbf{P}} &= \mathbf{P}, \\ \tilde{\mathbf{J}} &= \mathbf{J} + \mathbf{V}q_e, \\ \tilde{\mathbf{M}} &= \mathbf{M} - \frac{1}{c}\mathbf{V} \times \mathbf{P}, \\ \tilde{\mathbf{E}} &= \mathbf{E} - \frac{1}{c}\mathbf{V} \times \mathbf{B}, \\ \tilde{\mathbf{B}} &= \mathbf{B} + \frac{1}{c}\mathbf{V} \times \mathbf{E}, \\ \tilde{\mathbf{H}} &= \tilde{\mathbf{B}} - \tilde{\mathbf{M}}. \end{aligned} \quad (2.4)$$

Consequently, if one assumes that at each time, say t , the velocity of the matter in the moving frame $R_C(\mathbf{x}, t)$ is $\mathbf{V} = -\mathbf{v}$, then we have:

$$\begin{aligned} \tilde{q}_e &= q_e, \\ \tilde{\mathbf{P}} &= \mathbf{P}, \\ \tilde{\mathbf{J}} &= \mathbf{J} - \mathbf{v}q_e, \\ \tilde{\mathbf{M}} &= \mathbf{M} + \frac{1}{c}\mathbf{v} \times \mathbf{P}, \\ \tilde{\mathbf{E}} &= \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}, \\ \tilde{\mathbf{B}} &= \mathbf{B} - \frac{1}{c}\mathbf{v} \times \mathbf{E}, \end{aligned} \quad (2.5)$$

$$\tilde{\mathbf{H}} = \tilde{\mathbf{B}} - \tilde{\mathbf{M}}.$$

As a result, Maxwell's equations in the moving frame read [18]:

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{D} &= q_e, \\ \nabla \times \tilde{\mathbf{E}} + \frac{1}{c} \mathbf{B}^* &= \mathbf{0}, \\ \nabla \times \tilde{\mathbf{H}} - \frac{1}{c} \mathbf{D}^* &= \frac{1}{c} \mathbf{J},\end{aligned}\tag{2.6}$$

where

$$\begin{aligned}\mathbf{B}^* &= \dot{\mathbf{B}} + \mathbf{B}(\nabla \cdot \mathbf{v}) - (\mathbf{B} \cdot \nabla) \mathbf{v}, \\ \mathbf{D}^* &= \dot{\mathbf{D}} + \mathbf{D}(\nabla \cdot \mathbf{v}) - (\mathbf{D} \cdot \nabla) \mathbf{v},\end{aligned}\tag{2.7}$$

are convective derivatives, which are frame indifferent, see [18].

To consider the electromagnetic effects in a thermodynamic continuum, the balance laws — i.e., the balance of mass, balance of linear momentum, the balance of angular momentum, the first law of thermodynamics, and the second law of thermodynamics — need to be endowed with the contribution due to electromagnetic fields. The approach to account for such contributions is quite similar to the case where a continuum theory is built by passing from particle mechanics. In other words, the macroscopic contribution due to electromagnetic fields — i.e., electromagnetic force, electromagnetic couple, and electromagnetic power — are derived from microscopic ones by statistical averaging (see [18] for details). In this regard, after quite long manipulations, the generalised local form of balance laws can be written as [18]:

$$\begin{aligned}\dot{\rho} + \rho v_{i,i} &= 0, \\ \rho \dot{v}_i &= \sigma_{ji,j} + f_i + f_i^{em}, \\ \sigma_{[ij]} &= \tilde{E}_{[i} P_{j]} + B_{[i} \tilde{M}_{j]}, \\ \rho \dot{e} &= \sigma_{ij} v_{j,i} - q_{i,i} + \rho h + W_{in}^{em}, \\ -\rho(\dot{\psi} + \eta \dot{\theta}) + \sigma_{ij} v_{j,i} + \theta^{-1} q_i \theta_{,i} - P_i \tilde{E}_{,i} - \tilde{M}_i \dot{B}_i + \tilde{J}_i \tilde{E}_i &\geq 0,\end{aligned}\tag{2.8}$$

in which

$$\begin{aligned}
\mathbf{f}^{em} &= q_e \tilde{\mathbf{E}} + \frac{1}{c} (\tilde{\mathbf{J}} + \mathbf{P}^*) \times \mathbf{B} + (\mathbf{P} \cdot \nabla) \tilde{\mathbf{E}} + (\nabla \mathbf{B}) \cdot \tilde{\mathbf{M}}, \\
W_{in}^{em} &= \mathbf{f}^{em} \cdot \mathbf{v} + \rho \tilde{\mathbf{E}} \cdot \dot{\boldsymbol{\pi}} - \tilde{\mathbf{M}} \cdot \dot{\mathbf{B}} + \mathbf{J} \cdot \tilde{\mathbf{E}}, \\
\pi &= \frac{\mathbf{P}}{\rho}, \\
\mathbf{P}^* &= \dot{\mathbf{P}} + \mathbf{P}(\nabla \cdot \mathbf{v}) - (\mathbf{P} \cdot \nabla) \mathbf{v},
\end{aligned} \tag{2.9}$$

with electromagnetic jump or boundary conditions [18]:

$$\begin{aligned}
\mathbf{n} \cdot [\mathbf{D}] &= 0, \\
\mathbf{n} \cdot [\mathbf{B}] &= 0, \\
\mathbf{n} \times [\tilde{\mathbf{E}}] &= \mathbf{0}, \\
\mathbf{n} \times [\tilde{\mathbf{H}}] &= \mathbf{0},
\end{aligned} \tag{2.10}$$

with $[\mathbf{A}] \equiv \mathbf{A}^+ - \mathbf{A}^-$ in that $\mathbf{A}^+$ and $\mathbf{A}^-$, respectively, denote the values of $\mathbf{A}$ when approaching from the positive and negative sides of a surface characterized with normal vector $\mathbf{n}$ [18]. Zero surface current, zero surface free charge, and zero surface polarisation are assumed in (2.10) [18]. Also, σ_{ij} , $v_{i,j}$, q_i , v_i , and $t_{i(n)}^{em}$ are, respectively, components of Cauchy stress, gradient of velocity field, surface heat flux, velocity field, electromagnetic surface traction. In addition, e , h , ψ , η , and θ are the internal energy, the rate of heat source per unit volume, Helmholtz free energy density, entropy, and temperature, respectively. Furthermore, $\sigma_{[ij]}$ denotes the skew-symmetric part of the Cauchy stress tensor, i.e., $\sigma_{[ij]} = \frac{1}{2}(\sigma_{ij} - \sigma_{ji})$. Clearly, the total stress in the presence of electromagnetic field is no longer symmetric.

Eqations (2.1) and (2.8) together with boundary conditions give the governing equations of a deformable body in the presence of electromagnetic fields. However, similar to the case of classical continuum mechanics, one needs to further define constitutive equations to successfully resolve the intrinsic indeterminate nature of the problem. In this regard, the constitutive equations need to satisfy several axioms that we, herein, just enumerate them [18]: 1) causality, 2) determinism, 3) equipresence, 4) objectivity, 5) time reversal, 6) material invariance, 7) neighborhood, 8) memory,

and 9) admissibility. The systematic approach to obtaining constitutive equations is to define enough number of functionals, say $\mathfrak{I}^\alpha$, that meet the aforementioned axioms. To this end, following [18], the dependent physical quantities are taken as follows:

$$\mathcal{L} = \{ \mathbf{T}^E, \mathbf{Q}, \Psi, \eta, \Pi, \mathbf{M}, \mathbf{J} \} , \quad (2.11)$$

while the independent ones are:

$$\mathcal{Y} = \{ \tilde{\mathbf{E}}, \theta, \mathbf{B} \} \quad (2.12)$$

along with spatial coordinates and time parameter. (2.11) and (2.12) are represented in the material coordinate that can be related to the special coordinate through following relations [18]:

$$\begin{aligned} \rho &= J\rho_0, \quad T_{IJ}^E = JX_{I,i}X_{J,j,E}\sigma_{ij} , \\ Q_I &= JX_{I,i}q_i, \Pi_I = JX_{I,i}P_i, \quad M_I = JX_{I,i}\tilde{M}_i, \quad \tilde{J}_I = JX_{I,i}J_i, \quad J = \det x_{i,I} . \end{aligned} \quad (2.13)$$

$\mathbf{T}^E$, in (2.11), is the symmetric tensor in the reference configuration corresponding to σ . Next, first and second laws of thermodynamics in material coordinates can be formulated in the following form [18]:

$$\begin{aligned} \rho_0(\psi + \dot{\theta}\eta + \dot{\eta}\theta) &= \frac{1}{2}T_{IJ}^E\dot{C}_{IJ} + Q_{I,I} - \rho_0h + \Pi_I\dot{\tilde{E}}_I - M_I\dot{B}_I + \tilde{J}_I\dot{\tilde{E}}_I , \\ -\rho_0(\psi + \dot{\theta}\eta) + \frac{1}{2}T_{IJ}^E\dot{C}_{IJ} + \frac{1}{\theta}Q_I\theta_I - \Pi_I\dot{\tilde{E}}_I - M_I\dot{B}_I + \tilde{J}_I\dot{\tilde{E}}_I &\geq 0 . \end{aligned} \quad (2.14)$$

It is easy to infer that 20 more functionals are required to have the same number of unknowns and equations. In doing so, we define Helmholtz free energy for a deformable solid with heat, electrical, and mechanical coupling as follows [18]:

$$\rho_0\psi = \tilde{\psi}(\mathbf{C}, \tilde{\mathbf{E}}, \tilde{\mathbf{B}}, \nabla_R\theta; \theta, \mathbf{X}) . \quad (2.15)$$

By taking the derivative, we have:

$$\rho_0\dot{\psi} = \frac{\partial \tilde{\psi}}{\partial C_{IJ}}\dot{C}_{IJ} + \frac{\partial \tilde{\psi}}{\partial \tilde{E}_I}\dot{\tilde{E}}_I + \frac{\partial \tilde{\psi}}{\partial B_I}\dot{B}_I + \frac{\partial \tilde{\psi}}{\partial \theta_{,I}}\dot{\theta}_{,I} + \frac{\partial \tilde{\psi}}{\partial \theta}\dot{\theta} . \quad (2.16)$$

Substituting the result into (2.14) and considering that $\dot{\theta}$, $\dot{C}_{IJ}$, $\dot{\theta}_{,I}$, $\dot{\tilde{E}}_i$ and $\dot{B}_I$ are independent variables, the inequality holds when the following relations are true [18]:

$$\eta = -\frac{1}{\rho_0} \frac{\partial \tilde{\psi}}{\partial \theta}, T_{IJ}^E = 2 \frac{\partial \tilde{\psi}}{\partial C_{IJ}}, \frac{\partial \tilde{\psi}}{\partial \theta_{,I}} = 0, \Pi_I = -\frac{\partial \tilde{\psi}}{\partial \tilde{E}_I}, M_I = -\frac{\partial \tilde{\psi}}{\partial \tilde{B}_I}, \quad (2.17)$$

with [18]:

$$\begin{aligned} Q_I &= \tilde{Q}_I(\mathbf{C}, \tilde{\mathbf{E}}, \mathbf{B}, \nabla_R \theta; \theta \mathbf{X}), \\ \tilde{J}_I &= \tilde{J}_I(\mathbf{C}, \tilde{\mathbf{E}}, \mathbf{B}, \nabla_R \theta; \theta, \mathbf{X}), \end{aligned} \quad (2.18)$$

It is noticed that (2.18) satisfies [18]:

$$\frac{1}{\theta} Q_{,I} \theta_{,I} + J_I \tilde{E}_I \geq 0. \quad (2.19)$$

All the equations, up to this point, have been written in a general form, i.e., no assumption has been made within the continuum framework. However, in many applications, a linearized theory is sufficient to find accurate results. To this end, let assume that the natural state is free of fields, that is, it has the following form [60]:

$$x^0 = X, \Pi_I^0 = 0, E_{IJ}^0 = 0, E_I^0 = 0, B_I^0 = 0, \dots \quad (2.20)$$

In addition, suppose that all the fields are small enough in the sense that [18, 60]

$$\begin{aligned} \hat{\mathbf{u}} &= \delta \mathbf{u}, \hat{\mathbf{E}} = \delta \mathbf{E}, \hat{\mathbf{B}} = \delta \mathbf{B}, \hat{\Pi} = \delta \Pi, \\ \hat{\theta} - \hat{\theta}_0 &= \delta \theta, |\delta \hat{\theta}| \ll \hat{\theta}_0, \hat{\theta}_0 > 0, \nabla \hat{\theta} = \delta \nabla \theta \dots, \end{aligned} \quad (2.21)$$

where δ is a small number. Linearising nonlinear fields by expansion about the natural state and neglecting nonlinear terms, we find:

$$\begin{aligned} \rho \ddot{u}_i &= \sigma_{ij,j} + f_i, \\ \sigma_{ij} n_j &= t_{(n)i}, \\ \sigma_{[ij]} &= 0, \end{aligned} \quad (2.22)$$

$$\begin{aligned}
\rho \dot{e} &= \sigma_{ij} v_{i,j} - q_{i,i} + \rho h + W_{in}^{em}, \\
W_{in}^{em} &= J_i E_i + E_i \frac{\partial P_i}{\partial t} - M_i \frac{\partial B_i}{\partial t}, \\
-\rho(\dot{\psi} + \eta \dot{\theta}) + \sigma_{ij} v_{i,j} + \theta^{-1} q_i \theta_{,i} - P_i \dot{E}_i - M_i \dot{B}_i + J_i E_i &\geq 0,
\end{aligned}$$

while Maxwell's equations remain unchanged. Furthermore, due to the fact that all fields are small, a set of linear constitutive equations are sufficient. Hence, as a consequence of (2.16), $\tilde{\psi}$ has a quadratic form as follows [18]:

$$\begin{aligned}
\tilde{\psi} &= \tilde{\psi}_0 + \tilde{\psi}_{IJ} \varepsilon_{IJ} + \frac{1}{2} C_{IJKL} \varepsilon_{IJ} \varepsilon_{KL} - D_{IJK}^E E_I \varepsilon_{JK} - D_{IJK}^B B_I \varepsilon_{JK} - \chi_I^E E_I \\
&\quad - \frac{1}{2} \chi_{IJ}^E E_I E_J - \chi_I^B B_I - \frac{1}{2} \chi_{IJ}^B B_I B_J - \Lambda_{IJ} E_I B_J,
\end{aligned} \tag{2.23}$$

in which $\tilde{\psi}_0, \tilde{\psi}_{IJ}, C_{IJKL}, D_{IJK}^E, D_{IJK}^B, \chi_I^E, \chi_{IJ}^E, \chi_I^B, \chi_{IJ}^B$ and Λ_{IJ} are functions of the $\mathbf{X}$ and θ . It is noticed that ε_{IJ} denotes infinitesimal strain tensor. By making use of the assumption given in (2.19), $\tilde{\psi}_0, \tilde{\psi}_{IJ}, \chi_I^E$ and χ_I^B in (2.23) can be expressed in terms of temperature in the following form [18]:

$$\begin{aligned}
\tilde{\psi}_0 &= -\rho_0 \eta_0 \theta - \left(\frac{\rho_0 \gamma}{2\theta_0} \right) \theta^2, \quad \tilde{\psi}_{IJ} = -B_{IJ} \theta, \\
\chi_I^E &= \tilde{\omega}_I \theta, \quad \chi_I^B = \Gamma_I \theta,
\end{aligned} \tag{2.24}$$

Hence, based on (2.17), one can derive [18]:

$$\begin{aligned}
\eta &= \eta_0 + \frac{\gamma}{\theta_0} \theta + \frac{1}{\rho_0} (B_{IJ} \varepsilon_{IJ} + \tilde{\omega}_I E_I + \Gamma_I B_I), \\
\sigma_{IJ} &= -B_{IJ} \theta + C_{IJKL} \varepsilon_{KL} - D_{KIJ}^E E_K - D_{KIJ}^B B_K, \\
\Pi_I &= \tilde{\omega}_I \theta + \chi_{IJ}^E E_J + D_{IJK}^E \varepsilon_{JK} + \Lambda_{IJ} B_J, \\
M_I &= \Gamma_I \theta + \chi_{IJ}^B B_J + D_{IJK}^B \varepsilon_{JK} + \Lambda_{JI} E_J.
\end{aligned} \tag{2.25}$$

Analogously, by taking into account the Clausius–Duhem inequality, (2.18) can be written in the form of [18]:

$$\begin{aligned}
q_I &= \kappa_{IJ} \theta_{,J} + \kappa_{IJ}^E \tilde{E}_J, \\
J_I &= \sigma_{IJ} \tilde{E}_J + \sigma_{IJ}^\theta \theta_{,J},
\end{aligned} \tag{2.26}$$

where κ_{IJ} and σ_{IJ} are symmetric and semi-positive definite. Also, let assume that the material and spatial frames coincide. Therefore, we have:

$$x_i \approx X_J \delta_{iJ} , \quad (2.27)$$

where $\delta_{iJ} = \delta_{ij} = \delta_{Ij} = \delta_{IJ}$. Hence, in linear theory, there is no difference between the material and the spatial frame and each description can be interchangeably used.

Now, as it was mentioned earlier, one can derive various constitutive equations based on independent and dependent variables chosen in the problem. In the linear case, nevertheless, by employing an appropriate Legendre transformation, it is straightforward to obtain all possible forms from one form. In the following, we shall consider the following constitutive equations [74]:

$$\begin{aligned} \varepsilon_{ij} &= S_{ijkl} \sigma_{kl} + d_{kij}^E E_k + d_{kij}^H H_k + \alpha_{ij} \theta , \\ D_i &= d_{ikl}^E \sigma_{kl} + \chi_{ij}^E E_j + \chi_{ij}^{EH} H_j + \beta_i^E \theta , \\ B_i &= d_{ikl}^H \sigma_{kl} + \chi_{ji}^{EH} E_j + \chi_{ij}^H H_j + \beta_i^H \theta , \\ \eta &= \alpha_{ij} \sigma_{ij} + \beta_i^E E_i + \beta_i^H H_i + \alpha^s \theta , \end{aligned} \quad (2.28)$$

and

$$\begin{aligned} \sigma_{ij} &= C_{ijkl} \varepsilon_{kl} - e_{kij}^E E_k - e_{kij}^H H_k + \alpha_{ij} \theta , \\ D_i &= e_{ikl}^E \varepsilon_{kl} + \kappa_{ij}^E E_j + \kappa_{ij}^{EH} H_j + \beta_i^E \theta , \\ B_i &= e_{ikl}^H \varepsilon_{kl} + \kappa_{ji}^{EH} E_j + \kappa_{ij}^H H_j + \beta_i^H \theta , \\ \eta &= \alpha_{ij} \varepsilon_{ij} + \beta_i^E E_i + \beta_i^H H_i + \alpha^s \theta , \end{aligned} \quad (2.29)$$

in which, and hereafter, we use lowercase letters for indices. Clearly, each constitutive coefficient given in (2.28) and (2.29) has a physical meaning; S_{ijkl} , C_{ijkl} , d_{kij}^E , d_{kij}^H , e_{kij}^E , e_{kij}^H , χ_{ij}^E , χ_{ij}^H , κ_{ij}^E , κ_{ij}^H , χ_{ij}^{EH} , and κ_{ij}^{EH} denote components of the compliance tensor, stiffness tensor, direct piezoelectric tensor, direct piezomagnetic tensor, reverse piezoelectric tensor, reverse piezomagnetic tensor, permittivity under constant stress, permeability under constant stress, permittivity under constant strain, permeability under constant strain, magnetoelectric tensor under constant stress and magnetoelectric tensor under constant strain, respectively [2].

Chapter 3

Stress field formulation of Electromagneto-Elastic Materials ¹

3.1 Introduction

Attempts to design/produce multifunctional materials are receiving attention due to the growing demands in modern applied sciences and engineering. Among the materials, those having both mechanical and electromagnetic properties are more recognizable because of their particular applications in devices such as ultrasonic transducers and micro-actuators, thermal-imaging devices, health monitoring devices, biomedical devices, biomimetics (bionics), energy harvesting [56, 62, 11], among others.

As is implied by the name ‘multifunctional materials’, there are couplings between different physical quantities in such materials. In other words, changing of one physical quantity leads to vary another one and vice versa. Or from energy point of view, in such materials, the energy of one form is converted to the other form. For the case of materials with coupling between mechanical and electromagnetic fields, depending on the coupling between different physical quantities, several classes of materials can be defined: coupling between mechanical and electric fields (Piezoelectricity), coupling between mechanical and magnetic field (Piezomagnetic), or electric, magnetic and mechanical fields coupling (Electromagneto-elastic). The later one can be thought of as a material containing both piezoelectric and piezomagnetic phases. Obviously, due to the presence of both phases, in the electro-magneto-elastic materials, the electric and magnetic coupling needs to be accounted.

To capture the behavior of such materials, a mathematical framework for incorporating different boundary conditions is required. One way to achieve that goal is to develop a coupled-continuum

¹ Adapted with permission from [2]

framework resulting from classical continuum mechanics combined with electromagnetic fields. The general path to do so can be summarized throughout two steps: first, generalizing governing equations for each field, i.e. Maxwell's equations for electric and magnetic fields and equation of motion for mechanical part, to account for the presence of multi-physical quantities. Second, generalizing constitutive equations to capture the coupling effects while the second law of thermodynamics still holds true. The most prominent works in this regard include [25, 24, 81, 69, 13, 17, 80, 63, 64, 60, 18, 54].

In general, in continuum field theories, one can formulate the governing equations either in terms of the displacement field (i.e., Navier-Cauchy equations) or in terms of the stress field (Ignaczak equation). As is well known, the former is suitable for Dirichlet boundary conditions, while the latter is more convenient for Neumann boundary conditions [67]. While countless many studies were devoted to analyses of the Navier-Cauchy equations of elastodynamics, very few investigations were made of the Ignaczak stress equations of motion [43, 44]. The alternative governing equations of motion and convolutional variational principles in terms of stresses for the mixed and traction boundary-value problems have been developed in [26]. A generalization of that formulation for the case of thermo-elasticity was given in [65]. A reciprocity theorem and the corresponding integral representation for an inhomogeneous anisotropic elastic body under arbitrary initial stress and stress-rate fields have been formulated in [45]. Also, various elastodynamics studies have been conducted on the basis of the stress equations of motion: domain of influence theorem for isotropic inhomogeneous material; stress waves propagating in homogeneous anisotropic infinite elastic bodies; surface waves in an inhomogeneous isotropic elastic half-space; wave propagation in multifunctional materials (i.e., materials having coupled constitutive equations); interaction of the surface waves with body waves; electro-mechanical surface waves in piezoelectrics; see [67] for a review.

However, there is no rigorous stress language framework for the case of electromagneto-elastic materials yet. Hence, in this chapter, a rigorous mathematical approach is employed to formulate governing equations in terms of stress as the counter part of field equations in terms of displacement field. First, the geometry of the problem and the corresponding governing equations for

an electromagneto-elastic material for the quasi-static electromagnetic condition are defined in the section Problem Statement. Next, in the Stress Formulation section, generalized 3D stress equations of motion are obtained through Lemmas 1 & 2 and Theorems 1-3. A uniqueness result is presented in Theorem 4 to give a set of sufficient conditions by which the solution (if it exists) is unique. As the final part of this section, reciprocity theorem in terms of stress, electric and magnetic potentials is established within the context of Theorem 5. Finally, the stress formulation of planar elasticity is written for a homogeneous transversely isotropic case, which broadly captures properties of multifunctional materials, due to the importance of the stress approach for planar elasticity (e.g. Airy stress function, complex approach etc.). The results of the present chapter open up an alternative path for consideration of boundary value problems (BVPs) for such materials, leading to rather different analytical and numerical approaches, in particular, for 2d analysis.

3.2 Problem Statement

This section gives the geometry of the problem along with the basic concepts, notations, and governing equations for a linear electromagneto-elastic material. Throughout the chapter $\mathbf{x}$ and t represent, respectively, the position vector and time parameter; the standard index notation for Cartesian tensors is utilized.

Let $\bar{V}$ denote a closed and bounded subset of 3D Euclidean space, with interior V and boundary ∂V , occupied by a deformable electro-magneto-elastic material. Furthermore, let assume that V is a regular region in the sense of [26]. Since the problem at hand deals with a multi-physical nature, the boundary ∂V is partitioned as follows: Let ∂V_i denote a subset of ∂V over which $i = \mathbf{u}, \sigma, \varphi, \mathbf{D}, \psi$ and $\mathbf{B}$ is prescribed with the following conditions:

$$\begin{aligned} \partial V_{\mathbf{u}} \cup \partial V_{\sigma} &= \partial V & \partial V_{\mathbf{u}} \cap \partial V_{\sigma} &= \emptyset, \\ \partial V_{\varphi} \cup \partial V_{\mathbf{D}} &= \partial V & \partial V_{\varphi} \cap \partial V_{\mathbf{D}} &= \emptyset, \\ \partial V_{\psi} \cup \partial V_{\mathbf{B}} &= \partial V & \partial V_{\psi} \cap \partial V_{\mathbf{B}} &= \emptyset. \end{aligned} \tag{3.1}$$

Smoothness of a tensor (or scalar) function $\mathbf{f}$ (or f) is expressed mathematically by $C^{M,N}$, where M and N are non-negative integers, with the following definition: $\mathbf{f}$ (or f) $\in C^{M,N}$, in which $\mathbf{f}$ (or f) is a function of position and time defined on $\bar{V} \times (0, \infty)$, if and only if the functions $\underbrace{f_{,ij\dots k}^{(n)}}_{\text{mindices}} (m = 0, 1, \dots, M; n = 0, 1, \dots, N)$ exist and are continuous.

Also, by $A_{(i,j)}$, we mean the symmetric part of the second order tensor $\mathbf{A}$, i.e., $A_{(i,j)} = \frac{A_{ij} + A_{ji}}{2}$. For functions $f(\mathbf{x}, t)$ and $g(\mathbf{x}, t)$ defined on $\Re \times [0, \infty)$, with $\Re$ a subset of the Euclidean space, which are continuous timewise, the symbol $[f * g](\mathbf{x}, t)$ indicates the convolution of two functions in the sense of

$$[f * g](\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t - \lambda) g(\mathbf{x}, \lambda) d\lambda, \quad (\mathbf{x}, t) \in \Re \times [0, \infty). \quad (3.2)$$

By the Laplace transform of the function $f(\mathbf{x}, t)$, we mean:

$$\ell[f(\mathbf{x}, t)] = \tilde{f}(\mathbf{x}, \alpha) = \int_0^\infty e^{-\alpha t} f(\mathbf{x}, t) dt \quad (3.3)$$

The above has the following well-known properties:

$$\begin{aligned} \ell[\dot{f}(\mathbf{x}, t)] &= \alpha^2 \ell[f(\mathbf{x}, t)] - \alpha f(\mathbf{x}, 0^+) - \dot{f}(\mathbf{x}, 0^+) \\ \ell[[f * g](\mathbf{x}, t)] &= \ell[f(\mathbf{x}, t)] \ell[g(\mathbf{x}, t)] \end{aligned} \quad (3.4)$$

For an integer m , a square $m \times m$ matrix, say $[\mathbf{A}]_{m \times m}$, is called strictly negative-definite if the scalar $[\mathbf{Y}]^T [\mathbf{A}] [\mathbf{Y}]$ is negative for any non-zero array $[\mathbf{Y}]_{1 \times m}$. The second-order symmetric tensor A_{ij} and the fourth-order tensor B_{ijkl} with major symmetry are called positive-definite (positive semi-definite) if $X_i A_{ij} X_j > 0$ ($X_i A_{ij} X_j \geq 0$) and $Y_{ij} B_{ijkl} Y_{kl} > 0$ ($Y_{ij} B_{ijkl} Y_{kl} \geq 0$) hold true for any non-zero vector X_i and any second-order tensor Y_{ij} , respectively.

At any point $(\mathbf{x}, t) \in \bar{V} \times [0, \infty)$, $u_i(\mathbf{x}, t)$, $\sigma_{ij}(\mathbf{x}, t)$, $\varepsilon_{ij}(\mathbf{x}, t)$, $f_i(\mathbf{x}, t)$, $E_i(\mathbf{x}, t)$, $D_i(\mathbf{x}, t)$, $B_i(\mathbf{x}, t)$, and $H_i(\mathbf{x}, t)$ indicate, respectively, the components of the displacement vector $\mathbf{u}(\mathbf{x}, t)$, stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$, strain tensor $\boldsymbol{\varepsilon}(\mathbf{x}, t)$, body force $\mathbf{f}(\mathbf{x}, t)$, electric field $\mathbf{E}(\mathbf{x}, t)$, electric displacement field $\mathbf{D}(\mathbf{x}, t)$, magnetic field $\mathbf{B}(\mathbf{x}, t)$, and magnetic field strength $\mathbf{H}(\mathbf{x}, t)$. Likewise, scalar fields $\rho(\mathbf{x}, t)$, $q_f(\mathbf{x}, t)$, $\varphi(\mathbf{x}, t)$, and $\psi(\mathbf{x}, t)$ at any point $(\mathbf{x}, t) \in \bar{V} \times (0, \infty)$, denote the mass density, free charge

density, electric potential, and magnetic potential, respectively. Next, let us list the fundamental field equations for a linear coupled electro-magneto-elastic. When the rate of change of magnetic field is small (electric field is curl-free) and there is no electric current (the magnetic field is curl-free), the Maxwell-Faraday and Ampère equations are identically satisfied by definition of potential functions. Hence, the governing equations read:

$$\begin{aligned}\rho \ddot{u}_i &= \sigma_{ij,j} + f_i \text{ on } V \times (0, \infty), \\ D_{i,i} &= q_f \text{ on } V \times (0, \infty), \\ B_{i,i} &= 0 \text{ on } V \times (0, \infty),\end{aligned}\tag{3.5}$$

in which $\sigma_{ij} = \sigma_{ji}$ on $V \times (0, \infty)$, with “kinematic equations”

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ on } V \times (0, \infty), \\ E_i &= -\varphi_{,i} \text{ on } V \times (0, \infty), \\ H_i &= -\psi_{,i} \text{ on } V \times (0, \infty).\end{aligned}\tag{3.6}$$

These are augmented by the boundary conditions as follows

$$\begin{aligned}\text{Mechanical Boundary Conditions} & \left\{ \begin{aligned} u_i &= \hat{u}_i(\mathbf{x}, t) \quad \text{on } \partial V_{\mathbf{u}} \times [0, \infty), \\ t_i &= \hat{t}_i(\mathbf{x}, t) \quad \text{on } \partial V_{\boldsymbol{\sigma}} \times [0, \infty), \end{aligned} \right. \\ \text{Electric Boundary Conditions} & \left\{ \begin{aligned} \varphi &= \hat{\varphi}(\mathbf{x}, t) \quad \text{on } \partial V_{\varphi} \times [0, \infty), \\ d &= \hat{d}(\mathbf{x}, t) \quad \text{on } \partial V_{\mathbf{D}} \times [0, \infty), \end{aligned} \right. \\ \text{Magnetic Boundary Conditions} & \left\{ \begin{aligned} \psi &= \hat{\psi}(\mathbf{x}, t) \quad \text{on } \partial V_{\psi} \times [0, \infty), \\ b &= \hat{b}(\mathbf{x}, t) \quad \text{on } \partial V_{\mathbf{B}} \times [0, \infty), \end{aligned} \right.\end{aligned}\tag{3.7}$$

where $t_i = \sigma_{ij}n_j$, $d = D_i n_i$ and $b = B_i n_i$, along with the initial conditions

$$\begin{aligned} u_i(\mathbf{x}, 0) &= u_i^0(\mathbf{x}), \mathbf{x} \in \bar{V}, \\ \dot{u}_i(\mathbf{x}, 0) &= \dot{u}_i^0(\mathbf{x}), \mathbf{x} \in \bar{V}. \end{aligned} \quad (3.8)$$

Or

$$\begin{aligned} \sigma_{ij}(\mathbf{x}, 0) &= \sigma_{ij}^0(\mathbf{x}), \varphi(\mathbf{x}, 0) = \varphi^0(\mathbf{x}), \psi(\mathbf{x}, 0) = \psi^0(\mathbf{x}), \mathbf{x} \in \bar{V}, \\ \dot{\sigma}_{ij}(\mathbf{x}, 0) &= \dot{\sigma}_{ij}^0(\mathbf{x}), \dot{\varphi}(\mathbf{x}, 0) = \dot{\varphi}^0(\mathbf{x}), \dot{\psi}(\mathbf{x}, 0) = \dot{\psi}^0(\mathbf{x}), \mathbf{x} \in \bar{V}, \end{aligned} \quad (3.9)$$

in which $\hat{u}_i(\mathbf{x}, t)$, $\hat{t}_i(\mathbf{x}, t)$, $\hat{\varphi}(\mathbf{x}, t)$, $\hat{d}(\mathbf{x}, t)$, $\hat{\psi}(\mathbf{x}, t)$, and $\hat{b}(\mathbf{x}, t)$ are, respectively, prescribed displacement components, traction components, electric potential, electric displacement, magnetic potential and magnetic field on the boundary. Also, $u_i^0(\mathbf{x})$, $\dot{u}_i^0(\mathbf{x})$, $\sigma_{ij}^0(\mathbf{x})$, $\dot{\sigma}_{ij}^0(\mathbf{x})$, $\varphi^0(\mathbf{x})$, $\dot{\varphi}^0(\mathbf{x})$, $\psi^0(\mathbf{x})$, and $\dot{\psi}^0(\mathbf{x})$ are the corresponding prescribed initial conditions for the problem, assumed to be consistent in the sense of

$$\begin{aligned} S_{ijkl}\sigma_{kl}^0(\mathbf{x}) - d_{kij}^E \varphi_{,k}^0(\mathbf{x}) - d_{kij}^H \psi_{,k}^0(\mathbf{x}) &= \frac{1}{2}(u_{i,j}^0 + u_{j,i}^0), \mathbf{x} \in \bar{V}, \\ S_{ijkl}\dot{\sigma}_{kl}^0(\mathbf{x}) - d_{kij}^E \dot{\varphi}_{,k}^0(\mathbf{x}) - d_{kij}^H \dot{\psi}_{,k}^0(\mathbf{x}) &= \frac{1}{2}(\dot{u}_{i,j}^0 + \dot{u}_{j,i}^0), \mathbf{x} \in \bar{V}. \end{aligned} \quad (3.10)$$

with

$$\begin{aligned} (d_{ikl}^E \sigma_{kl}^0 - \chi_{ij}^E \varphi_{,j}^0 - \chi_{ij}^{EH} \psi_{,j}^0)_{,i} &= q_e(\mathbf{x}, 0), \text{ on } \bar{V}, \\ (d_{ikl}^H \sigma_{kl}^0 - \chi_{ij}^{EH} \varphi_{,j}^0 - \chi_{ij}^H \psi_{,j}^0)_{,i} &= 0, \text{ on } \bar{V}, \\ (d_{ikl}^E \dot{\sigma}_{kl}^0 - \chi_{ij}^E \dot{\varphi}_{,j}^0 - \chi_{ij}^{EH} \dot{\psi}_{,j}^0)_{,i} &= \dot{q}_e(\mathbf{x}, 0), \text{ on } \bar{V}, \\ (d_{ikl}^H \dot{\sigma}_{kl}^0 - \chi_{ij}^{EH} \dot{\varphi}_{,j}^0 - \chi_{ij}^H \dot{\psi}_{,j}^0)_{,i} &= 0, \text{ on } \bar{V}, \end{aligned} \quad (3.11)$$

such that $\sigma_{kl}^0(\mathbf{x})$, $\dot{\sigma}_{kl}^0(\mathbf{x})$, $\varphi^0(\mathbf{x})$, $\dot{\varphi}^0(\mathbf{x})$, $\psi^0(\mathbf{x})$, and $\dot{\psi}^0(\mathbf{x})$ are uniquely obtained from (3.10) and (3.11) along with boundary conditions at $t = 0$. That is, when the right-hand sides of (3.10) and (3.11) along with the boundary conditions at $t = 0$ are zero, $\sigma_{kl}^0(\mathbf{x})$, $\dot{\sigma}_{kl}^0(\mathbf{x})$, $\varphi^0(\mathbf{x})$, $\dot{\varphi}^0(\mathbf{x})$, $\psi^0(\mathbf{x})$, and $\dot{\psi}^0(\mathbf{x})$ are all vanished. The coefficients in (3.10) and (3.11) shall be defined clearly in the following.

Next, we need to define constitutive equations, as the number of equations is not sufficient to obtain unknown fields. In this regard, based on the independent variables taken into account in the problem,

different versions of constitutive equations can be formulated. These forms in the linear regime can be readily obtained from one another by means of an appropriate Legendre transformation [74]. Assuming a hyperelastic material under an isothermal condition, we have the following constitutive equations:

$$\begin{aligned}\sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e_{kij}^E E_k - e_{kij}^H H_k \quad \text{on } V \times (0, \infty), \\ D_i &= e_{ikl}^E \varepsilon_{kl} + \kappa_{ij}^E E_j + \kappa_{ij}^{EH} H_j \quad \text{on } V \times (0, \infty), \\ B_i &= e_{ikl}^H \varepsilon_{kl} + \kappa_{ij}^{EH} E_j + \kappa_{ij}^H H_j \quad \text{on } V \times (0, \infty),\end{aligned}\tag{3.12}$$

with the associated symmetry conditions:

$$\begin{aligned}C_{ijkl} &= C_{klij} = C_{jikl} = C_{ijlk} \quad \text{on } V, \\ e_{kij}^E &= e_{kji}^E \quad \text{on } V, \\ e_{kij}^H &= e_{kji}^H \quad \text{on } V, \\ \kappa_{ij}^E &= \kappa_{ji}^E, \kappa_{ij}^{EH} = \kappa_{ji}^{EH}, \kappa_{ij}^H = \kappa_{ji}^H \quad \text{on } V.\end{aligned}\tag{3.13}$$

Or

$$\begin{aligned}\varepsilon_{ij} &= S_{ijkl}\sigma_{kl} + d_{kij}^E E_k + d_{kij}^H H_k \quad \text{on } V \times (0, \infty), \\ D_i &= d_{ikl}^E \sigma_{kl} + \chi_{ij}^E E_j + \chi_{ij}^{EH} H_j \quad \text{on } V \times (0, \infty), \\ B_i &= d_{ikl}^H \sigma_{kl} + \chi_{ij}^{EH} E_j + \chi_{ij}^H H_j \quad \text{on } V \times (0, \infty),\end{aligned}\tag{3.14}$$

with the associated symmetry conditions:

$$\begin{aligned}S_{ijkl} &= S_{klij} = S_{jikl} = S_{ijlk} \quad \text{on } V, \\ d_{kij}^E &= d_{kji}^E \quad \text{on } V, \\ d_{kij}^H &= d_{kji}^H \quad \text{on } V, \\ \chi_{ij}^E &= \chi_{ji}^E, \chi_{ij}^{EH} = \chi_{ji}^{EH}, \chi_{ij}^H = \chi_{ji}^H \quad \text{on } V.\end{aligned}\tag{3.15}$$

The relation between the aforementioned coefficients can be written as follows:

$$\begin{aligned}
S_{ijkl}C_{ijpq} &= \delta_{kp}\delta_{lq} \text{ on } \bar{V}, \\
d_{kij}^E &= S_{pqij}e_{kpq}^E \text{ on } \bar{V}, \\
d_{kij}^H &= S_{pqij}e_{kpq}^H \text{ on } \bar{V}, \\
\chi_{ij}^E &= S_{pqrs}e_{ipq}^E e_{jrs}^E + \kappa_{ij}^E \text{ on } \bar{V}, \\
\chi_{ij}^H &= S_{pqrs}e_{ipq}^H e_{jrs}^H + \kappa_{ij}^H \text{ on } \bar{V}, \\
\chi_{ij}^{EH} &= S_{pqrs}e_{ipq}^H e_{jrs}^E + \kappa_{ij}^{EH} \text{ on } \bar{V}.
\end{aligned} \tag{3.16}$$

Similar to [26], for reference in the remainder of the chapter, the following regularity conditions are assumed throughout the chapter:

- i. $\rho > 0$ is continuously differentiable on $\bar{V}$;
- ii. $\mathbf{C}$, $\mathbf{e}$, κ and $\mathbf{S}$, $\mathbf{d}$, χ are continuously differentiable on $\bar{V}$ and meet (3.13) and (3.15), respectively;
- iii. $\mathbf{u}^0(\mathbf{x})$ is continuously differentiable on $\bar{V}$;
- iv. $\mathbf{v}^0(\mathbf{x})$ is continuously differentiable on $\bar{V}$;
- v. $\mathbf{f}$ and q_e are continuously differentiable on $\bar{V}$;
- vi. $\hat{\mathbf{u}}$, $\hat{\phi}$, and $\hat{\psi}$ are continuous on $\overline{\partial V_{\mathbf{u}}} \times [0, \infty)$, $\overline{\partial V_{\mathbf{D}}} \times [0, \infty)$ and $\overline{\partial V_{\mathbf{B}}} \times [0, \infty)$, respectively;
- vii. $\hat{t}$, $\hat{d}$ and $\hat{b}$ are piecewise continuous on $\overline{\partial V_{\sigma}} \times [0, \infty)$, $\overline{\partial V_{\mathbf{D}}} \times [0, \infty)$ and $\overline{\partial V_{\mathbf{B}}} \times [0, \infty)$, respectively.

To proceed further, the following is needed.

Definition 1. An ordered array $S = [\mathbf{u}, \varepsilon, \sigma, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \phi, \psi]$ is called an admissible process on $\bar{V} \times [0, \infty)$ provided that

$$\begin{aligned}
u_i &\in C^{1,2}, \quad \varepsilon_{ij} \in C^{0,0}, \quad \sigma_{ij} \in C^{1,0}, \quad D_i \in C^{1,0}, \quad B_i \in C^{1,0}, \quad \phi \in C^{1,0}, \quad \psi \in C^{1,0}, \quad E_i \in C^{0,0}, \quad H_i \in C^{0,0} \\
\varepsilon_{ij} &= \varepsilon_{ji}, \quad \sigma_{ij} = \sigma_{ji}.
\end{aligned} \tag{3.17}$$

In addition, a solution of the initial mixed-boundary value problem (i.e., IBVP) is an admissible process which meets (3.5), (3.6), (3.7), (3.8), and (3.12).

3.3 Stress framework

As described in the previous section, a solution to the problem is an admissible process that satisfies (3.5), (3.6), (3.7), (3.8), and (3.12). However, an equivalent set of governing equations can be established. For pure elasticity when the boundary condition is entirely traction-type, an alternative set of governing equations was proposed [44], which is in terms of stress. Following that, [26] obtained a more general framework, which is suitable for general mixed boundary conditions. For the present case, since the equation of motion (3.5)₁, is the same as the pure elasticity problems, one can generalize that method for the case of electromagneto-elastic materials. The idea behind the approach can be described as follows: taking the Laplace transform of both sides of (3.5)₁, we have

$$\rho \ell[u_i] = \rho \left(\frac{u_i^0}{\alpha} + \frac{\dot{u}_i^0}{\alpha^2} \right) + \frac{\ell[\sigma_{ij,j} + f_i]}{\alpha^2}, \quad (3.18)$$

or

$$\begin{aligned} \rho u_i &= \rho(u_i^0 + t \dot{u}_i^0) + t * (\sigma_{ij,j} + f_i) \\ \text{or} \\ \rho u_i &= \rho(u_i^0 + t \dot{u}_i^0) + t * f_i + t * \sigma_{ij,j}. \end{aligned} \quad (3.19)$$

Hence, one can define

$$\begin{aligned} h(t) &= t, \quad 0 \leq t < \infty, \\ f_i^b(\mathbf{x}, t) &= \rho(u_i^0 + t \dot{u}_i^0) + h * f_i(\mathbf{x}, t). \end{aligned} \quad (3.20)$$

Consequently, we have the following theorem

Theorem 1. *Let $u_i \in C^{0,2}$ and $\sigma_{ij} \in C^{1,0}$ be a vector field and a second-order symmetric tensor field, respectively. Then $\mathbf{u}$ and $\boldsymbol{\sigma}$ meet (3.5)₁ and the associated initial conditions (3.8) if and only if*

$$\rho \mathbf{u} = h * (\nabla \cdot \boldsymbol{\sigma}) + \mathbf{f}^b. \quad (3.21)$$

Proof. See [26]. □

Consequently, with the help of the following Theorem, which is the direct result of Theorem 1, one can define the alternative integro-partial-differential field equations of the mixed IBVP:

Theorem 2. *The admissible process $S = [\mathbf{u}, \varepsilon, \sigma, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \varphi, \psi]$ is a solution of the initial mixed-boundary value problem if and only if it satisfies (3.21), (3.5)₂₋₃, (3.6), (3.7), and (3.12).*

As shown comprehensively in [26], (3.21) is well-suited for a general boundary condition when , leading to an integro-partial differential equations. Nevertheless, when the mechanical boundary condition is only traction type (present case) – which actually encompasses a certain class of the IBVPs – one can obtain a set of explicit PDE's provided that a more stronger continuity conditions are met. Hence, the explicit form of governing equations is obtained throughout some Lemmas and a Theorem. Subsequently, Uniqueness Reciprocity Theorems corresponding to such formulation are derived and proved in details. First, we have the following Lemmas

Lemma 1. *Let $u_i(\mathbf{x}, t)$, $\varepsilon_{ij}(\mathbf{x}, t)$, $\sigma_{ij}(\mathbf{x}, t)$, $E_i(\mathbf{x}, t)$, $H_i(\mathbf{x}, t)$, $\varphi(\mathbf{x}, t)$, and $\psi(\mathbf{x}, t)$ satisfy (3.21), (3.6), and (3.14)₁ with*

$$\sigma \in C^{2,2}, \varphi \in C^{1,2}, \psi \in C^{1,2}. \quad (3.22)$$

Then, we have

$$S_{ijkl}\ddot{\sigma}_{kl} = \left(\frac{1}{\rho}\sigma_{(ik,k),j}\right) + \left(\frac{1}{\rho}f_{(i),j}\right) + d_{kij}^E\ddot{\varphi}_{,k} + d_{kij}^H\ddot{\psi}_{,k}. \quad (3.23)$$

Proof. Taking the derivative of (3.21) with respect to time twice and with respect to space variables once, employing (3.6) and (3.14)₁ , we obtain the desired result. This completes the proof. □

Lemma 2. *Let (3.22) hold true together with $\sigma_{ij} = \sigma_{ji}$. Define u_i with (3.21), ε_{ij} with (3.14)₁, E_i with (3.6)₂ and H_i with (3.6)₃. Let σ , φ and ψ meet (3.23) and (3.10). Then the kinematic equation (3.6)₁ holds true.*

Proof. Taking the derivative of (3.21) with respect to time twice and with respect to space variables once and making use of (3.6)₂₋₃ and (3.23), we find

$$S_{ijkl}\ddot{\sigma}_{kl} - d_{kij}^E\ddot{\varphi}_{,k} - d_{kij}^H\ddot{\psi}_{,k} = \frac{1}{2}(\ddot{u}_{i,j} + \ddot{u}_{j,i}). \quad (3.24)$$

Now, taking the integration twice with respect to time, on account of (3.10) and (3.14)₁, we obtain the desired result. $\square$

Next, using Lemmas 1 & 2, the stress coupled governing equation of motion for an electromagneto-elastic material, which is the counterpart of governing equations in terms of displacement fields, is expressed in the following theorem.

Theorem 3. *Let $\sigma \in C^{2,2}$, $\varphi \in C^{2,2}$, $\psi \in C^{2,2}$ with $\sigma_{ij} = \sigma_{ji}$. An ordered array $S = [\sigma, \varphi, \psi]$ is a solution of the initial mixed IBVP if and only if*

$$\begin{aligned}
S_{ijkl}\ddot{\sigma}_{kl} &= \left(\frac{1}{\rho}\sigma_{(ik,k),j}\right) + \left(\frac{1}{\rho}f(i),j\right) + d_{kij}^E\ddot{\varphi}_{,k} + d_{kij}^H\ddot{\psi}_{,k}, \text{ on } V \times (0,\infty), \\
(d_{ikl}^E\sigma_{kl} - \chi_{ij}^E\varphi_{,j} - \chi_{ij}^{EH}\psi_{,j}),_i &= q_f, \text{ on } V \times (0,\infty), \\
(d_{ikl}^H\sigma_{kl} - \chi_{ij}^{EH}\varphi_{,j} - \chi_{ij}^H\psi_{,j}),_i &= 0, \text{ } V \times (0,\infty), \\
\sigma_{ij}n_j &= \hat{t}_i(\mathbf{x},t), \text{ on } \partial V \times [0,\infty), \\
\varphi &= \hat{\varphi}(\mathbf{x},t), \text{ on } \partial V_\varphi \times [0,\infty), \\
(d_{ikl}^E\sigma_{kl} - \chi_{ij}^E\varphi_{,j} - \chi_{ij}^{EH}\psi_{,j})n_i &= \hat{d}(\mathbf{x},t), \text{ on } \partial V_{\mathbf{D}} \times [0,\infty), \\
\psi &= \hat{\psi}(\mathbf{x},t), \text{ on } \partial V_\psi \times [0,\infty), \\
(d_{ikl}^H\sigma_{kl} - \chi_{ij}^{EH}\varphi_{,j} - \chi_{ij}^H\psi_{,j})n_i &= \hat{b}(\mathbf{x},t), \text{ on } \partial V_{\mathbf{B}} \times [0,\infty) \\
\sigma_{ij}(\mathbf{x},0) &= \sigma_{ij}^0(\mathbf{x}), \varphi(\mathbf{x},0) = \varphi^0(\mathbf{x}), \psi(\mathbf{x},0) = \psi^0(\mathbf{x}), \mathbf{x} \in \bar{V}, \\
\dot{\sigma}_{ij}(\mathbf{x},0) &= \dot{\sigma}_{ij}^0(\mathbf{x}), \dot{\varphi}(\mathbf{x},0) = \dot{\varphi}^0(\mathbf{x}), \dot{\psi}(\mathbf{x},0) = \dot{\psi}^0(\mathbf{x}), \mathbf{x} \in \bar{V}.
\end{aligned} \tag{3.25}$$

Proof. Suppose that (3.25) holds true. Define $\mathbf{u}$, $\mathbf{E}$ and $\mathbf{H}$ through (3.21), (3.6)₂ and (3.6)₃, respectively. Also, define ε , $\mathbf{D}$ and $\mathbf{B}$ through (3.14). Then, considering symmetry condition (3.15), the boundary and initial conditions (3.7) and (3.8) are satisfied identically; Lemma 2 gives (3.6)₁; (3.25)₂₋₃ together with the above defined $\mathbf{D}$ and $\mathbf{B}$, and the symmetry condition (3.15) gives (3.5)₂₋₃. Hence, by Theorem 2, $[\sigma, \varphi, \psi]$ is a solution of the IBVP. Conversely, (3.21), (3.5)₂₋₃, (3.6), Lemma 1, (3.12), (3.16), (3.7), and (3.8) imply (3.25). This completes the proof. $\square$

Note that at this point, when $\partial V_{\mathbf{u}} = \emptyset$ in (3.7), together with initial conditions, one can use (3.25) as a set of coupled governing equations to derive stress fields, electric and magnetic potentials directly. In turn, the displacement field can be obtained using Theorem 1. Thus, the important

question here is whether the solution is unique. The following gives an answer.

Theorem 4. Consider (3.25). Let $\mathbf{S}$ be a positive-definite fourth order tensor and $\begin{bmatrix} \chi^E & \chi^{EH} \\ \chi^{EH} & \chi^H \end{bmatrix}$ is a negative-definite matrix, respectively. Then, if a solution exists for (3.25), it is unique.

Proof. It is sufficient to show that (3.25) results in $\sigma_{ij}(\mathbf{x}, t) = 0$, $\phi(\mathbf{x}, t) = 0$, and $\psi(\mathbf{x}, t) = 0$ for the case of zero body force, charge density, boundary and initial conditions. To do so, multiply (3.25)₁ by $\dot{\sigma}_{ij}$ and take integrals over the volume and time

$$\int_V \int_0^t (2S_{ijkl}\ddot{\sigma}_{kl}\dot{\sigma}_{ij} - (\frac{1}{\rho}\sigma_{ik,k})_{,j}\dot{\sigma}_{ij} - (\frac{1}{\rho}\sigma_{jk,k})_{,i}\dot{\sigma}_{ij} - 2d_{kij}^E\ddot{\phi}_{,k}\dot{\sigma}_{ij} - 2d_{kij}^H\ddot{\psi}_{,k}\dot{\sigma}_{ij})dt dV = 0 \quad (3.26)$$

To further simplify the above integral, define $D_i = d_{ikl}^E\sigma_{kl} - \chi_{ij}^E\phi_{,j} - \chi_{ij}^{EH}\psi_{,j}$. Then the following useful equations (which can be obtained from the rate of change of D_i (or B_i) multiplied by $\dot{\phi}_{,i}$ (or $\dot{\psi}_{,i}$) and applying (3.25)₂ (or (3.25)₃) can be utilized

$$\begin{aligned} d_{ikl}^E\dot{\sigma}_{kl}\dot{\phi}_{,i} &= (\dot{D}_i\dot{\phi})_{,i} + \chi_{ij}^E\dot{\phi}_{,j}\dot{\phi}_{,i} + \chi_{ij}^{EH}\dot{\psi}_{,j}\dot{\phi}_{,i}, \\ d_{ikl}^H\dot{\sigma}_{kl}\dot{\psi}_{,i} &= (\dot{B}_i\dot{\psi})_{,i} + \chi_{ij}^{EH}\dot{\phi}_{,j}\dot{\psi}_{,i} + \chi_{ij}^H\dot{\psi}_{,j}\dot{\psi}_{,i}. \end{aligned} \quad (3.27)$$

Substituting (3.27) into (3.26), employing the Divergence Theorem once more and applying zero electric and magnetic boundary conditions, we find

$$\begin{aligned} &\int_V \int_0^t (\frac{d}{d\tau}(S_{ijkl}\dot{\sigma}_{kl}\dot{\sigma}_{ij}) + \frac{d}{d\tau}(\frac{1}{\rho}\sigma_{ik,k}\sigma_{ij,j}))d\tau dV \\ &- \int_V \int_0^t (\chi_{ij}^E\frac{d}{d\tau}(\dot{\phi}_{,j}\dot{\phi}_{,i}) + 2\chi_{ij}^{EH}\frac{d}{d\tau}(\dot{\psi}_{,j}\dot{\phi}_{,i}) + \chi_{ij}^H\frac{d}{d\tau}(\dot{\psi}_{,j}\dot{\psi}_{,i}))d\tau dV = 0, \end{aligned} \quad (3.28)$$

or

$$\int_V (-(\chi_{ij}^E\dot{\phi}_{,j}\dot{\phi}_{,i} + 2\chi_{ij}^{EH}\dot{\psi}_{,j}\dot{\phi}_{,i} + \chi_{ij}^H\dot{\psi}_{,j}\dot{\psi}_{,i}) + \frac{1}{\rho}\sigma_{ik,k}\sigma_{ij,j} + S_{ijkl}\dot{\sigma}_{kl}\dot{\sigma}_{ij})dV = 0. \quad (3.29)$$

Now, since S_{ijkl} is a positive-definite fourth order tensor and $\begin{bmatrix} \chi^E & \chi^{EH} \\ \chi^{EH} & \chi^H \end{bmatrix}$ is a negative-definite matrix, we conclude

$$\sigma_{ik,k} = 0, \dot{\sigma}_{kl} = 0, \dot{\psi}_{,j} = 0, \dot{\phi}_{,i} = 0, \quad (3.30)$$

which, together with the vanishing initial and boundary conditions, imply the desired result. $\square$

In the following, a reciprocity theorem is established for an electromagneto-elastic material when the mechanical boundary condition is pure traction.

Theorem 5. *Let $S^{(n)} = [\sigma^{(n)}, \varphi^{(n)}, \psi^{(n)}]$ with $n = 1, 2$ be two sets of solutions to (3.25) with the initial conditions, boundary conditions, body force and free charge density:*

$$\begin{aligned} \text{Initial Conditions : } & [\sigma_{ij}^{(n)(0)}(\mathbf{x}), \dot{\sigma}_{ij}^{(n)(0)}(\mathbf{x}), \varphi^{(n)(0)}(\mathbf{x}), \dot{\varphi}^{(n)(0)}(\mathbf{x}), \psi^{(n)(0)}(\mathbf{x}), \dot{\psi}^{(n)(0)}(\mathbf{x})], \\ \text{Boundary Conditions : } & [\hat{t}_i^{(n)}(\mathbf{x}, t), \hat{\phi}^{(n)}(\mathbf{x}, t), \hat{\psi}^{(n)}(\mathbf{x}, t), \hat{d}^{(n)}(\mathbf{x}, t), \hat{b}^{(n)}(\mathbf{x}, t)], \\ & \mathbf{f}^{(n)}(\mathbf{x}, t), q_f^{(n)}(\mathbf{x}, t) \end{aligned} \quad (3.31)$$

Then the following relation holds true:

$$\begin{aligned} & \int_V S_{ijkl} (\dot{\sigma}_{kl}^{(1)} \sigma_{ij}^{(2)(0)} - \dot{\sigma}_{kl}^{(1)(0)} \sigma_{ij}^{(2)} - \dot{\sigma}_{kl}^{(2)} \sigma_{ij}^{(1)(0)} + \dot{\sigma}_{kl}^{(2)(0)} \sigma_{ij}^{(1)}) d\mathbf{x} \\ & - \int_{\partial V} \left(\frac{1}{\rho} \sigma_{ik,k}^{(1)} * \hat{t}_i^{(2)} - \frac{1}{\rho} \sigma_{ik,k}^{(2)} * \hat{t}_i^{(1)} \right) d\mathbf{x} - \int_V \left(\left(\frac{1}{\rho} f_{(i)}^{(1)} \right)_{,j} * \sigma_{ij}^{(2)} - \left(\frac{1}{\rho} f_{(i)}^{(2)} \right)_{,j} * \sigma_{ij}^{(1)} \right) d\mathbf{x} \\ & - \int_{\partial V} (\hat{d}^{(2)(0)} \dot{\phi}^{(1)} - \hat{d}^{(2)} \dot{\phi}^{(1)(0)} + \dot{\hat{d}}^{(2)} * \dot{\phi}^{(1)} - \hat{d}^{(1)(0)} \dot{\phi}^{(2)} + \hat{d}^{(1)} \dot{\phi}^{(2)(0)} - \dot{\hat{d}}^{(1)} * \dot{\phi}^{(2)}) d\mathbf{x} \\ & - \int_V (\chi_{ij}^E \varphi_{,j}^{(2)(0)} \dot{\phi}_{,i}^{(1)} - \chi_{ij}^E \dot{\phi}_{,i}^{(1)(0)} \varphi_{,j}^{(2)} - \chi_{ij}^E \varphi_{,j}^{(1)(0)} \dot{\phi}_{,i}^{(2)} + \chi_{ij}^E \dot{\phi}_{,i}^{(2)(0)} \varphi_{,j}^{(1)}) d\mathbf{x} \\ & - \int_V (\chi_{ij}^{EH} \psi_{,j}^{(2)(0)} \dot{\phi}_{,i}^{(1)} - \chi_{ij}^{EH} \dot{\phi}_{,i}^{(1)(0)} \psi_{,j}^{(2)} - \chi_{ij}^{EH} \psi_{,j}^{(1)(0)} \dot{\phi}_{,i}^{(2)} + \chi_{ij}^{EH} \dot{\phi}_{,i}^{(2)(0)} \psi_{,j}^{(1)}) d\mathbf{x} \\ & + \int_V (q_f^{(2)(0)} \dot{\phi}^{(1)} - q_f^{(2)} \dot{\phi}^{(1)(0)} + \dot{q}_f^{(2)} * \dot{\phi}^{(1)} - q_f^{(1)(0)} \dot{\phi}^{(2)} + q_f^{(1)} \dot{\phi}^{(2)(0)} - \dot{q}_f^{(1)} * \dot{\phi}^{(2)}) d\mathbf{x} \end{aligned} \quad (3.32)$$

$$\begin{aligned}
& - \int_{\partial V} (\hat{b}^{(2)(0)} \dot{\psi}^{(1)} - \hat{b}^{(2)} \dot{\psi}^{(1)(0)} + \dot{b}^{(2)} * \dot{\psi}^{(1)} - \hat{b}^{(1)(0)} \dot{\psi}^{(2)} + \hat{b}^{(1)} \dot{\psi}^{(2)(0)} - \dot{b}^{(1)} * \dot{\psi}^{(2)}) d\mathbf{x} \\
& - \int_V (\chi_{ij}^H \psi_{,j}^{(2)(0)} \dot{\psi}_{,i}^{(1)} - \chi_{ij}^H \dot{\psi}_{,i}^{(1)(0)} \psi_{,j}^{(2)} - \chi_{ij}^H \psi_{,j}^{(1)(0)} \dot{\psi}_{,i}^{(2)} + \chi_{ij}^H \dot{\psi}_{,i}^{(2)(0)} \psi_{,j}^{(1)}) d\mathbf{x} \\
& - \int_V (\chi_{ij}^{EH} \phi_{,j}^{(2)(0)} \dot{\psi}_{,i}^{(1)} - \chi_{ij}^{EH} \dot{\psi}_{,i}^{(1)(0)} \phi_{,j}^{(2)} - \chi_{ij}^{EH} \phi_{,j}^{(1)(0)} \dot{\psi}_{,i}^{(2)} + \chi_{ij}^{EH} \dot{\psi}_{,i}^{(2)(0)} \phi_{,j}^{(1)}) d\mathbf{x} = 0.
\end{aligned}$$

Proof. For S^1 (S^2), applying the convolution product to both sides of (3.25)₁ with $\sigma_{ij}^{(2)}(\sigma_{ij}^{(1)})$ and doing some manipulations including integrating by parts with respect to the dummy variable of the convolution product, employing (3.25)₂₋₃ and Divergence Theorem, we obtain:

$$\begin{aligned}
& \int_V S_{ijkl} (\dot{\sigma}_{kl}^{(1)} * \dot{\sigma}_{ij}^{(2)} + \dot{\sigma}_{kl}^{(1)} \sigma_{ij}^{(2)(0)} - \dot{\sigma}_{kl}^{(1)(0)} \sigma_{ij}^{(2)}) d\mathbf{x} = \int_{\partial V} \frac{1}{\rho} \sigma_{ik,k}^{(1)} * \hat{t}_i^{(2)} d\mathbf{x} + \int_V \left(\frac{1}{\rho} f_{(i,j)}^{(1)} \right) * \sigma_{ij}^{(2)} d\mathbf{x} \\
& - \int_V \frac{1}{\rho} \sigma_{ik,k}^{(1)} * \sigma_{ij,j}^{(2)} d\mathbf{x} + \int_V (\chi_{ij}^E \phi_{,j}^{(2)(0)} \dot{\phi}_{,i}^{(1)} - \chi_{ij}^E \dot{\phi}_{,i}^{(1)(0)} \phi_{,j}^{(2)} + \chi_{ij}^E \dot{\phi}_{,j}^{(2)} * \dot{\phi}_{,i}^{(1)}) d\mathbf{x} \\
& + \int_V (\chi_{ij}^{EH} \psi_{,j}^{(2)(0)} \dot{\phi}_{,i}^{(1)} - \chi_{ij}^{EH} \dot{\phi}_{,i}^{(1)(0)} \psi_{,j}^{(2)} + \chi_{ij}^{EH} \psi_{,j}^{(2)} * \dot{\phi}_{,i}^{(1)}) d\mathbf{x} \\
& + \int_{\partial V} (\hat{d}^{(2)(0)} \dot{\phi}^{(1)} - \hat{d}^{(2)} \dot{\phi}^{(1)(0)} + \dot{d}^{(2)} * \dot{\phi}^{(1)}) d\mathbf{x} + \int_{\partial V} (\hat{b}^{(2)(0)} \dot{\psi}^{(1)} - \hat{b}^{(2)} \dot{\psi}^{(1)(0)} + \dot{b}^{(2)} * \dot{\psi}^{(1)}) d\mathbf{x} \\
& + \int_V (\chi_{ij}^H \psi_{,j}^{(2)(0)} \dot{\psi}_{,i}^{(1)} - \chi_{ij}^H \dot{\psi}_{,i}^{(1)(0)} \psi_{,j}^{(2)} + \chi_{ij}^H \psi_{,j}^{(2)} * \dot{\psi}_{,i}^{(1)}) d\mathbf{x} \\
& + \int_V (\chi_{ij}^{EH} \phi_{,j}^{(2)(0)} \dot{\psi}_{,i}^{(1)} - \chi_{ij}^{EH} \dot{\psi}_{,i}^{(1)(0)} \phi_{,j}^{(2)} + \chi_{ij}^{EH} \dot{\phi}_{,j}^{(2)} * \dot{\psi}_{,i}^{(1)}) d\mathbf{x} \\
& - \int_V (q_f^{(2)(0)} \dot{\phi}^{(1)} - q_f^{(2)} \dot{\phi}^{(1)(0)} + \dot{q}_f^{(2)} * \dot{\phi}^{(1)}) d\mathbf{x},
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
& \int_V S_{ijkl} (\dot{\sigma}_{kl}^{(2)} * \dot{\sigma}_{ij}^{(1)} + \dot{\sigma}_{kl}^{(2)} \sigma_{ij}^{(1)(0)} - \dot{\sigma}_{kl}^{(2)(0)} \sigma_{ij}^{(1)}) d\mathbf{x} = \int_{\partial V} \frac{1}{\rho} \sigma_{ik,k}^{(2)} * \hat{t}_i^{(1)} d\mathbf{x} + \int_V \left(\frac{1}{\rho} f_{(i,j)}^{(2)} \right) * \sigma_{ij}^{(1)} d\mathbf{x} \\
& - \int_V \frac{1}{\rho} \sigma_{ik,k}^{(2)} * \sigma_{ij,j}^{(1)} d\mathbf{x} + \int_V (\chi_{ij}^E \phi_{,j}^{(1)(0)} \dot{\phi}_{,i}^{(2)} - \chi_{ij}^E \dot{\phi}_{,i}^{(2)(0)} \phi_{,j}^{(1)} + \chi_{ij}^E \dot{\phi}_{,j}^{(1)} * \dot{\phi}_{,i}^{(2)}) d\mathbf{x}
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
& + \int_{\partial V} (\hat{d}^{(1)(0)} \dot{\phi}^{(2)} - \hat{d}^{(1)} \dot{\phi}^{(2)(0)} + \dot{d}^{(1)} * \dot{\phi}^{(2)}) d\mathbf{x} + \int_{\partial V} (\hat{b}^{(1)(0)} \dot{\psi}^{(2)} - \hat{b}^{(1)} \dot{\psi}^{(2)(0)} + \dot{b}^{(1)} * \dot{\psi}^{(2)}) d\mathbf{x} \\
& + \int_V (\chi_{ij}^{EH} \psi_{,j}^{(1)(0)} \dot{\phi}_{,i}^{(2)} - \chi_{ij}^{EH} \dot{\phi}_{,i}^{(2)(0)} \psi_{,j}^{(1)} + \chi_{ij}^{EH} \dot{\psi}_{,j}^{(1)} * \dot{\phi}_{,i}^{(2)}) d\mathbf{x} \\
& + \int_V (\chi_{ij}^H \psi_{,j}^{(1)(0)} \dot{\psi}_{,i}^{(2)} - \chi_{ij}^H \dot{\psi}_{,i}^{(2)(0)} \psi_{,j}^{(1)} + \chi_{ij}^H \dot{\psi}_{,j}^{(1)} * \dot{\psi}_{,i}^{(2)}) d\mathbf{x} \\
& + \int_V (\chi_{ij}^{EH} \phi_{,j}^{(1)(0)} \dot{\psi}_{,i}^{(2)} - \chi_{ij}^{EH} \dot{\psi}_{,i}^{(2)(0)} \phi_{,j}^{(1)} + \chi_{ij}^{EH} \dot{\phi}_{,j}^{(1)} * \dot{\psi}_{,i}^{(2)}) d\mathbf{x} \\
& - \int_V (q_f^{(1)(0)} \dot{\phi}^{(2)} - q_f^{(1)} \dot{\phi}^{(2)(0)} + \dot{q}_f^{(1)} * \dot{\phi}^{(2)}) d\mathbf{x}.
\end{aligned}$$

Then the statement is obtained by Subtracting (3.34) from (3.33). This completes the proof. $\square$

3.4 Planar stress formulation of homogenous electromagneto-elastic material

Due to the significance of 2d problems in solid mechanics, e.g. analysis of piezoelectric and/or piezomagnetic composites, shells etc., and numerous application of the stress approach (e.g. Airy stress function), herein, it is worthwhile to degenerate general alternative governing equations (3.25) for the planar problems. To this end, it is sufficient to obtain the governing equations (i.e., without boundary/initial conditions). Nevertheless, for the sake of completeness, we define the problem; the formulation is expressed for transversely isotropic electromagneto-elastic materials, which captures accurately many multifunctional materials' behavior.

Consider a homogenous transversely isotropic electro-magneto-elastic 2d domain (designated by $\bar{S}$, a 2d Euclidean space with boundary ∂S) whose poling direction and axis of symmetry are both along x_2 -axis. Likewise in the 3D case, we assume the following boundary conditions:

$$\begin{aligned}
& \text{Mechanical Boundary Conditions } \left\{ t_i = \hat{t}_i(\mathbf{x}, t) \text{ on } \overline{\partial S_\sigma} \times [0, \infty), \partial S_{\mathbf{u}} = \emptyset, \right. \\
& \text{Electric Boundary Conditions } \left\{ \begin{aligned} \varphi &= \hat{\varphi}(\mathbf{x}, t) \text{ on } \partial S_\varphi \times [0, \infty), \\ d &= \hat{d}(\mathbf{x}, t) \text{ on } \partial S_{\mathbf{D}} \times [0, \infty), \end{aligned} \right. \quad (3.35)
\end{aligned}$$

$$\text{Magnetic Boundary Conditions} \begin{cases} \psi = \hat{\psi}(\mathbf{x}, t) \text{ on } \partial S_{\psi} \times [0, \infty), \\ b = \hat{b}(\mathbf{x}, t) \text{ on } \partial S_{\mathbf{B}} \times [0, \infty), \end{cases}$$

$$\text{in which } \begin{cases} \partial S_{\mathbf{u}} \cap \partial S_{\sigma} = \emptyset & \partial S_{\varphi} \cap \partial S_{\mathbf{D}} = \emptyset & \partial S_{\psi} \cap \partial S_{\mathbf{B}} = \emptyset \\ \partial S_{\mathbf{u}} \cup \partial S_{\sigma} = \partial S & \partial S_{\varphi} \cup \partial S_{\mathbf{D}} = \partial S & \partial S_{\psi} \cup \partial S_{\mathbf{B}} = \partial S \end{cases}, \text{ and the initial conditions}$$

$$\begin{aligned} u_i(\mathbf{x}, 0) &= u_i^0(\mathbf{x}), \sigma_{ij}(\mathbf{x}, 0) = \sigma_{ij}^0(\mathbf{x}), \varphi(\mathbf{x}, 0) = \varphi^0(\mathbf{x}), \psi(\mathbf{x}, 0) = \psi^0(\mathbf{x}), \mathbf{x} \in \bar{V}, i, j = 1, 2, \\ \dot{u}_i(\mathbf{x}, 0) &= \dot{u}_i^0(\mathbf{x}), \dot{\sigma}_{ij}(\mathbf{x}, 0) = \dot{\sigma}_{ij}^0(\mathbf{x}), \dot{\varphi}(\mathbf{x}, 0) = \dot{\varphi}^0(\mathbf{x}), \dot{\psi}(\mathbf{x}, 0) = \dot{\psi}^0(\mathbf{x}), \mathbf{x} \in \bar{V}, i, j = 1, 2, \end{aligned} \quad (3.36)$$

which are consistent in the sense of (3.10) and (3.11).

3.4.1 Plane strain

In this case, we have $u_3 = 0$ and $\frac{\partial}{\partial x_3} = 0$. Hence, the constitutive equations in the plane $x_3 = 0$ can be rearranged as follows

$$\begin{aligned} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{13} & 0 \\ C_{13} & C_{33} & 0 \\ 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} - \begin{bmatrix} 0 & e_{31}^E \\ 0 & e_{33}^E \\ e_{15}^E & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} - \begin{bmatrix} 0 & e_{31}^H \\ 0 & e_{33}^H \\ e_{15}^H & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \\ \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} &= \begin{bmatrix} 0 & e_{31}^E \\ 0 & e_{33}^E \\ e_{15}^E & 0 \end{bmatrix}^T \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} + \begin{bmatrix} \kappa_{11}^E & 0 \\ 0 & \kappa_{33}^E \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \kappa_{11}^{EH} & 0 \\ 0 & \kappa_{33}^{EH} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} &= \begin{bmatrix} 0 & e_{31}^H \\ 0 & e_{33}^H \\ e_{15}^H & 0 \end{bmatrix}^T \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} + \begin{bmatrix} \kappa_{11}^{EH} & 0 \\ 0 & \kappa_{33}^{EH} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \kappa_{11}^H & 0 \\ 0 & \kappa_{33}^H \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}. \end{aligned} \quad (3.37)$$

Or alternatively by using (3.16):

$$\begin{aligned}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} &= \begin{bmatrix} S_{11} & S_{13} & 0 \\ S_{13} & S_{33} & 0 \\ 0 & 0 & S_{44} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} 0 & d_{31}^E \\ 0 & d_{33}^E \\ d_{15}^E & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} 0 & d_{31}^H \\ 0 & d_{33}^H \\ d_{15}^H & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \\
\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} &= \begin{bmatrix} 0 & d_{31}^E \\ 0 & d_{33}^E \\ d_{15}^E & 0 \end{bmatrix}^T \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} \chi_{11}^E & 0 \\ 0 & \chi_{33}^E \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \chi_{11}^{EH} & 0 \\ 0 & \chi_{33}^{EH} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \\
\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} &= \begin{bmatrix} 0 & d_{31}^H \\ 0 & d_{33}^H \\ d_{15}^H & 0 \end{bmatrix}^T \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} \chi_{11}^{EH} & 0 \\ 0 & \chi_{33}^{EH} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \chi_{11}^H & 0 \\ 0 & \chi_{33}^H \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.
\end{aligned} \tag{3.38}$$

Therefore, a solution of the plane strain IBVP is an admissible process $S = [\sigma_{11}, \sigma_{22}, \sigma_{12}, \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, u_1, u_2, E_1, E_2, D_1, D_2, H_1, H_2, B_1, B_2, \varphi, \psi]$ that meets (3.5), (3.6) ($i, j = 1, 2$), (3.35), (3.36), and (3.37).

Hence, we have the following corollary of Theorem 3:

Corollary 1. *Let $\sigma \in C^{2,2}$, $\varphi \in C^{2,2}$, $\psi \in C^{2,2}$, $\sigma_{ij} = \sigma_{ji}$, then $S = [\sigma_{11}, \sigma_{22}, \sigma_{12}, \varphi, \psi]$ is a solution of the plane strain IBVP if and only if:*

$$\begin{aligned}
&\sigma_{11,11} + \sigma_{12,21} + f_{1,1} - \rho (S_{11}\ddot{\sigma}_{11} + S_{13}\ddot{\sigma}_{22} - d_{31}^E\ddot{\varphi}_{,2} - d_{31}^H\ddot{\psi}_{,2}) = 0, \quad \text{on } S \times (0, \infty), \\
&\sigma_{22,22} + \sigma_{12,21} + f_{2,2} - \rho (S_{13}\ddot{\sigma}_{11} + S_{33}\ddot{\sigma}_{22} - d_{33}^E\ddot{\varphi}_{,2} - d_{33}^H\ddot{\psi}_{,2}) = 0, \quad \text{on } S \times (0, \infty), \\
&(\sigma_{11} + \sigma_{22})_{,12} + \nabla^2 \sigma_{12} + 2f_{(1,2)} - \rho (S_{44}\ddot{\sigma}_{12} - d_{15}^E\ddot{\varphi}_{,1} - d_{15}^H\ddot{\psi}_{,1}) = 0, \quad \text{on } S \times (0, \infty), \\
&d_{15}^E\sigma_{12,1} + d_{31}^E\sigma_{11,2} + d_{33}^E\sigma_{22,2} - \chi_{11}^E\varphi_{,11} - \chi_{33}^E\varphi_{,22} - \chi_{11}^{EH}\psi_{,11} - \chi_{33}^{EH}\psi_{,22} = q_f, \quad \text{on } S \times (0, \infty), \\
&d_{15}^H\sigma_{12,1} + d_{31}^H\sigma_{11,2} + d_{33}^H\sigma_{22,2} - \chi_{11}^{EH}\varphi_{,11} - \chi_{33}^{EH}\varphi_{,22} - \chi_{11}^H\psi_{,11} - \chi_{33}^H\psi_{,22} = 0, \quad \text{on } S \times (0, \infty), \\
&\sigma_{ij}n_j = \hat{t}_i(\mathbf{x}, t), \quad \text{on } \partial S \times [0, \infty), \\
&\varphi = \hat{\varphi}(\mathbf{x}, t), \quad \text{on } \partial S_\varphi \times [0, \infty), \\
&(d_{15}^E\sigma_{12} - \chi_{11}^E\varphi_{,1} - \chi_{11}^{EH}\psi_{,1})n_1 + (d_{31}^E\sigma_{11} + d_{33}^E\sigma_{22} - \chi_{33}^E\varphi_{,2} - \chi_{33}^{EH}\psi_{,2})n_2 = \hat{d}(\mathbf{x}, t), \quad \text{on } \partial S_D \times [0, \infty),
\end{aligned} \tag{3.39}$$

$$\psi = \hat{\psi}(\mathbf{x}, t), \text{ on } \partial S_\psi \times [0, \infty),$$

$$(d_{15}^H \sigma_{12} - \chi_{11}^{EH} \phi_{,1} - \chi_{11}^H \psi_{,1}) n_1 + (d_{31}^H \sigma_{11} + d_{33}^H \sigma_{22} - \chi_{33}^{EH} \phi_{,2} - \chi_{33}^H \psi_{,2}) n_2 = \hat{b}(\mathbf{x}, t), \text{ on } \partial S_B \times [0, \infty),$$

$$\sigma_{ij}(\mathbf{x}, 0) = \sigma_{ij}^0(\mathbf{x}), \phi(\mathbf{x}, 0) = \phi^0(\mathbf{x}), \psi(\mathbf{x}, 0) = \psi^0(\mathbf{x}), \mathbf{x} \in \bar{S}, i, j = 1, 2,$$

$$\dot{\sigma}_{ij}(\mathbf{x}, 0) = \dot{\sigma}_{ij}^0(\mathbf{x}), \dot{\phi}(\mathbf{x}, 0) = \dot{\phi}^0(\mathbf{x}), \dot{\psi}(\mathbf{x}, 0) = \dot{\psi}^0(\mathbf{x}), \mathbf{x} \in \bar{S}, i, j = 1, 2.$$

Proof. It is sufficient to show that (3.25) reduces to (3.39) for the plane strain condition. Expanding (3.25) and substituting (3.38), one can obtain the desired relation. $\square$

3.4.2 Generalized plane stress

To derive the stress formulation for 2d generalized plane stress problems (which itself is an approximation in the sense that, theoretically, there is no exact generalized plane stress condition and it is satisfactory for the case of thin plate), one needs to modify the constitutive equations (3.37) and (3.38). It is noted that all fields, in this part, are average values over the thickness of the plate.

Assuming $\sigma_{33} = 0$, we obtain

$$\begin{aligned} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} &= \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{13} & 0 \\ \tilde{C}_{13} & \tilde{C}_{33} & 0 \\ 0 & 0 & \tilde{C}_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} - \begin{bmatrix} 0 & \tilde{e}_{31}^E \\ 0 & \tilde{e}_{33}^E \\ \tilde{e}_{15}^E & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{e}_{31}^H \\ 0 & \tilde{e}_{33}^H \\ \tilde{e}_{15}^H & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \\ \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} &= \begin{bmatrix} 0 & \tilde{e}_{31}^E \\ 0 & \tilde{e}_{33}^E \\ \tilde{e}_{15}^E & 0 \end{bmatrix}^T \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} + \begin{bmatrix} \tilde{\kappa}_{11}^E & 0 \\ 0 & \tilde{\kappa}_{33}^E \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \tilde{\kappa}_{11}^{EH} & 0 \\ 0 & \tilde{\kappa}_{33}^{EH} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} &= \begin{bmatrix} 0 & \tilde{e}_{31}^H \\ 0 & \tilde{e}_{33}^H \\ \tilde{e}_{15}^H & 0 \end{bmatrix}^T \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} + \begin{bmatrix} \tilde{\kappa}_{11}^{EH} & 0 \\ 0 & \tilde{\kappa}_{33}^{EH} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \tilde{\kappa}_{11}^H & 0 \\ 0 & \tilde{\kappa}_{33}^H \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \end{aligned} \quad (3.40)$$

in which

$$\begin{aligned}
\tilde{C}_{11} &= C_{11} - \frac{C_{12}^2}{C_{11}}, \tilde{C}_{13} = C_{13} - \frac{C_{12}C_{13}}{C_{11}}, \tilde{C}_{33} = C_{33} - \frac{C_{13}^2}{C_{11}}, \tilde{C}_{44} = C_{44}, \\
\tilde{e}_{31}^E &= e_{31}^E \left(\frac{C_{12}}{C_{11}} - 1 \right), \tilde{e}_{31}^H = e_{31}^H \left(\frac{C_{12}}{C_{11}} - 1 \right), e_{33}^E = e_{33}^E \left(\frac{C_{13}}{C_{11}} - 1 \right), e_{33}^H = e_{33}^H \left(\frac{C_{13}}{C_{11}} - 1 \right), \tilde{e}_{15}^E = e_{15}^E, \tilde{e}_{15}^H = e_{15}^H, \\
\tilde{\kappa}_{11}^E &= \kappa_{11}^E, \tilde{\kappa}_{33}^E = \kappa_{33}^E, \tilde{\kappa}_{11}^{EH} = \kappa_{11}^{EH}, \tilde{\kappa}_{33}^{EH} = \kappa_{33}^{EH}, \tilde{\kappa}_{11}^E = \kappa_{11}^E, \tilde{\kappa}_{33}^E = \kappa_{33}^E, \tilde{\kappa}_{11}^H = \kappa_{11}^H, \tilde{\kappa}_{33}^H = \kappa_{33}^H.
\end{aligned} \tag{3.41}$$

Likewise the plane strain problem, using (3.16), one can write:

$$\begin{aligned}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} &= \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{13} & 0 \\ \tilde{S}_{13} & \tilde{S}_{33} & 0 \\ 0 & 0 & \tilde{S}_{44} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} 0 & \tilde{d}_{31}^E \\ 0 & \tilde{d}_{33}^E \\ \tilde{d}_{15}^E & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} 0 & \tilde{d}_{31}^H \\ 0 & \tilde{d}_{33}^H \\ \tilde{d}_{15}^H & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \\
\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} &= \begin{bmatrix} 0 & \tilde{d}_{31}^E \\ 0 & \tilde{d}_{33}^E \\ \tilde{d}_{15}^E & 0 \end{bmatrix}^T \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} \tilde{\chi}_{11}^E & 0 \\ 0 & \tilde{\chi}_{33}^E \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \tilde{\chi}_{11}^{EH} & 0 \\ 0 & \tilde{\chi}_{33}^{EH} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \\
\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} &= \begin{bmatrix} 0 & \tilde{d}_{31}^H \\ 0 & \tilde{d}_{33}^H \\ \tilde{d}_{15}^H & 0 \end{bmatrix}^T \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} \tilde{\chi}_{11}^{EH} & 0 \\ 0 & \tilde{\chi}_{33}^{EH} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \tilde{\chi}_{11}^H & 0 \\ 0 & \tilde{\chi}_{33}^H \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.
\end{aligned} \tag{3.42}$$

Similarly, a solution of the generalized plane stress IBVP is an admissible process $S = [\sigma_{11}, \sigma_{22}, \sigma_{12}, \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, u_1, u_2, E_1, E_2, D_1, D_2, H_1, H_2, B_1, B_2, \varphi, \psi]$ that meets (3.5), (3.6) ($i, j = 1, 2$), (3.35), (3.36), and (3.40).

Comparing (3.38) with (3.42), it is straightforward to obtain the stress equations of motion for plane stress case. Hence, we have the following Corollary of Theorem 3

Corollary 2. *Let $\sigma \in C^{2,2}$, $\varphi \in C^{2,2}$, $\psi \in C^{2,2}$, $\sigma_{ij} = \sigma_{ji}$, then $S = [\sigma_{11}, \sigma_{22}, \sigma_{12}, \varphi, \psi]$ is a solution of the plane strain IBVP if and only if:*

$$\begin{aligned}
\sigma_{11,11} + \sigma_{12,21} + f_{1,1} - \rho (\tilde{S}_{11} \ddot{\sigma}_{11} + \tilde{S}_{13} \ddot{\sigma}_{22} - \tilde{d}_{31}^E \ddot{\varphi}_{,2} - \tilde{d}_{31}^H \ddot{\psi}_{,2}) &= 0, \quad \text{on } S \times (0, \infty), \\
\sigma_{22,22} + \sigma_{12,21} + f_{2,2} - \rho (\tilde{S}_{13} \ddot{\sigma}_{11} + \tilde{S}_{33} \ddot{\sigma}_{22} - \tilde{d}_{33}^E \ddot{\varphi}_{,2} - \tilde{d}_{33}^H \ddot{\psi}_{,2}) &= 0, \quad \text{on } S \times (0, \infty),
\end{aligned} \tag{3.43}$$

$$\begin{aligned}
& (\sigma_{11} + \sigma_{22})_{,12} + \nabla^2 \sigma_{12} + 2f_{(1,2)} - \rho (\tilde{S}_{44} \ddot{\sigma}_{12} - \tilde{d}_{15}^E \ddot{\phi}_{,1} - \tilde{d}_{15}^H \ddot{\psi}_{,1}) = 0, \quad \text{on } S \times (0, \infty), \\
& \tilde{d}_{15}^E \sigma_{12,1} + \tilde{d}_{31}^E \sigma_{11,2} + \tilde{d}_{33}^E \sigma_{22,2} - \tilde{\chi}_{11}^E \phi_{,11} - \tilde{\chi}_{33}^E \phi_{,22} - \tilde{\chi}_{11}^{EH} \psi_{,11} - \tilde{\chi}_{33}^{EH} \psi_{,22} = q_f, \quad \text{on } S \times (0, \infty), \\
& \tilde{d}_{15}^H \sigma_{12,1} + \tilde{d}_{31}^H \sigma_{11,2} + \tilde{d}_{33}^H \sigma_{22,2} - \tilde{\chi}_{11}^{EH} \phi_{,11} - \tilde{\chi}_{33}^{EH} \phi_{,22} - \tilde{\chi}_{11}^H \psi_{,11} - \tilde{\chi}_{33}^H \psi_{,22} = 0, \quad \text{on } S \times (0, \infty), \\
& \sigma_{ij} n_j = \hat{t}_i(\mathbf{x}, t), \quad \text{on } \partial S \times [0, \infty), \\
& \phi = \hat{\phi}(\mathbf{x}, t), \quad \text{on } \partial S_\phi \times [0, \infty), \\
& (\tilde{d}_{15}^E \sigma_{12} - \tilde{\chi}_{11}^E \phi_{,1} - \tilde{\chi}_{11}^{EH} \psi_{,1}) n_1 + (\tilde{d}_{31}^E \sigma_{11} + \tilde{d}_{33}^E \sigma_{22} - \tilde{\chi}_{33}^E \phi_{,2} - \tilde{\chi}_{33}^{EH} \psi_{,2}) n_2 = \hat{d}(\mathbf{x}, t), \quad \text{on } \partial S_{\mathbf{D}} \times [0, \infty), \\
& \psi = \hat{\psi}(\mathbf{x}, t), \quad \text{on } \partial S_\psi \times [0, \infty), \\
& (\tilde{d}_{15}^H \sigma_{12} - \tilde{\chi}_{11}^{EH} \phi_{,1} - \tilde{\chi}_{11}^H \psi_{,1}) n_1 + (\tilde{d}_{31}^H \sigma_{11} + \tilde{d}_{33}^H \sigma_{22} - \tilde{\chi}_{33}^{EH} \phi_{,2} - \tilde{\chi}_{33}^H \psi_{,2}) n_2 = \hat{b}(\mathbf{x}, t), \quad \text{on } \partial S_{\mathbf{B}} \times [0, \infty), \\
& \sigma_{ij}(\mathbf{x}, 0) = \sigma_{ij}^0(\mathbf{x}), \quad \phi(\mathbf{x}, 0) = \phi^0(\mathbf{x}), \quad \psi(\mathbf{x}, 0) = \psi^0(\mathbf{x}), \quad \mathbf{x} \in \bar{V}, \quad i, j = 1, 2, \\
& \dot{\sigma}_{ij}(\mathbf{x}, 0) = \dot{\sigma}_{ij}^0(\mathbf{x}), \quad \dot{\phi}(\mathbf{x}, 0) = \dot{\phi}^0(\mathbf{x}), \quad \dot{\psi}(\mathbf{x}, 0) = \dot{\psi}^0(\mathbf{x}), \quad \mathbf{x} \in \bar{V}, \quad i, j = 1, 2.
\end{aligned}$$

Chapter 4

Convolutional Variational Principles for Electromagneto-elastic Materials¹

4.1 Introduction

Owing to the fact that the analytical methods are only sufficient tools for problems with simple geometry and rather strict assumptions, variational principles are of great importance in engineering sciences as they pave the way for developing numerical approaches to solve PDEs with either more relaxed assumptions or arbitrary/complicated boundary conditions. The finite element method, mesh-free particle, and Ritz methods are examples stemming from variational principles. As an alternative application of such principles, we note the homogenization theory which supplies bounds for properties of materials (e.g., [27]). In the case of solid mechanics, however, the classical variants of seminal work of [83] are not applicable to the case of a mixed problem of elastodynamics since the prescribed initial velocity is not realized and the knowledge of displacement field at a later time is only presupposed [26]. Therefore, starting with [26], as an alternative approach appropriate to elastodynamics, convolutional variational principles have been developed. The elegance of the approach consists in imposing the initial conditions implicitly in the form of a body force and thus assuring appropriate satisfaction of them.

Concerning the variational principles for electromagneto-elastic materials, several studies have been carried out dating back to [81]. Variational principles for various problems including piezoelectric ceramics have been proposed in [28, 29, 30, 31]. Convolutional region-wise variational principles for thermo-piezoelectric media can be found in [12]. Also, for the nonlinear case, some studies have been done, e.g., by making use of the first law of thermodynamics in [49]. Convolutional variational principles have then been proposed for the case of nonlinear electromagneto-elastic

¹Adapted with permission from[3]

solids in [82]. Recently, on the basis of incremental variational principles, a general framework for functional dissipative materials has been obtained in [62] and employed in [61] to analyze piezoelectric ceramics.

To the best of authors' knowledge, a comprehensive and systematic generalization of the results initially developed in [26] for the case of linear electromagneto-elastic materials is not available in the literature and this challenge defines the focus of the present study. Indeed, it is of interest to enrich the numerical framework relevant to the analysis of electromagneto-elastic materials because of the progressive increase in the application of such materials in the realm of structural mechanics; see, for example, a recent review [46]. As an example of the recent development in the use of smart materials, one can mention the recent paper [77] in which the design of piezoelectric devices controlling the level of stress in thin bars has been discussed. The methodology to form and prove the results obtained in this study, similar to the presentation given in [65], is based on [26]. For the sake of completeness of the presentation, we collect in the Appendix 4.2 the mathematical concepts and lemmas originally proved in [26] along with a corollary obtained in [65]. Some mathematical concepts are mentioned in section 4.2. Alternative field equations, the main ground for establishing the corresponding convolutional variational principles, for electromagneto-elastic materials are described in section 4.3. Subsequently, by analogy to [26], the convolutional variational forms of the alternative integro-partial-differential field equations are obtained and proved comprehensively in Section 4.4. As mentioned earlier, the derivations in the main body of this chapter can be useful in the sense of analysis and design of electromagneto-elastic materials in both practice and research.

4.2 Mathematical Background

Here, for the sake of completeness, we summarize the basic concepts, originally developed in [26] and [65], that are employed in the main body of the paper. For a comprehensive discussion, the readers are referred to those references.

The pair $(\mathbf{x}, t) \in \overline{\partial V} \times [0, \infty)$ is called a regular point if the unit outward normal $\mathbf{n}$ at $\mathbf{x}$, and at any time, is continuous. Moreover, the function f is called a piecewise regular function on boundary

$\overline{\partial V_i} \times [0, \infty)$ with $(i = \mathbf{u}, \sigma, \varphi, \mathbf{D}, \psi, \mathbf{B})$ if and only if it is piecewise continuous on $\overline{\partial V_i} \times [0, \infty)$ and continuous on every regular point of that region. Additionally, for piecewise regular functions f and $\hat{f}$ on $\partial \bar{V}_i \times [0, \infty)$, we say $f = \hat{f}$ if and only if the equality holds true for any regular point $(\mathbf{x}, t) \in \overline{\partial V_i} \times [0, \infty)$.

The symbol $f * g$, in which f and g are functions of the position and continuous functions of time defined on $\Re \times [0, \infty)$, with $\Re$ a subset of the Euclidean space, indicates the convolution of two functions in the sense of

$$[f * g](\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t - \lambda) g(\mathbf{x}, \lambda) d\lambda, \quad (\mathbf{x}, t) \in \Re \times [0, \infty). \quad (4.1)$$

In this regard, one can show that the following properties hold true:

$$f * g = g * f, \quad (4.2)$$

$$f * g = 0 \Leftrightarrow f = 0 \vee g = 0, \quad (4.3)$$

$$f * (g * h) = (f * g) * h = f * g * h. \quad (4.4)$$

A functional is a real-valued function on a subset of a linear space. Denoting a linear space by L , a subset of L by K , and defining $\Phi(S)$ as a functional on K , we define

$$\delta_{\tilde{S}} \Phi(S) = \left. \frac{d}{d\lambda} \Phi(S + \lambda \tilde{S}) \right|_{\lambda=0} \quad (4.5)$$

for all real numbers λ , where $S, \tilde{S} \in L$, and $S + \lambda \tilde{S} \in K$. And we say the variation of $\Phi(S)$ is zero and write $\delta \Phi(S) = 0$ over L if and only if $\delta_{\tilde{S}} \Phi(S)$ exists and equals zero for all $\tilde{S}$ such that $S, \tilde{S} \in L$, and $S + \lambda \tilde{S} \in K$.

Now, we list four lemmas and one corollary proved in [26, 65]. However, we write them in such a way that they are applicable to the present study.

Lemma 3. [26], let ϑ be a continuous function on $\bar{V} \times [0, \infty)$ and suppose

$$\int_V \vartheta * \omega(\mathbf{x}, t) d\mathbf{x} = 0, \quad 0 \leq t < \infty \quad (4.6)$$

for every $\omega \in C^{\infty, \infty}$ which, together with its space derivatives, vanishes on $\partial V \times [0, \infty)$. Then

$$\vartheta = 0 \quad \text{on } \bar{V} \times [0, \infty). \quad (4.7)$$

Lemma 4. [26], let ϑ be a piecewise regular function on $\bar{\partial V_i} \times [0, \infty)$ with $i = \sigma, \mathbf{D}, \mathbf{B}$, and suppose

$$\int_{\partial V_i} \vartheta * \omega(\mathbf{x}, t) d\mathbf{x} = 0, \quad 0 \leq t < \infty \quad (4.8)$$

for every $\omega \in C^{\infty, \infty}$ that vanishes on $\partial V_j \times [0, \infty)$ with $j = \mathbf{u}, \varphi, \psi$. Then

$$\vartheta = 0 \quad \text{on } \bar{\partial V_i} \times [0, \infty). \quad (4.9)$$

Lemma 5. [26], let ϑ_i be continuous on $\bar{\partial V_{\mathbf{u}}} \times [0, \infty)$, and suppose we have:

$$\int_{\partial V_{\mathbf{u}}} \vartheta_i * (\omega_{ij} n_j)(\mathbf{x}, t) d\mathbf{x} = 0, \quad 0 \leq t < \infty \quad (4.10)$$

for every $\omega_{ij} \in C^{\infty, \infty}$ which, together with all of its space derivatives, vanishes on $\partial V_{\sigma} \times [0, \infty)$ and has the property $\omega_{ij} = \omega_{ji}$. Then

$$\vartheta_i = 0 \quad \text{on } \bar{\partial V_{\mathbf{u}}} \times [0, \infty). \quad (4.11)$$

The following statement is a corollary of Lemma 5

Corollary 3. [65], let ϑ be continuous on $\overline{\partial V_\varphi}$ (or $\overline{\partial V_\psi}$) $\times [0, \infty)$ and suppose

$$\int_{\partial V_\varphi \text{ (or } \partial V_\psi)} [\vartheta * (\omega_i n_i)](\mathbf{x}, t) d\mathbf{x} = 0, \quad 0 \leq t < \infty \quad (4.12)$$

for all $\omega_i \in C^{\infty, \infty}$ which, together with its space derivatives, vanishes on $\partial V_{\mathbf{D}}$ (or $\partial V_{\mathbf{B}}$) $\times [0, \infty)$. Then

$$\vartheta = 0 \quad \text{on} \quad \overline{\partial V_\varphi} \text{ (or } \overline{\partial V_\psi}) \times [0, \infty) \quad (4.13)$$

Lemma 6. [26], let ϑ_i be a piecewise regular function on $\overline{\partial V_\sigma} \times [0, \infty)$, and suppose

$$\int_{\partial V_\sigma} \vartheta_i * (\omega_{ij,j})(\mathbf{x}, t) d\mathbf{x} = 0, \quad 0 \leq t < \infty \quad (4.14)$$

for all $\omega_{ij} \in C^{\infty, \infty}$ with $\omega_{ij} = \omega_{ji}$. Then

$$\vartheta_i = 0 \quad \text{on} \quad \overline{\partial V_\sigma} \times [0, \infty). \quad (4.15)$$

4.3 Integro-Partial-Differential Field Equations

The alternative integro-partial-differential field equations of motion of an electromagneto-elastic body are derived in this part. To start with, let define the following functions [26]:

$$\begin{aligned} h(t) &= t, \quad 0 \leq t < \infty \\ f_i^b(\mathbf{x}, t) &= h * f(\mathbf{x}, t) + \rho(\mathbf{x}) (tv_i^0(\mathbf{x}) + u_i^0(\mathbf{x})) \end{aligned} \quad (4.16)$$

In which f_i^b is a vector field obtained from the prescribed data (3.8) and the body force. We now have the following alternative formulation of the equation (3.5)₁:

Theorem 6. Let $u_i \in C^{0,2}$ and $\sigma_{ij} \in C^{1,0}$ be a vector field and a second-order symmetric tensor field, respectively. Then $\mathbf{u}$ and σ meet (3.5)₁ and the associated initial boundary conditions (3.8) if and only if

$$\rho \mathbf{u} = h * \nabla \cdot \sigma + \mathbf{f}^b \quad \text{on } V \times [0, \infty) \quad (4.17)$$

Proof. See [26]. □

Now, with the help of the following Theorem, which is the direct result of Theorem 6, one can define alternative field equations of the mixed initial-boundary value problem:

Theorem 7. *The admissible process $S = [\mathbf{u}, \varepsilon, \sigma, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \varphi, \psi]$ is a solution of the mixed initial-boundary value problem if and only if it satisfies (4.17), (3.5)₂₋₃, (3.6), (3.12) and (3.7).*

Now, through the next two Theorems, we obtain two variants of field equations for electromagneto-elastic materials.

Theorem 8. *Let $u_i \in C^{2,2}$, $\varphi \in C^{2,0}$ and $\psi \in C^{2,0}$. Then the ordered array $[\mathbf{u}, \varphi, \psi]$ corresponds to a solution of the mixed initial-boundary value problem if and only if the following equations hold true:*

$$\begin{aligned}
u_i &= h * (C_{ijkl}u_{k,l} + e_{kij}^E \varphi_{,k} + e_{kij}^H \psi_{,k})_{,j} + f_i^b & \text{on } V \times [0, \infty), \\
(e_{ikl}^E u_{k,l} - \kappa_{ij}^E \varphi_{,j} - \kappa_{ij}^{EH} \psi_{,j})_{,i} &= q_e & \text{on } V \times [0, \infty), \\
(e_{ikl}^H u_{k,l} - \kappa_{ji}^{EH} \varphi_{,j} - \kappa_{ij}^H \psi_{,j})_{,i} &= 0 & \text{on } V \times [0, \infty), \\
u_i &= \hat{u}_i(\mathbf{x}, t) & \text{on } \partial V_{\mathbf{u}} \times [0, \infty), \\
(C_{ijkl}u_{k,l} + e_{kij}^E \varphi_{,k} + e_{kij}^H \psi_{,k})n_j &= \hat{t}_i(\mathbf{x}, t) & \text{on } \partial V_{\sigma} \times [0, \infty), \\
\varphi &= \hat{\varphi}(\mathbf{x}, t) & \text{on } \partial V_{\varphi} \times [0, \infty), \\
(e_{ikl}^E u_{k,l} - \kappa_{ij}^E \varphi_{,j} - \kappa_{ij}^{EH} \psi_{,j})n_i &= \hat{d}(\mathbf{x}, t) & \text{on } \partial V_{\mathbf{D}} \times [0, \infty), \\
\psi &= \hat{\psi}(\mathbf{x}, t) & \text{on } \partial V_{\psi} \times [0, \infty), \\
(e_{ikl}^H u_{k,l} - \kappa_{ji}^{EH} \varphi_{,j} - \kappa_{ij}^H \psi_{,j})n_i &= \hat{b}(\mathbf{x}, t) & \text{on } \partial V_{\mathbf{B}} \times [0, \infty).
\end{aligned} \tag{4.18}$$

Proof. First, suppose that relations (4.18) holds true. Thus, (3.7)_{1,3,5} are automatically satisfied. Define ε , $\mathbf{E}$, and $\mathbf{H}$ through (3.6). Also, define σ , $\mathbf{D}$, and $\mathbf{B}$ via (3.12). Then, (3.7)_{2,4,6} are identically satisfied due to the symmetry (3.13); the above-defined ε , σ , and (3.13) together with (4.18)₁ give (4.17); (4.18)₂₋₃ together with the above-defined $\mathbf{D}$, $\mathbf{B}$, ε , $\mathbf{E}$, $\mathbf{H}$, and symmetry (3.13) give (3.5)₂₋₃. Hence, by Theorem 7, (4.18) is a solution to the mixed initial-boundary value problem. On the other hand, (4.17), (3.5)₂₋₃, (3.6), (3.12), (3.13), and (3.7) conclude (4.18) and the proof is complete. □

Theorem 9. Let $\sigma_{ij} \in C^{2,0}$, $\varphi \in C^{2,0}$ and $\psi \in C^{2,0}$ with $\sigma_{ij} = \sigma_{ji}$. Then the ordered array $[\sigma, \varphi, \psi]$ is a solution to the mixed initial-boundary value problem if and only if the following equations hold true:

$$\begin{aligned}
S_{ijkl}\sigma_{kl} &= \left(\frac{h}{\rho} * \sigma_{(ik,k),j}\right) + \left(\frac{1}{\rho} f_{(i),j}^b\right) + d_{kij}^E \varphi_{,k} + d_{kij}^H \psi_{,k} && \text{on } V \times [0, \infty), \\
(d_{ikl}^E \sigma_{kl} - \chi_{ij}^E \varphi_{,j} - \chi_{ij}^{EH} \psi_{,j})_{,i} &= q_e && \text{on } V \times [0, \infty), \\
(d_{ikl}^H \sigma_{kl} - \chi_{ji}^{EH} \varphi_{,j} - \chi_{ij}^H \psi_{,j})_{,i} &= 0 && \text{on } V \times [0, \infty), \\
\frac{h}{\rho} * \sigma_{ik,k} + \frac{1}{\rho} f_i^b &= \hat{u}_i(\mathbf{x}, t) && \text{on } \partial V_{\mathbf{u}} \times [0, \infty), \\
\sigma_{ij} n_j &= \hat{t}_i(\mathbf{x}, t) && \text{on } \partial V_{\sigma} \times [0, \infty), \\
\varphi &= \hat{\varphi}(\mathbf{x}, t) && \text{on } \partial V_{\varphi} \times [0, \infty), \\
(d_{ikl}^E \sigma_{kl} - \chi_{ij}^E \varphi_{,j} - \chi_{ij}^{EH} \psi_{,j}) n_i &= \hat{d}(\mathbf{x}, t) && \text{on } \partial V_{\mathbf{D}} \times [0, \infty), \\
\psi &= \hat{\psi}(\mathbf{x}, t) && \text{on } \partial V_{\psi} \times [0, \infty), \\
(d_{ikl}^H \sigma_{kl} - \chi_{ji}^{EH} \varphi_{,j} - \chi_{ij}^H \psi_{,j}) n_i &= \hat{b}(\mathbf{x}, t) && \text{on } \partial V_{\mathbf{B}} \times [0, \infty).
\end{aligned} \tag{4.19}$$

Proof. First, suppose that (4.19) hold. Hence, (3.7)_{2,3,5} are automatically satisfied. Define $\mathbf{u}$, $\mathbf{E}$, and $\mathbf{H}$ through (4.17), (3.6)₂, and (3.6)₃, respectively. Also, define ε , $\mathbf{D}$, and $\mathbf{B}$ via (3.14). Then, (3.7)_{1,4,6} are identically satisfied; (3.14) and (3.16) imply (3.12); (3.6)₁ holds because of (4.19)₁, (4.19)₄ and the above-defined $\mathbf{u}$ and ε ; (4.19)₂₋₃ together with the above-defined $\mathbf{D}$, $\mathbf{B}$, $\mathbf{E}$, and $\mathbf{H}$ give (3.5)₂₋₃. Hence, by Theorem 7, (4.19) is a solution to the mixed initial-boundary value problem. On the other hand, (4.17), (3.5)₂₋₃, (3.6), (3.12), (3.16), and (3.7) imply (4.19), and the proof is complete. $\square$

4.4 Variational Principles

Now, for an electromagneto-elastic material, convolutional variational forms, originally developed for the case of pure elasticity in [26], will be derived. The first part of the results, i.e., Theorem 10, Theorem 11, and Theorem 12, are devoted to the characterization of $[\mathbf{u}, \varepsilon, \sigma, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \varphi, \psi]$. In the second part, as a corollary of Theorem 12, a variational form characterizing $[\mathbf{u}, \varphi, \psi]$ is set.

Finally, the results of the third part are applied to the characterization of $[\sigma, \varphi, \psi]$. In the results, $t_i, \tilde{t}_i, d, \tilde{d}, b, \tilde{b}$ will be consistently used in place of $\sigma_{ij}n_j, \tilde{\sigma}_{ij}n_j, D_i n_i, \tilde{D}_i n_i, B_i n_i$ and $\tilde{B}_i n_i$, respectively. Additionally, the definitions of $h(t)$ and $f_i^b(\mathbf{x}, t)$, which shall be used in the sequel, have been given in (4.16).

4.4.1 Variational Principles Characterizing $[\mathbf{u}, \varepsilon, \sigma, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \varphi, \psi]$:

First, let derive a general form which imposes no restriction on the fields:

Theorem 10. *Let Ω denote the set of all admissible processes. Let $S = [\mathbf{u}, \varepsilon, \sigma, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \varphi, \psi]$ be an element of Ω and define the functional ϑ_t on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$\begin{aligned}
\vartheta_t(S) = & \frac{1}{2} \int_V C_{ijkl}(\mathbf{x}) [h * \varepsilon_{ij} * \varepsilon_{kl}](\mathbf{x}, t) d\mathbf{x} - \int_V [h * \sigma_{ij} * \varepsilon_{ij}](\mathbf{x}, t) d\mathbf{x} \\
& - \int_V e_{kij}^E(\mathbf{x}) [h * \varepsilon_{ij} * E_k](\mathbf{x}, t) d\mathbf{x} - \int_V e_{kij}^H(\mathbf{x}) [h * \varepsilon_{ij} * H_k](\mathbf{x}, t) d\mathbf{x} \\
& - \frac{1}{2} \int_V \kappa_{ij}^E(\mathbf{x}) [h * E_i * E_j](\mathbf{x}, t) d\mathbf{x} - \frac{1}{2} \int_V \kappa_{ij}^H(\mathbf{x}) [h * H_i * H_j](\mathbf{x}, t) d\mathbf{x} \\
& - \int_V \kappa_{ij}^{EH}(\mathbf{x}) [h * E_i * H_j](\mathbf{x}, t) d\mathbf{x} + \int_V [h * D_i * E_i](\mathbf{x}, t) d\mathbf{x} + \int_V [h * H_i * B_i](\mathbf{x}, t) d\mathbf{x} \\
& - \int_V [h * (D_{i,i} - q_e) * \varphi](\mathbf{x}, t) d\mathbf{x} - \int_V [h * B_{i,i} * \psi](\mathbf{x}, t) d\mathbf{x} + \frac{1}{2} \int_V \rho(\mathbf{x}) [u_i * u_i](\mathbf{x}, t) d\mathbf{x} \\
& - \int_V [(h * \sigma_{ij,j} + f_i^b) * u_i](\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_\sigma} [h * (t_i - \hat{t}_i) * u_i](\mathbf{x}, t) d\mathbf{x} \\
& + \int_{\partial V_u} [h * t_i * \hat{u}_i](\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_\varphi} [h * d * \hat{\varphi}](\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_\psi} [h * b * \hat{\psi}](\mathbf{x}, t) d\mathbf{x} \\
& + \int_{\partial V_D} [h * (d - \hat{d}) * \varphi](\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_B} [h * (b - \hat{b}) * \psi](\mathbf{x}, t) d\mathbf{x}
\end{aligned} \tag{4.20}$$

Then, S is a solution of the mixed initial-boundary value problem if and only if $\delta \vartheta_t(S) = 0$ over Ω , within the time interval $t \in [0, \infty)$.

Proof. Let $\tilde{S} = [\tilde{\mathbf{u}}, \tilde{\varepsilon}, \tilde{\sigma}, \tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{D}}, \tilde{\mathbf{B}}, \tilde{\varphi}, \tilde{\psi}] \in \Omega$ be an admissible process and suppose that $S + \lambda \tilde{S} \in \Omega$ for all real values of λ . Using (4.20), (4.5), the symmetry condition (3.13), the divergence

Theorem, and properties of convolution product given in appendix 4.2, we obtain

$$\begin{aligned}
\delta_{\tilde{S}} \vartheta_t(S) = & \int_V \left[h * (C_{ijkl} \varepsilon_{kl} - e_{kij}^E E_k - e_{kij}^H H_k - \sigma_{ij}) * \tilde{\varepsilon}_{ij} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[h * (-e_{kij}^E \varepsilon_{ij} - \kappa_{ik}^E E_i - \kappa_{ki}^{EH} H_i + D_k) * \tilde{E}_k \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[h * (-e_{kij}^H \varepsilon_{ij} - \kappa_{ik}^{EH} E_i - \kappa_{ki}^H H_i + B_k) * \tilde{H}_k \right] (\mathbf{x}, t) d\mathbf{x} + \int_V \left[h * (E_i + \varphi_{,i}) * \tilde{D}_i \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[h * (H_i + \psi_{,i}) * \tilde{B}_i \right] (\mathbf{x}, t) d\mathbf{x} - \int_V \left[h * (D_{i,i} - q_e) * \tilde{\varphi} \right] (\mathbf{x}, t) d\mathbf{x} - \int_V \left[h * B_{i,i} * \tilde{\psi} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[h * (u_{(i,j)} - \varepsilon_{ij}) * \tilde{\sigma}_{ij} \right] (\mathbf{x}, t) d\mathbf{x} - \int_V \left[(h * \sigma_{ij,j} + f_i^b - \rho u_i) * \tilde{u}_i \right] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_{\partial V_{\mathbf{u}}} \left[h * (u_i - \hat{u}_i) * \tilde{t}_i \right] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_{\sigma}} \left[h * (t_i - \hat{t}_i) * \tilde{u}_i \right] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_{\partial V_{\varphi}} \left[h * (\varphi - \hat{\varphi}) * \tilde{d} \right] (\mathbf{x}, t) d\mathbf{x} - \int_{\partial V_{\psi}} \left[h * (\psi - \hat{\psi}) * \tilde{b} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_{\partial V_{\mathbf{D}}} \left[h * (d - \hat{d}) * \tilde{\varphi} \right] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_{\mathbf{B}}} \left[h * (b - \hat{b}) * \tilde{\psi} \right] (\mathbf{x}, t) d\mathbf{x}.
\end{aligned} \tag{4.21}$$

First, based on Theorem 7, for every $\tilde{S} \in \Omega$ ($0 \leq t < \infty$) we immediately find $\delta_{\tilde{S}} \vartheta_t(S) = 0$ when S is a solution of the IBVP, implying $\delta \vartheta_t(S) = 0$. Conversely, suppose $\delta \vartheta_t(S) = 0$. Let $\tilde{S} = [\tilde{\mathbf{u}}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, 0] \in \Omega$ where $\tilde{\mathbf{u}}$ and all its space derivatives are identical to zero on $\partial V \times [0, \infty)$. Then, based on $\delta \vartheta_t(S) = 0$, (4.21), and Lemma 3, we obtain $h * \sigma_{ij,j} + f_i^b - u_i = 0$ on $V \times [0, \infty)$. Next, suppose $\tilde{\mathbf{u}}$ and all its space derivatives are identical to zero on $\partial V_{\mathbf{u}} \times [0, \infty)$. Based on Lemma 4, $h * \sigma_{ij,j} + f_i^b - u_i = 0$ on $V \times [0, \infty)$, $\delta \vartheta_t(S) = 0$, and (4.21), we have $h * (t_i - \hat{t}_i) = 0$ on $\partial V_{\sigma} \times [0, \infty)$. Since $h \neq 0$, the property of convolution reads $(t_i - \hat{t}_i) = 0$ on $\partial V_{\sigma} \times [0, \infty)$. Considering (3.13), by the same token, $-e_{kij}^E \varepsilon_{ij} - \kappa_{ik}^E E_i - \kappa_{ki}^{EH} H_i + D_k = 0$ on $V \times (0, \infty)$ and $-e_{kij}^H \varepsilon_{ij} - \kappa_{ik}^{EH} E_i - \kappa_{ki}^H H_i + B_k = 0$ on $V \times (0, \infty)$ can be obtained. With the same logic so far mentioned, one can readily deduce (3.5)₂₋₃, (3.6)₂₋₃, and (3.7)_{4,6}. Next, let $\tilde{S} = [\mathbf{0}, \tilde{\varepsilon}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, 0] \in \Omega$ in which $\tilde{\varepsilon}$ is a symmetric second-order tensor and zero-valued on the whole boundary at all times. Thus, the symmetry of the constitutive equations and symmetry of

$\sigma, \delta \vartheta_t(S) = 0$, (4.21), and Lemma 3 imply $h * (C_{ijkl} \varepsilon_{kl} - e_{kij}^E E_k - e_{kij}^H H_k - \sigma_{ij}) = 0$ on $V \times [0, \infty)$, leading to (3.12)₁. Similarly, by taking $\tilde{S} = [\mathbf{0}, \mathbf{0}, \tilde{\sigma}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, 0]$ in which $\tilde{\sigma}$ is a symmetric second-order tensor and zero-valued on the whole boundary at all times, considering symmetry of ε_{ij} , we conclude (3.6)₁. Moreover, let us define $\tilde{S} = [\mathbf{0}, \mathbf{0}, \tilde{\sigma}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, 0] \in \Omega$ in which $\tilde{\sigma}$ is a symmetric second-order tensor and zero-valued on the boundary ∂V_σ at all times. By taking into account (3.6)₁, $\delta \vartheta_t(S) = 0$, (4.21), and Lemma 5, we immediately find $u_i - \hat{u}_i = 0$ on $\partial V_u \times [0, \infty)$. Also, by having $\tilde{S} = [\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \tilde{\mathbf{D}}, \mathbf{0}, 0, 0]$ in which $\tilde{\mathbf{D}}$ is zero on ∂V_D at all times, using (3.6)₂, $\delta \vartheta_t(S) = 0$, (4.21), and the Corollary 3, we conclude $h * (\varphi - \hat{\varphi}) = 0$ on $\partial V_\varphi \times [0, \infty)$, which implies (3.7)₃. In a similar fashion, we have (3.7)₅. Hence, based on Theorem 7, $\delta \vartheta_t(S) = 0$ yields a solution of the mixed initial-boundary value problem, and the proof is complete. $\square$

Next, as the first example in which there is a restriction on fields, analogous to [26], we obtain a variational form of the mixed initial-boundary value problem of the admissible process for which the kinematic equation (3.6)₁ is identically satisfied.

Theorem 11. *Let Ω denote the set of all admissible processes which satisfy (3.6)₁. Let $S = [\mathbf{u}, \varepsilon, \sigma, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \varphi, \psi]$ be an element of Ω and define the functional Ξ_t on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$\begin{aligned}
\Xi_t(S) = & \int_V [h * \sigma_{ij} * \varepsilon_{ij}] (\mathbf{x}, t) d\mathbf{x} - \frac{1}{2} \int_V S_{ijkl}(\mathbf{x}) [h * \sigma_{ij} * \sigma_{kl}] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_V d_{kij}^E(\mathbf{x}) [h * \sigma_{ij} * E_k] (\mathbf{x}, t) d\mathbf{x} - \int_V d_{kij}^H(\mathbf{x}) [h * \sigma_{ij} * H_k] (\mathbf{x}, t) d\mathbf{x} \\
& - \frac{1}{2} \int_V \chi_{ij}^E(\mathbf{x}) [h * E_i * E_j] (\mathbf{x}, t) d\mathbf{x} - \frac{1}{2} \int_V \chi_{ij}^H(\mathbf{x}) [h * H_i * H_j] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_V \chi_{ij}^{EH}(\mathbf{x}) [h * E_i * H_j] (\mathbf{x}, t) d\mathbf{x} + \int_V [h * D_i * E_i] (\mathbf{x}, t) d\mathbf{x} + \int_V [h * B_i * H_i] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_V [h * (D_{i,i} - q_e) * \varphi] (\mathbf{x}, t) d\mathbf{x} - \int_V [h * B_{i,i} * \psi] (\mathbf{x}, t) d\mathbf{x} + \frac{1}{2} \int_V \rho(\mathbf{x}) [u_i * u_i] (\mathbf{x}, t) d\mathbf{x}
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
& - \int_V [f_i^b * u_i] (\mathbf{x}, t) d\mathbf{x} - \int_{\partial V_{\mathbf{u}}} [h * t_i * (u_i - \hat{u}_i)] (\mathbf{x}, t) d\mathbf{x} - \int_{\partial V_{\sigma}} [h * \hat{t}_i * u_i] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_{\partial V_{\phi}} [h * d * \hat{\phi}] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_{\mathbf{D}}} [h * (d - \hat{d}) * \phi] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_{\psi}} [h * b * \hat{\psi}] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_{\partial V_{\mathbf{B}}} [h * (b - \hat{b}) * \psi] (\mathbf{x}, t) d\mathbf{x}.
\end{aligned}$$

Then, S is a solution of the mixed initial-boundary value problem if and only if $\delta \Xi_t(S) = 0$ over Ω , within the time interval $t \in [0, \infty)$.

Proof. Let $\tilde{S} = [\tilde{\mathbf{u}}, \tilde{\varepsilon}, \tilde{\sigma}, \tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{D}}, \tilde{\mathbf{B}}, \tilde{\phi}, \tilde{\psi}] \in \Omega$ be an admissible process and suppose that $S + \lambda \tilde{S} \in \Omega$ for all real values of λ . By employing (4.22), using (4.5), the compatibility equation (3.6)₁, the symmetry condition (3.15), the divergence Theorem, and properties of convolution product given in Appendix 4.2, one can find

$$\begin{aligned}
\delta_{\tilde{S}} \Xi_t(S) = & - \int_V \left[h * (S_{ijkl} \sigma_{kl} + d_{kij}^E E_k + d_{kij}^H H_k - \varepsilon_{ij}) * \tilde{\sigma}_{ij} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[h * (D_k - d_{kij}^E \sigma_{ij} - \chi_{ik}^E E_i - \chi_{ki}^{EH} H_i) * \tilde{E}_k \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[h * (B_k - d_{kij}^H \sigma_{ij} - \chi_{ik}^{EH} E_i - \chi_{ki}^H H_i) * \tilde{H}_k \right] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_V \left[(h * \sigma_{ij,j} + f_i^b - \rho u_i) * \tilde{u}_i \right] (\mathbf{x}, t) d\mathbf{x} + \int_V [h * (E_i + \phi_{,i}) * \tilde{D}_i] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V [h * (H_i + \psi_{,i}) * \tilde{B}_i] (\mathbf{x}, t) d\mathbf{x} - \int_V [h * (D_{i,i} - q_e) * \tilde{\phi}] (\mathbf{x}, t) d\mathbf{x} - \int_V [h * B_{i,i} * \tilde{\psi}] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_{\partial V_{\mathbf{u}}} [h * (u_i - \hat{u}_i) * \tilde{t}_i] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_{\sigma}} [h * (t_i - \hat{t}_i) * \tilde{u}_i] (\mathbf{x}, t) d\mathbf{x} - \int_{\partial V_{\phi}} [h * (\phi - \hat{\phi}) * \tilde{d}] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_{\partial V_{\psi}} [h * (\psi - \hat{\psi}) * \tilde{b}] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_{\mathbf{D}}} [h * (d - \hat{d}) * \tilde{\phi}] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_{\mathbf{B}}} [h * (b - \hat{b}) * \tilde{\psi}] (\mathbf{x}, t) d\mathbf{x}.
\end{aligned} \tag{4.23}$$

Because of Theorem 7, if S is a solution to the IBVP, then we conclude $\delta_{\tilde{S}} \Xi_t(S) = 0$ for every $\tilde{S} \in \Omega$ ($0 \leq t < \infty$), leading us to $\delta \Xi_t(S) = 0$. Also, with the same path given in Theorem 10, by using Lemma 3, Lemma 4, Lemma 5, Corollary 3, (4.23), $\delta \Xi_t(S) = 0$, properties of convolution

product, considering (3.14), (3.15), (3.16), and Theorem 7, the implication in the other direction is proved. $\square$

Theorem 10 is the most general variational form giving a solution of the elastodynamics IBVP for electromagneto-elastic materials. In Theorem 11, the displacement-strain kinematic equation is employed as the only restriction. Hence, one can further restrict the admissible process by which it automatically satisfies some of the field equations and boundary conditions. In doing so, let define a kinematically admissible process and consequently obtain a relevant variational form.

Definition 2. *An admissible process is called a kinematically admissible process if it satisfies the kinematic equations (3.6), the constitutive equations (3.12), and the displacement-potential boundary conditions.*

Theorem 12. *Let Ω denote the set of all kinematically admissible processes. Let $S = [\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \varphi, \psi]$ be an element of Ω and define the functional $\Sigma_t(S)$ on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$\begin{aligned} \Sigma_t(S) = & \frac{1}{2} \int_V [h * \sigma_{ij} * \varepsilon_{ij}] (\mathbf{x}, t) d\mathbf{x} + \frac{1}{2} \int_V \rho(\mathbf{x}) [u_i * u_i] (\mathbf{x}, t) d\mathbf{x} - \int_V [f_i^b * u_i] (\mathbf{x}, t) d\mathbf{x} \\ & - \frac{1}{2} \int_V [h * D_i * E_i] (\mathbf{x}, t) d\mathbf{x} - \frac{1}{2} \int_V [h * B_i * H_i] (\mathbf{x}, t) d\mathbf{x} + \int_V [h * q_e * \varphi] (\mathbf{x}, t) d\mathbf{x} \\ & - \int_{\partial V_\sigma} [h * \hat{t}_i * u_i] (\mathbf{x}, t) d\mathbf{x} - \int_{\partial V_D} [h * \hat{d} * \varphi] (\mathbf{x}, t) d\mathbf{x} - \int_{\partial V_B} [h * \hat{b} * \psi] (\mathbf{x}, t) d\mathbf{x} \end{aligned} \quad (4.24)$$

Then, S is a solution of the mixed initial- boundary value problem if and only if $\delta \Sigma_t(S) = 0$ over Ω , within the time interval $t \in [0, \infty)$.

Proof. Let $\tilde{S} = [\tilde{\mathbf{u}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{D}}, \tilde{\mathbf{B}}, \tilde{\varphi}, \tilde{\psi}] \in \Omega$ be an admissible process and suppose that $S + \lambda \tilde{S} \in \Omega$ for all real values of λ . Obviously, it infers $\tilde{u}_i = 0$ on $\partial V_{\mathbf{u}} \times [0, \infty)$, $\tilde{\varphi} = 0$ on $\partial V_\varphi \times [0, \infty)$ and $\tilde{\psi} = 0$ on $\partial V_\psi \times [0, \infty)$. By making use of (4.24), (4.5), Kinematic equation (3.6), the constitutive

equation (3.12), the symmetry condition (3.13), and the divergence Theorem, we obtain:

$$\begin{aligned}
\delta_{\tilde{S}} \Sigma_t(S) = & - \int_V \left[(h * \sigma_{ij,j} + f_i^b - \rho u_i) * \tilde{u}_i \right] (\mathbf{x}, t) d\mathbf{x} - \int_V [h * (D_{i,i} - q_e) * \tilde{\varphi}] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_V [h * B_{i,i} * \tilde{\psi}] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_\sigma} [h * (t_i - \hat{t}_i) * \tilde{u}_i] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_{\partial V_D} [h * (d - \hat{d}) * \tilde{\varphi}] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_B} [h * (b - \hat{b}) * \tilde{\psi}] (\mathbf{x}, t) d\mathbf{x}.
\end{aligned} \tag{4.25}$$

As is clear when S is a solution of the mixed initial-boundary value problem, then $\delta_{\tilde{S}} \Sigma_t(S) = 0$ for every admissible $\tilde{S}$ ($0 \leq t < \infty$), leading to $\delta \Sigma_t(S) = 0$. Conversely, similar to the Theorem 10, since the array $[\tilde{\mathbf{u}}, \tilde{\varphi}, \tilde{\psi}]$ can be defined arbitrarily, for every $t \in [0, \infty)$, on the domain and the boundary ∂V , then by employing $\delta \Sigma_t(S) = 0$, (4.25), Lemma 3, Lemma 4, properties of convolution product, and Theorem 7, we obtain the desired result. $\square$

4.4.2 Variational Principles Characterizing $[\mathbf{u}, \varphi, \psi]$

With the aid of Theorem 12, it is straightforward to obtain a variational form in terms of the displacement field, electric potential, and magnetic potential. In this regard, define an admissible displacement-potential process and a kinematically admissible displacement-potential as follows:

Definition 3. An array $S = [\mathbf{u}, \varphi, \psi]$ is called an *admissible displacement-potential process* if $\mathbf{u} \in C^{1,2}$, $\varphi \in C^{1,0}$, $\psi \in C^{1,0}$.

Definition 4. An array $S = [\mathbf{u}, \varphi, \psi]$ is called a *kinematically admissible displacement-potential process* if it is an *admissible displacement-potential process* and meets the *displacement-potential boundary conditions*.

Now, to obtain the desired variational form as a corollary of Theorem 12, the constitutive equations (3.12) and kinematic relations (3.6) need to be employed into (4.24). Doing so, one can easily obtain the variational form corresponding to a kinematically admissible displacement-potential process:

Theorem 13. Let Ω denote the set of all kinematically admissible displacement-potential processes. Let $S = [\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\psi}]$ be an element of Ω and define the functional Θ_t on Ω at each time, say $t \in [0, \infty)$, in the form of

$$\begin{aligned}
\Theta_t(S) = & \frac{1}{2} \int_V \left[h * (C_{ijkl} u_{k,l} + e_{kij}^E \varphi_{,k} + e_{kij}^H \psi_{,k}) * u_{i,j} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \frac{1}{2} \int_V \left[h * (e_{ikl}^E u_{k,l} - \kappa_{ij}^E \varphi_{,j} - \kappa_{ij}^{EH} \psi_{,j}) * \varphi_{,i} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \frac{1}{2} \int_V \left[h * (e_{ikl}^H u_{k,l} - \kappa_{ji}^{EH} \varphi_{,j} - \kappa_{ji}^H \psi_{,j}) * \psi_{,i} \right] (\mathbf{x}, t) d\mathbf{x} - \int_V \left[f_i^b * u_i \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[h * q_e * \varphi \right] (\mathbf{x}, t) d\mathbf{x} + \frac{1}{2} \int_V \rho(\mathbf{x}) \left[u_i * u_i \right] (\mathbf{x}, t) d\mathbf{x} - \int_{\partial V_\sigma} \left[h * \hat{t}_i * u_i \right] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_{\partial V_D} \left[h * \hat{d} * \varphi \right] (\mathbf{x}, t) d\mathbf{x} - \int_{\partial V_B} \left[h * \hat{b} * \psi \right] (\mathbf{x}, t) d\mathbf{x}.
\end{aligned} \tag{4.26}$$

Then, $S = [\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\psi}]$ is a solution of the mixed initial-boundary value problem if and only if $\delta \Theta_t(S) = 0$ over Ω , within the time interval $t \in [0, \infty)$.

4.4.3 Variational Principles Characterizing $[\boldsymbol{\sigma}, \boldsymbol{\varphi}, \boldsymbol{\psi}]$:

The Theorem 9 motivates us to develop variational forms in terms of the stress field rather than the displacement field, which is more desirable when the mechanical boundary conditions are traction-type. In other words, it is of interest to obtain conditions by which the array $S = [\boldsymbol{\sigma}, \boldsymbol{\varphi}, \boldsymbol{\psi}]$ is a solution to the mixed initial-boundary value problems. To this end, let first define what we mean by an admissible electro-magneto-stress process:

Definition 5. An array $[\boldsymbol{\sigma}, \boldsymbol{\varphi}, \boldsymbol{\psi}]$ in which $\boldsymbol{\sigma}$ is second-order symmetric tensor and $\boldsymbol{\sigma} \in C^{2,0}$, $\boldsymbol{\varphi} \in C^{2,0}$, $\boldsymbol{\psi} \in C^{2,0}$ is called admissible electro-magneto-stress process if $\boldsymbol{\varphi} = \hat{\boldsymbol{\varphi}}(\mathbf{x}, t)$ on $\partial V_\varphi \times [0, \infty)$ and $\boldsymbol{\psi} = \hat{\boldsymbol{\psi}}(\mathbf{x}, t)$ on $\partial V_\psi \times [0, \infty)$

Now, the following statement for the electro-magneto-stress processes holds true:

Theorem 14. Let Ω denote the set of all admissible electro-magneto-stress processes. Let $S = [\boldsymbol{\sigma}, \boldsymbol{\varphi}, \boldsymbol{\psi}]$ be an element of Ω and define the functional Υ_t on Ω at each time, say $t \in [0, \infty)$, in the

form of

$$\begin{aligned}
\Upsilon_t(S) = & \frac{1}{2} \int_V \left[\frac{h}{\rho} * \sigma_{ij,j} * \sigma_{ik,k} \right] (\mathbf{x}, t) d\mathbf{x} - \int_V \left[\left(\frac{1}{\rho} f_{(i),j}^b \right) * \sigma_{ij} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \frac{1}{2} \int_V [S_{ijkl} \sigma_{ij} * \sigma_{kl} + \chi_{ij}^E \varphi_{,i} * \varphi_{,j} + \chi_{ij}^H \psi_{,i} * \psi_{,j}] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V [-d_{ikl}^E \sigma_{kl} * \varphi_{,i} - d_{ikl}^H \sigma_{kl} * \psi_{,i} + \chi_{ij}^{EH} \psi_{,j} * \varphi_{,i}] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_V [q_e * \varphi] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_u} \left[\left(\frac{f_i^b}{\rho} - \hat{u}_i \right) * t_i \right] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_\sigma} \left[\frac{h}{\rho} * (\hat{t}_i - t_i) * \sigma_{ij,j} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_{\partial V_D} [\hat{d} * \varphi] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_B} [\hat{b} * \psi] (\mathbf{x}, t) d\mathbf{x}.
\end{aligned} \tag{4.27}$$

Then, S is a solution of the mixed initial- boundary value problem if and only if $\delta \Upsilon_t(S) = 0$ over Ω , within the time interval $t \in [0, \infty)$.

Proof. Let $\tilde{S} = [\tilde{\sigma}, \tilde{\varphi}, \tilde{\psi}]$ be an ordered array in which $\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji}$, $\tilde{\sigma} \in C^{2,0}$, $\tilde{\varphi} \in C^{2,0}$, $\tilde{\psi} \in C^{2,0}$ such that $S + \lambda \tilde{S} \in \Omega$ for all real values of λ —which infers $\tilde{\varphi} = 0$ on $\partial V_\varphi \times [0, \infty)$ and $\tilde{\psi} = 0$ on $\partial V_\psi \times [0, \infty)$. By making use of (4.27), (4.5), symmetry condition (3.15), applying the divergence Theorem, properties of convolution, and the above mentioned restriction, we find

$$\begin{aligned}
\delta_{\tilde{S}} \Upsilon_t(S) = & \int_V \left[-\left(\frac{h}{\rho} * \sigma_{(ik,k),j} \right) + S_{ijkl} \sigma_{kl} - d_{kij}^E \varphi_{,k} - d_{kij}^H \psi_{,k} - \left(\frac{1}{\rho} f_{(i),j}^b \right) * \tilde{\sigma}_{ij} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[((d_{ikl}^E \sigma_{kl} - \chi_{ij}^E \varphi_{,j} - \chi_{ij}^{EH} \psi_{,j})_{,i} - q_e) * \tilde{\varphi} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_V \left[(d_{ikl}^H \sigma_{kl} - \chi_{ji}^{EH} \varphi_{,j} - \chi_{ij}^H \psi_{,j})_{,i} * \tilde{\psi} \right] (\mathbf{x}, t) d\mathbf{x} \\
& + \int_{\partial V_u} \left[\left(\frac{h}{\rho} * \sigma_{ik,k} + \frac{f_i^b}{\rho} - \hat{u}_i \right) * \tilde{t}_i \right] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_\sigma} \left[\frac{h}{\rho} * (\hat{t}_i - t_i) * \tilde{\sigma}_{ij,j} \right] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_{\partial V_D} \left[((d_{ikl}^E \sigma_{kl} - \chi_{ij}^E \varphi_{,j} - \chi_{ij}^{EH} \psi_{,j}) n_i - \hat{d}) * \tilde{\varphi} \right] (\mathbf{x}, t) d\mathbf{x} \\
& - \int_{\partial V_B} \left[((d_{ikl}^H \sigma_{kl} - \chi_{ji}^{EH} \varphi_{,j} - \chi_{ij}^H \psi_{,j}) n_i - \hat{b}) * \tilde{\psi} \right] (\mathbf{x}, t) d\mathbf{x}.
\end{aligned} \tag{4.28}$$

Obviously, if S is a solution of the mixed initial-boundary value problem, then $\delta_{\tilde{S}} \Upsilon_t(S) = 0$ for every above-defined $\tilde{S}$ ($0 \leq t < \infty$) is implied from Theorem 9, resulting $\delta \Upsilon_t(S) = 0$. Conversely, if $\delta \Upsilon_t(S) = 0$, then, by utilizing Lemma 3, Lemma 4, Lemma 5, Lemma 6, and Theorem 9, we obtain the desired result. $\square$

When the mechanical boundary condition is entirely traction-type, say traction problems, one can establish a more convenient variational form in terms of the ordered array $S = [\sigma, \varphi, \psi]$.

Definition 6. *An admissible electro-magneto-stress process is called a dynamically admissible electro-magneto-stress process if $\sigma_{ij}n_j = \hat{t}_i(\mathbf{x}, t)$ on $\partial V_\sigma \times [0, \infty)$.*

Now, based on Theorem 14, it is straightforward to obtain the variational framework for the dynamical electro-magneto-stress processes:

Theorem 15. *Let Ω denote the set of all dynamically admissible electro-magneto-stress processes. Let $S = [\sigma, \varphi, \psi]$ be an element of Ω and define the functional $\mathfrak{S}_t$ on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$\begin{aligned} \mathfrak{S}_t(S) = & \frac{1}{2} \int_V \left[\frac{h}{\rho} * \sigma_{ij,j} * \sigma_{ik,k} \right] (\mathbf{x}, t) d\mathbf{x} - \int_V \left[\left(\frac{1}{\rho} f_{(i),j}^b \right) * \sigma_{ij} \right] (\mathbf{x}, t) d\mathbf{x} \\ & + \frac{1}{2} \int_V [S_{ijkl} \sigma_{ij} * \sigma_{kl} + \chi_{ij}^E \varphi_{,i} * \varphi_{,j} + \chi_{ij}^H \psi_{,i} * \psi_{,j}] (\mathbf{x}, t) d\mathbf{x} \\ & + \int_V [-d_{ikl}^E \sigma_{kl} * \varphi_{,i} - d_{ikl}^H \sigma_{kl} * \psi_{,i} + \chi_{ij}^{EH} \psi_{,j} * \varphi_{,i}] (\mathbf{x}, t) d\mathbf{x} \\ & - \int_V [q_e * \varphi] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_D} [\hat{d} * \varphi] (\mathbf{x}, t) d\mathbf{x} + \int_{\partial V_B} [\hat{b} * \psi] (\mathbf{x}, t) d\mathbf{x}. \end{aligned} \quad (4.29)$$

Then, S is a solution of the traction problem (i.e., $\partial V_u = \emptyset$) if and only if $\delta \mathfrak{S}_t(S) = 0$ over Ω , within the time interval $t \in [0, \infty)$.

Proof. The proof is analogous to that of Theorem 14. $\square$

Chapter 5

Convolution Finite Element Method¹

5.1 Introduction

Theoretical mechanics underscores the importance of developing efficient methods to solve partial differential equations (PDEs). Indeed, the desired results of almost every time-dependent physical process lie in the solution of initial boundary value problems (IBVPs). From a practical standpoint, developing numerical methods is of great significance to deal with mathematical difficulties accompanied by abnormal geometry and non-linearity. Among numerical methods for PDEs, perhaps the Finite Element Method (FEM) is the most popular tool, thanks to its ease of computer programming implementation and mathematical foundation. While the FEM follows standard steps in stationary problems (boundary value problems), in IBVPs, several approaches have been developed, falling mainly into two main categories [79]: (i) space-time decoupled approaches and (ii) space-time coupled approaches. In what follows, as the main concern of the present section is the wave propagation in solid media, we mainly mention relevant works to the second order evolutionary equations.

In the first of these, the solution is assumed as the product of functions of spatial and time variables. Subsequently, governing equations are reduced into a set of time-wise ODEs. These ODEs are then typically solved by explicit/implicit time integration methods or methods of time discretization [75, 22, 21]. The time integration methods have gained popularity mainly because there is only one set of calculations required to obtain the solution at each step. Also, the algorithms are closely similar to FEM for stationary problems. The most well-known approaches in this vein are [79]: (1) Central Difference method (2) Houbolt method, (3) Wilson's θ method, (4) Newmark family of

¹Adapted with permission from [4]

methods.

Mathematical principles such as consistency, stability, convergence, and accuracy determine the superiority in time integration approaches. Several studies have been carried out to investigate those characteristics (see, for example, [9, 32]). Among the methods, the Newmark average acceleration algorithm (the trapezoidal rule) has received more attention since it is unconditionally stable, second-order accurate that conserves the energy in the linear regime [48]. The stability offers the possibility of taking large time steps, which is a crucial factor for practical purposes as the response of structures often involves high frequencies.

There are noticeable studies available in the literature (e.g., see [33, 84, 34, 35, 16]) which improve some features of the time integration methods. In these studies, a time integration method with controllable algorithmic damping has been established; the technique has a negligible effect on lower modes while the higher ones are eliminated. This property excludes spurious parts of the solution coming from the discretization of the problem. As shown in [33], the numerical damping of the trapezoidal rule is zero, and, on the other hand, the Houbolt method has high damping property. In this regard, as compared to the other relevant techniques, the family of HHT α method has adjustable algorithmic damping and efficiently removes higher-order spurious modes that artifact of discretization. It is worth mentioning that there are new approaches in the literature developing new time step algorithms with higher order accuracy (see, for example, [41]).

As the name suggests, the assumption of separation of time and space variables is no longer presumed in the second category. As a consequence, the relevant algebraic equations contain the concurrent effect of time and spatial variables. These approaches can be either based on variational principles, e.g., Hamiltonian's principles employed in [66], or integral form directly obtained from localized governing equations. They can be utilized for the whole time interval, or a time-marching algorithm—by which the time interval is split into several time slabs and solved consecutively—can be followed. The earlier case is mostly accompanied by computational cost, and thus, in some instances, it may be prohibitive [79].

The resulting coupled integral forms in the second category are typically dealt with the space-time FEM: time variable is mainly treated similar to spatial ones, leading to the simultaneous discretiza-

tion of time and spatial variables. Continuous Galerkin (e.g., see [59]) (CG) and Discontinuous Galerkin (e.g., see [39]) (DG) are the two main methods employed in the space-time FEM. While both the CG and DG are capable of approximating the solution accurately, the continuity of fields in the earlier may enforce algebraic equations that are computationally intensive. The latter seems to be efficient as the solution is considered on the level of element, and instead, the space-wise and/or time-wise continuity of fields are weakly met. In this vein, for the elastodynamic case, the Time-discontinuous Galerkin method combined with the least square approach has been developed in [39]. Two separate spaces are defined for displacement and velocity fields in this method. The continuity of displacement and velocity in time direction is enforced in a weak sense (see [39]). Subsequently, several methods, namely, Time-discontinuous Galerkin, Galerkin/least-squares, and Discontinuity-capturing formulations, have been proposed in [40] to accurately capture discontinuities or sharp gradients. As reported in this reference, time integration methods, specifically without controllable algorithmic damping ones, lead to a noisy solution even faraway from points with sharp gradients or discontinuities. The further development of the discontinuous-Galerkin method can be found for example in [1, 42], among others.

However, the concurrent discretization of time and spatial variables may increase the computational cost even though a higher-order of accuracy is gained. For example, one can compare the computational efforts of 1d problems with those of 2d ones. This fact makes their application justifiable when the solution contains irregularities such as discontinuity and sharp gradients. Moreover, the solution resulting from all the algorithms mentioned above (time integration and time-marching methods) satisfies the governing equations discretely. Hence, it is of more interest to develop a framework to approximate the solution across the given time-interval. Additionally, increasing the time step in time integration methods gives rise to the loss of accuracy thanks to the underlying finite-difference approach.

Hence, it is still desirable to develop an algorithm having the following characteristics:

- it preferably follows simple steps analogous to time integration methods.
- it is versatile to gain accurate approximation for any given time interval.

- it preferably converges to the solution at almost all time-instances.
- it has the potential of detaching the spurious part of the response similar to algorithmic damping methods, e.g., HHT α method.
- it does not require time-space discretization to obtain a higher-order accurate result.
- it is computationally justifiable if compared to other methods.

It has been shown in [26] that there exists an equivalent set of equations, so-called alternative field equations, to the equations of motion. Upon that, a new variational framework, convolutional variational principles, has been developed in [26]. Later on, similar formulations have been established for coupled problems, e.g., in thermoelasticity [65] and, recently, in electromagneto-elastic materials [3]. These frameworks could be an appropriate candidate for implementing numerical methods for time-dependent problems. As can be seen in the main body, the alternative formulation is a set of coupled integro-partial differential equations involving convolution products rather than a time derivative. Hence, the corresponding variational principles, which have been derived in [26], are also written in terms of convolution products. This characteristic circumvents dealing with time derivatives, which is an inevitable step in the existing algorithms. Also, in this formulation, initial conditions are imposed as a specific form of a forcing term, securing their systematic satisfaction.

By employing the alternative framework and the relevant variational principles, one can expand the solution in terms of the time-wise series. For instance, by direct application of those variational principles, [73, 71, 72] have developed a so-called 'space time semi-analytical method.' Further development of the technique for some specific problems can also be found in [47, 70]. As reported by authors in [72], this method can achieve accurate results provided that proper time series are chosen. The time series in these studies are mainly based on prior knowledge obtained from similar analytical solutions; for example, the solution obtained from the separation of variables corresponding to a similar problem. This prior knowledge makes the application of the method somewhat restricted.

In the present chapter, a new method stemming from an alternative formulation given in [26]

is proposed. Since the technique is based on a weak formulation closely related to variational principles involving the convolution product (see [26]), it is called the 'convolution finite element method.' Although the algorithm is versatile enough to adapt it to solve some other physical processes, we herein mainly focus on linear elastodynamics. To illustrate this point, in numerical results, we briefly consider the mass-spring-damper problem, a prototype of structural mechanics, and highlight its applicability to elastodynamics. Finally, by numerical examples or mathematical analysis, we show that the present approach inherits the enumerated desirable characteristics for the case of elastodynamics.

The remainder of the chapter is presented with the following structure: first, we list the well-known governing equations and boundary and initial conditions. Then, the concept of alternative governing equations is explained. Next, we establish the convolution weighted residual relevant to the alternative governing equations. Analogous to the classical finite element method, the convolution Galerkin formulation is then constructed. Subsequently, a particular form of shape functions is introduced, reducing the problem into a set of algebraic equations whose matrix form has a desirable property for elastodynamics. Several numerical examples, including a mass-spring, mass-spring-damper, the rod problem, and plane strain, are then considered to demonstrate the method's numerical capability.

5.2 Problem Statement

Let Ω denote the domain of linear elastic body, an open subset of 3D Euclidean space whose boundary, Γ , is Lipschitz. Let Γ_{gi} and Γ_{hi} denote open subsets of Γ ($i = 1, 2, 3$) with:

$$\begin{aligned}\Gamma_{gi} \cap \Gamma_{hi} &= \emptyset, \\ \overline{\Gamma_{gi} \cup \Gamma_{hi}} &= \Gamma.\end{aligned}\tag{5.1}$$

Also, $\overline{\Omega}$, $\overline{\Gamma_{gi}}$, and $\overline{\Gamma_{hi}}$ denote, respectively, closure of Ω , Γ_{gi} , and Γ_{hi} .

In the linear regime, the motion of the elastic body is governed by the following field equations and the initial and boundary conditions:

1. Equations of Motion:

$$\sigma_{ij,j} + \rho b_i = \rho \ddot{u}_i \quad \text{on } \Omega \times (0, T). \quad (5.2)$$

2. Kinematic Equations:

$$\varepsilon_{ij} = u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{on } \Omega \times (0, T). \quad (5.3)$$

3. Constitutive Equations:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad \text{on } \Omega \times (0, T), \quad (5.4)$$

where symmetry and positive definiteness of C_{ijkl} are assumed.

4. Boundary Conditions:

$$\begin{aligned} u_i &= g_i(\mathbf{x}, t) & \text{on } \Gamma_{gi} \times (0, T), \\ \sigma_{ij} n_j &= h_i(\mathbf{x}, t) & \text{on } \Gamma_{hi} \times (0, T). \end{aligned} \quad (5.5)$$

5. Initial Conditions:

$$\begin{aligned} u_i(\mathbf{x}, 0) &= u_i^0(\mathbf{x}) & \text{on } \overline{\Omega}, \\ \dot{u}_i(\mathbf{x}, 0) &= v_i^0(\mathbf{x}) & \text{on } \overline{\Omega}. \end{aligned} \quad (5.6)$$

Now, following [37], let us formally state what we mean by a ‘strong form’ of the initial boundary value problem. In doing so, we have the following regularity conditions:

Definition 7. An ordered array $[\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ is called an admissible process on $\overline{V} \times (0, T)$ if

$$u_i \in C^{1,2}, \varepsilon_{ij} \in C^{0,0}, \sigma_{ij} \in C^{1,0} \quad (5.7)$$

Definition 8. An admissible process $[\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ is called a strong solution of the IBVP if (6.2), (6.3), (6.4), (6.5), and (6.6) hold.

In the linear regime, for traction problems, there is an equivalent set of equations, in terms of stress, assuring the satisfaction of the equation of motion [43, 44]. Stemming from those equations, a necessary and sufficient condition equivalent to the equation of motion and the initial conditions is that the following set of equations holds [26]:

$$\rho \mathbf{u}(\mathbf{x}, t) = [t * (\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f})](\mathbf{x}, t) + \rho(\mathbf{u}^0(\mathbf{x}) + t \mathbf{v}^0(\mathbf{x})), \quad (5.8)$$

in which

$$[f * g](\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t - \tau) g(\mathbf{x}, \tau) d\tau. \quad (5.9)$$

Also, we shall use the following notation:

$$[f * g * h](\mathbf{x}, t) = \int_0^t \left(\int_0^\lambda f(\mathbf{x}, \lambda - \tau) g(\mathbf{x}, \tau) d\tau \right) h(\mathbf{x}, t - \lambda) d\lambda \quad (5.10)$$

(6.8) is obtained by applying the Laplace transformation to the equation of motion, inferring a systematic procedure for derivation of such equations. Conversely, a direct application of the Leibnitz formula shows that (6.8) implies (6.2) and (6.6). It is worth mentioning a larger class of functions for the displacement field can be considered under (6.8) where there is no time derivative. Subsequently, the following statement [26] is an alternative representation of the above-mentioned strong form:

Theorem 16. *An admissible process $[\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ is a strong solution of the IBVP if and only if (6.3), (6.4), (6.5), and (6.8) hold.*

Now, let us reiterate some lemmas initially proved in [26], along with some corollaries which shall be utilized in the sequel.

Lemma 7. *let ϑ be a continuous function on $\overline{\Omega} \times [0, \infty)$ and suppose*

$$\int_{\Omega} [\vartheta * \boldsymbol{\omega}](\mathbf{x}, t) dV = 0, \quad 0 \leq t < \infty, \quad (5.11)$$

for every $\omega \in C^{\infty, \infty}$ which, together with its space derivatives, vanishes on $\Gamma \times [0, \infty)$. Then

$$\vartheta = 0 \quad \text{on } \overline{\Omega} \times [0, \infty). \quad (5.12)$$

The previous statement is based on the continuity of function and infinite time interval premises. To form a rigorous numerical foundation, nevertheless, one needs to furnish the problem with complete function spaces defined over the domain of the problem:

Definition 9. Denote $L^2(0, T; H^1(\Omega))$ as a Hilbert space with inner product

$$\int_0^T (u(t), v(t)) dt, \quad (5.13)$$

whose elements are members of the Hilbert space $H^1(\Omega)$ for $0 \leq t \leq T$ and

$$\int_0^T \|u(t)\|^2 dt < \infty. \quad (5.14)$$

Additionally, in (5.13) and (5.14), $(,)$ and $\| \cdot \|$ represent the inner product and norm of $H^1(\Omega)$, respectively.

Now, we have the following statement:

Lemma 8. let ϑ be an element of $L^2(0, T; H^1(\Omega))$, and the following holds

$$\int_{\Omega} \omega(\mathbf{x}) dV \int_0^T \vartheta(\mathbf{x}, T - \tau) \cos\left(\frac{i\pi\tau}{T}\right) d\tau = 0, \quad i = 0, \dots, \infty, \quad (5.15)$$

for every $\omega \in H^1(\Omega)$. Then

$$\vartheta = 0 \quad \text{on } \overline{\Omega} \times [0, T], \quad (5.16)$$

in the sense of

$$\int_{\Omega} \int_0^T \vartheta^2 dt dV = 0. \quad (5.17)$$

Proof. Let

$$S_N^\vartheta(\mathbf{x}, t) = \sum_{j=0}^N A_j(\mathbf{x}) \cos\left(\frac{j\pi t}{T}\right) \quad (5.18)$$

denote the finite Fourier cosine expansion of ϑ where

$$\lim_{N \rightarrow \infty} \int_0^T \left(\vartheta - S_N^\vartheta \right)^2 dt = 0. \quad (5.19)$$

Substitution of (5.18) into (5.15) and letting $N \rightarrow \infty$ result in

$$\int_{\Omega} \omega(\mathbf{x}) A_j(\mathbf{x}) dV = 0, \quad j = 0, \dots, \infty, \quad (5.20)$$

for every $\omega \in H^1(\Omega)$. Now, taking $\omega(\mathbf{x}) = A_j(\mathbf{x})$ in (5.20), it follows that $\int_{\Omega} A_j^2 d\mathbf{x} = 0$. Then, the desired result follows from spatial integration of (5.19). $\square$

Remark 1. Obviously, the statement mentioned in lemma 8 is still true when

$$\int_{\Omega} \omega(\mathbf{x}) dV \int_0^T \vartheta(\mathbf{x}, \tau) \cos\left(\frac{j\pi\tau}{T}\right) d\tau = 0, \quad j = 0, \dots, \infty, \quad (5.21)$$

is met for every $\omega \in H^1(\Omega)$.

Remark 2. Let $\vartheta \in L^2(0, T; H^1(\Omega))$ be such that

$$\int_{\Omega} dV \int_0^T \vartheta(\mathbf{x}, T - \tau) w(\mathbf{x}, \tau) d\tau = 0, \quad (5.22)$$

for every $w \in L^2(0, T; H^1(\Omega))$. Then $\vartheta = 0$ in the sense of (5.17).

Remark 3. A direct consequence of remark 2: $\vartheta \in L^2(0, T; H^1(\Omega))$ is zero in the sense of (5.17) if

$$\int_{\Omega} [\vartheta * w](\mathbf{x}, t) dV = 0, \quad 0 < t \leq T, \quad (5.23)$$

for every $w \in L^2(0, T; H^1(\Omega))$.

It is noticed that, in a similar fashion, the same results hold for the boundaries Γ_{hi} , $i = 1, 2, 3$:

Remark 4. $\vartheta \in L^2((0, T) \times \Gamma_{hi})$ is zero in the sense of (5.17) if

$$\int_{\Gamma_{hi}} [\vartheta * w](\mathbf{x}, t) d\Gamma = 0, \quad 0 < t \leq T, \quad (5.24)$$

for every $w \in L^2((0, T) \times \Gamma_{hi})$.

Remark 5. Based on remark 1, similar results corresponding to remarks 3 and 4 hold when (5.23) and (5.24) are, respectively, replaced by:

$$\int_0^T \int_{\Omega} \vartheta(\mathbf{x}, t) w(\mathbf{x}, t) dV dt = 0, \quad (5.25)$$

and

$$\int_0^T \int_{\Gamma_{hi}} \vartheta(\mathbf{x}, t) w(\mathbf{x}, t) d\Gamma dt = 0 \quad (5.26)$$

Thus, conceptually, one can define a weak formulation corresponding to (6.8) where either classical dot product or convolution product in time is utilized.

In what follows, a weak solution in terms of convolution product is defined. This formulation highlights space-time convolutional variational principles without having further computational cost since based on lemma 8 it suffices to consider the case $t = T$ where both products result in the same form of equations.

5.3 Convolution Weighted Residual

Similar to the concept of the weighted residual defined in classical FEM for stationary problems, one can introduce a weak formulation relevant to (6.8):

$$\int_{\Omega} [\rho u_i * w_i] dV = \int_{\Omega} [t * (\sigma_{ij,j} + \rho f_i) * w_i] dV + \int_{\Omega} [\rho (u_i^0 + t v_i^0) * w_i] dV, \quad (5.27)$$

in which w_i represents weighting functions, and $(\mathbf{x}, t)$ has been omitted for brevity. It is noted that the convolution in time is associative, hence, there is no need to indicate the order of time convolutions under the space integrals. According to remark 2, (6.8) is satisfied in the sense of (5.17) provided that (5.27) is met for all weight functions $\mathbf{w} \in L^2(0, T; \mathbf{H}^1(\Omega))$. Next, analogous to [37], we define the solution and the variation spaces:

Definition 10. $\mathbb{S}_i$ and $\mathbb{V}_i$ stand for, respectively, the solution space and the variation space whose elements have the following properties:

$$\mathbb{S}_i = \{u_i \in L^2(0, T; \mathbf{H}^1(\Omega)) | u_i = g_i(\mathbf{x}, t) \text{ on } \Gamma_{g_i} \times (0, T)\}, \quad (5.28)$$

$$\mathbb{V}_i = \{w_i \in L^2(0, T; \mathbf{H}^1(\Omega)) | w_i = 0 \text{ on } \Gamma_{g_i} \times (0, T)\}, \quad i = 1, 2, 3. \quad (5.29)$$

By applying (6.3) and (6.4), using the divergence theorem, and noting $w_i \in \mathbb{V}_i$, (5.27) reduces to:

$$\begin{aligned} & \int_{\Omega} [\rho u_i * w_i] dV + \int_{\Omega} [C_{ijkl} t * u_{i,j} * w_{k,l}] dV = \\ & \sum_{i=1}^3 \int_{\Gamma_{fi}} [t * h_i * w_i] d\Gamma + \int_{\Omega} [t * \rho f_i * w_i] dV + \int_{\Omega} [\rho (u_i^0 + t v_i^0) * w_i] dV. \end{aligned} \quad (5.30)$$

It should be mentioned that (5.30) corresponds to $\delta \varphi_t \{u_i^c\} = 0$ in the sense of [26] with [26]

$$\varphi_t \{\mathbf{u}^c\} = \frac{1}{2} \int_{\Omega} [t * C_{ijkl} u_{i,j}^c * u_{k,l}^c] dV + \frac{1}{2} \int_{\Omega} [\rho u_i^c * u_i^c] dV - \int_{\Omega} [\rho f_i * u_i^c] dV - \sum_{i=1}^3 \int_{\Gamma_{fi}} [t * h_i * u_i^c] d\Gamma, \quad (5.31)$$

implying the procedure has a variational foundation. Now, with (6.11), (6.12), and (5.30), a weak solution pertinent to the strong solution can be formally defined as:

Definition 11. Let u_i be an element of $\mathbb{S}_i$. Then, u_i is called a weak solution relevant to the strong solution if (5.30) is satisfied for all $w_i \in \mathbb{V}_i$.

It is a significant step to examine whether the weak and strong solutions are equivalent:

Theorem 17. Let all the necessary conditions validating the derivation of (5.30) hold. Then, the

weak and strong solutions are equivalent.

Proof. The first part of the theorem, that is, a strong solution satisfies (5.30) has been already shown. Now, we prove that a weak solution meets (6.3), (6.4), (6.5), and (6.8). Consider (5.30). Define $\epsilon_{ij} = u_{(i,j)}$ and $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$; the symmetry condition implies $\sigma_{ij} = C_{ijkl}u_{k,l}$ and $\sigma_{ij} = \sigma_{ji}$. Hence, after some manipulations, we obtain:

$$\int_{\Omega} [(\rho u_i - t * (\sigma_{ij,j} + \rho f_i) - \rho(u_i^0 + tv_i^0)) * w_i] dV = \sum_{i=1}^3 \int_{\Gamma_{hi}} [t * (h_i - \sigma_{ij}n_j) * w_i] d\Gamma, \quad (5.32)$$

Now, (5.32) and remarks 3 and 4 imply that the defined weak solution satisfies (6.3), (6.4), (6.5), and (6.8). $\square$

As can be seen, the framework (5.30) is constructed so that only the displacement field appears in the formulation. As a result, there is no need to define velocity and acceleration fields to obtain the corresponding weak solution. Although the convolution product is involved, it can be reduced to only spatial integrals with the aid of lemma 8 and the similar results hold for the case of the boundary integrals (see, for example, remark 4). Moreover, initial conditions in this formulation are treated as a forcing term. Hence, by increasing the accuracy of the numerical solution, initial conditions are uniformly satisfied.

5.4 Convolution Galerkin Formulation

Now, we establish the Galerkin formulation corresponding to the Convolution Weighted Residual. By taking the following form for u_i^h , an element of $\mathbb{S}_i$,

$$u_i^h(\mathbf{x}, t) = v_i^h(\mathbf{x}, t) + \tilde{g}_i^h(\mathbf{x}, t), \quad (5.33)$$

with

$$\begin{aligned} v_i^h(\mathbf{x}, t) &= 0 & \text{on } \Gamma_{gi} \times (0, T), \\ \tilde{g}_i^h(\mathbf{x}, t) &= g_i(\mathbf{x}, t) & \text{on } \Gamma_{gi} \times (0, T), \end{aligned} \quad (5.34)$$

and substituting (5.33) into (5.27), one can find:

$$\begin{aligned}
& \int_{\Omega} [\rho v_i^h * w_i] dV + \int_{\Omega} [t * C_{ijkl} v_{i,j}^h * w_{k,l}] dV = - \int_{\Omega} [t * C_{ijkl} \tilde{g}_{i,j}^h * w_{i,j}] dV \\
& - \int_{\Omega} [\rho \tilde{g}_i^h * w_i] dV + \sum_{i=1}^3 \int_{\Gamma_{hi}} [t * h_i * w_i] d\Gamma + \int_{\Omega} [t * \rho f_i * w_i] dV \\
& + \int_{\Omega} [\rho (u_i^0 + t v_i^0) * w_i] dV,
\end{aligned} \tag{5.35}$$

where w_i and v_i^h are elements of $\mathbb{V}_i$. Alternatively, the above form can be rearranged as:

$$\begin{aligned}
& \int_{\Omega} [\rho \gamma_i^h * w_i] dV + \int_{\Omega} [t * C_{ijkl} \gamma_{i,j}^h * w_{k,l}] dV = - \int_{\Omega} [t * C_{ijkl} g_{i,j}^h * w_{k,l}] dV \\
& - \int_{\Omega} [\rho g_i^h * w_i] dV + \sum_{i=1}^3 \int_{\Gamma_{hi}} [t * h_i * w_i] d\Gamma + \int_{\Omega} [t * \rho f_i * w_i] dV \\
& - \int_{\Omega} [t * C_{ijkl} (u_{i,j}^0 + t v_{i,j}^0) * w_{k,l}] dV,
\end{aligned} \tag{5.36}$$

where

$$\gamma_i^h(\mathbf{x}, t) = u_i^h(\mathbf{x}, t) - u_i^0(\mathbf{x}) - t v_i^0(\mathbf{x}) - g_i^h(\mathbf{x}, t), \tag{5.37}$$

in which $g_i^h(\mathbf{x}, t)$ satisfies:

$$\begin{aligned}
g_i^h(\mathbf{x}, t) &= g_i(\mathbf{x}, t) - u_i^0(\mathbf{x}) - t v_i^0(\mathbf{x}) \quad \text{on } \Gamma_{gi} \times (0, T), \\
g_i^h(\mathbf{x}, 0) &= \dot{g}_i^h(\mathbf{x}, 0) = 0.
\end{aligned} \tag{5.38}$$

(5.36) is a computationally proper formulation. Indeed, by making use of (5.36) and (5.37), the following homogeneous conditions need to be satisfied:

$$\begin{aligned}
\gamma_i^h(\mathbf{x}, 0) &= 0 \quad \text{on } \Omega, \\
\dot{\gamma}_i^h(\mathbf{x}, 0) &= 0 \quad \text{on } \Omega, \\
\gamma_i^h(\mathbf{x}, t) &= 0 \quad \text{on } \Gamma_{gi} \times (0, T).
\end{aligned} \tag{5.39}$$

It is noticed that by increasing the accuracy of the approximation, $(5.39)_{1-2}$ are indirectly satisfied; That is, the initial conditions have been built into the weak formulation and, therefore, are met in the limit. However, as shall be seen, by introducing a new form of shape function, $(5.39)_{2-3}$ are identically satisfied, and only $(5.39)_1$ is secured by increasing the accuracy.

Now, suppose it is desired to obtain the solution within a time interval $(0, T)$. In this regard, according to lemma 8 or remark 3, it suffices to consider (5.36) when $t = T$. To this end, by $\mathbb{S}_i^{h,x}$ and $\mathbb{V}_i^{h,x}$ we denote finite dimensional subsets of, respectively, $H^1(\Omega)$ and $H_0^1(\Omega)$. Moreover, let us construct finite function spaces $\mathbb{S}_i^{h,N}$ and $\mathbb{V}_i^{h,N}$ whose elements are the product of function spaces over fields $\mathbf{x} \in \bar{\Omega}$ and $t \in (0, T)$ indicated by:

$$\begin{aligned}\mathbb{S}_i^{h,N} &= \mathbb{S}_i^{h,x} \otimes \mathbb{F}_N^{h,t}, \\ \mathbb{V}_i^{h,N} &= \mathbb{V}_i^{h,x} \otimes \mathbb{F}_N^{h,t},\end{aligned}\tag{5.40}$$

with

$$\mathbb{F}_N^{h,t} = \left\{ 1, \cos \frac{\pi t}{T}, \dots, \cos \frac{(N-1)\pi t}{T} \right\}.\tag{5.41}$$

As a spetial case, following [37], let $\eta = \{1, 2, \dots, n_{np}\}$ denote the set of all nodal points specifying the discretized domain of the problem. Furthermore, η_{gi} represents the set of all nodal points on the boundary Γ_{gi} . Then, $\mathcal{V}_i^h$, an element of $\mathbb{V}_i$, can be expressed as:

$$\mathcal{V}^h(\mathbf{x}, t) \approx \sum_{P=0}^{N-1} \sum_{j=1}^{n_{dof}} \sum_{\alpha \in \eta - \eta_{gj}} c_{Pj\alpha} \psi_\alpha(\mathbf{x}) \cos\left(\frac{P\pi t}{T}\right) \mathbf{e}_j,\tag{5.42}$$

where ψ_α 's denote Lagrangian shape functions, $c_{Pj\alpha}$'s are unknown coefficients to be determined from (5.36), and $\mathbf{e}_j$, $j = 1, \dots, n_{dof}$ denotes unit vectors of Cartesian coordinate. Obviously, (5.42) leads to the satisfaction of $(5.39)_{2-3}$ while $(5.39)_1$ is satisfied by increasing N . In a similar fashion, we write:

$$\begin{aligned}\mathbf{f}(\mathbf{x}, t) &\approx \sum_{P=0}^{N-1} \hat{\mathbf{f}}_P \cos\left(\frac{P\pi t}{T}\right), \\ \mathcal{J}^h(\mathbf{x}, t) &\approx \sum_{P=0}^{N-1} \sum_{j=1}^{n_{dof}} \sum_{\alpha \in \eta_{gj}} \hat{g}_{Pj\alpha} \psi_\alpha(\mathbf{x}) \cos\left(\frac{P\pi t}{T}\right) \mathbf{e}_j,\end{aligned}\tag{5.43}$$

$$\mathbf{h}(\mathbf{x}, t) \approx \sum_{p=0}^{N-1} \hat{\mathbf{h}}_p(\mathbf{x}) \cos\left(\frac{P\pi t}{T}\right),$$

and to embody the Galerkin form, we consider:

$$\mathbf{w}(\mathbf{x}, t) \approx \sum_{Q=0}^{N-1} \sum_{i=1}^{n_{dof}} \sum_{\beta \in \eta - \eta_{gi}} d_{Qi\beta} \psi_{\beta}(\mathbf{x}) \cos\left(\frac{Q\pi t}{T}\right) \mathbf{e}_i. \quad (5.44)$$

Also, we shall make use of the following identities in which the upper limits of convolution product is set $t = T$:

$$\begin{aligned} \mathcal{A}_{ij} &= t * \cos\left(\frac{j\pi t}{T}\right) * \cos\left(\frac{i\pi t}{T}\right) \Big|_{t=T} = \begin{cases} \frac{T^3}{i^2 \pi^2} & i \neq 0, j = 0 \\ \frac{T^3}{j^2 \pi^2} & i = 0, j \neq 0 \\ (-1)^{i+1} \frac{T^3}{2i^2 \pi^2} & i = j \neq 0 \\ \frac{T^3}{6} & i = j = 0 \\ 0 & \text{otherwise} \end{cases}, \\ \mathcal{B}_{ij} &= 1 * \cos\left(\frac{j\pi t}{T}\right) * \cos\left(\frac{i\pi t}{T}\right) \Big|_{t=T} = \begin{cases} \frac{T^2((-1)^{i+1} + (-1)^j)}{(i^2 - j^2)\pi^2} & i \neq j \\ \frac{T^2(1 + (-1)^{i+1})}{i^2 \pi^2} & i \neq j = 0 \\ \frac{T^2(1 + (-1)^{j+1})}{j^2 \pi^2} & j \neq i = 0 \\ \frac{T^2}{2} & i = j = 0 \\ 0 & i = j \neq 0 \end{cases} \quad (5.45) \\ \mathcal{C}_i &= 1 * \cos\left(\frac{i\pi t}{T}\right) \Big|_{t=T} = \begin{cases} 0 & i \neq 0 \\ T & i = 0 \end{cases}, \\ \mathcal{D}_i &= t * \cos\left(\frac{i\pi t}{T}\right) \Big|_{t=T} = \begin{cases} -\frac{T^2(-1 + (-1)^i)}{i^2 \pi^2} & i \neq 0 \\ \frac{T^2}{2} & i = 0 \end{cases}, \\ \mathcal{F}_{ij} &= \cos\left(\frac{i\pi t}{T}\right) * \cos\left(\frac{j\pi t}{T}\right) \Big|_{t=T} = \begin{cases} (-1)^i \frac{T}{2} & i = j \neq 0 \\ T & i = j = 0 \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

$$\mathcal{E}_i = t * t * \cos\left(\frac{i\pi t}{T}\right) \Big|_{t=T} = \begin{cases} -\frac{T^4(2+2(-1)^{m-1}-i^2\pi^2)}{2i^4\pi^4} & i \neq 0 \\ \frac{T^4}{24} & i = 0 \end{cases}.$$

where $\mathcal{C}_i$ and $\mathcal{D}_i$ can be obtained, respectively, from $\mathcal{F}_{ij}$ and $\mathcal{B}_{ij}$ when $i = 0$ (or $j = 0$). Substituting (5.42), (5.43), and (5.44) into (5.36), employing (5.45), one can obtain:

$$\begin{aligned} & \sum_{P=0}^{N-1} \sum_{j=1}^{n_{dof}} \sum_{\alpha \in \eta - \eta_{gi}} c_{Pj\alpha} \left[\mathcal{F}_{PQ}(\rho \psi_\alpha \mathbf{e}_j, \psi_\beta \mathbf{e}_i)_\Omega + \mathcal{A}_{PQ} a(\psi_\alpha \mathbf{e}_j, \psi_\beta \mathbf{e}_i) \right] = \\ & - \sum_{P=0}^{N-1} \sum_{j=1}^{n_{dof}} \sum_{\alpha \in \eta_{gi}} \widehat{g}_{Pj\alpha} \left[\mathcal{F}_{PQ}(\rho \psi_\alpha \mathbf{e}_j, \psi_\beta \mathbf{e}_i)_\Omega + \mathcal{A}_{PQ} a(\psi_\alpha \mathbf{e}_j, \psi_\beta \mathbf{e}_i) \right] \\ & + \sum_{P=0}^{N-1} \mathcal{A}_{PQ} \left[(\widehat{\mathbf{h}}_P, \psi_\beta \mathbf{e}_i)_{\Gamma_h} + (\rho \widehat{\mathbf{f}}, \psi_\beta \mathbf{e}_i)_\Omega \right] + \mathcal{A}_{0Q} a(\mathbf{u}^0, \psi_\beta \mathbf{e}_i) \\ & + \mathcal{E}_Q a(\mathbf{v}^0, \psi_\beta \mathbf{e}_i), \quad (i = 1, \dots, n_{dof}), \quad (Q = 0, \dots, N-1), \quad (\beta \in \eta - \eta_{gi}), \end{aligned} \quad (5.46)$$

in which symbols $a(\mathbf{u}, \mathbf{w})$ and $(\mathbf{u}, \mathbf{w})_\Omega$, and $(\mathbf{u}, \mathbf{w})_\Gamma$ denote [37]:

$$\begin{aligned} a(\mathbf{u}, \mathbf{w}) &= \int_{\Omega} C_{ijkl} u_{i,j} w_{k,l} dV, \\ (\mathbf{u}, \mathbf{w})_\Omega &= \int_{\Omega} u_i w_i dV, \\ (\mathbf{u}, \mathbf{w})_\Gamma &= \sum_{i=1}^{n_{dof}} \int_{\Gamma_{hi}} u_i w_i d\Gamma, \end{aligned} \quad (5.47)$$

for every vectors $\mathbf{u}$ and $\mathbf{w}$. n_{dof} indicates number of degree of freedom which could be $n_{dof} = 1, 2, 3$ depending on whether the problem is, respectively, $1d$, $2d$ or $3d$. Now, analogous to the classical finite element method, one can write (6.35) in the matrix form:

$$\sum_{P=0}^{N-1} (F_{PQ}[\mathbf{M}] + A_{PQ}[\mathbf{K}])[\mathbf{c}]_P = [\mathbf{f}_{eq}]_Q \quad Q = 0, \dots, N-1, \quad (5.48)$$

in which $[\mathbf{f}_{eq}]_Q$ represents the matrix form of the right hand side of (6.35); $[\mathbf{M}]$ and $[\mathbf{K}]$ stands for mass and stiffness matrices defined in the classical FEM (see[37]). Solving the set of algebraic equations (6.37), the displacement field can be obtained from (5.37), (5.38) and (5.42). Having found the displacement field, one can attain the velocity and acceleration by taking the solution's

time-wise derivative once and twice, respectively. Clearly, (6.37)) is a set of algebraic equation whose size is N times of the similar elastostatics one. However, as shall be seen, these equations inherit a proper property, avoiding direct calculation of inverse matrix. This fact leads us to propose an alternative algorithm for obtaining a numerical solution to elastodynamics problems.

5.5 Properties of the method

In this section, we discuss some properties of the above method. In this regard, let us first elaborate on a different approach to derivation (6.37). This manner of presentation paves the way to propose a new and simple algorithm for the elastodynamics problems. By discretizing the problem (spatial discretization), the equations of motion are reduced to a set of time-wise ODEs:

$$[\mathbf{M}] [\ddot{\mathbf{U}}] + [\mathbf{K}] [\mathbf{U}] = [\mathbf{f}], \quad (5.49)$$

with initial conditions:

$$[\mathbf{U}(0)] = [\mathbf{U}_0], [\dot{\mathbf{U}}(0)] = [\dot{\mathbf{U}}_0]. \quad (5.50)$$

(6.38) and (5.50) can alternatively be written as follows:

$$[\mathbf{M}] [\mathbf{U}(t)] = -[\mathbf{K}] t * [\mathbf{U}(t)] + t * [\mathbf{f}(t)] + [\mathbf{M}] [\mathbf{U}_0] + [\mathbf{M}] [\dot{\mathbf{U}}_0] \quad (5.51)$$

or

$$[\mathbf{M}] [\mathbf{V}(t)] = -[\mathbf{K}] t * [\mathbf{V}(t)] + t * [\mathbf{f}(t)] - [\mathbf{K}] [\mathbf{U}_0] t * 1 - [\mathbf{K}] [\dot{\mathbf{U}}_0] t * t, \quad (5.52)$$

where

$$[\mathbf{U}(t)] = [\mathbf{V}(t)] + [\mathbf{U}_0] + t [\dot{\mathbf{U}}_0]. \quad (5.53)$$

Now, by expanding $[\mathbf{V}(t)]$ in terms of the Fourier-Cosine series—i.e., $[\mathbf{V}(t)] = \sum_{i=0}^{N-1} [\mathbf{V}_i] \cos \frac{i\pi t}{T}$ —substituting into (6.41), making use of the orthogonality of Fourier-Cosine in $[0, T]$, we end up with the set of algebraic equations (6.37). In a similar fashion, one can develop the method for a more

general case:

$$[\mathbf{M}] [\ddot{\mathbf{U}}(t)] + [\mathbf{C}] [\dot{\mathbf{U}}(t)] + [\mathbf{K}] [\mathbf{U}(t)] = [\mathbf{f}(t)], \quad (5.54)$$

that can be rearranged as:

$$[\mathbf{M}] [\mathbf{U}(t)] = -[\mathbf{K}] t * [\mathbf{U}(t)] - [\mathbf{C}] 1 * [\mathbf{U}(t)] + t * [\mathbf{f}(t)] + t [\mathbf{C}] [\mathbf{U}_0] + [\mathbf{M}] [\mathbf{U}_0] + t [\mathbf{M}] [\dot{\mathbf{U}}_0]. \quad (5.55)$$

Analogous to governing equations of mass-spring system (6.38), (5.55) can be written as:

$$[\mathbf{M}] [\mathbf{V}(t)] = -[\mathbf{K}] t * [\mathbf{V}(t)] - [\mathbf{C}] 1 * [\mathbf{V}(t)] + t * [\mathbf{f}(t)] - [\mathbf{K}] [\mathbf{U}_0] t * 1 - [\mathbf{K}] [\dot{\mathbf{U}}_0] t * t - [\mathbf{C}] [\dot{\mathbf{U}}_0] t * 1, \quad (5.56)$$

leading to the algebraic equations:

$$\sum_{P=0}^{N-1} (\mathcal{F}_{PQ}[\mathbf{M}] + \mathcal{A}_{PQ}[\mathbf{K}] + \mathcal{B}_{PQ}[\mathbf{C}])[\mathbf{c}]_P = [\mathbf{f}_{eq}]_Q, \quad Q = 0, \dots, N-1, \quad (5.57)$$

where $[\mathbf{f}_{eq}]_Q$ represents the contribution of the body forces and the initial conditions.

Moreover, the reason behind choosing Fourier-Cosine expansion rather than other the Fourier forms can be explained as follows. By considering (5.45), it is noted that given matrices $[\mathcal{F}]$ and $[\mathcal{A}]$ are, respectively, diagonal and arrowhead matrices. As a consequence, the matrix representation of the left hand side of (6.37) is a block arrowhead matrix:

$$\begin{bmatrix} \mathcal{F}_{00}[\mathbf{M}] + \mathcal{A}_{00}[\mathbf{K}] & \mathcal{A}_{01}[\mathbf{K}] & \dots & \mathcal{A}_{0(N-1)}[\mathbf{K}] \\ \mathcal{A}_{01}[\mathbf{K}] & \mathcal{F}_{11}[\mathbf{M}] + \mathcal{A}_{11}[\mathbf{K}] & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{A}_{0m}[\mathbf{K}] & \mathbf{0} & \dots & \mathcal{F}_{mm}[\mathbf{M}] + \mathcal{A}_{mm}[\mathbf{K}] & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots & \ddots & & \end{bmatrix} \begin{bmatrix} [\mathbf{c}]_0 \\ [\mathbf{c}]_1 \\ \vdots \\ [\mathbf{c}]_m \\ \vdots \end{bmatrix}, \quad (5.58)$$

which considerably reduces the computational costs. That is, instead of directly inverting the left-hand-side matrix, one can straightforwardly obtain $[\mathbf{c}]_0$ by writing all unknown arrays $[\mathbf{c}]_m$ $m = 1, \dots, N-1$ in terms of $[\mathbf{c}]_0$. This procedure is handled efficiently in parallel programming as

the operations are mostly independent. This is a significant step in which, without relaxing the time-wise continuity condition, yet the solution procedure can be performed in parallel. Hence, given N , to solve (6.37), the main computation is $2N - 1$ matrix inversions and $5N - 4$ matrix multiplications whose sizes are equal to those of elastostatic problems. After some manipulations, one can observe that other Fourier series forms do not inherit that appropriate property. It is worth mentioning that this is not the case for the general form (5.54).

Also, it is interesting to have an insight concerning appropriate number of time-wise terms N . In this regard, for simplicity, let us consider (6.38) and (5.50) in the absence of $[\mathbf{f}]$. The solution of the system of the ODEs (6.38) can then be written as:

$$[\mathbf{U}] = \sum_{i=1}^n (A_i \sin(\omega_i t) + B_i \cos(\omega_i t)) [\Phi]_i, \quad (5.59)$$

in which $[\Phi]_i$'s and ω_i 's are, respectively, eigenvectors and eigenvalues of the generalized eigenvalue problem:

$$[\mathbf{K}] [\Phi]_i = \omega_i^2 [\mathbf{M}] [\Phi]_i, i = 1, \dots, n_{dof}. \quad (5.60)$$

Now (6.42) results in:

$$[\mathbf{V}] = \sum_{i=1}^{n_{dof}} (A_i \sin(\omega_i t) + B_i \cos(\omega_i t)) [\Phi]_i - [\mathbf{U}_0] - t [\dot{\mathbf{U}}_0]. \quad (5.61)$$

As a consequence, to capture the solution (5.59), the present method has to be converged to the Fourier cosine expansion of (5.61). A straightforward calculation gives:

$$\begin{aligned} S_{j \neq 0}^{sin} &= \begin{cases} \frac{2T\omega_i(-1+(-1)^j \cos(T\omega_i))}{j^2\pi^2 - T^2\omega_i^2} \\ 0 \end{cases} & \begin{aligned} \omega_i &\neq \frac{j\pi}{T} \\ \omega_i &= \frac{j\pi}{T} \end{aligned}, \quad S_{j=0}^{sin} = \frac{1 - \cos(T\omega_i)}{T\omega_i}, \\ S_{j \neq 0}^{cos} &= \begin{cases} -\frac{2(-1)^j T\omega_i \sin(T\omega_i)}{j^2\pi^2 - T^2\omega_i^2} \\ 1 \end{cases} & \begin{aligned} \omega_i &\neq \frac{j\pi}{T} \\ \omega_i &= \frac{n\pi}{T} \end{aligned}, \quad S_{j=0}^{cos} = \frac{\sin(T\omega_i)}{T\omega_i}, \end{aligned} \quad (5.62)$$

$$S_j^t = \begin{cases} \frac{2T(-1+(-1)^j)}{j^2\pi^2} & j \neq 0 \\ \frac{T}{2} & j = 0 \end{cases}, \quad S_j^1 = \begin{cases} 0 & j \neq 0 \\ 1 & j = 0 \end{cases} \quad j = 1, \dots, N-1,$$

in which S_j^{sin} , S_j^{cos} , S_j^t , and S_j^1 are, respectively, the Fourier cosine coefficients of $\sin(\omega_i t)$, $\cos(\omega_i t)$, t , and 1. Apparently, S_j^{sin} and S_j^{cos} indicate that to capture each vibration mode, the Fourier expansion should contain terms with $j \in (\frac{T\omega_i}{\pi} - T\lambda_i, \frac{T\omega_i}{\pi} + T\lambda_i)$ where λ_i depends on ω_i . Denoting $\omega_{\max}$ and $\lambda_{\max}$ as the highest frequency and the corresponding λ , we therefore have $N \geq \frac{T\omega_{\max}}{\pi} + \lambda_{\max}$. Also, S_j^t shows that small values of j is necessary for capturing the effect of initial conditions.

Furthermore, in contradistinction to the time integration methods, Lemma 8 implies that the present approach converges at almost all time-instances, provided that a sufficient number of terms is taken. Apparently the present form of the velocity at the time $t = T$, which is obtained by calculating the time-wise derivative of (5.37), does not converge to the solution regardless of number of terms as $\dot{\mathcal{V}}^h(\mathbf{x}, T) = 0$. This shortcoming may result in undershooting or overshooting of the velocity field at times close to $t = T$. One way to address that issue is the substitution of the resulting solution from (5.57) (or (6.37) with $[\mathbf{C}] = [\mathbf{0}]$) into (5.56):

$$\begin{aligned} [\mathbf{U}(t)] &= -[\mathbf{M}]^{-1} [\mathbf{K}] \left(\sum_{i=1}^{N-1} [\mathbf{V}]_i \frac{T^2(1 - \cos(\frac{i\pi t}{T}))}{i^2\pi^2} + ([\mathbf{V}]_0 + [\mathbf{U}_0]) \frac{t^2}{2} + [\dot{\mathbf{U}}_0] \frac{t^3}{6} \right) \\ &\quad + [\mathbf{M}]^{-1} (-[\mathbf{C}] \left(\sum_{i=1}^{N-1} [\mathbf{V}]_i \frac{T \sin(\frac{i\pi t}{T})}{i\pi} + [\mathbf{V}]_0 t + [\dot{\mathbf{U}}_0] \frac{t^2}{2} \right) + t * [\mathbf{f}(t)]) + [\mathbf{U}_0] + t [\dot{\mathbf{U}}_0], \\ [\dot{\mathbf{U}}(t)] &= -[\mathbf{M}]^{-1} [\mathbf{K}] \left(\sum_{i=1}^{N-1} [\mathbf{V}]_i \frac{T \sin \frac{i\pi t}{T}}{i\pi} + ([\mathbf{V}]_0 + [\mathbf{U}_0])t + [\dot{\mathbf{U}}_0] \frac{t^2}{2} \right) \\ &\quad + [\mathbf{M}]^{-1} (-[\mathbf{C}] \left(\sum_{i=0}^{N-1} [\mathbf{V}]_i \cos\left(\frac{i\pi t}{T}\right) + [\dot{\mathbf{U}}_0] t + 1 * [\mathbf{f}(t)] \right) + [\dot{\mathbf{U}}_0], \end{aligned} \quad (5.63)$$

resulting

$$\begin{aligned} [\mathbf{U}(T)] &= -[\mathbf{M}]^{-1} [\mathbf{K}] \left(\sum_{i=0}^{N-1} \mathcal{D}_i [\mathbf{V}]_i + \mathcal{D}_0 [\mathbf{U}_0] + \mathcal{A}_{00} [\dot{\mathbf{U}}_0] \right) \\ &\quad + [\mathbf{M}]^{-1} (-[\mathbf{C}] (\mathcal{C}_0 [\mathbf{V}]_0 + \mathcal{D}_0 [\dot{\mathbf{U}}_0]) + t * [\mathbf{f}(t)]|_T) + [\mathbf{U}_0] + \mathcal{C}_0 [\dot{\mathbf{U}}_0], \end{aligned} \quad (5.64)$$

$$[\mathbf{U}(0)] = [\mathbf{U}_0],$$

and

$$\begin{aligned}
[\dot{\mathbf{U}}(T)] &= -[\mathbf{M}]^{-1} [\mathbf{K}] (\mathcal{C}_0[\mathbf{V}]_0 + \mathcal{C}_0[\mathbf{U}_0] + \mathcal{D}_0[\dot{\mathbf{U}}_0]) \\
&\quad + [\mathbf{M}]^{-1} (-[\mathbf{C}] (\sum_{i=0}^{N-1} (-1)^i [\mathbf{V}]_i + \mathcal{C}_0[\dot{\mathbf{U}}_0]) + 1 * [\mathbf{f}(t)]|_T) + [\dot{\mathbf{U}}_0], \\
[\dot{\mathbf{U}}(0)] &= [\dot{\mathbf{U}}_0],
\end{aligned} \tag{5.65}$$

where

$$\begin{aligned}
t * [\mathbf{f}(t)]|_T &= \int_0^T (T-t) * [\mathbf{f}(t)] dt, \\
1 * [\mathbf{f}(t)]|_T &= \int_0^T \mathbf{f}(t) dt.
\end{aligned} \tag{5.66}$$

It should be noticed that an insufficient number of time-wise terms in (5.63) amplifies the error for $t \approx T$. Nevertheless, sufficient numbers lead to an accurate solution at all time-instances. In the next section, we show numerically through a mass-spring-damper system that $[\mathbf{U}(t)]$ and $[\dot{\mathbf{U}}(t)]$, $t \in [0, T]$, in (5.63) converge to the exact solution. As can be seen, however, (5.63) contains extra calculations. On the other hand, as demonstrated in the numerical results, the time interval of the error due to non-convergence issue may be reduced by increasing time-wise terms. Hence, in this study, we just numerically show that (5.63) does not contain the above-mentioned issue.

The above solution procedure for elastodynamics problems can be summarized in the following steps:

1. Spatial discretization of the problem.
2. Choose a set of appropriate spatial shape function for a given problem.
3. Assign initial and boundary conditions and body forces in the same fashion calculated in classical FEM.
4. Calculate matrices $[\mathbf{M}]$ and $[\mathbf{K}]$ in the same manner done in classical FEM.
5. Define coefficients given in (5.45).
6. Express $[\mathbf{c}]_P$, $P = 1, \dots, N-1$ in terms of $[\mathbf{c}]_0$ based on (6.50) and (6.37).

7. Obtain $[\mathbf{c}]_0$ from (6.37).
8. Calculate $[\mathbf{c}]_P, P = 1, \dots, N - 1$ from step 6.
9. Compute $\mathcal{V}^h(\mathbf{x}, t)$ from (5.42).
10. Obtain $u_i^h(\mathbf{x}, t)$ from (5.37).

The method can be employed as a time-step algorithm by partitioning the desired total time into several time intervals and applying the present technique consecutively. It has to be mentioned that the term 'time-step' here does not convey the traditional sense of the time step algorithm. In fact, in the standard time step algorithms, the finite difference method is employed, while here, only the total time is split into several shorter time intervals. As a result, the new time-step form has three characteristics: firstly, lower time-wise terms are required for each time step. Secondly, the time-step algorithm is more flexible in choosing the time step, say Δt , as one can choose a larger time step by increasing the number of time-wise terms while maintaining the accuracy. Thirdly, in contrast to traditional time integration methods, it is possible to obtain arbitrary number of solution points at each time step because of the explicit time-wise formulation solution obtained in this method. Hence, in the time-step method, there exist three parameters: 1) number of terms, N 2) length of time step, Δt 3) a parameter, γ , such that $u(\mathbf{x}, \mathcal{S}(1 - \gamma)T), \dot{u}(\mathbf{x}, \mathcal{S}(1 - \gamma)T)$, where $0 < \gamma < 1, T = \frac{\Delta t}{1 - \gamma}$, and $\mathcal{S} \geq 1$, are used as initial values for $(\mathcal{S} + 1)^{th}$ step. As shall be seen in the numerical results, each parameter's value may affect the other one's appropriate choice.

To get a better picture regarding the computational cost, one can compare the computational complexity of algorithms. In this regard, observe that the main computations based on the trapezoidal rule are $\mathcal{M}$ matrix inversions and $3\mathcal{M}$ matrix multiplications, $\mathcal{M} = \frac{T^{total}}{\Delta t}$. Thus, given $n_{eq} \times n_{eq}$ is the size of $[\mathbf{M}]$ and $[\mathbf{K}]$, $\mathcal{K}$ is the number of steps in the present algorithm, the computational complexity for the new approach and the trapezoidal method lies in the range of $\mathcal{O}(\mathcal{K} N n_{eq}^3)$ and $\mathcal{O}(\mathcal{M} n_{eq}^3)$, respectively. Hence, $\mathcal{K} N$ and $\mathcal{M}$ are critical factors in the performance of each method. However, the new approach has two desirable features compared to time integration methods: i) larger time-step is feasible without losing the accuracy ii) computations at each time-step can be

performed in parallel. These features effectively improve the performance of the new method , in particular for large time intervals.

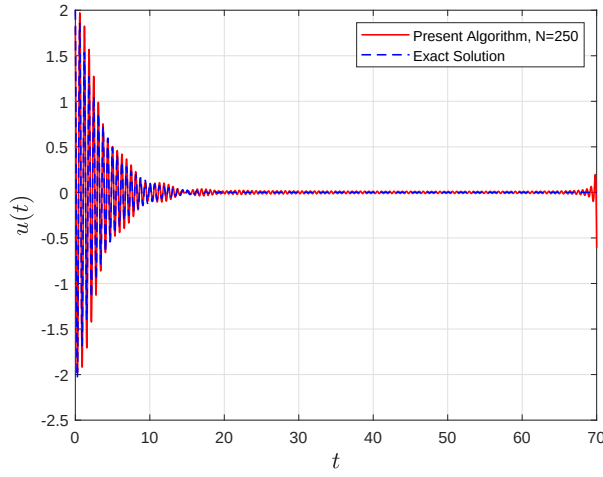
5.6 Numerical Results

Several examples are considered to reveal properties of the above method. In the first example, we analyze a mass-spring-damper single degree of freedom (SDOF) to show that the method successfully works for the case (5.54). The governing equation and characteristics of the problem are:

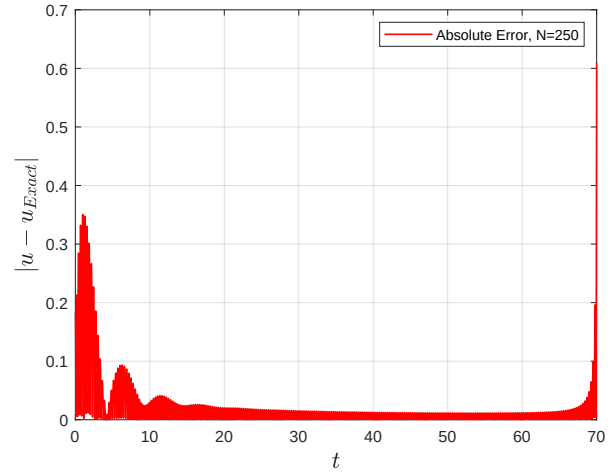
$$\begin{aligned} m\ddot{u}(t) + \zeta\dot{u}(t) + ku(t) &= 0, \\ u(0) &= 2(mm), \\ \dot{u}(0) &= -10(mm/s), \end{aligned} \tag{5.67}$$

with $m = 1Kg$, $\zeta = 0.6 Nmm/s$, and $k = 100N/mm$. Figures 6.2 and 5.2 show, respectively, the comparison of displacements and velocity when $N = 250$. To mitigate the error, one can increase N as shown in Fig. 6.4 and Fig. 5.4. As mentioned earlier, there is an error at $t = 70s$, leading to an overshooting in its neighborhood. In particular, the form (5.42) does not converge to the solution at $t = 70s$; it converges to zeros, and thus, u reaches at $\dot{u}(0)$. As shown in Fig. 6.4 and Fig. 5.4 by increasing the number of time-wise terms, the length of the neighborhood and maximum error can be reduced. As explained earlier, this source of error may be eliminated if, for example, (5.63) is applied. as can be seen, the non-convergence issue has been addressed in Figs. 5.5-5.6. However, Figs. 6.4-5.4 imply that the amount of propagation of that error can become smaller when the number of time-wise terms is increased. Also, one can calculate the result for slightly larger time interval, say $t \in [0, T + \Delta t_0]$, to reduce or eliminate the aforementioned source of error. Hence, unless otherwise stated, the procedure of the calculation given in (5.63) is not employed for the rest of the results.

Next, conservation of the energy is investigated. In this regard, the following example is carried out:

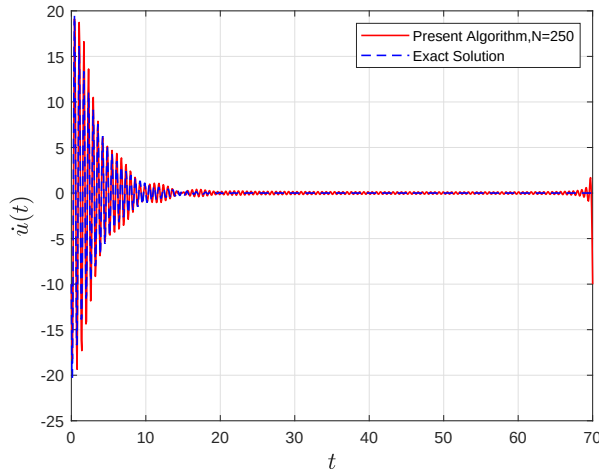


(a) comparison of displacement for a SDOF mass-spring-damper system

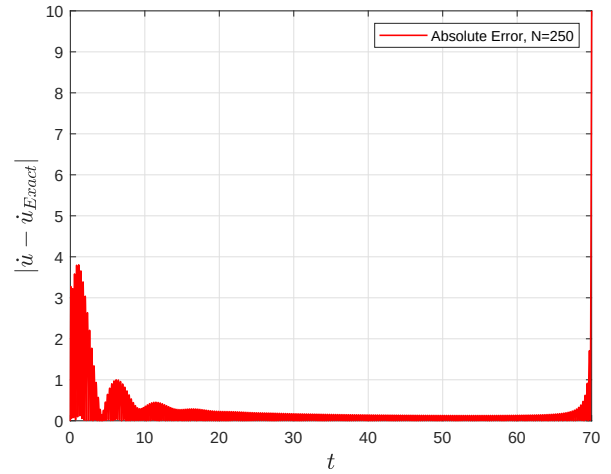


(b) error for a SDOF mass-spring-damper system

Figure 5.1: Displacements for a SDOF mass-spring-damper system ($T = 70s$)

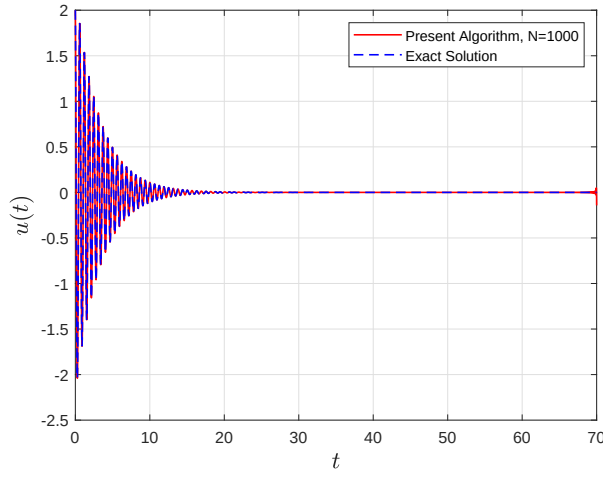


(a) comparison of velocity for a SDOF mass-spring-damper system

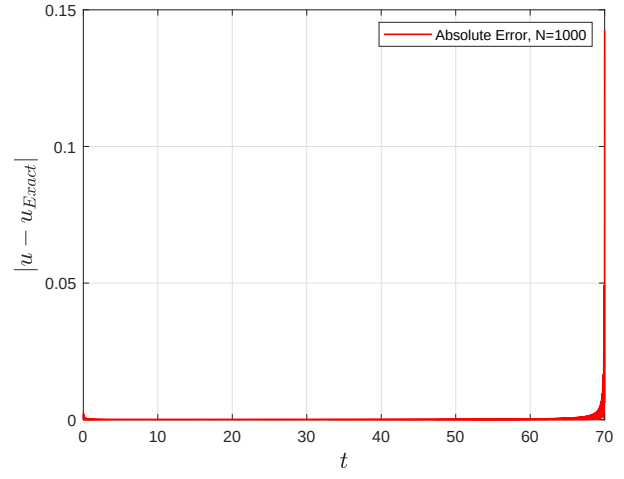


(b) error for a SDOF mass-spring-damper system

Figure 5.2: Velocity for a SDOF mass-spring-damper system ($T = 70s$)

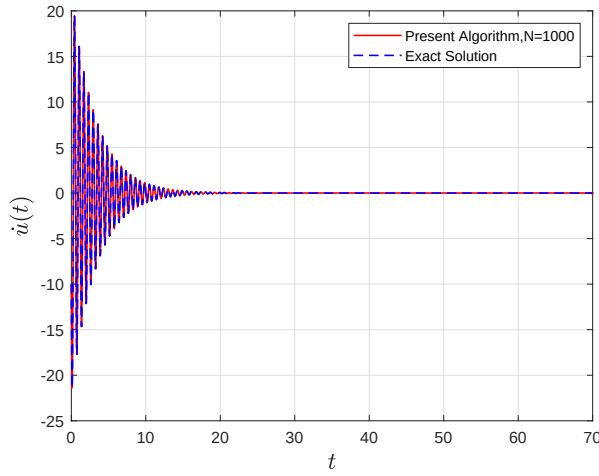


(a) comparison of Displacement for a SDOF mass-spring-damper system

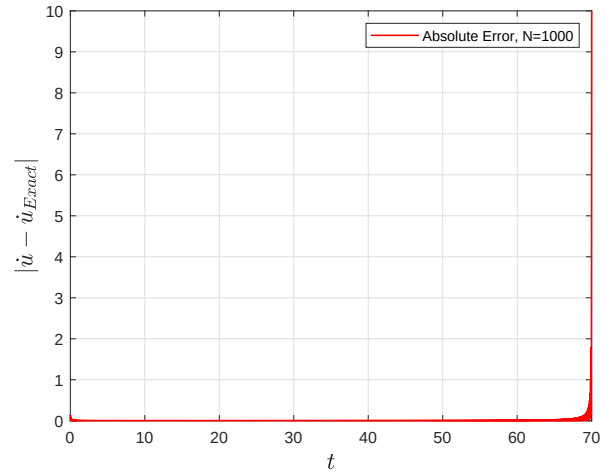


(b) error for SDOF mass-spring-damper system

Figure 5.3: Displacement for SDOF mass-spring-damper system ($T = 70s$)

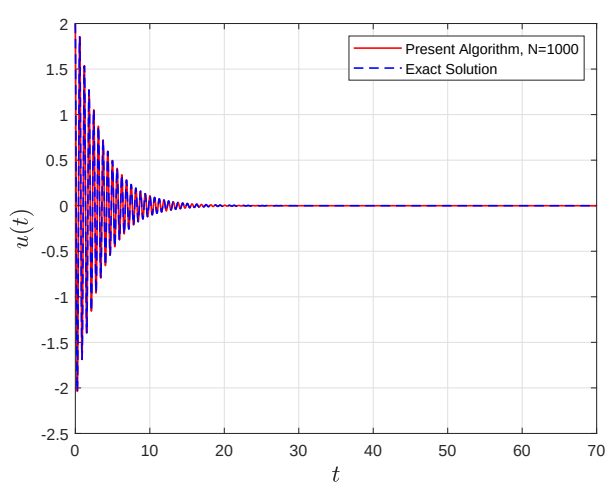


(a) comparison of velocity for SDOF mass-spring-damper system

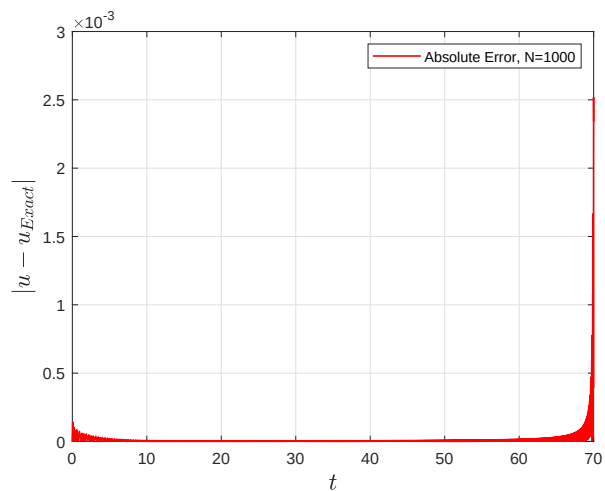


(b) error for SDOF mass-spring-damper system

Figure 5.4: Velocity for SDOF mass-spring-damper system ($T = 70s$)

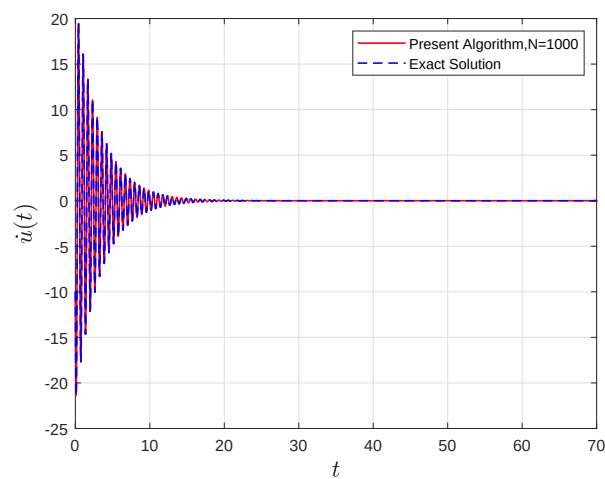


(a) comparison of velocity for SDOF mass-spring-damper system

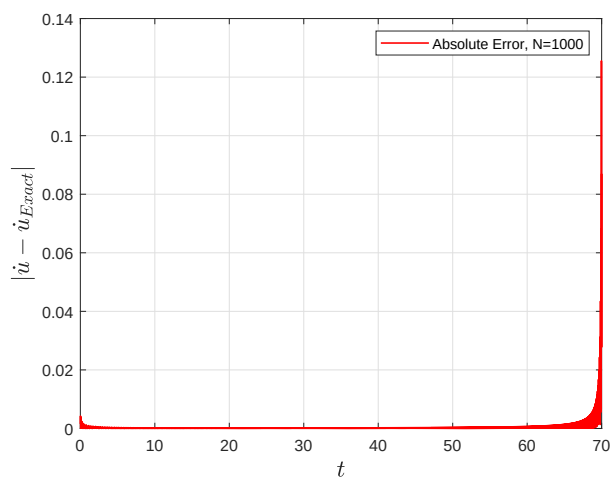


(b) error for SDOF mass-spring-damper system

Figure 5.5: Displacement for SDOF mass-spring-damper system obtained from (5.63) ($T = 70s$)



(a) comparison of velocity for SDOF mass-spring-damper system



(b) error for SDOF mass-spring-damper system

Figure 5.6: Velocity for SDOF mass-spring-damper system obtained from (5.63) ($T = 70s$)

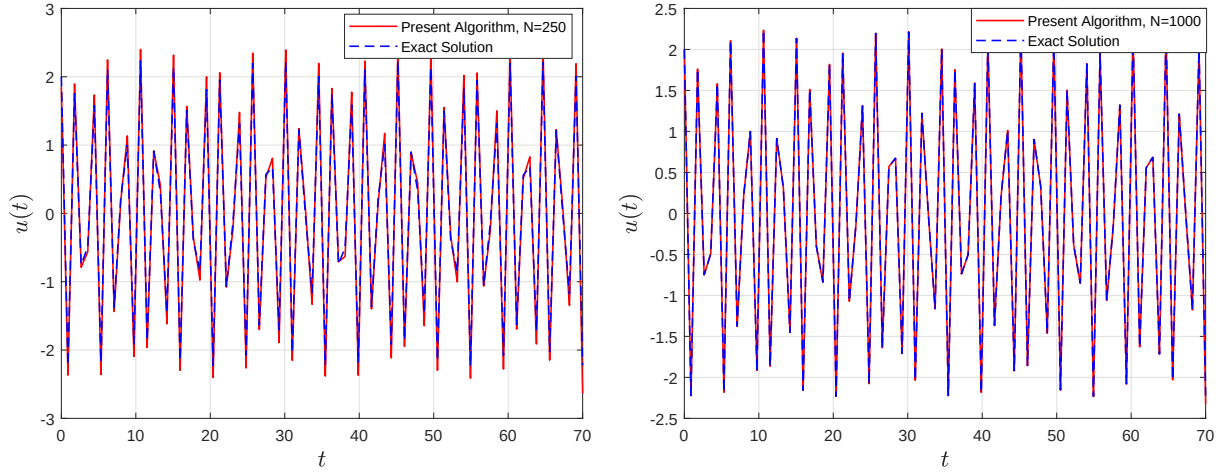


Figure 5.7: Displacement corresponding to the SDOF mass-spring system for $N = 250$ and $N = 1000$ ($T = 70s$)

$$\begin{aligned}
 m\ddot{u}(t) + ku(t) &= 0, \\
 u(0) &= 20(mm), \\
 \dot{u}(0) &= -100(mm/s),
 \end{aligned} \tag{5.68}$$

with $m = 1\text{ kg}$ and $k = 100\text{ N/mm}$. Two cases are considered: 1) $N = 250$ 2) $N = 1000$. To be visible, in Figs. 6.10-6.11, we only show the solution for specific times while the method can accurately obtain the solution for any $t(s) \in [0, 70)$. Based on these figures, to accurately approximate the solution in this example, $N = 250$ is not enough, while the case $N = 1000$ contains adequate terms to capture the exact solution. Moreover, as can be seen in Fig. 6.13, by increasing the number of time-wise terms, i.e., N , the conservation of the energy is satisfied in a better fashion. The distribution of amplitude versus $i = 0, \dots, N - 1$ are shown for the case (5.68) when $N = 100$, $N = 250$, $N = 500$, and $N = 1000$. As shown in Fig. 6.14a, there is no obvious decay in the amplitudes of each term. This is in agreement with the exact solution having a natural frequency $\omega_n = \sqrt{K/M} = 10$. That is, considering $T = 70s$ and $20\cos 10t$, as a part of the exact solution, one can see at least $N = 224$ is required. This fact is manifested in the Fig. 6.14b, in which there is a jump close to $i = 223$. In the Fig. 6.5a and 6.5b, the similar results are shown, respectively, for the case $N = 500$ and $N = 1000$. The details of the amplitude have also been depicted in these figures.

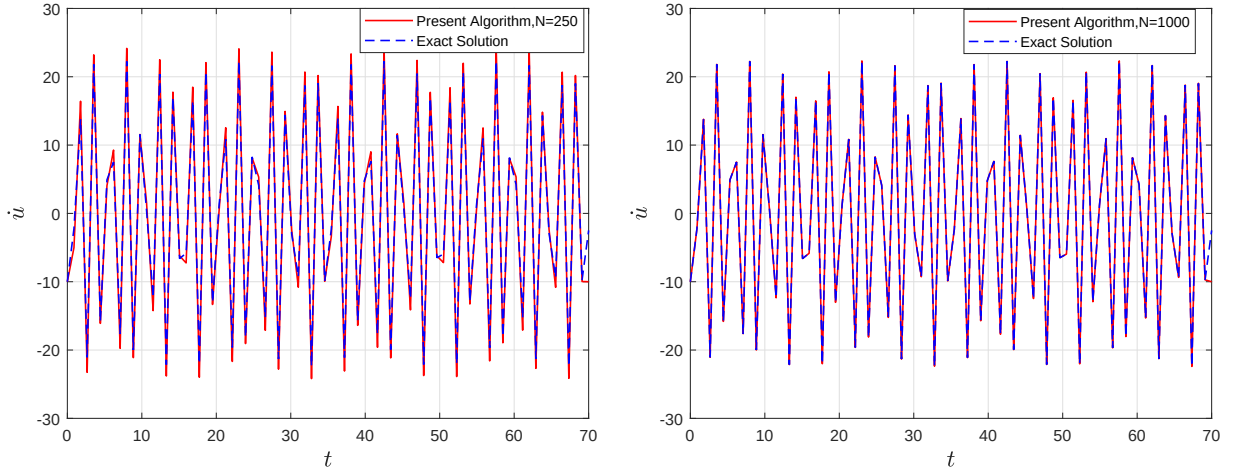


Figure 5.8: Velocity corresponding to the SDOF mass-spring system for $N = 250$ and $N = 1000$ ($T = 70s$)

One can observe that there is a better convergence for the case $N = 1000$ if compared to the case $N = 500$; that is, the order of amplitudes of higher-order terms for the case $N = 1000$ is about 10^{-3} while it is of order 10^{-2} when $N = 500$ (see Fig.6.5).

As the first case of finite element examples, the elastodynamics of the 1d rod is considered. The characteristics of the model have been specified in Fig. 5.12a. Element-wise, three cases are considered. In the first case, the geometry is discretized to two uniform elements. For the second case, on the other hand, ten uniform elements are considered. Eventually, the problem is solved for 300 uniform elements. In all cases, linear Lagrangian shape functions are used. Also, to better compare the convergence and the accuracy of the method, a semi-analytic solution—which can be readily obtained by employing the method of separation of variables—and the results relevant to the Newmark average acceleration method (or to the HHT α method) have been depicted. Although the separation of variables still contains truncation error, the terms 'semi-analytic' and 'analytical' for the relevant solution have been used interchangeably since the method is precise when compared to FEM.

For two uniform elements, a comparison of the present algorithm for different values of time-wise terms with the Newmark approach when $\Delta t = 1$ has been presented in Figs. 5.13-5.15. A comparison of figures indicates that $N = 100$ obtains a good approximation of the solution corresponding to two

elements, while this is not the case when $N = 20$. Growth in displacement and velocity is obvious. This fact may cause error-growth even when N is increased as long as it is below the adequate number. Also, the discrepancy between the present algorithm results and the semi-analytic solution can be attributed to insufficient elements.

Numerical solutions corresponding to ten uniform elements can be found in Fig. 5.16-5.18. It is required to consider more time-wise terms for a greater number of elements, which is a direct consequence of higher vibration modes. One can notice that, for instance, in Fig. 5.16b, the velocity has not been converged at larger times. This lack of convergence can be predicted from Fig. 5.16a in that there is an overshooting in the displacement profile at $t = 60s$ while such errors have been mitigated in cases $N = 500$ and $N = 1000$ (see Figs. 5.16-5.18). As discussed earlier, for ten elements, as an example of inadequate time-wise terms, more overshooting and undershooting can be seen if compared to Fig. 5.13. The present algorithm does not match the semi-analytic solution in the velocity profile. This difference shows that more elements, and more time-wise terms, are required to capture the velocity profile.

This fact can be manifested in Figs. 5.19-5.21 in which 300 uniform elements have been considered. This time $N = 1000$ does not contain enough number of terms (see Fig. 5.19). In Fig. 5.20, present study with $N = 10,000$ and the Newmark average acceleration method when $\Delta t = 0.0075$ have been compared. As shown in Fig. 5.20b, the present method has resulted in greater values in the apexes of the velocity profile than the Newmark method. Nonetheless, decreasing Δt in the next figures reveals that $N = 10,000$ captures precisely the Finite Element approximation. Figure 5.22 pictures displacement and velocity fields across the bar for $N = 14,000$ and 300 uniform linear elements. Also, Fig. 5.23 shows energy conservation for ten and 300 uniform elements. The divergence trend close to $t = 60s$ due to failure in convergence for velocity regardless of N can be seen. However, this problem can be reduced by increasing the number of time-wise terms. For example, as shown in Fig. 5.23a, when $N = 1,000$, the conservation of energy is attained in almost all time instances, while this number of terms is not adequate when the geometry is discretized into 300 elements (see Fig. 5.23b). This property contrasts with the Newmark average acceleration method in which the conservation of energy, in the linear regime, is satisfied identically. Also, it

is noted that decreasing N triggers violation of energy conservation at lower times. This property, i.e., the satisfaction of energy provided that an adequate number of time-wise terms are considered, could be a good indicator for convergence of the method. For instance, an acceptable approximation has been attained in Fig. 5.19 when $0 \leq t \leq 50s$.

Reconsidering Fig. 5.22a, one can observe abrupt changes in the velocity field, leading to a serrated shape that cannot be seen in the analytical solution. A similar circumstance occurs when a piece-wise differentiable function has been expanded in the Fourier series; so-called Gibbs phenomenon [20]. Considering (6.50), one can notice that unknown coefficients form a set of coupled algebraic equations in the present method. In effect, to obtain an accurate expansion, a higher number of time-wise terms needs to be considered if compared to the Fourier-Cosine series of the solution in that each term is obtained independently. One can also see the spurious part of the solution due to the inevitable artificial frequency resulting from the discretization. To address this issue in time integration algorithms, one can point out, for instance, the well-known HHT α method [33] in which the Newmark algorithm has been equipped with an additional α parameter for producing algorithmic damping, discarding spurious part of the solution. To resolve those issues, there exist three options:

Firstly, one can employ various filters to modify the obtained coefficients, increasing the vanishing rate of Fourier series without reducing the accuracy [20]. These filters are applicable for the current method as the unknowns converge to Fourier-Cosine coefficients of the solution. In this study, Fourier space filters are utilized [20]. Given Fourier expansion of a periodic function with the period 2π , say $f(t)$:

$$f_N(t) = \sum_{k=-N}^{k=N} \hat{f}_k e^{i\pi t}, \quad (5.69)$$

one can write [20]:

$$f_N^\beta(t) = \sum_{k=-N}^{k=N} \hat{f}_k \beta_{filter}(k/N) e^{i\pi t}. \quad (5.70)$$

where $\beta_{filter}(k/N)$ smooths $f(t)$ obtained from Fourier expansion. There are several filters; in this

study, the following filters are used [20]:

$$\begin{aligned}
\beta_1^f(\eta) &= \frac{\sin(\pi\eta)}{\pi\eta}, \\
\beta_2^f(\eta) &= \frac{1}{2}(1 + \cos(\pi\eta)), \\
\beta_3^f(\eta) &= \beta_2^{f^4}(35 - 84\beta_2^f + 70\beta_2^{f^2} - 20\beta_2^{f^3}), \\
\beta_4^f(\eta) &= e^{-\alpha\eta^p},
\end{aligned} \tag{5.71}$$

with $0 \leq \eta \leq 1$. β_1^f , β_2^f , β_3^f , and β_4^f are, respectively, the first-order, the second-order, the eighth-order, and the p^{th} -order in the sense of [20]. It is assumed that (5.70) increases the accuracy and converges faster than (5.69) [20]. In the meantime, This operation does not increase the computational cost as it is applied in the post-processing part. Fig. 5.28 shows graphical representation of (5.71).

Secondly, one can partition the desired time domain into several timespans and employ the present method for each time interval. This version of the approach is called the "time-step version of the algorithm." The parameters in this respect are N , T , and γ .

Thirdly, a combination of the approaches mentioned above can be employed. As discussed in the sequel, one can develop a solution procedure for any arbitrary accurate time step.

Among β_i^f , $i = 1, 2, 3, 4$, only β_4^f has parameters to be defined. As mentioned in [20], p is an even number. Also, to eliminate truncation error, α is chosen in such a way that $e^{-\alpha}$ is placed in the range of computers' round-off error [20]. Fig. 5.28, (5.70), and (5.71) show that the choice of the parameter p is projected in the solution for a given N . As a rule of thumb, p should be chosen inversely proportional to the number of time-wise terms; that is, the more time-wise terms are considered, the lesser values of p can be assigned (see Fig. 5.28).

By choosing appropriate β_i^f , $i = 1, \dots, 4$ and N , one can remove higher frequencies while the lower ones change negligibly. Indeed, by considering (5.62), we immediately find out that the contribution due to each modal frequencies in the present approach is mainly projected to the terms whose frequencies are close to the modal frequency. This fact implies that the contribution of the higher modes can be excluded by employing those filters. This property is the same as the desired

controllable algorithmic damping in time integration methods, e.g., HHT α method.

The rod example with the characteristics given in Fig. 5.13a is again considered to show the effect of filtering. In doing so, (5.71)₄ with the parameters $\alpha = 700$ and different values of p are utilized. The results for $T = 60s$ have been shown in Figs. 5.24-5.26. To compare, in Fig. 5.24, the solution corresponding to HHT α method has also been shown. These figures indicate that employing filters may result in smoother solutions, particularly in the velocity profile.

As the first example of the time-step version of the algorithm, the displacement and velocity profile, with $T = 1s$, have been illustrated in Fig. 5.27. As can be observed, applying the time-step version of the algorithm furnished with a filter may exclude the noisy (spurious) part of the velocity response, whereas the displacement field almost remains unchanged. It is noticed that in Figs. 5.25b and 5.26b, when N is a large number, the exponential filter with $p = 2$ has been employed while, for the time-step version of the algorithm, β_1^f has been utilized.

The solution in the range of $t(s) \in [0, 2, 000]$ for various filters, along with those of the HHT α method ($\Delta t = 0.1s$ and $\Delta t = 0.01s$) and the analytical solution, has been depicted in Figs. 5.29-5.30. Since the span of solution is large, the results only for $t = 0, 20, 40, \dots$ have been reported. The reduction in the amplitude of displacement for all filters and HHT α methods is apparent. Strong damping in the solution corresponding to β_2^f is observable, indicating that $N = 100$ does not contain adequate terms for $p = 2$, and lower modes have been affected. This negative effect has been gradually weakened for other filters. Fig. 5.31 shows the distribution of filters over the Fourier cosine expansion. It is clear from the figure that the decaying rate for β_4^f , $p = 2$ is fast if compared to the others. In other words, the choice of $N = 100$, $T = 10s$, $\alpha = 700$, and $p = 2$ obtains poor solution when natural frequencies are approximately greater than two. However, in this example, a better approximation has been obtained for larger values of p . On the other hand, Fig. 5.24 manifests that the case $N = 10,000$, $T = 60s$, $\alpha = 700$, and $p = 2$, in contrast to the Newmark average acceleration, could capture the solution with less spurious part. These outcomes imply that a family of algorithms mitigating spurious solutions is defined with the parameters N , T , γ , α , and p . Also, through the HHT α method, good results have been obtained when $\Delta t = 0.01$, while for the case $\Delta t = 0.1s$, the period error has occurred. A close approximation from the present algorithm to that

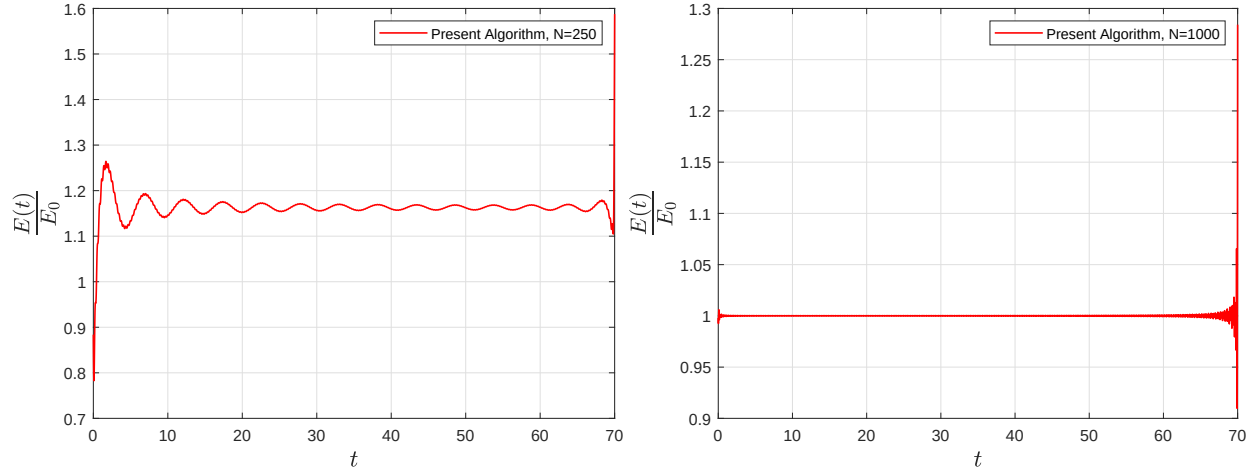
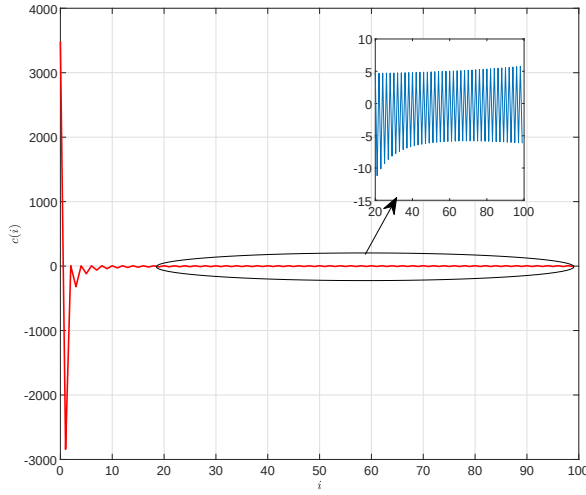


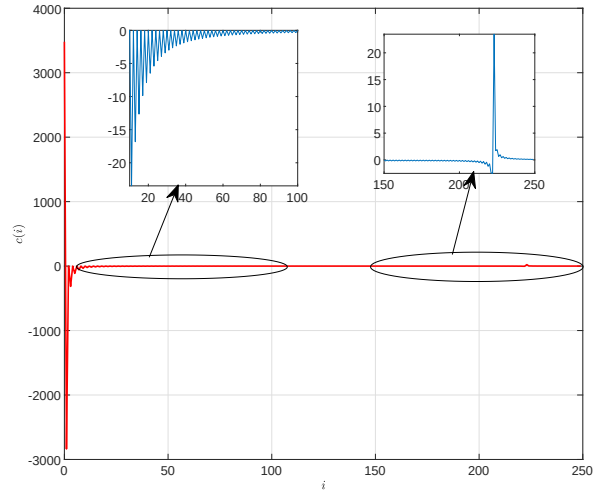
Figure 5.9: Energy versus time corresponding to SDOF mass-spring system for $N = 250$ and $N = 1,000$ ($T = 70s$)

of HHT α ($\Delta t = 0.01s$) is observable. However, with the present algorithm, T (which is equivalent to Δt in time integration methods) can be increased without loss of accuracy, leading to considerable computational cost savings.

It should be pointed out that without the use of filters, the case $N = 100$ for both $T = 1s$ and $T = 10s$ blows up, showing a faster convergence of the method when filters are employed. However, this does not mean the current approach is conditionally stable in a sense utilized for time integration methods. Indeed, as explained earlier, provided that adequate numbers of time-wise terms N is taken (see lemma 2), the time-step version of the algorithm can be stable and convergent for arbitrary T . Finally, a plane strain problem with the characteristics given in Fig.5.30 has been considered. The number of time-wise terms is $N = 1000$, and 25×25 uniform quadrilateral elements with linear Lagrange shape functions are employed. The results corresponding to displacement, velocity, and stress fields have been obtained both with and without filters. In this respect, filter $(5.71)_4$ with the parameters $\alpha = 700$ and $p = 4$ has been utilized. Analogous to 1d example, it is observable that employing filters does not affect the results of the displacement field, whereas detaching noisy parts of both velocity and stress fields. Fig. 5.35a shows the energy when different filters are applied, implying filters may lead to better fulfillment of the energy conservation. Similar results have also been obtained corresponding to the time-step version of the algorithm.

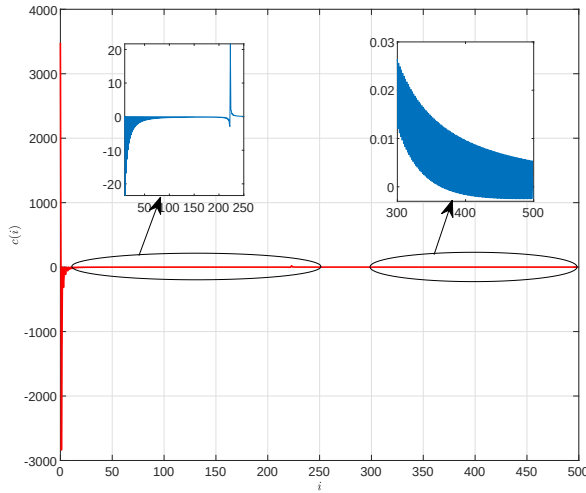


(a) $N = 100$

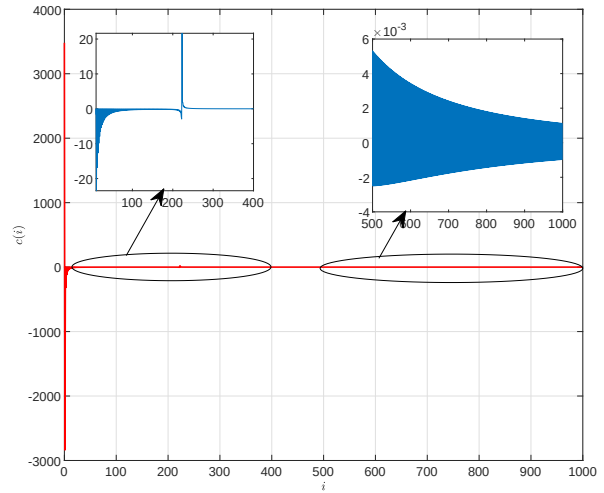


(b) $N = 250$

Figure 5.10: Amplitudes of time-wise terms corresponding to SDOF mass-spring system for $N = 100$ and $N = 250$ ($T = 70s$)

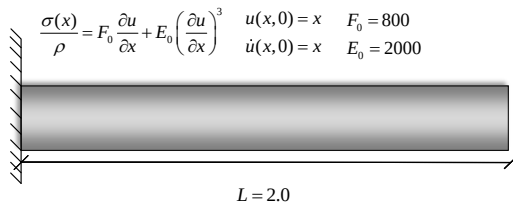


(a) $N = 500$

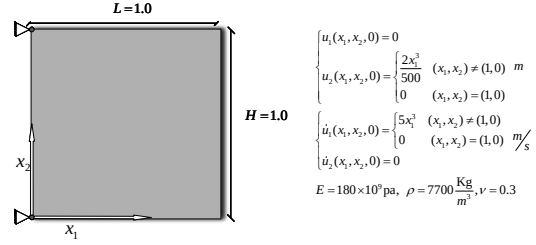


(b) $N = 10,000$

Figure 5.11: Amplitudes of time-wise terms corresponding to SDOF mass-spring system for $N = 500$ and $N = 1,000$ ($T = 70s$)

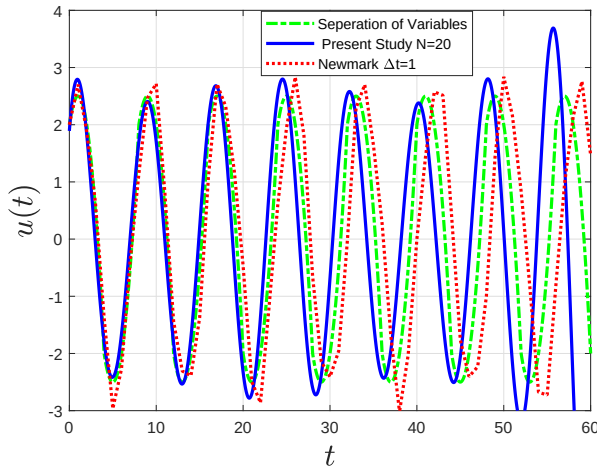


(a) Characteristics of bar example

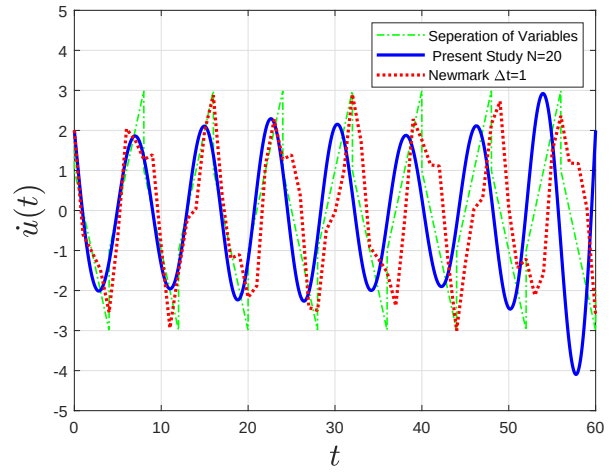


(b) Plane strain problem

Figure 5.12: Characteristics of 1d and 2d examples

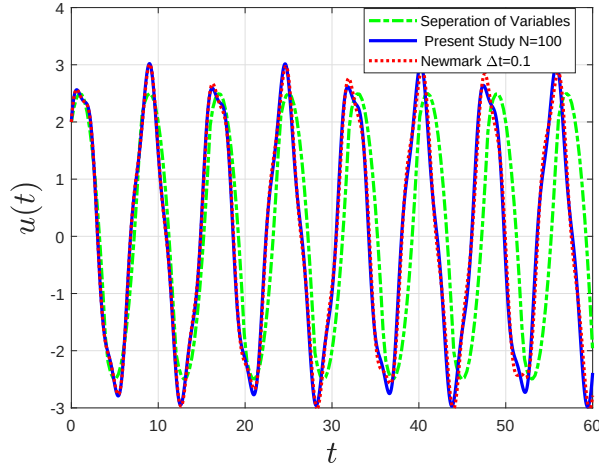


(a) displacement at $x = L$

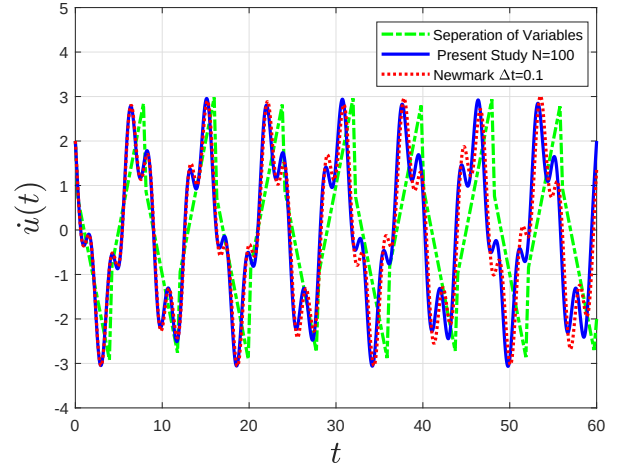


(b) velocity at $x = L$

Figure 5.13: Comparison of present algorithm with Newmark average acceleration method for 2 uniform linear elements ($T = 60s$)

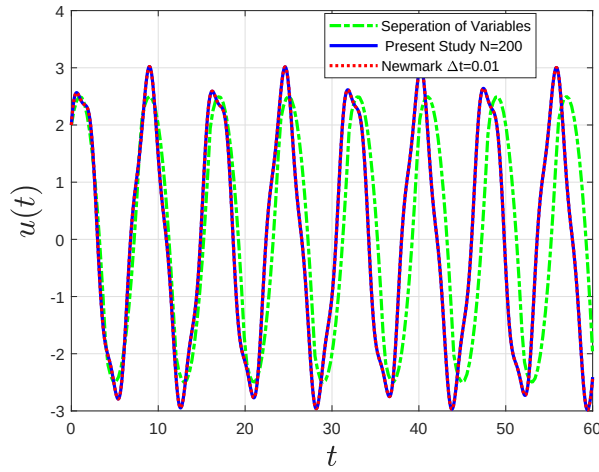


(a) displacement at $x = L$

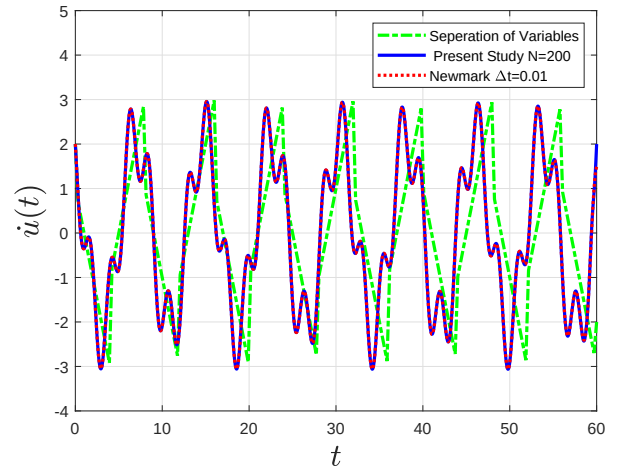


(b) velocity at $x = L$

Figure 5.14: Comparison of present algorithm with Newmark average acceleration method for 2 uniform linear elements ($T = 60s$)

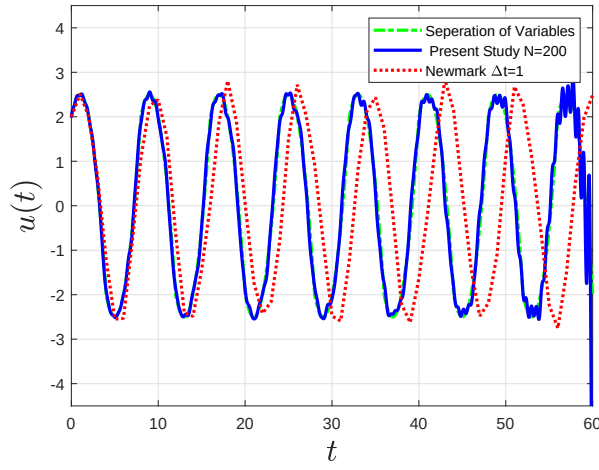


(a) displacement at $x = L$

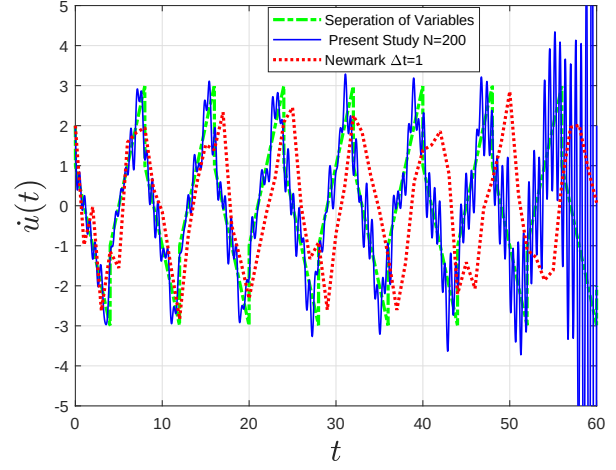


(b) velocity at $x = L$

Figure 5.15: Comparison of present algorithm with Newmark average acceleration method for 2 uniform linear elements ($T = 60s$)

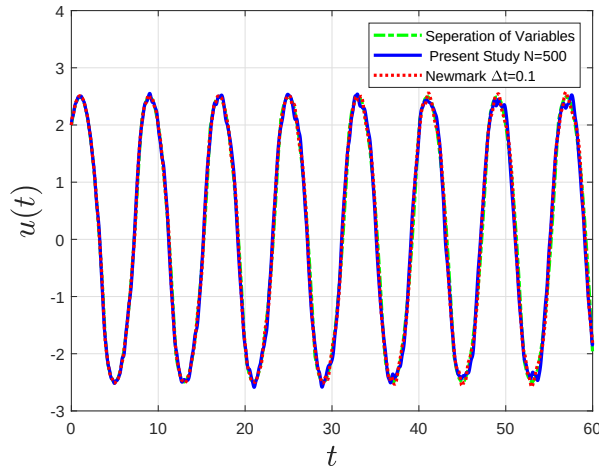


(a) displacement at $x = L$

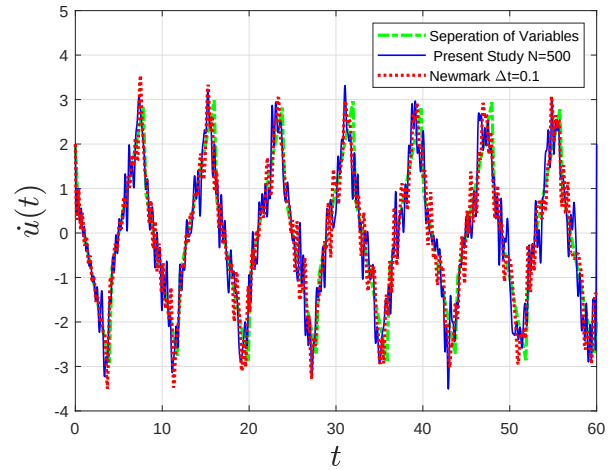


(b) velocity at $x = L$

Figure 5.16: Comparison of present algorithm with Newmark average acceleration method for 10 uniform linear elements ($T = 60s$)

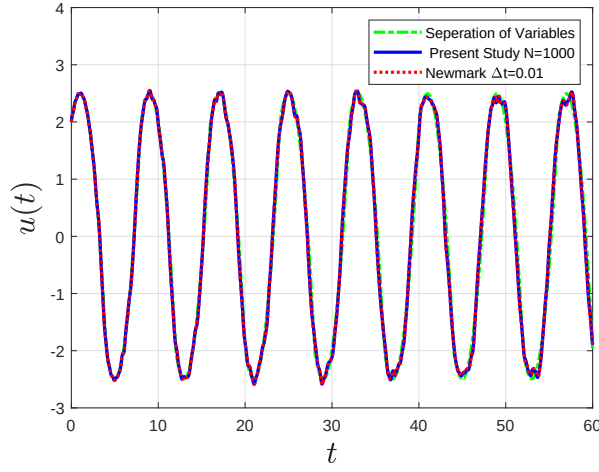


(a) displacement at $x = L$

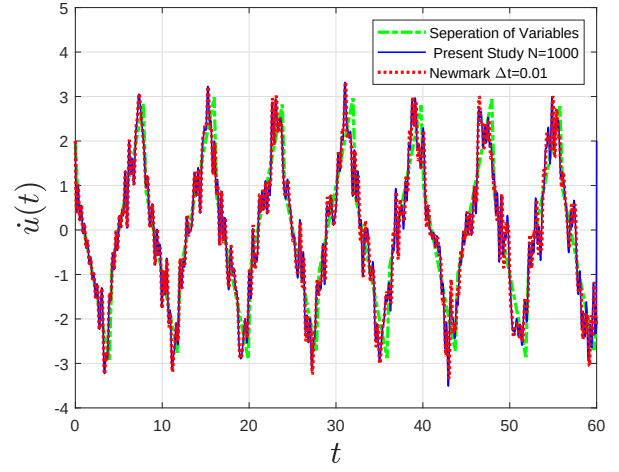


(b) velocity at $x = L$

Figure 5.17: Comparison of present algorithm with Newmark average acceleration method for 10 uniform linear elements ($T = 60s$)

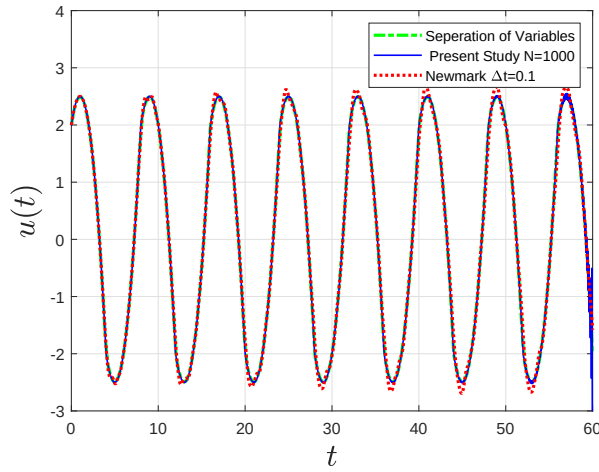


(a) displacement at $x = L$

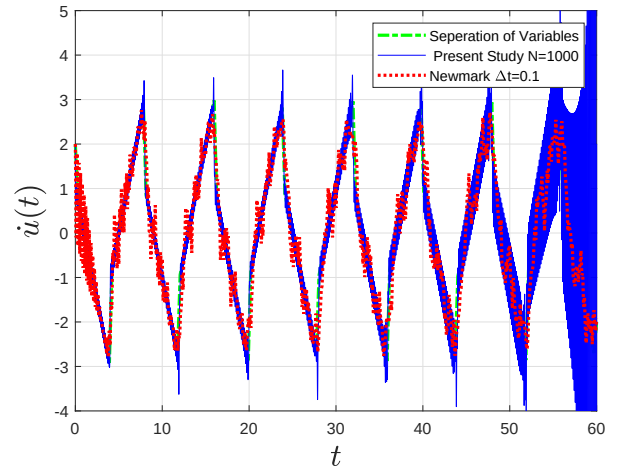


(b) velocity at $x = L$

Figure 5.18: Comparison of present algorithm with Newmark average acceleration method for 10 uniform linear elements ($T = 60s$)

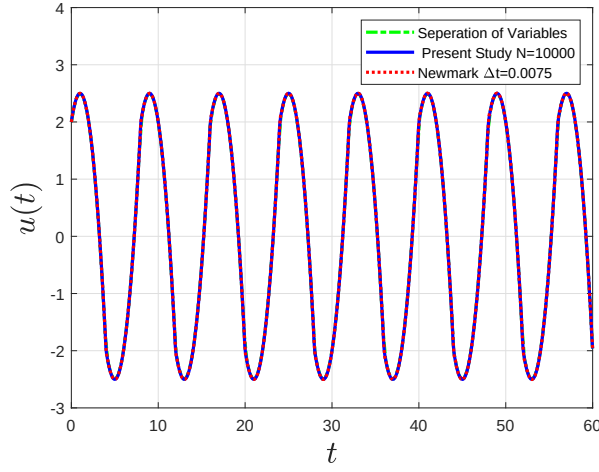


(a) displacement at $x = L$

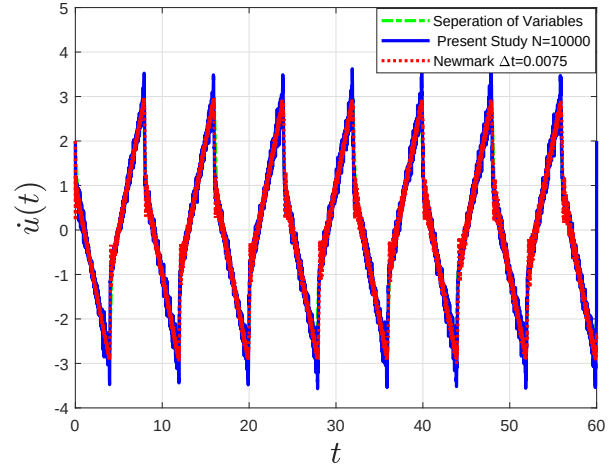


(b) velocity at $x = L$

Figure 5.19: Comparison of present algorithm with Newmark average acceleration method for 300 uniform linear elements ($T = 60s$)

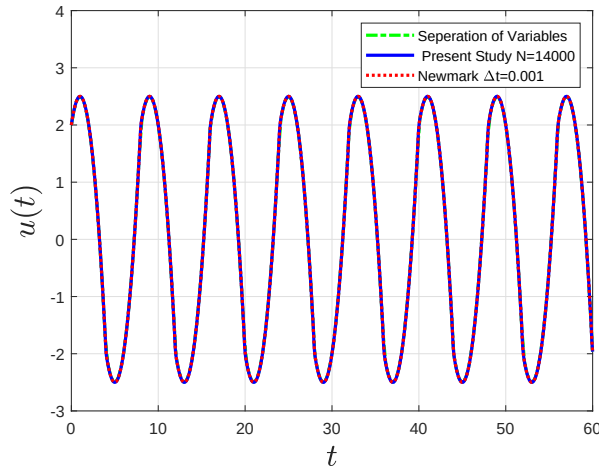


(a) displacement at $x = L$

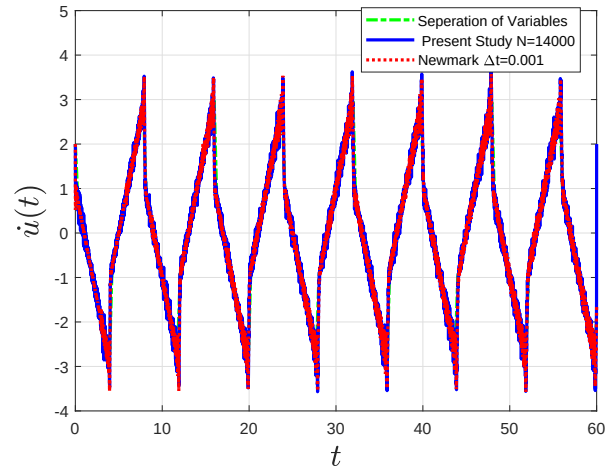


(b) velocity at $x = L$

Figure 5.20: Comparison of present algorithm with Newmark average acceleration method for 300 uniform linear elements ($T = 60s$)



(a) displacement at $x = L$



(b) velocity at $x = L$

Figure 5.21: Comparison of present algorithm with Newmark average acceleration method for 300 uniform linear elements ($T = 60s$)

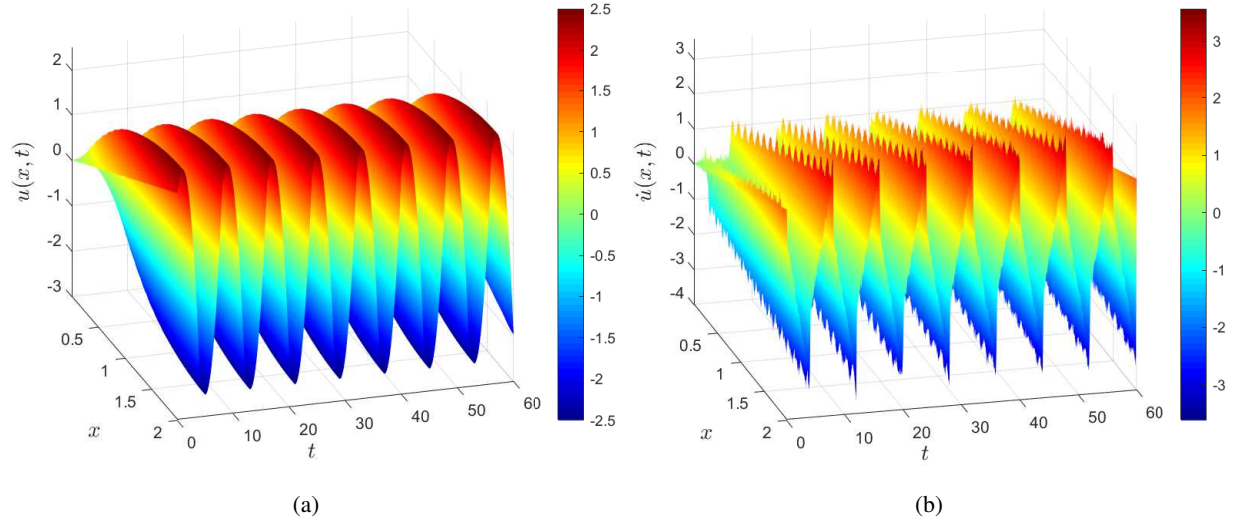


Figure 5.22: Displacement and velocity fields of the bar for $N = 14,000$ ($T = 60s$)

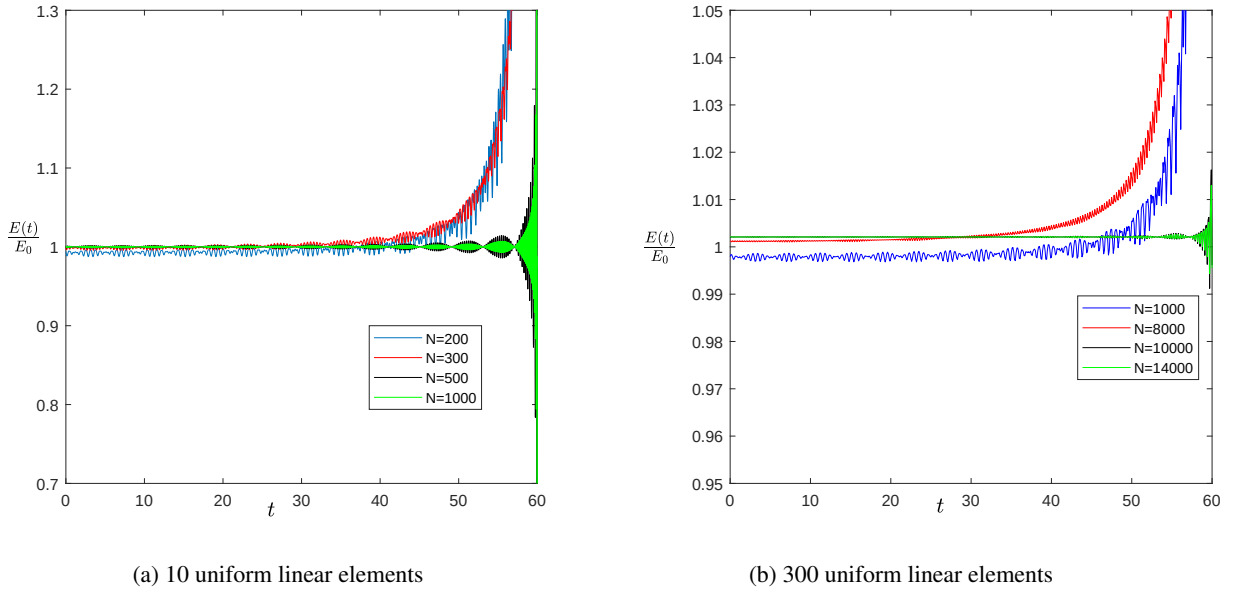
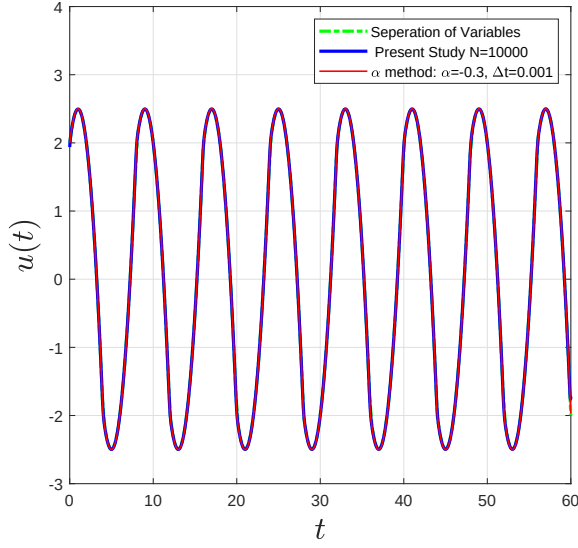
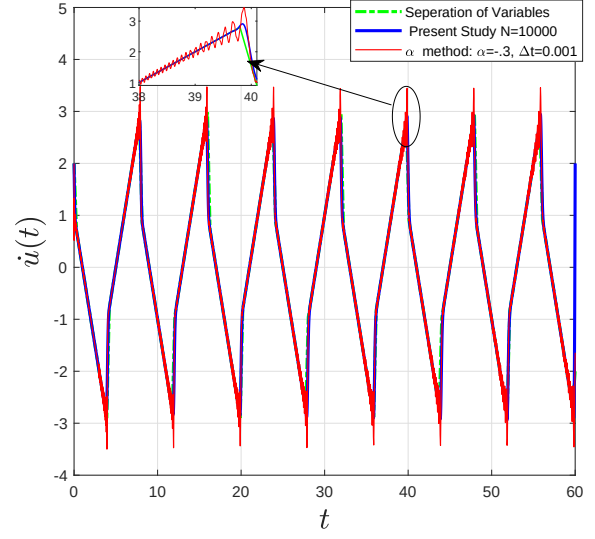


Figure 5.23: Energy versus time obtained from present method for different values of N ($T = 60s$)

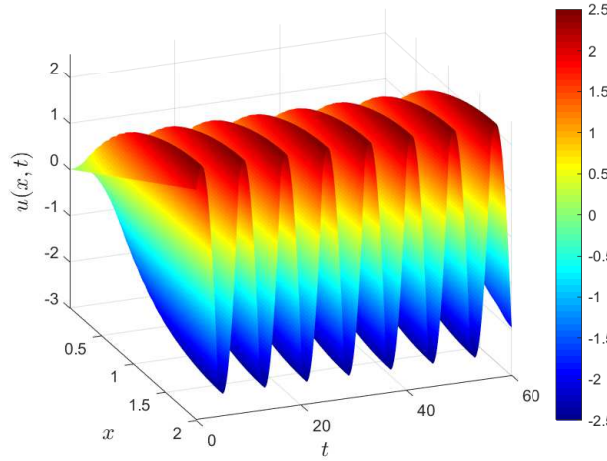


(a) displacement at $x = L$

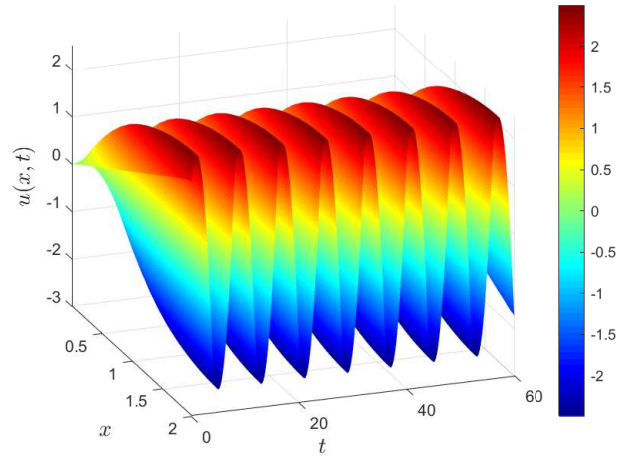


(b) velocity profile at $x = L$

Figure 5.24: Present algorithm (with filter $\beta_4^f(\eta) = e^{-700(\frac{\eta}{N-1})^2}$ and $(T = 60s)$) and HHT α method for 300 uniform elements and separation of variables

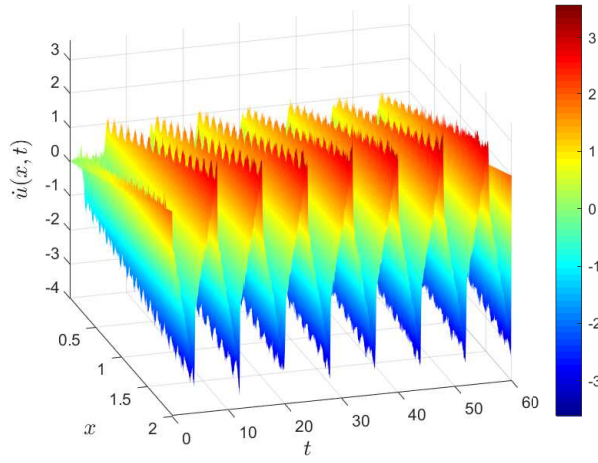


(a) displacement profile without filter

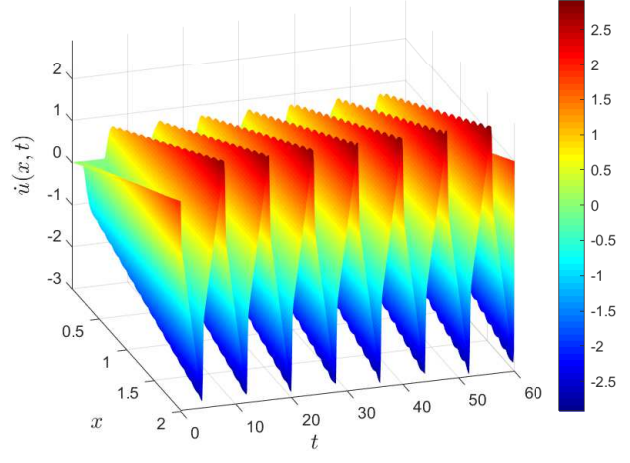


(b) displacement profile with filter $\beta_4^f(\eta) = e^{-700(\frac{\eta}{N-1})^2}$

Figure 5.25: Present algorithm for 300 uniform elements, $N = 10,000$ ($T = 60s$)

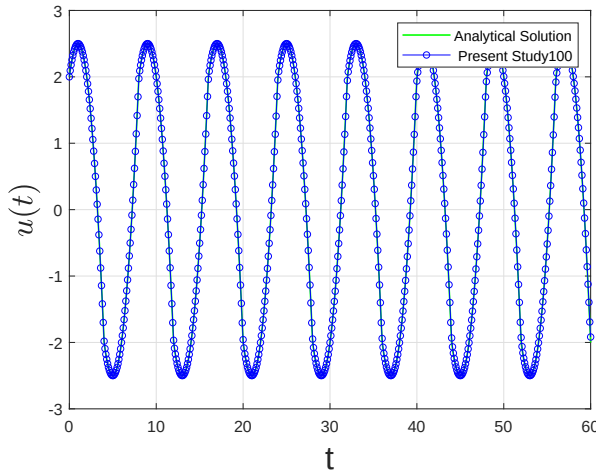


(a) velocity profile without filter

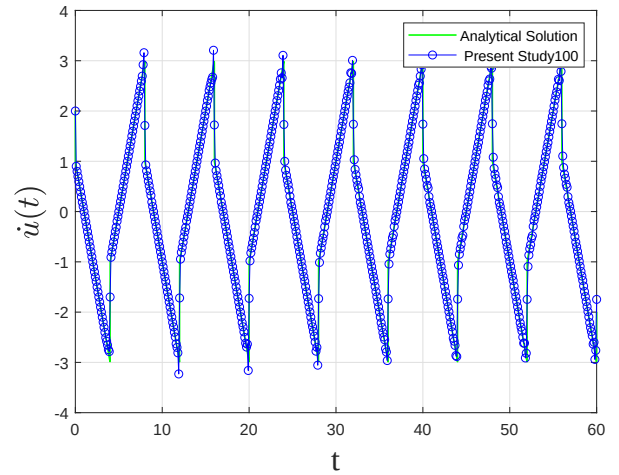


(b) velocity profile with filter $\beta_4^f(\eta) = e^{-700(\frac{\eta}{N-1})^2}$

Figure 5.26: Present algorithm for 300 uniform elements, $N = 10,000$ ($T = 60s$)



(a) displacement at $x = L$ with filter $\beta_1^f(\eta)$, $N = 100$



(b) velocity at $x = L$ with filter $\beta_1^f(\eta)$, $N = 100$

Figure 5.27: Time step version of the algorithm for 300 uniform elements with $T = 1s$, $\gamma = 0.2$

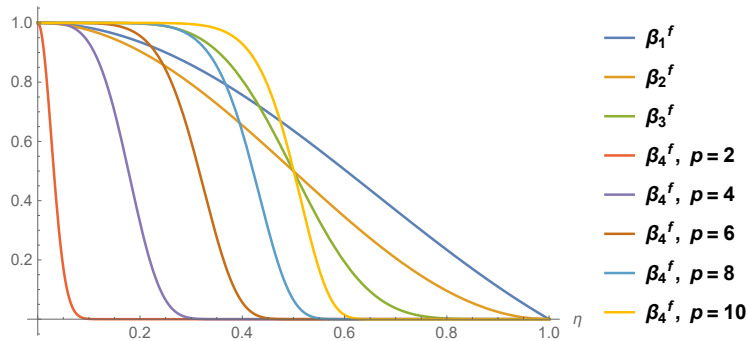
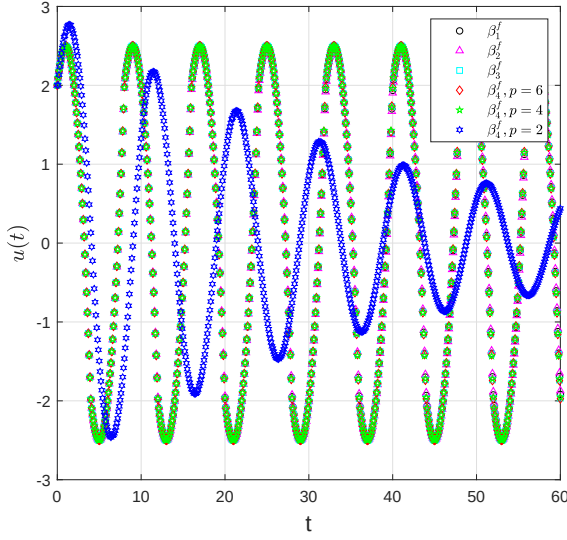
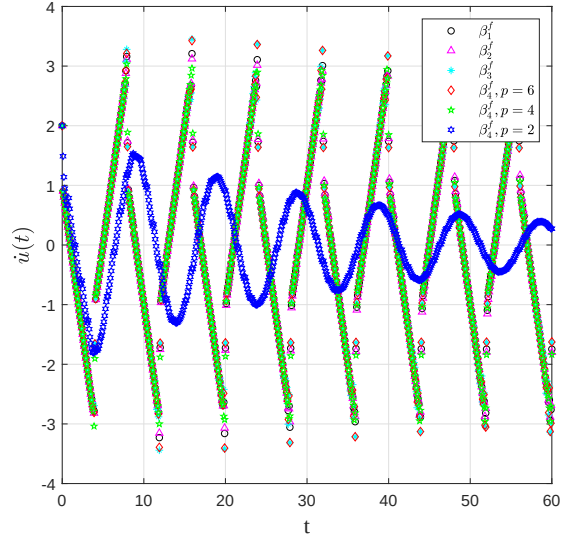


Figure 5.28: Graphical representation of the filters used in this study

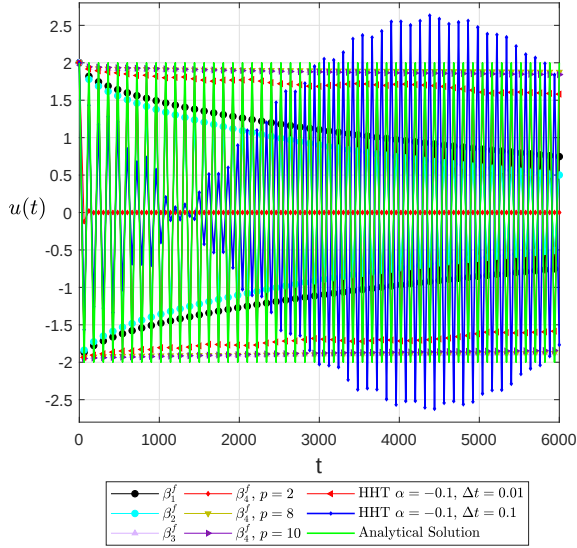


(a) displacement at $x = L$

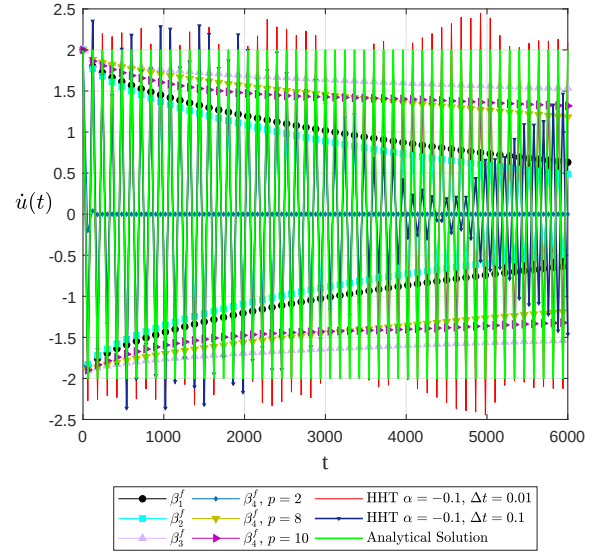


(b) velocity at $x = L$

Figure 5.29: Time step version of the algorithm for 300 uniform elements with $T = 1s$, $N = 100$, $\gamma = 0.2$

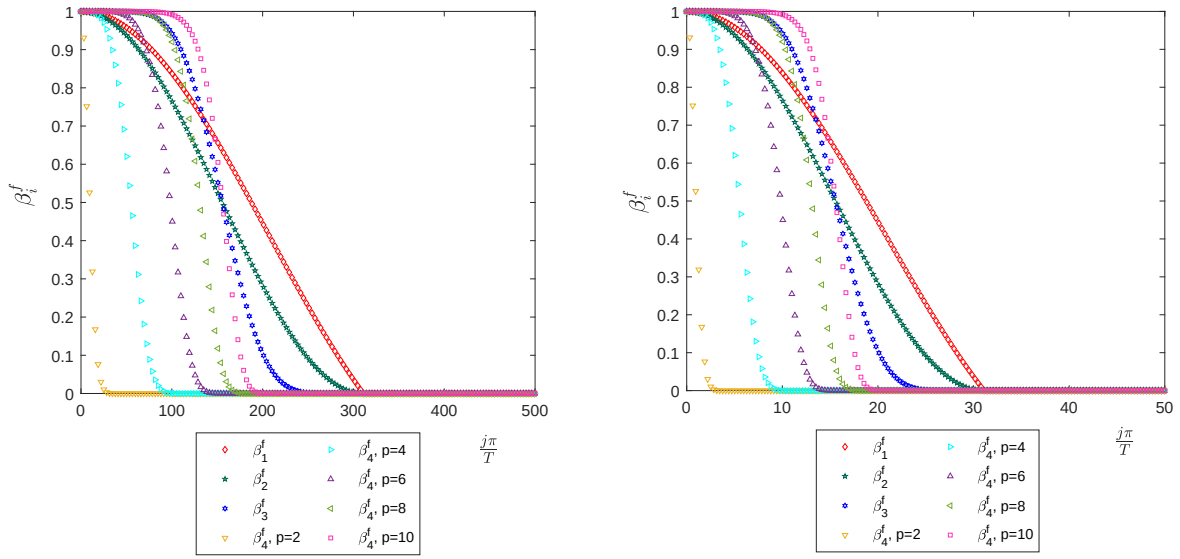


(a) displacement at $x = L$



(b) velocity at $x = L$

Figure 5.30: Time step version of the algorithm for 300 uniform elements with $T = 10s$, $N = 100$, $\gamma = 0.2$



(a) distribution of the filters over the Fourier cosine expansion when $N = 100$, $T = 10$

(b) distribution of the filters over the Fourier cosine expansion when $N = 100$, $T = 1$

Figure 5.31: Distribution of filters

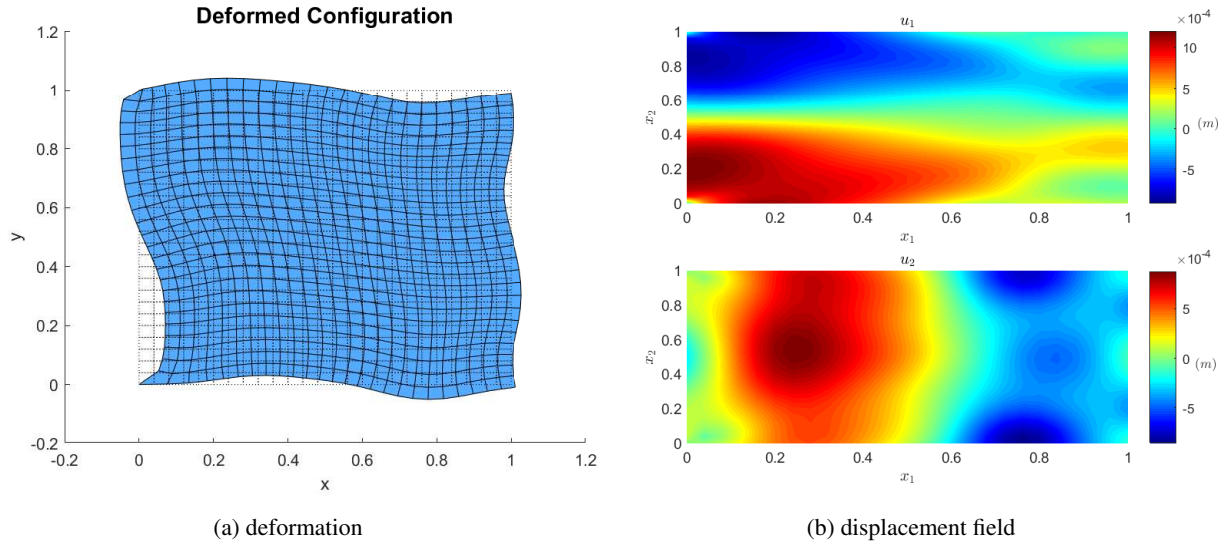


Figure 5.32: Plane strain problem: $t = 0.005991984s$ with filter $\beta_4(\eta) = e^{-700(\frac{\eta}{N-1})^4}$, $N = 1101$

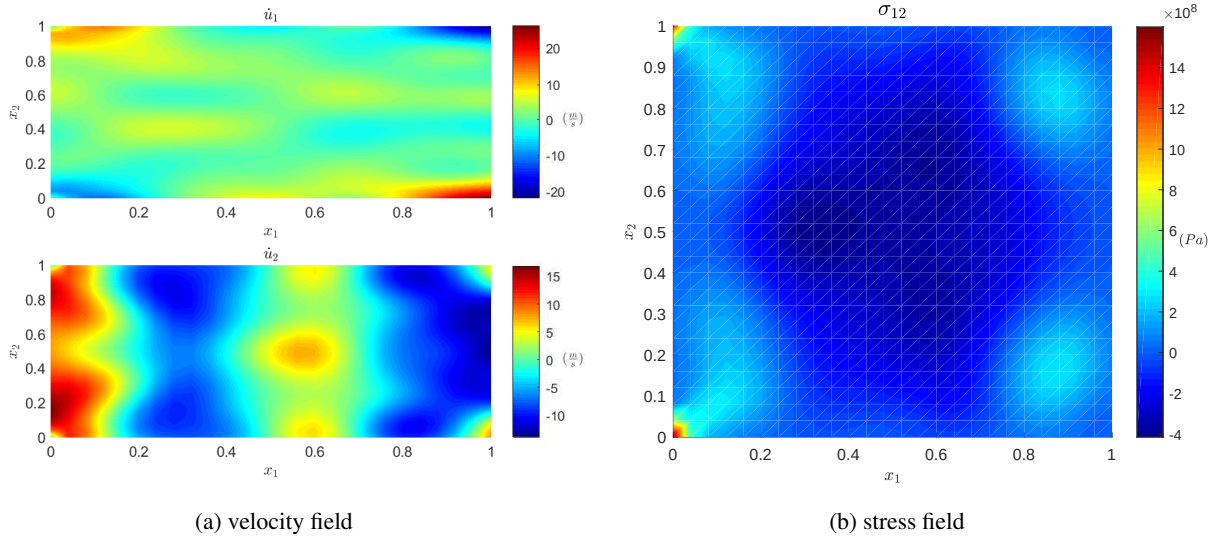


Figure 5.33: Plane strain problem: $t = 0.005991984s$ with filter $\beta_4(\eta) = e^{-700(\frac{\eta}{N-1})^4}$, $N = 1101$

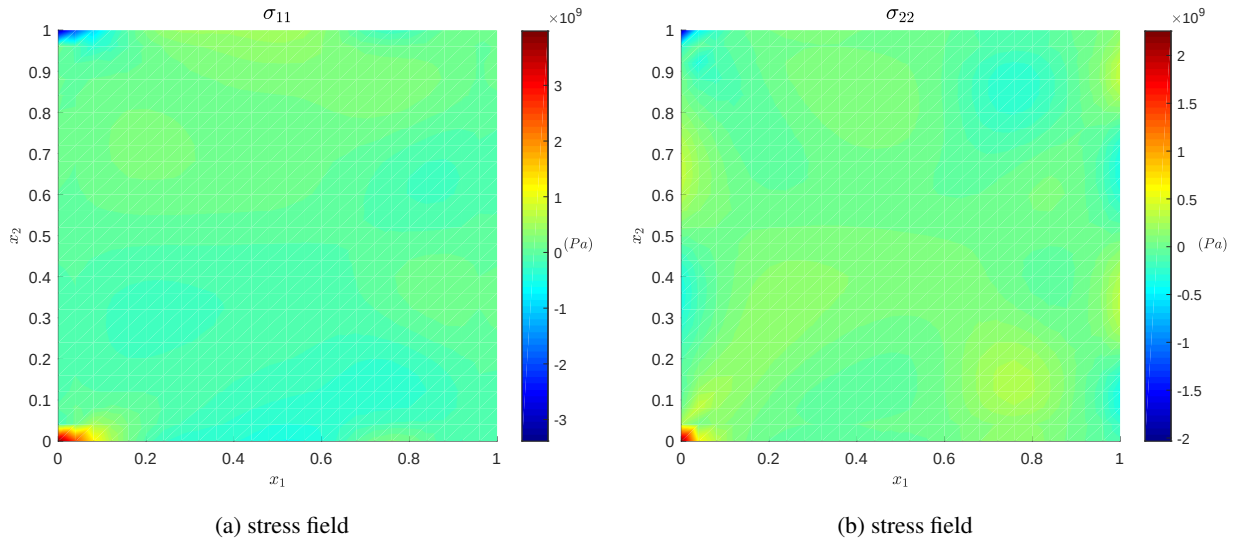
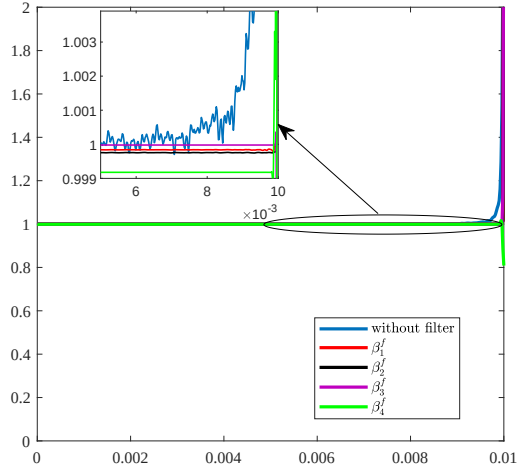
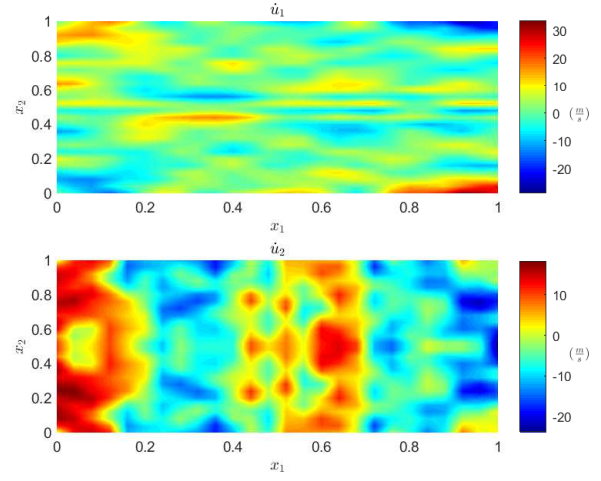


Figure 5.34: Plane strain problem: $t = 0.005991984s$ with filter $\beta_4(\eta) = e^{-700(\frac{\eta}{N-1})^4}$, $N = 1101$

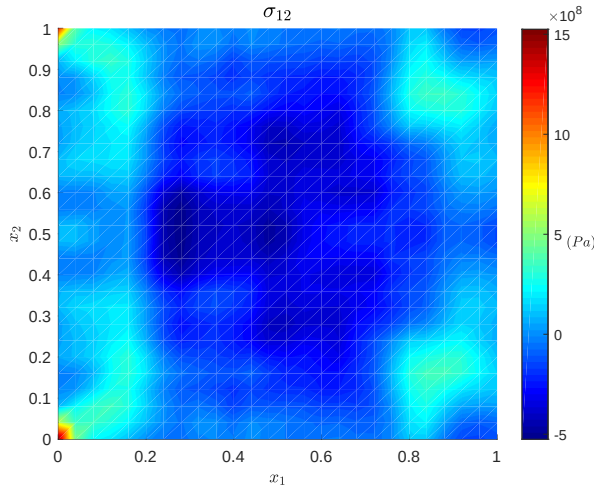


(a) energy versus time for filters given in (5.71) , $N = 1101$

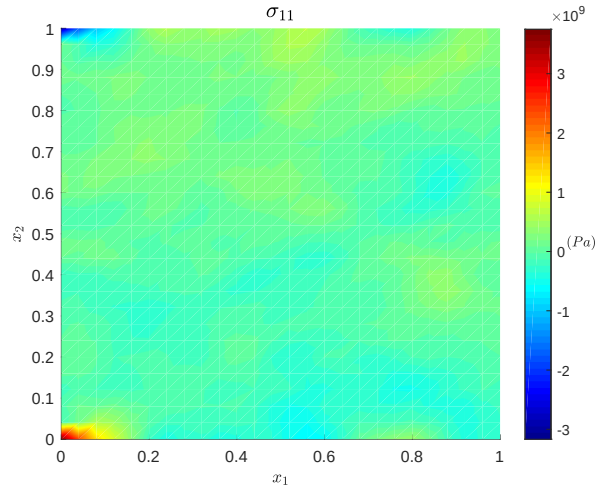


(b) velocity field: $t = 0.005991984s$ without filter, $N = 1101$

Figure 5.35: Plane strain problem



(a) stress field



(b) stress field

Figure 5.36: Plane strain problem: $t = 0.005991984s$ without filter, $N = 1101$

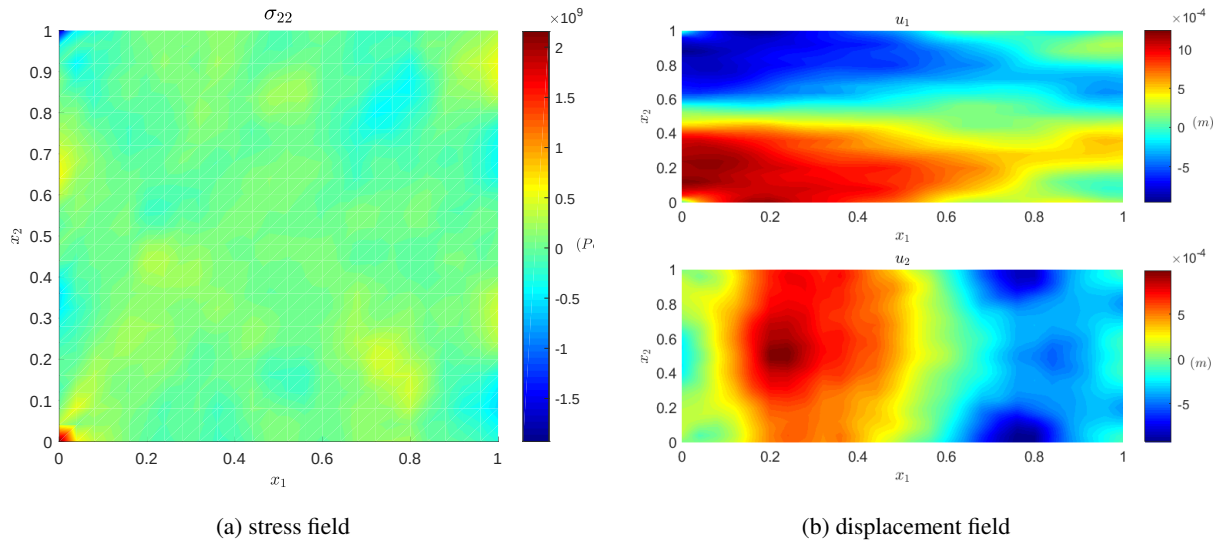


Figure 5.37: Plane strain problem: $t = 0.005991984s$ without filter, $N = 1101$

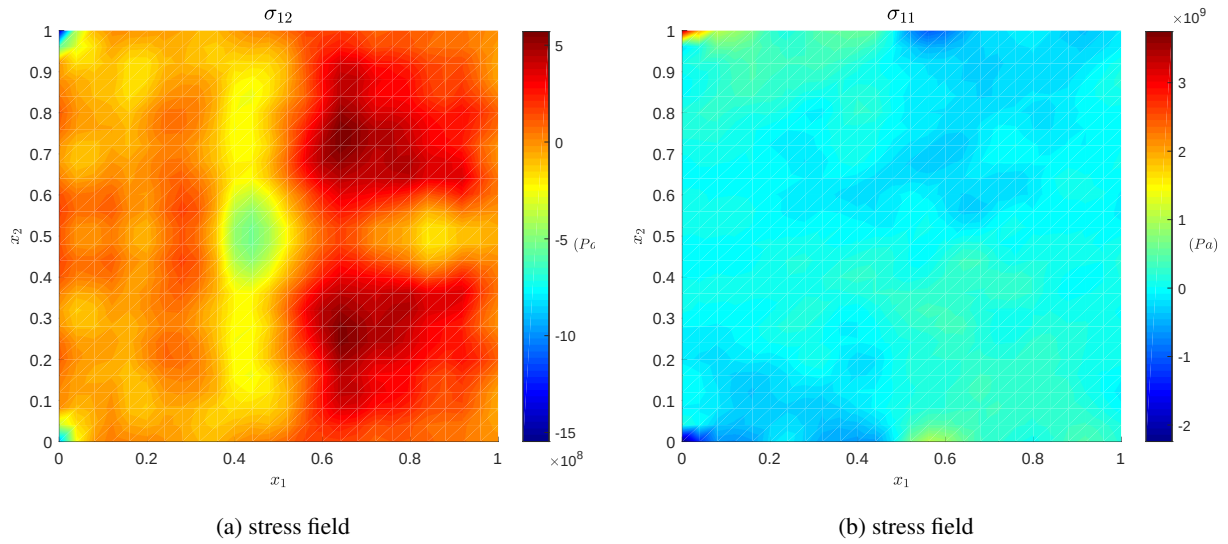
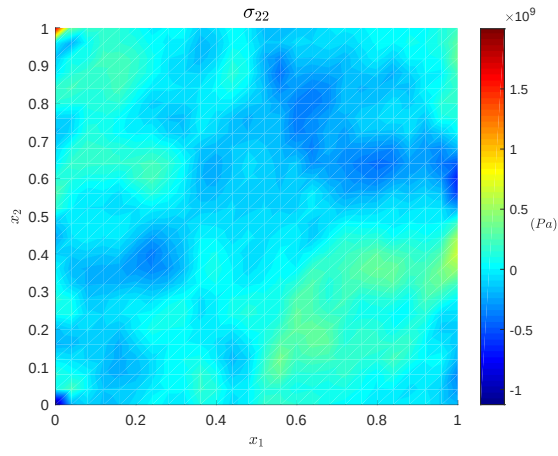
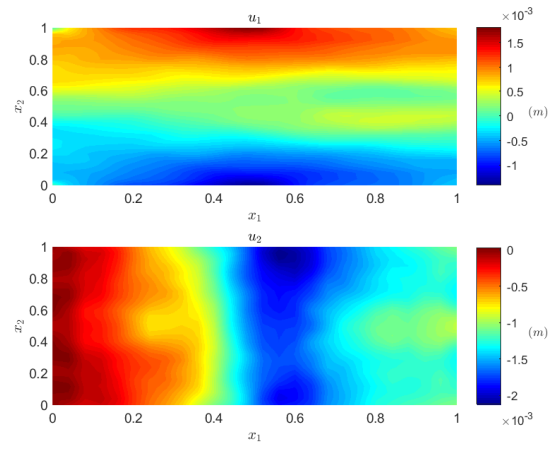


Figure 5.38: Time step version of the method for plane strain problem with $t = 0.012$, $T = 0.0001s$, $\gamma = 0.2$, and $N = 500$

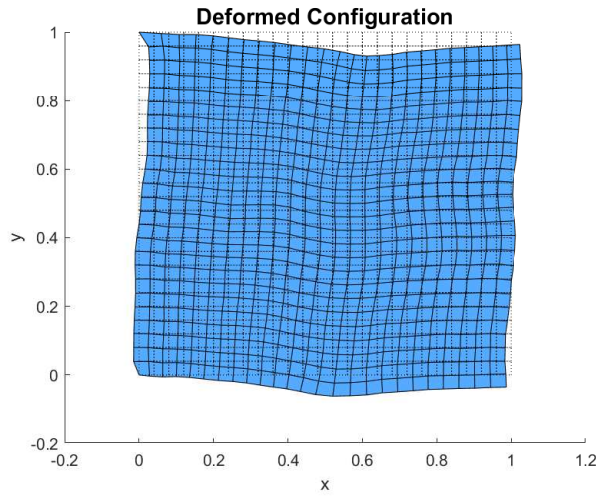


(a) stress field

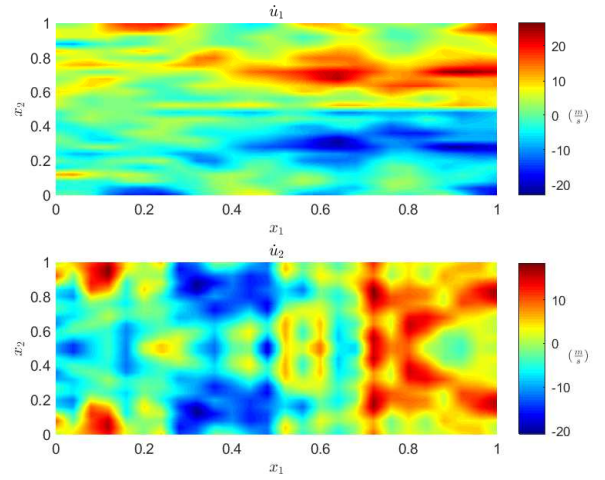


(b) displacement field

Figure 5.39: Time step version of the method for plane strain problem with $t = 0.012$, $T = 0.0001s$, $\gamma = 0.2$, and $N = 500$



(a) deformation



(b) velocity field

Figure 5.40: Time step version of the method for plane strain problem with $t = 0.012$, $T = 0.0001s$, $\gamma = 0.2$, and $N = 500$

Chapter 6

Convolution-Newton-Raphson Algorithm for Nonlinear Elastodynamics

6.1 Introduction

It is well-understood that, in general, the Newmark family of algorithms falls short of satisfying energy identity and conservation of angular momentum in the nonlinear elastodynamics and nonlinear dynamics of rigid bodies. In more details, As demonstrated in [36], when there is a negligible error in the Newton Raphson method, the energy at each step varies in the following form (colored lines) [36]:

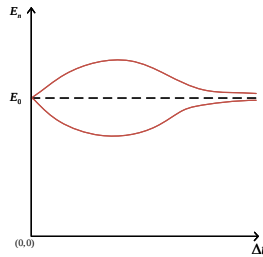


Figure 6.1: Failure of the average acceleration method in conserving energy of free vibration of nonlinear elastodynamics problems [36].

This non-conserving behavior, particularly in the energy, results in the violation of the unconditional stability of the method. As mentioned in [50], the algorithms that address this issue belong to one of three categories: 1) algorithmic damping methods 2) approaches through which conservation laws are imposed as additional restrictions to the solution procedure 3) algorithmic energy-momentum conserving approaches.

Algorithmic damping approaches have been employed to solve nonlinear problems since they cut off the contribution of higher modes by introducing controllable algorithmic damping in the linear

regime. This desirable property excludes the spurious part of the solution, which may improve the stability of the method in the nonlinear analysis. However, as pointed out in [50], in such scenarios, the energy dissipation is not ensured for all parameters in the nonlinear regime.

The conservation laws are enforced as additional restrictions in the second category. The central methodology to achieve this goal is the use of the method of the Lagrange multipliers. This method was introduced as an energy-conserving algorithm (i.e., the discrete form of the energy identity is satisfied) in [38]. Nevertheless, [51], based on numerical observations, concluded: "constraint conservation of energy alone does not guarantee a stable time integration in nonlinear dynamics." Consequently, in that study, through Lagrange-multipliers, they generalized the method by enforcing conservation of linear and angular momentum in addition to energy conservation. Although, as demonstrated numerically, the algorithm works suitably, the process is accompanied by a computational cost.

In the third class, often through variational integrators, the conservation of energy and momentums is achieved algorithmically. Variational integrators are a category of numerical integration used in the Hamiltonian mechanics that may conserve characteristics (constants) of the motion (see [58]). According to a statement in [85], symplectic algorithms—a category of variational integrators that conserves two-form in the phase space— with constant time step fails to exactly conserve the energy. Although for some computational purposes, the energy at each time step in the symplectic methods may be satisfactory, this is not the case in general [78]. Hence, in [78], a two-steps finite difference approach has been proposed. In the first step, by applying the midpoint rule, a symplectic integrator, exact conservation of angular and linear momentum is satisfied. On the other hand, the energy conservation is met by solving a nonlinear equation by which stress of midpoint is written in terms of initial and end times (second step). It has been shown that the equation has an exact solution for Saint Venant-Kirchhoff material, giving the stress of midpoint as the average values of the initial and the endpoint at each time step.

However, it would be better to develop a method that conserves constants of motion through which there is no need to solve an extra equation. With this goal in mind, [19] introduced a discrete-time derivative, proposing a similar second-order accurate approach that satisfies invariants of motion

for general hyperelastic materials. The main idea behind the energy-conserving methods is defining a consistent discrete form of stress where it precisely resembles the time rate of change of potential energy. Also, as reported in [55], the technique developed initially in [78] neglects the coupling of the weak form and the above-mentioned nonlinear equations. This fact may result in divergence of the method [55]. Hence, in [55], the original algorithm has been improved by considering the coupling effect. Another progress in that vein, similar to linear elastodynamics, is developing algorithms damping out the spurious part of the solution in the nonlinear regimes. The noticeable works in this regard can be found in the comprehensive works [5, 6]. In these studies, the exclusion of high frequencies has been achieved by introducing dissipative terms in the original energy-momentum conserving algorithms in [52, 53] and [19]. It should be mentioned that, in effect, there is a family of energy-momentum conserving algorithms since the definition of the new form of stress used in this category is not unique. This fact has been shown in [76], concluding that "there are indeed infinite ways of obtaining second order accurate, energy and momentum preserving methods." In addition to these studies, in [8], a combined two-step finite difference method has been unutilized for conservation of energy and momenta.

The above-mentioned methods are based on finite difference methods. However, in [10], the authors have developed an energy-momentum conserving space-time FEM based on Hamiltonian's principle. In particular, the dynamical system resulting from the spatial discretization of elastodynamics problems has been written as a system of first-order ODEs, where a time finite element approach has been introduced to solve these equations. While the time finite element method conserves energy theoretically, special attention is required for numerical implementation since the formulation involves integrals that cannot be evaluated exactly [10]. Accordingly, they supplemented the method with a criterion fulfilling the conservation of energy. Also, for the linear shape functions, the preservation of angular momentum further imposes a stronger restriction in the choice of numerical quadrature [10]. In continuation of this study, the authors have shown that higher-order accuracy could be obtained through their method [23].

Overall, in most of the aforementioned studies, it has been attempted to propose new methods recovering unconditional stability and higher-order accuracy in the nonlinear problems . The use

of the finite difference methods leads to define either new parameters or a new definition of stress in the discrete format. For example, in the third category, the key point is to define a new form of stress to secure satisfaction of the energy and angular momentum, leading to a more intricate nonlinear set of equations than the ones resulting from conventional time integration methods such as the trapezoidal method. For instance, in contrast to the Newmark method, as we shall see in the numerical section, special attention should be paid to the energy-momentum conservation methods when the denominator of the corresponding equations goes to zero. Hence, it is more convenient to introduce a method that preserves the aforementioned constants while following the simple steps as in the standard methods.

Furthermore, although the conservation of energy and angular momentum are sufficient criteria in developing a stable method, accuracy is yet another crucial factor to be considered. As mentioned in [76], the family of energy-momentum conserving methods defined with discrete derivative combined with mid-point rule is at most second order accurate. Hence, analogous to the linear problems, developing methods capable of capturing the solution when larger time steps are used is of great importance in the nonlinear regime. Accordingly, it is advantageous to enrich the literature by introducing a method containing the following attributes:

- a. It deals with the continuous form of the equations of motion rather than discrete ones.
- b. The form of resulting nonlinear equations is similar to those obtained from applying standard time integration methods.
- c. Higher-order accurate solution is achievable as long as the nonlinear solver converges.
- d. The conservation of the invariants of motion is achievable.
- e. The computational cost of the method is comparable with that of the existing methods.

In this chapter, we aim to suggest a new approach that satisfies the above items. The method is based on an alternative formulation and the corresponding convolution variational form introduced in [26]. This framework has received less attention than other methods since, at first glance, it seems to be a more complicated formulation than original equations of motion. However, in [4],

by introducing a new finite element method, it has been shown that this framework inherits some numerical features that cannot be found in the classical numerical methods:

- 1 Although it stems from space-time coupled variational form, its implementation is similar to classical time integration methods.
- 2 Higher-order accuracy for any given time interval is achievable.
- 3 Provided that a sufficient number of terms is considered, it uniformly converges to the exact solution in any given time interval.
- 4 The spurious part of the response, similar to algorithmic damping methods (e.g., HHT α method), can be excluded by applying filters.
- 5 Time-space discretization to increase the accuracy is not required.
- 6 Algorithm's computational cost is justifiable if compared to the classical methods.

Also, it has been demonstrated in a number of works that the framework can be developed for various types of physical processes (see [65], [2] & [3], among others). These desired characteristics motivated us to develop the approach for nonlinear elastodynamics. The present chapter is organized as follows. First, alternative equations pertaining to the nonlinear governing equations of motion are established. Subsequently, a combination of Newton-Raphson and the convolution finite element method are applied to solve those equations in sections 6.3 & 6.4. As a result, in each step of the iteration, the algorithm inherits the characteristics 1-6 mentioned earlier. Next, the properties of the method are investigated in section 6.5. In particular, we show that the method preserves constants of motion in an adaptive manner. That is, by increasing the number of time-wise terms, the satisfaction of those identities is met regardless of the size of time-steps. Finally, to show the plausibility of the method, we consider several examples in section 6.6: 1. free vibration of bilinear stiffening and softening mass-spring systems are used to show that the new method conserves energy and obtains accurate solution when an adequate number of time-wise terms is assumed 2. a particular form of 2d mass-spring system is considered for testing the method for multi-degree of freedom systems 3. a bar problem is analyzed as the prototype of elastodynamics.

6.2 Problem Statement

We introduce the method in the framework of infinitesimal strain theory. That is, we assume there is a negligible difference between Lagrangian and Eulerian descriptions of motion:

$$x_i \approx \delta_{iI} X_I, (i = 1, 2, 3; I = 1, 2, 3) \quad (6.1)$$

where x_i and X_I denote, respectively, spatial and material coordinate systems, and δ_{iI} is the Kronecker delta. As a result, we consider material nonlinearity. Hence, in what follows, similar to the linear regime, lowercase letters are used.

The domain of the problem is denoted by Ω , an open and Lipschitz subset of 3d Euclidean space with the boundary Γ . Let Γ_{gi} and Γ_{hi} denote open subsets of Γ ($i = 1, 2, 3$) with $\Gamma_{gi} \cap \Gamma_{hi} = \emptyset$ and $\Gamma_{gi} \cup \Gamma_{hi} = \Gamma$. Also, $\bar{\Omega}$, $\bar{\Gamma}_{gi}$, and $\bar{\Gamma}_{hi}$ denote, respectively, closure of Ω , Γ_{gi} , and Γ_{hi} . Then, the motion of the nonlinear elastic body is governed by the following field equations and the initial and boundary conditions:

1. Equations of Motion:

$$\sigma_{ij,j} + \rho b_i = \rho \ddot{u}_i, \quad \sigma_{ij} = \sigma_{ji} \quad \text{on } \Omega \times (0, T^{total}). \quad (6.2)$$

2. Kinematic Equations:

$$\epsilon_{ij}(\mathbf{u}) = u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{on } \Omega \times (0, T^{total}). \quad (6.3)$$

3. Constitutive Equations:

$$\sigma_{ij} = \sigma_{ij}(\boldsymbol{\epsilon}) \quad \text{on } \Omega \times (0, T^{total}), \quad (6.4)$$

4. Boundary Conditions:

$$\begin{aligned}
u_i &= g_i(\mathbf{x}, t) & \text{on } \Gamma_{gi} \times (0, T^{total}), \\
\sigma_{ij} n_j &= f_i(\mathbf{x}, t) & \text{on } \Gamma_{fi} \times (0, T^{total}).
\end{aligned} \tag{6.5}$$

5. Initial Conditions:

$$\begin{aligned}
u_i(\mathbf{x}, 0) &= u_i^0(\mathbf{x}) & \text{on } \overline{\Omega}, \\
\dot{u}_i(\mathbf{x}, 0) &= \dot{u}_i^0(\mathbf{x}) & \text{on } \overline{\Omega}.
\end{aligned} \tag{6.6}$$

Next, we define the ‘strong solution’ of the initial boundary value problem. In doing so, we have the following regularity conditions:

Definition 12. An ordered array $[\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ is called an admissible process on $\overline{V} \times (0, T^{total})$ if

$$u_i \in C^{1,2}, \varepsilon_{ij} \in C^{0,0}, \sigma_{ij} \in C^{1,0}, \tag{6.7}$$

Definition 13. An admissible process $[\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ is called a strong solution of the initial boundary value problem (IBVP) if (6.2), (6.3), (6.4), (6.5), and (6.6) hold.

By using the Laplace transform, it can be shown that the following equations are equivalent to the equations of motion and the initial conditions:

$$\rho u_i(\mathbf{x}, t) = [t * (\sigma_{ij,j} + \rho f_i)](\mathbf{x}, t) + [\rho(u_i^0 + t \dot{u}_i^0)](\mathbf{x}, t), \quad \sigma_{ij}(\mathbf{x}, t) = \sigma_{ji}(\mathbf{x}, t), \tag{6.8}$$

in which the symbol $[h * g](\mathbf{x}, t)$ for given functions $h(\mathbf{x}, t)$ and $g(\mathbf{x}, t)$ denotes the convolution product in the sense of

$$[f * g](\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t - \tau) g(\mathbf{x}, \tau) d\tau. \tag{6.9}$$

Also, the following notation is used in sequel:

$$[f * g * h](\mathbf{x}, t) = \int_0^t \left(\int_0^\lambda f(\mathbf{x}, \lambda - \tau) g(\mathbf{x}, \tau) d\tau \right) h(\mathbf{x}, t - \lambda) d\lambda. \tag{6.10}$$

Hence, one can conclude:

Theorem 18. *An ordered array $[\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}]$ is a strong solution of the IBVP if and only if (6.3), (6.4), (6.5), and (6.8) hold.*

6.3 Convolution Weak Formulation

The procedure of obtaining a weak solution is elaborated in this section. First, let us define the solution and variation spaces:

Definition 14. $\mathbb{S}_i$ and $\mathbb{V}_i$ stand for, respectively, the solution space and the variation space whose elements have the following properties:

$$\mathbb{S}_i = \{u_i \in L^2(0, T; H^1(\Omega)) | u_i = g_i(\mathbf{x}, t) \text{ on } \Gamma_{g_i} \times (0, T)\}, \quad (6.11)$$

$$\mathbb{V}_i = \{w_i \in L^2(0, T; H^1(\Omega)) | w_i = 0 \text{ on } \Gamma_{g_i} \times (0, T)\}, \quad i = 1, 2, 3. \quad (6.12)$$

Also, we rewrite (6.8) as:

$$\rho v_i(\mathbf{x}, t) = t * \sigma_{ij,j}(\mathbf{v} + \mathbf{A}(\mathbf{x}, t)) + t * \rho f_i(\mathbf{x}, t), \quad (6.13)$$

with:

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) - \mathbf{A}(\mathbf{x}, t), \quad (6.14)$$

$$\mathbf{A}(x, t) = \mathbf{u}^0(x) + t \dot{\mathbf{u}}^0(x).$$

Now, considering the statements in the appendix, applying the divergence theorem, one can obtain the following integral form:

$$\begin{aligned} \int_{\Omega} \rho [v_i * w_i](\mathbf{x}, t) dV + \int_{\Omega} [t * \sigma_{ij,j}(\mathbf{v} + \mathbf{A}) * w_{i,j}](\mathbf{x}, t) dV = \\ \int_{\Gamma_{h_i}} [t * h_i * w_i](\mathbf{x}, t) d\Gamma + \int_{\Omega} [t * \rho f_i * w_i](\mathbf{x}, t) dV, \end{aligned} \quad (6.15)$$

in which w_i , an element of $\mathbb{V}_i$, represents weighting functions. It is noted that the convolution in

time is associative, so that there is no need to indicate the order of time convolutions under the space integrals. In view of the Remark 2 (see Appendix), satisfaction of (6.15) for all weight functions $w_i \in \mathbb{V}_i$ results in satisfaction of (6.8) in the sense of (5.17). Hence, we define:

Definition 15. $u_i(\mathbf{x}, t)$, and element of $\mathbb{S}_i$, is called a weak solution corresponding to the strong solution if (6.15) holds true for all $w_i \in \mathbb{V}_i$.

As can be seen, in contrast to classical formulations, the velocity and acceleration fields are no longer present in the formulation. Moreover, initial conditions need no special attention as they are satisfied as forcing terms.

6.4 Newton-Raphson Procedure

Since the integral form (6.15) consists of (6.4), a nonlinear solver has to be employed. Hence, to proceed, the modified Newton-Raphson procedure is implemented in this section. To this end, we partition the total time T^{total} into $0 = T_0 < T_1 < \dots < T_n = T^{total} = n(1 - \gamma)T$ ($0 < \gamma < 1$ is a parameter which shall be explained in sequel), and solve the problem consecutively. Now, in view of the Lemma 8 and (6.15), to obtain solution for $t = T_{pa+1}$, $pa = 0, \dots, n-1$, we consider the following:

$$\begin{aligned} & \int_{\Omega} \rho [v_i * w_i](\mathbf{x}, T) dV + \int_{\Omega} [t * \sigma_{ij}(\mathbf{v} + \mathbf{A}_{pa}) * w_{i,j}](\mathbf{x}, T) dV \\ & - \int_{\Gamma h_i} [t * h_i^{pa} * w_i](\mathbf{x}, T) d\Gamma - \int_{\Omega} [t * \rho f_i^{pa} * w_i](\mathbf{x}, T) dV = 0, \end{aligned} \quad (6.16)$$

with

$$\mathbf{A}_{pa}(\mathbf{x}, t) = \mathbf{u}^{T_{pa}}(\mathbf{x}) + t \dot{\mathbf{u}}^{T_{pa}}(\mathbf{x}), \quad f_i^{pa}(\mathbf{x}, t) = f_i(\mathbf{x}, t + T_{pa}), \quad h_i^{pa}(\mathbf{x}, t) = h_i(\mathbf{x}, t + T_{pa}), \quad (6.17)$$

where $\mathbf{u}^{T_{pa}}(\mathbf{x})$ and $\dot{\mathbf{u}}^{T_{pa}}(\mathbf{x})$ denote the solutions of the previous step which are used as the initial data for the current step. Equation (6.16) can be alternatively written as:

$$\int_{\Omega} \rho [\mathcal{V}_i^h * w_i](\mathbf{x}, T) dV + \int_{\Omega} [t * \sigma_{ij}(\mathcal{V}^h + \tilde{\mathcal{G}}^{pa} + \mathbf{A}_{pa}) * w_{i,j}](\mathbf{x}, T) dV \quad (6.18)$$

$$-\int_{\Gamma_{h_i}} [t * h_i^{pa} * w_i] (\mathbf{x}, T) d\Gamma - \int_{\Omega} [t * \rho f_i^{pa} * w_i] (\mathbf{x}, T) dV = 0,$$

with

$$\begin{aligned} \mathcal{V}_i^h(\mathbf{x}, t) &= 0 \text{ on } \Gamma_{gi} \times (0, T), \\ \mathcal{V}_i^h(\mathbf{x}, 0) &= \dot{\mathcal{V}}_i^h(\mathbf{x}, 0) = 0 \text{ on } \Omega, \\ \tilde{g}_i^{pa}(\mathbf{x}, t) &= g_i(\mathbf{x}, t + T_{pa}) - A_{pa i}(\mathbf{x}, t) \text{ on } \Gamma_{gi} \times (0, T), \\ \tilde{g}_i^{pa}(\mathbf{x}, 0) &= \dot{\tilde{g}}_i(\mathbf{x}, 0) = 0 \text{ on } \Omega \end{aligned} \quad (6.19)$$

Next, we define:

$$\begin{aligned} \mathcal{V}^{h(m+1)}(\mathbf{x}, t) &= \mathcal{V}^{h(m)}(\mathbf{x}, t) + \Delta \mathcal{V}^{h(m)}(\mathbf{x}, t), \quad m = 0, \dots, L-1, \\ \mathcal{V}^{h(0)}(\mathbf{x}, t) &= \mathbf{0}, \end{aligned} \quad (6.20)$$

in that $\Delta \mathcal{V}^{h(m)}$, $m = 0, \dots, L-1$, denotes as the m^{th} displacement increment added to the m^{th} iteration, $\mathcal{V}^{h(m)}$. Now, one can write for $m = 0$:

$$\begin{aligned} &\int_{\Omega} [\rho \Delta \mathcal{V}_i^{h(0)} * w_i] (\mathbf{x}, T) dV + \int_{\Omega} \left[t * \left(D^{(0)} \sigma(\tilde{\mathcal{G}}^{pa}(\mathbf{x}, T) + \mathbf{A}_{pa}(\mathbf{x}, T)) \right)_{ij} * w_{i,j} \right] (\mathbf{x}, T) dV \\ &+ R_{pa}^{(0)}(\mathbf{w}(\mathbf{x}, t)) \Big|_{t=T} = 0, \end{aligned} \quad (6.21)$$

where

$$\begin{aligned} R_{pa}^{(0)}(\mathbf{w}(\mathbf{x}, t)) \Big|_{t=T} &= \int_{\Omega} [t * \sigma_{ij}(\tilde{\mathcal{G}}^{pa} + \mathbf{A}_{pa}) * w_{i,j}] (\mathbf{x}, T) dV - \int_{\Omega} [t * \rho f_i^{pa} * w_i] (\mathbf{x}, T) dV \\ &- \sum_{i=1}^3 \int_{\Gamma_{h_i}} [t * h_i^{pa} * w_i] (\mathbf{x}, T) d\Gamma, \\ \left(D^{(0)} \sigma(\tilde{\mathcal{G}}^{pa} + \mathbf{A}_{pa}) \right)_{ij} &= \left(\lim_{\alpha \rightarrow 0} \frac{d\sigma(\alpha \hat{\mathbf{u}} + \tilde{\mathcal{G}}^{pa} + \mathbf{A}_{pa})}{d\alpha} \Big|_{\hat{\mathbf{u}} = \Delta \mathcal{V}^{h(0)}} \right)_{ij}. \end{aligned} \quad (6.22)$$

In (6.22)₁, while $t = T$ has been utilized, any time in the range of $0 \leq t \leq T$ is acceptable provided that the Newton-Raphson converges. Clearly, the above equation is equivalent to the linear variational form in terms of convolution product. Obtaining $\Delta \mathcal{V}^{h(0)}$, the second iteration can be established. In doing so, given the following integral form is not less than an appropriate tolerance

for all $w_i \in \mathbb{V}_i$:

$$\begin{aligned} R_{pa}^{(1)}(\mathbf{w}(\mathbf{x}, t)) \Big|_{t=T} &= \int_{\Omega} \left[t * \sigma_{ij}(\mathcal{V}^{h(1)} + \tilde{\mathbf{g}}^{pa} + \mathbf{A}_{pa}) * w_{i,j} \right] (\mathbf{x}, T) dV + \int_{\Omega} \left[\rho \mathcal{V}_i^{(1)} * w_i \right] (\mathbf{x}, T) dV \\ &- \int_{\Omega} \left[t * \rho f_i^{pa} * w_i \right] (\mathbf{x}, T) dV - \sum_{i=1}^3 \int_{\Gamma h_i} \left[t * h_i^{pa} * w_i \right] (\mathbf{x}, T) d\Gamma \end{aligned} \quad (6.23)$$

by subtracting (6.21) from (6.16), the second iteration can then be written as:

$$\begin{aligned} &\int_{\Omega} \left[\rho \Delta \mathcal{V}_i^{h(1)} * w_i \right] (\mathbf{x}, T) dV + \int_{\Omega} \left[t * \left(D^{(1)} \sigma(\tilde{\mathbf{g}}^{pa}(\mathbf{x}, T) + \mathbf{A}_{pa}(\mathbf{x}, T)) \right)_{ij} * w_{i,j} \right] (\mathbf{x}, T) dV \\ &+ R_{pa}^{(1)}(\mathbf{w}(\mathbf{x}, t)) \Big|_{t=T} = 0 \end{aligned} \quad (6.24)$$

in which:

$$\left(D^{(1)} \sigma(\tilde{\mathbf{g}}^{pa} + \mathbf{A}_{pa}) \right)_{ij} = \left(\lim_{\alpha \rightarrow 0} \frac{d\sigma(\alpha \hat{\mathbf{u}} + \tilde{\mathbf{g}}^{pa} + \mathbf{A}_{pa})}{d\alpha} \Big|_{\hat{\mathbf{u}} = \Delta \mathcal{V}^{h(1)}} \right)_{ij}. \quad (6.25)$$

solving (6.24) and using (6.20)₁, we find $\mathbf{v}^{(2)} = \mathbf{v}^{(1)} + \Delta \mathbf{v}^{(1)}$. In an analogous manner, the m^{th} iteration can then be found as:

$$\begin{aligned} &\int_{\Omega} \left[\rho \Delta \mathcal{V}_i^{h(m)} * w_i \right] (\mathbf{x}, T) dV + \int_{\Omega} \left[t * \left(D^{(m)} \sigma(\tilde{\mathbf{g}}^{pa}(\mathbf{x}, T) + \mathbf{A}_{pa}(\mathbf{x}, T)) \right)_{ij} * w_{i,j} \right] (\mathbf{x}, T) dV \\ &+ R_{pa}^{(m)}(\mathbf{w}(\mathbf{x}, t)) \Big|_{t=T} = 0 \end{aligned} \quad (6.26)$$

with

$$\begin{aligned} R_{pa}^{(m)}(\mathbf{w}(\mathbf{x}, t)) \Big|_{t=T} &= \int_{\Omega} \left[t * \sigma_{ij}(\mathcal{V}^{h(m)} + \tilde{\mathbf{g}}^{pa} + \mathbf{A}_{pa}) * w_{i,j} \right] (\mathbf{x}, T) dV + \int_{\Omega} \left[\rho \mathcal{V}_i^{(m)} * w_i \right] (\mathbf{x}, T) dV \\ &- \int_{\Omega} \left[t * \rho f_i^{pa} * w_i \right] (\mathbf{x}, T) dV - \sum_{i=1}^3 \int_{\Gamma h_i} \left[t * h_i^{pa} * w_i \right] (\mathbf{x}, T) d\Gamma \\ &\left(D^{(m)} \sigma(\tilde{\mathbf{g}}^{pa} + \mathbf{A}_{pa}) \right)_{ij} = \left(\lim_{\alpha \rightarrow 0} \frac{d\sigma(\alpha \hat{\mathbf{u}} + \tilde{\mathbf{g}}^{pa} + \mathbf{A}_{pa})}{d\alpha} \Big|_{\hat{\mathbf{u}} = \Delta \mathcal{V}^{h(m)}} \right)_{ij}. \end{aligned} \quad (6.27)$$

This procedure should be continued as long as the error, e.g., the norm of the residual, is greater than the tolerance.

6.4.1 Convolution FEM Solver

In this part, we employ the convolution finite element introduced recently in [4] to obtain the solution regarding each iteration of the above-mentioned procedure. To establish a Galerkin formulation, we define [4] :

$$\begin{aligned}\mathbb{S}_i^{h,N} &= \mathbb{S}_i^{h,x} \otimes \mathbb{F}_N^{h,t}, \\ \mathbb{V}_i^{h,N} &= \mathbb{V}_i^{h,x} \otimes \mathbb{F}_N^{h,t},\end{aligned}\tag{6.28}$$

where $\mathbb{S}_i^{h,x}$ and $\mathbb{V}_i^{h,x}$ denote the following finite dimensional subsets of $H^1(\Omega)$:

$$\begin{aligned}\mathbb{S}_i^{h,x} &= \{f(\mathbf{x}) | f(\mathbf{x}) \in H^1(\Omega)\}, \\ \mathbb{V}_i^{h,x} &= \{g(\mathbf{x}) | g(\mathbf{x}) \in H^1(\Omega), g(\mathbf{x}) = 0 \text{ on } \Gamma_{gi}\},\end{aligned}\tag{6.29}$$

and

$$\mathbb{F}_N^{h,t} = \left\{ 1, \cos \frac{\pi t}{T}, \dots, \cos \frac{(N-1)\pi t}{T} \right\}.\tag{6.30}$$

Considering (6.19), (6.20), and (6.28), we write:

$$\begin{aligned}\mathcal{V}^{h(m)}(\mathbf{x}, t) &= \sum_{P=0}^{N-1} \sum_{j=1}^{n_{dof}} \sum_{\lambda \in \eta - \eta_{gj}} c_{Pj\lambda}^{(m)} \psi_{\lambda}(\mathbf{x}) \cos\left(\frac{P\pi t}{T}\right) \mathbf{e}_j, \\ c_{Pj\lambda}^{(0)} &= 0,\end{aligned}\tag{6.31}$$

and

$$\Delta \mathcal{V}^{h(m)}(\mathbf{x}, t) = \sum_{P=0}^{N-1} \sum_{j=1}^{n_{dof}} \sum_{\lambda \in \eta - \eta_{gj}} \Delta c_{Pj\lambda}^{(m)} \psi_{\lambda}(\mathbf{x}) \cos\left(\frac{P\pi t}{T}\right) \mathbf{e}_j,\tag{6.32}$$

in which $\eta = \{1, 2, \dots, n_{np}\}$ stands for the set of all nodal points defining geometry of the problem; η_{gi} indicates the set of all nodal points on the boundary Γ_{gi} ; ψ_{λ} 's show Lagrangian shape functions; $\Delta c_{Pj\lambda}^{(m)}$'s represent unknown coefficients of m^{th} increments, and $\mathbf{e}_j$, $j = 1, 2, 3$, denotes unit vectors of Cartesian coordinate system. Analogous to basis function $\Delta \mathcal{V}^{h(m)}(\mathbf{x}, t)$, the test function $\mathbf{w}(\mathbf{x}, t)$ is set as:

$$\mathbf{w}(\mathbf{x}, t) = \sum_{Q=0}^{N-1} \sum_{j=1}^{n_{dof}} \sum_{\lambda \in \eta - \eta_{gi}} d_{Qj\beta} \psi_{\lambda}(\mathbf{x}) \cos\left(\frac{Q\pi t}{T}\right) \mathbf{e}_i.\tag{6.33}$$

Moreover, since only the case $t = T$ in (6.26) is considered, the relevant convolution products, except the first integral of the right-hand-side of (6.27)₁, can be evaluated exactly:

$$\begin{aligned} \mathcal{A}_{ij} = t * \cos\left(\frac{j\pi t}{T}\right) * \cos\left(\frac{i\pi t}{T}\right) \Big|_{t=T} &= \begin{cases} \frac{T^3}{i^2\pi^2} & i \neq 0, j = 0 \\ \frac{T^3}{j^2\pi^2} & i = 0, j \neq 0 \\ (-1)^{i+1} \frac{T^3}{2i^2\pi^2} & i = j \neq 0 \\ \frac{T^3}{6} & i = j = 0 \\ 0 & otherwise \end{cases}, \\ \mathcal{F}_{ij} = \cos\left(\frac{i\pi t}{T}\right) * \cos\left(\frac{j\pi t}{T}\right) \Big|_{t=T} &= \begin{cases} (-1)^i \frac{T}{2} & i = j \neq 0 \\ T & i = j = 0 \\ 0 & otherwise \end{cases}. \end{aligned} \quad (6.34)$$

Substituting (6.32), (6.33), employing (6.34), we find after some manipulations:

$$\begin{aligned} \sum_{P=0}^{N-1} \sum_{j=1}^{n_{dof}} \sum_{\alpha \in \eta - \eta_{gi}} \Delta c_{Pj\alpha}^{(m)} \left[\mathcal{F}_{PQ}(\rho \psi_\alpha \mathbf{e}_j, \psi_\beta \mathbf{e}_i)_\Omega + \mathcal{A}_{PQ} a(\psi_\alpha \mathbf{e}_j, \psi_\beta \mathbf{e}_i) \right] = \\ - R_{pa}^{(m)}(\psi_\beta(\mathbf{x}) \cos\left(\frac{Q\pi t}{T}\right) \mathbf{e}_i) \Big|_{t=T}, \quad (i = 1, \dots, n_{dof}), \quad (Q = 0, \dots, N-1), \quad (\beta \in \eta - \eta_{gi}). \end{aligned} \quad (6.35)$$

in which $a(\mathbf{u}, \mathbf{w})$ and $(\mathbf{u}, \mathbf{w})_\Omega$ represent the following:

$$\begin{aligned} a(\mathbf{u}(\mathbf{x}), \mathbf{w}(\mathbf{x})) &= \int_{\Omega} (D\sigma|_{\nabla \mathbf{u}})_{ij} w_{i,j} dV, \\ (\mathbf{u}(\mathbf{x}), \mathbf{w}(\mathbf{x}))_\Omega &= \int_{\Omega} u_i w_i dV, \end{aligned} \quad (6.36)$$

for every vectors $\mathbf{u}$ and $\mathbf{w}$; n_{dof} indicates the number of degrees of freedom which could be $n_{dof} = 1, 2, 3$. Equation(6.35) can then be written in the following matrix form:

$$\sum_{P=0}^{N-1} (\mathcal{F}_{PQ}[\mathbf{M}] + \mathcal{A}_{PQ}[D\mathbf{K}])[\mathbf{c}]_P = [\mathbf{f}_{eq}]_Q, \quad Q = 0, \dots, N-1 \quad (6.37)$$

in which $[\mathbf{f}_{eq}]_Q$ represents the contribution of the right hand side of (6.35); $[\mathbf{M}]$ and $[D\mathbf{K}]$ represent mass and the consistent tangent stiffness matrices. By solving (6.37), the m^{th} iteration is followed

from (6.20),(6.31)₁ and (6.32). Obtaining the displacement field, the velocity and acceleration fields can then be computed by taking the time-wise derivative of the solution once and twice, respectively. In what follows, we consider the characteristics of the above method.

6.5 Properties of Solution Procedure

The convolution solver inherits the desirable characteristics of time-space decoupled and coupled finite element methods. In other words, as explained in [4], this solver follows the same straightforward steps relevant to time integration method while stemming from coupled space-time variational formulation. Herein by saying the same ‘step,’ we mean in the convolution solver, analogous to the time integration methods, spatial discretization can be initially done and then the associated nonlinear ODEs are solved. To explain properties of the new method, we first explain some facts regarding semi-discrete equations of motion obtained from (6.2). The formulation is represented only for one time step which should be applied consecutively with updated initial data and external force to find the solution for any desired time interval. In doing so, after spatial discretization, one obtains the following initial values nonlinear ODEs:

$$\begin{aligned} [\mathbf{M}] [\ddot{\mathbf{U}}(t)] + [\mathbf{K}([\mathbf{U}(t)])] &= [\mathbf{f}(t)], \quad 0 \leq t \leq \Delta t = (1 - \gamma)T \\ [\mathbf{U}(0)] &= [\mathbf{U}_0], \quad [\dot{\mathbf{U}}(0)] = [\dot{\mathbf{U}}_0], \end{aligned} \quad (6.38)$$

where $[\mathbf{U}(t)]$ represents the nodal values, that is:

$$[\mathbf{U}(t)] = \left[\underbrace{[U_1^1(t), U_2^1(t), U_3^1(t)]}_{:=\mathbf{U}^1(t)}, \dots, \underbrace{[U_1^{n_{nod}}(t), U_2^{n_{nod}}(t), U_3^{n_{nod}}(t)]}_{:=\mathbf{U}^{n_{nod}}(t)} \right]^T, \quad (6.39)$$

and n_{nod} stands for the number of spatial nodes.

6.5.1 New Method

(6.38) is equivalent to:

$$[\mathbf{M}] [\mathbf{U}(t)] = -t * [\mathbf{K}([\mathbf{U}(t)])] + t * [\mathbf{f}(t)] + [\mathbf{M}] [\mathbf{U}_0] + [\mathbf{M}] t [\dot{\mathbf{U}}_0], \quad 0 \leq t \leq \Delta t = (1 - \gamma)T \quad (6.40)$$

or

$$[\mathbf{M}] [\mathbf{V}(t)] + t * [\mathbf{K}([\mathbf{V}(t)] + [\mathbf{U}_0] + t [\dot{\mathbf{U}}_0])] - t * [\mathbf{f}(t)] = [\mathbf{0}], \quad 0 \leq t \leq \Delta t = (1 - \gamma)T \quad (6.41)$$

where

$$[\mathbf{V}(t)] = [\mathbf{U}(t)] - [\mathbf{U}_0] - t [\dot{\mathbf{U}}_0]. \quad (6.42)$$

The set of nonlinear ODEs (6.41) is equivalent to:

$$([\mathbf{M}] [\mathbf{V}(t)] + t * [\mathbf{K}([\mathbf{V}(t)] + [\mathbf{U}_0] + t [\dot{\mathbf{U}}_0])] - t * [\mathbf{f}(t)]) * \cos\left(\frac{Q\pi t}{T}\right) \Big|_{t=T} = [\mathbf{0}], \quad Q = 0, \dots, \infty. \quad (6.43)$$

Now, approximating the function $[\mathbf{V}(t)]$ by:

$$[\mathbf{V}(t)] = \sum_{P=0}^{\infty} [\mathbf{c}] \cos\left(\frac{P\pi t}{T}\right), \quad (6.44)$$

substituting into (6.43), we obtain the following nonlinear equations which have to be solved:

$$\left([\mathbf{M}] \left[\sum_{P=0}^{\infty} [\mathbf{c}] \cos\left(\frac{P\pi t}{T}\right) \right] + t * \left[\mathbf{K} \left(\sum_{P=0}^{\infty} [\mathbf{c}] \cos\left(\frac{P\pi t}{T}\right) + \mathbf{U}_0 + t \dot{\mathbf{U}}_0 \right) \right] - t * [\mathbf{f}(t)] \right) * \cos\left(\frac{Q\pi t}{T}\right) \Big|_{t=T} = [\mathbf{0}], \quad Q = 0, \dots, \infty. \quad (6.45)$$

In what follows, we replace ∞ in the above formulation with a large enough positive integer $N - 1$ such that the sense of infinity is approximately imposed. Now, let us define:

$$[\mathbf{V}(t)]^{(m+1)} = [\mathbf{V}(t)]^{(m)} + [\Delta \mathbf{V}(t)]^{(m)}, \quad m = 0, \dots, L - 1, \quad (6.46)$$

with

$$\begin{aligned}
[\mathbf{V}(t)]^{(m)} &= \sum_{P=0}^{N-1} \left([\mathbf{c}]_P^{(m)} \right) \cos \left(\frac{P\pi t}{T} \right), \\
[\Delta \mathbf{V}(t)]^{(m)} &= \sum_{P=0}^{N-1} \left(\Delta [\mathbf{c}]_P^{(m)} \right) \cos \left(\frac{P\pi t}{T} \right), \\
[\mathbf{c}]_P^{(0)} &= [\mathbf{0}], \quad P = 0, \dots, N-1.
\end{aligned} \tag{6.47}$$

Substituting (6.46) into (6.45), applying the Newton-Raphson method, one can find the governing equations corresponding to the m^{th} iteration:

$$\begin{aligned}
[\mathbf{M}] \left[\sum_{P=0}^{N-1} \left(\Delta [\mathbf{c}]_P^{(m)} \right) \cos \left(\frac{P\pi t}{T} \right) \right] * \cos \left(\frac{Q\pi t}{T} \right) + t * [\mathbf{DK}] \left[\sum_{P=0}^{N-1} \left(\Delta [\mathbf{c}]_P^{(m)} \right) \cos \left(\frac{P\pi t}{T} \right) \right] * \cos \left(\frac{Q\pi t}{T} \right) \\
+ [\mathbf{R}]^{(m,Q)} = [\mathbf{0}], \quad Q = 0, \dots, N-1.
\end{aligned} \tag{6.48}$$

with:

$$[\mathbf{R}]^{(m,Q)} = \left([\mathbf{M}] [\mathbf{V}(t)]^{(m)} + t * [\mathbf{K}([\mathbf{V}(t)]^{(m)} + [\mathbf{U}_0] + t [\dot{\mathbf{U}}_0])] - t * [\mathbf{f}(t)] \right) * \cos \left(\frac{Q\pi t}{T} \right) \Big|_{t=T}, \tag{6.49}$$

where $[\mathbf{DK}]$ represents the tangent matrix. The above equations can be written in the matrix form as:

$$\begin{bmatrix} \mathcal{F}_{00}[\mathbf{M}] + \mathcal{A}_{00}[\mathbf{DK}] & \mathcal{A}_{01}[\mathbf{DK}] & \dots & \mathcal{A}_{0N}[\mathbf{DK}] \\ \mathcal{A}_{01}[\mathbf{DK}] & \mathcal{F}_{11}[\mathbf{M}] + \mathcal{A}_{11}[\mathbf{DK}] & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{A}_{0r}[\mathbf{DK}] & \mathbf{0} & \dots & \mathcal{F}_{rr}[\mathbf{M}] + \mathcal{A}_{rr}[\mathbf{DK}] & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots & \ddots & & \end{bmatrix} \begin{bmatrix} [\Delta \mathbf{c}]_0^{(m)} \\ [\Delta \mathbf{c}]_1^{(m)} \\ \vdots \\ [\Delta \mathbf{c}]_r^{(m)} \\ \vdots \end{bmatrix} = - \begin{bmatrix} [\mathbf{R}]^{(m,0)} \\ [\mathbf{R}]^{(m,1)} \\ \vdots \\ [\mathbf{R}]^{(m,r)} \\ \vdots \end{bmatrix}, \tag{6.50}$$

where the left hand side is an arrowhead block matrix. This matrix form paves the way for considering an arbitrary number of time-wise term while maintaining the computational cost affordable. More specifically, to obtain the inverse of the large matrix given in (6.50), one can write $[\Delta \mathbf{c}]_r^{(m)}$, $r = 1, \dots, N-1$, in terms of $[\Delta \mathbf{c}]_0^{(m)}$, calculating $[\Delta \mathbf{c}]_0^{(m)}$ from the first set of equations. Then, the other $[\Delta \mathbf{c}]_r^{(m)}$ $r = 1, \dots, N-1$ can be calculated by knowing $[\Delta \mathbf{c}]_0^{(m)}$. Apparently, this procedure is suitable for parallel programming as most of the calculations are performed independently. Thus,

based on previous explanations, the linear solver is similar to time integration methods while, as analyzed in [4], achieving accurate solution for any time span. Also, given time step T , the linear solver may converge to the solution at all times, say $t \in [0, T)$, provided that sufficient number of time-wise terms are considered. Thus, this solver has three parameters: 1) number of time-wise terms N 2) time sub-interval T 3) a parameter, γ , such that $u(\mathbf{x}, (s - \gamma)T)$, $\dot{u}(\mathbf{x}, (s - \gamma)T)$, where $0 < \gamma < 1$ and $s \geq 1$, are used as initial values for $(s + 1)^{th}$ step. For instance, for a given T the larger N , the lesser γ can be selected. It should be mentioned that it is required to define γ in this method since, regardless of N , the resulting solution does not converge to the exact values at $t = T$ (see [4]). In view of (6.43), one can conclude that the solution resulting from this method converges in $t \in [0, T^{total}]$. This is in contradistinction to finite difference methods where the solution is obtained at discrete time instances.

Furthermore, as demonstrated in [4], due to its convergence to the Fourier cosine of the exact solution, filters can be adapted in this solution procedure. These filters can mitigate the spurious part of the solution and expedite the rate of convergence (see [20]).

The evaluation of residuals is a significant step in this method. As can be seen, contrary to classical forms, the residuals in this approach involve both time and spatial integrals. In this respect, quadrature methods, e.g., Gauss-Legendre, can be employed to deal with such integrals. However, given that the Newton-Raphson approach converges, for large values of time step T , evaluating the integrals with quadrature methods could be an uphill task. However, this problem can be efficiently alleviated by the method initially introduced in [7]. The time integrals of the residuals has the following form:

$$\int_0^T f(t) \cos(\omega t) dt. \quad (6.51)$$

This type of integrals can be approximated by using [68]:

$$\int_{-1}^1 f(t) e^{i\omega t} dt \approx \left(\frac{\pi}{2\omega} \right)^{1/2} \int_{-1}^1 f(x) \sum_{\kappa=1}^r (2\kappa + 1) i^\kappa J_{\kappa+1/2}(\omega) P_\kappa(t) dt, \quad (6.52)$$

or

$$\int_{-1}^1 f(t) \cos(\omega t) dt = \left(\frac{\pi}{2\omega}\right)^{1/2} \int_{-1}^1 f(t) \sum_{\kappa=1}^{\lceil r/2 \rceil} (2\kappa+1)(-1)^\kappa J_{2\kappa+\frac{1}{2}}(\omega) P_{2\kappa}(t) dt, \quad (6.53)$$

in which J_α and P_α denote Bessel functions of the first kind of order α and Legendre functions of order α , respectively. Now, the right-hand-side of (6.52) and (6.53) can be calculated by applying r -points Gauss-Legendre, where it is exact when $f(t)$ is a polynomial of order r [68]. The merit of (6.52) (or (6.53)) is that the major part of computation for a problem is performed and saved once for all time-steps. As another approach, one can also approximate the Fourier cosine expansion of the corresponding integrand with the Fast Fourier Transform (FFT) algorithm. Having found the Fourier cosine expansion, one can straightforwardly evaluate such integrals with the aid of (6.34). It should be noticed that the larger T , the more sample points for FFT algorithm are required.

6.6 Numerical Results

In this section, several examples are considered. To highlight the characteristics of the new method, we also show the same results corresponding to two existing methods: 1. trapezoidal method 2. discrete energy-momentum conserving method utilized in [5], which has been introduced in [52, 53]. To proceed, we first mention the algorithms:

1. *Trapezoidal rule*: in this method, (6.38) is satisfied at discrete time instances $t_1 = \Delta t, \dots, t_n = n\Delta t$, with Δt represents time step. Denoting $[\mathbf{U}(t_n)] \approx [\mathbf{U}_n]$, $[\dot{\mathbf{U}}(t_n)] \approx [\dot{\mathbf{U}}_n]$, and $[\ddot{\mathbf{U}}(t_n)] \approx [\ddot{\mathbf{U}}_n]$, one solves the following equations with Newton-Raphson method [36]:

$$\frac{4}{\Delta t^2} [\mathbf{M}] [\mathbf{U}_{n+1}] + [\mathbf{K}(\mathbf{U}_{n+1})] = [\mathbf{f}_{n+1}] + [\mathbf{G}_{n+1}], \quad (6.54)$$

with

$$\begin{aligned}
[\mathbf{G}_{n+1}] &= \frac{4}{\Delta t^2} [\mathbf{M}] [\mathbf{U}_n] + \frac{4}{\Delta t} [\mathbf{M}] [\dot{\mathbf{U}}_n] + [\mathbf{M}] [\ddot{\mathbf{U}}_n], \\
[\dot{\mathbf{U}}_{n+1}] &= \frac{2}{\Delta t} ([\mathbf{U}_{n+1}] - [\mathbf{U}_n]) - [\dot{\mathbf{U}}_n], \\
[\ddot{\mathbf{U}}_{n+1}] &= \frac{2}{\Delta t} ([\dot{\mathbf{U}}_{n+1}] - [\dot{\mathbf{U}}_n]) - [\ddot{\mathbf{U}}_n], \quad [\ddot{\mathbf{U}}_0] = [\mathbf{M}]^{-1}([\mathbf{f}_0] - [\mathbf{K}(\mathbf{U}_0)]),
\end{aligned} \tag{6.55}$$

where the initial values are considered for the case $n = 0$.

2. *Discrete energy-momentum conserving method:* As we shall use the algorithm for mass-spring examples, we utilize the formulation mentioned [5]. Hence, in this case, we drop the brackets for displacements and velocities used for general elastodynamics problems to emphasize that the algorithm is for problems of single (two) degree(s) of freedom whose potential function is a function of the relative distance. Find $\mathbf{U}(t_n) \approx \mathbf{U}_n$, $\dot{\mathbf{U}}(t_n) \approx \dot{\mathbf{U}}_n$ from the following discrete nonlinear equations:

$$\begin{aligned}
\frac{\mathbf{U}_{n+1} - \mathbf{U}_n}{\Delta t} &= \frac{\dot{\mathbf{U}}_{n+1} + \dot{\mathbf{U}}_n}{2}, \\
m \frac{\dot{\mathbf{U}}_{n+1} - \dot{\mathbf{U}}_n}{\Delta t} &= - \frac{\mathcal{V}'(\|\mathbf{U}_{n+1}\|) - \mathcal{V}'(\|\mathbf{U}_n\|)}{\|\mathbf{U}_{n+1}\| - \|\mathbf{U}_n\|} \frac{\mathbf{U}_{n+1} - \mathbf{U}_n}{\|\mathbf{U}_{n+1}\| + \|\mathbf{U}_n\|},
\end{aligned} \tag{6.56}$$

where $\frac{\mathcal{V}'(\|\mathbf{U}_{n+1}\|) - \mathcal{V}'(\|\mathbf{U}_n\|)}{\|\mathbf{U}_{n+1}\| - \|\mathbf{U}_n\|}$ is replaced with $\mathcal{V}''(\frac{1}{2}(\|\mathbf{U}_{n+1}\| + \|\mathbf{U}_n\|))$ as $\|\mathbf{U}_{n+1}\| \rightarrow \|\mathbf{U}_n\|$ (see [5]), and $\|\cdot\|$ denotes the Euclidean norm.

As the first, second, and third examples, we analyze the hardening and the softening bilinear mass-spring systems analyzed in [36] and a nonlinear mass-spring system with a hyperbolic nonlinearity. The characteristics of examples are:

$$\begin{cases} \ddot{u} + K(u) = 0 \\ u(0) = 4, \\ \dot{u}(0) = 0 \end{cases} \tag{6.57}$$

- Bilinear hardening

$$k(u) = \begin{cases} 100u & |u| \leq 2 \\ -1800 \operatorname{sgn}(u) + 1000u & |u| > 2 \end{cases}, \quad (6.58)$$

- Bilinear softening

$$k(u) = \begin{cases} 100u & |u| \leq 2 \\ 180 \operatorname{sgn}(u) + 10u & |u| > 2 \end{cases}, \quad (6.59)$$

As shown in Fig 6.4, the present algorithm successfully conserves the energy for softening and hardening cases. For the hardening and softening examples, to converge the Newton-Raphson method at each time step, we either used $DK(u)|_{u=u(0)}$ or $DK(u)|_{u=u(0)+T\dot{u}(0)}$ to maximize the norm of the left-hand-side of (6.50). It is observed that in addition to fulfillment of energy conservation, the new method converges to the solution resulting from the trapezoidal method with $\Delta t = 0.0001$ (see Figs 6.2 & 5.2). These results clearly show that the new method is a higher-order accurate method. It is worth mentioning that since the nonlinear equations are solved iteratively—thanks to the time integrals involved in the new method—the larger the time step, the greater the number of iterations is required to converge to the solution. Also, the main part of computation in this method is the evaluation of residuals, which consist of spatial and time integrals. Hence, in practice, to save computational cost, the time step should not be selected beyond some values even though the Newton-Raphson method converges.

Next, we analyze the nonlinear elastic 2d mass-spring system investigated in [5]: a nonlinear spring fixed at one end while, from the other side, it is connected to a mass. The governing equations of the motion read:

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -\frac{\mathcal{V}'(l)}{\sqrt{x^2 + y^2}} \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{k(\sqrt{x^2 + y^2} - l_0)}{\sqrt{x^2 + y^2}} \begin{bmatrix} x \\ y \end{bmatrix} = -k\left(1 - \frac{l_0}{\sqrt{x^2 + y^2}}\right) \begin{bmatrix} x \\ y \end{bmatrix}, \quad (6.60)$$

with the following parameters:

$$\begin{aligned} \mathcal{V}(l) &= \frac{1}{2}k(l-l_0)^2, \quad l = \sqrt{x^2+y^2}, \quad k = 15, \quad l_0 = 10, m = 2, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \\ \begin{bmatrix} \dot{x}(0) \\ \dot{y}(0) \end{bmatrix} &= \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \end{aligned} \quad (6.61)$$

As pointed out in [5], this example contains the geometric nonlinearity because of finite rotation and material nonlinearity. Hence, it could be a good representative of nonlinear elastodynamics. However, we only mention this example to indicate that while the new algorithm can successfully conserve the angular momentum and energy, it also yields accurate results. Figures 6.6 & 6.7, respectively, show the results corresponding to the trapezoidal rule and the present algorithm. To accelerate the convergence, we employed the following filter (see [4]):

$$\beta^f(\eta) = \left(\frac{1 + \cos(\pi\eta)}{2} \right)^4 (35 - 84 \left(\frac{1 + \cos(\pi\eta)}{2} \right) + 70 \left(\frac{1 + \cos(\pi\eta)}{2} \right)^2 - 20 \left(\frac{1 + \cos(\pi\eta)}{2} \right)^3), \quad (6.62)$$

As can be seen, while the trapezoidal rule fails to conserve energy and angular momentum, those constants have been preserved in this method. To compare the accuracy of the results, we also compare solutions obtained from different methods in Figs 6.6 & 6.7. From these figures, we draw a similar conclusion: the present approach is a higher-order accurate solution procedure in the nonlinear regime.

As the final example, we consider wave motion in a nonlinear elastic bar (see Fig 6.5b):

$$\begin{aligned} \ddot{u} &= \frac{\partial}{\partial x} \left(\frac{\sigma(x)}{\rho} \right), \\ \frac{\sigma(x)}{\rho} &= F_0 \frac{\partial u}{\partial x} + E_0 \left(\frac{\partial u}{\partial x} \right)^3, \\ u(x, 0) &= x, \quad \dot{u}(x, 0) = x, \\ F_0 &= 800, \quad E_0 = 2000. \end{aligned} \quad (6.63)$$

100 elements and linear Lagrangian spatial shape functions are taken. Also, we consider $\gamma = 0.2$,

apply the FFT method to evaluate the residuals. In the linear regime, the suitable values of N depend on the number of elements, time interval T^{total} , the initial and boundary conditions, and body forces. For example, the larger the time interval and/or the number of elements, the larger N is required. The same trends exist in the nonlinear regime as the corresponding solution is attained by employing the linearized problem where the body force is the residual calculated from the previous iteration. Furthermore, as explained in [4], in the linear convolution finite element, one can employ filters to detach the spurious part of the solution without extra computational cost. These filters may accelerate the convergence of the method provided that sufficiently many time-wise terms are considered. Hence, the numerical results for this example are shown for both cases. For the case of filter, we use $\beta(\eta) = e^{-700\eta^4}$. In Fig 6.11, the displacement and the velocity corresponding to the endpoint of the bar have been shown. A convergence trend can be tracked in the displacement and the velocity when increasing N . This fact is also observable for energy (Fig.6.11a): energy is not conserved for the case of $N = 10$ while it is approximately secured when $N = 200$. Fig.6.11b shows the energy when the filter has been employed. The growth and decay of the energy are noticeable when insufficient N is selected. However, for larger N , better conservation of the energy is recognizable than for the cases without filters (Fig 6.11a). Figs 6.13 & 6.14 show the displacement and velocity across the bar for $0 \leq t \leq 0.3$ without and with filter, respectively. To obtain the results $N = 200$ has been considered. The colors in these graphs represent values of displacement and velocity. From these figures, it is clear that applying filters results in a smoother solution in the velocity while the displacement field remains unchanged.

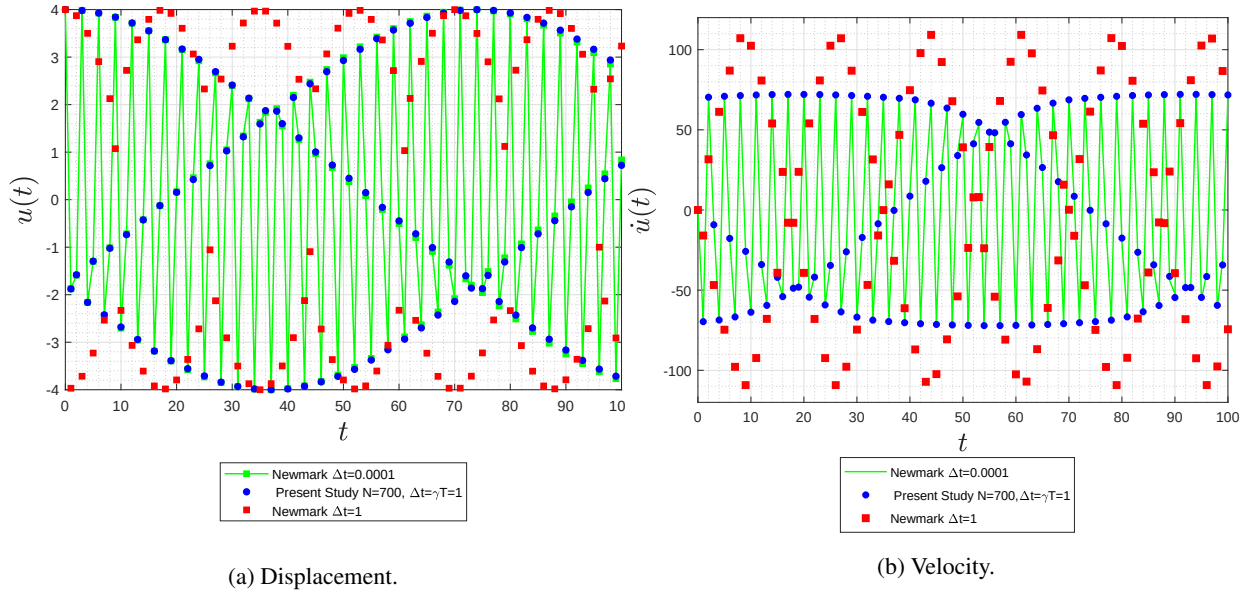


Figure 6.2: Comparison of displacement and velocity for the hardening problem.

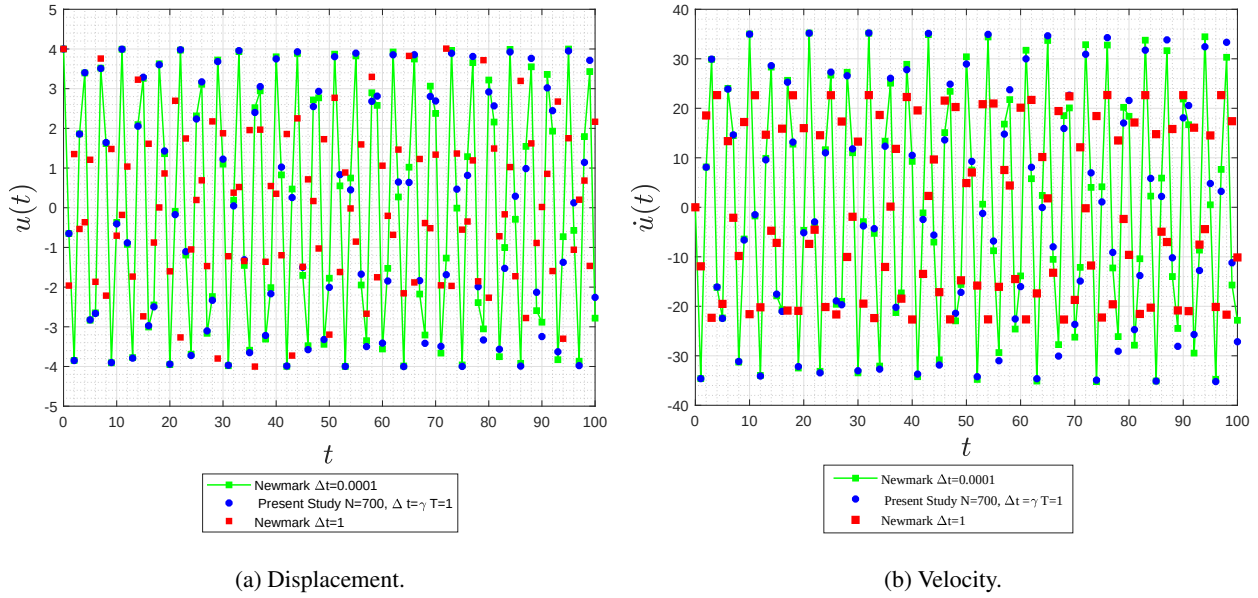
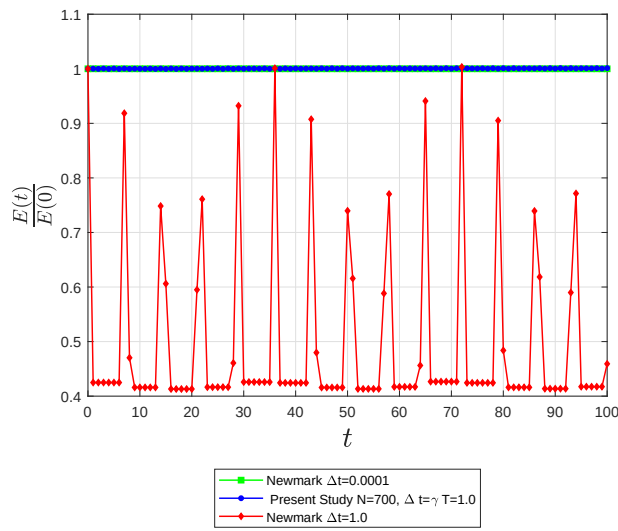
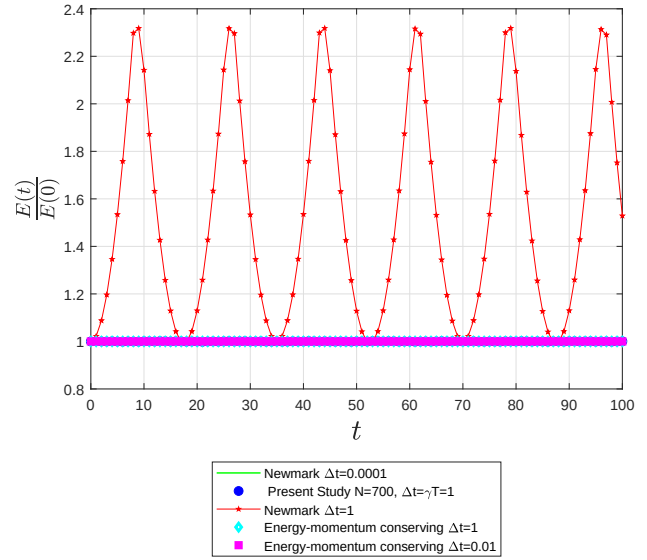


Figure 6.3: Comparison of the solution for the softening problem, $\gamma = 0.5$, $T = 2$.

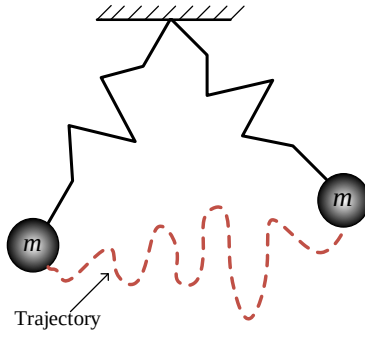


(a) Softening.

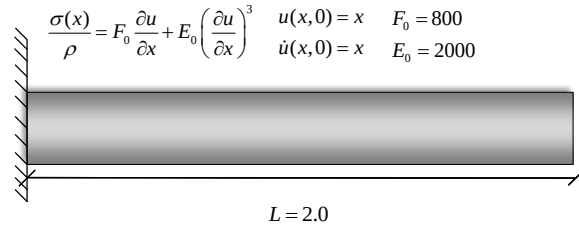


(b) Hardening.

Figure 6.4: conservation of the energy for the softening and the hardening problems, $\gamma = 0.5$, $T = 2$.



(a) 2d mass-spring system.



(b) Nonlinear elastic bar.

Figure 6.5: The 2d mass-spring system and the bar examples.

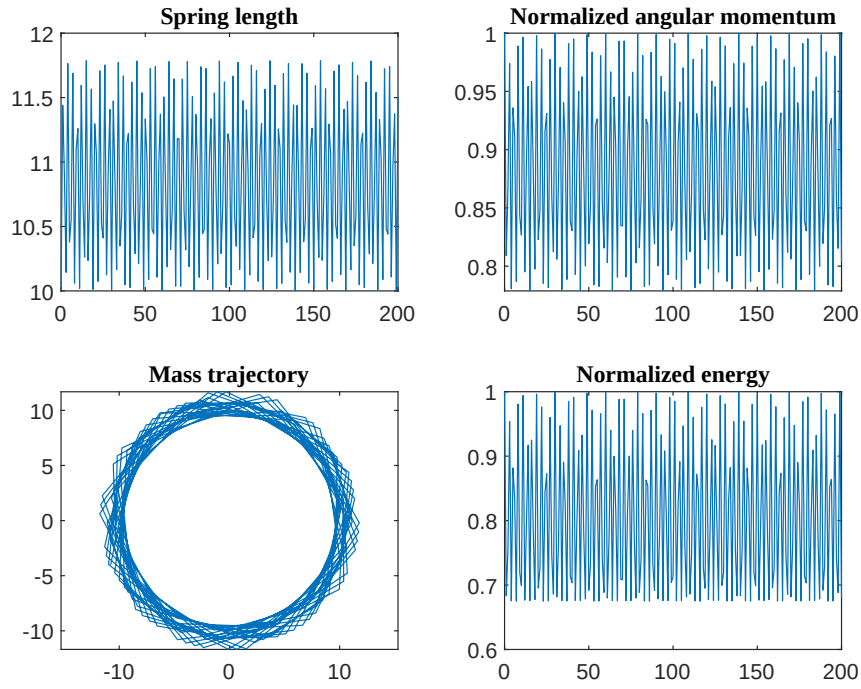


Figure 6.6: Solution of the 2d mass-spring system with the trapezoidal rule: $\Delta t = 1$.

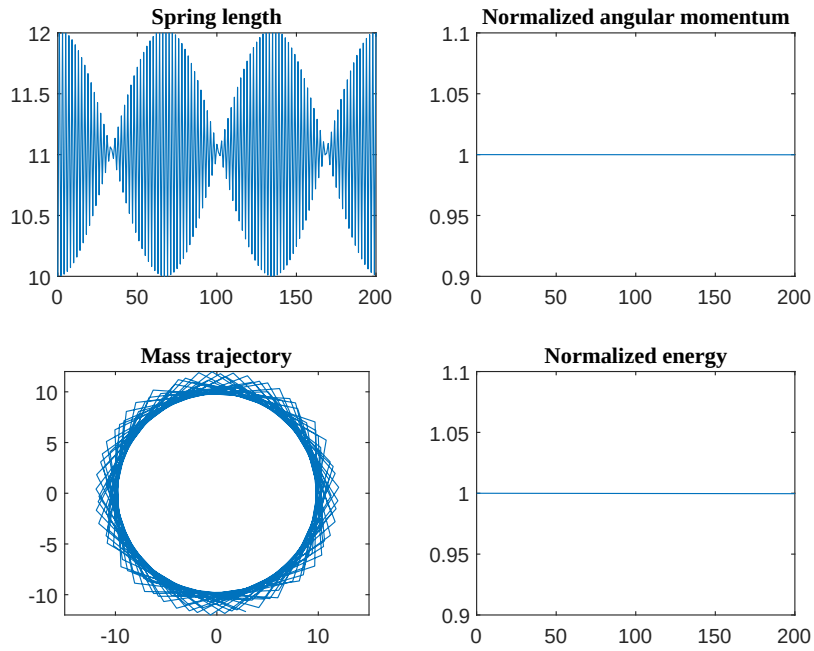


Figure 6.7: Solution of the 2d mass-spring system with the new method: $\Delta t = (1 - \gamma)T = 1$, $N = 400$ with the filter (6.62).

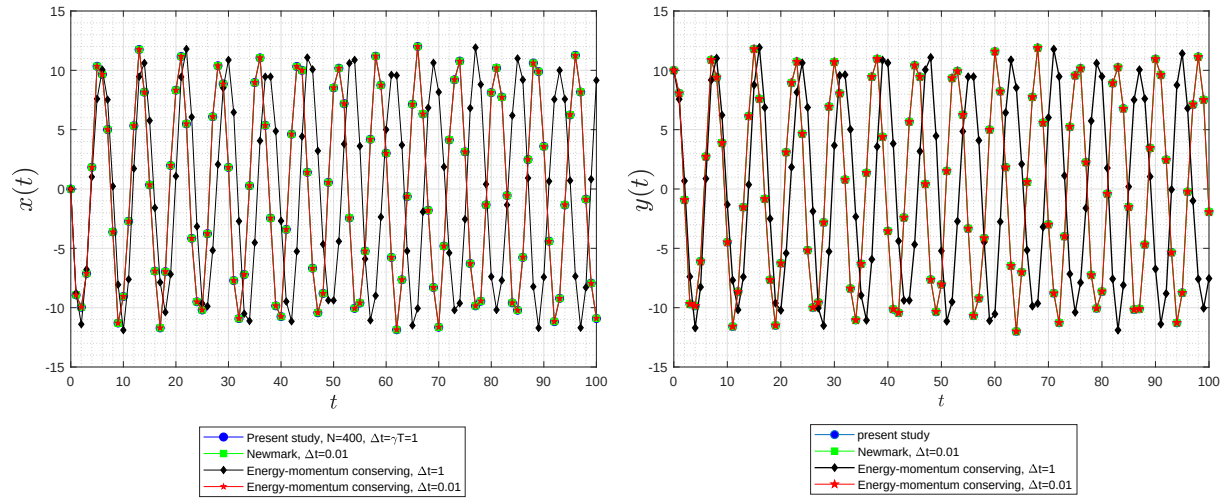


Figure 6.8: Comparison of the displacements (for the 2d mass-spring system) resulted from different methods.

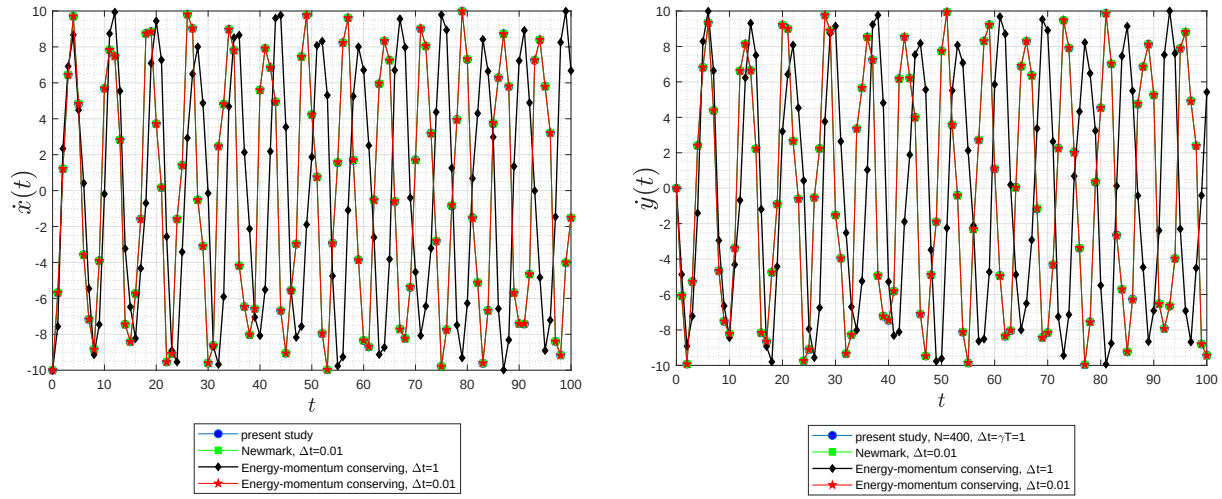
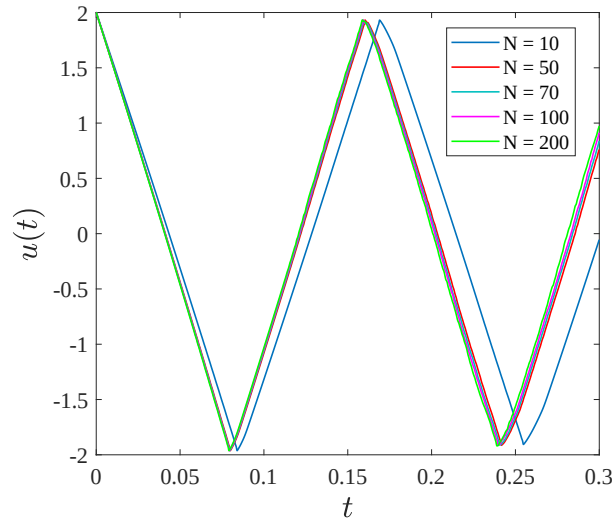
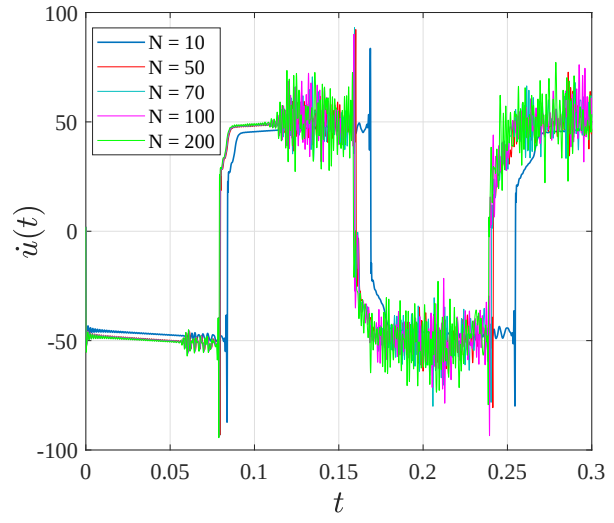


Figure 6.9: Comparison of the velocities (for the 2d mass-spring system) resulted from different methods.

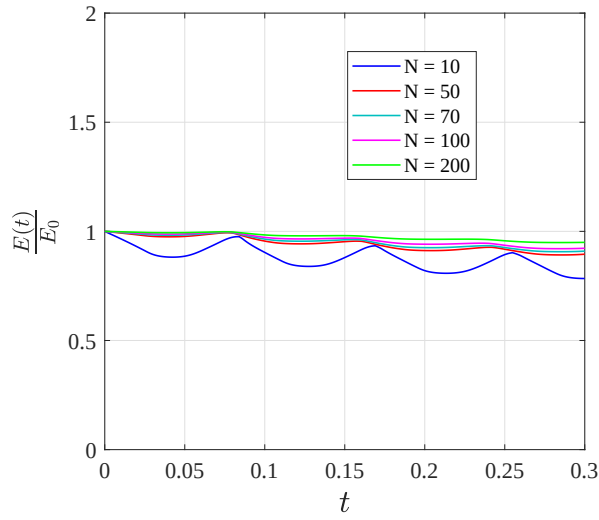


(a) Displacement.

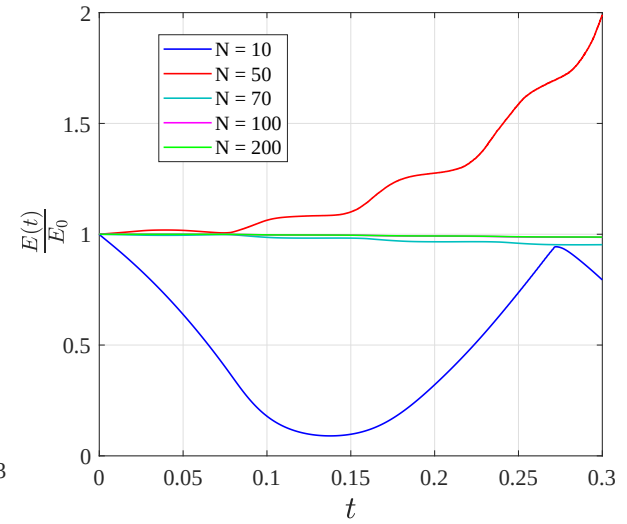


(b) Velocity.

Figure 6.10: Comparison of the solution of the bar problem (endpoint) for different values of N .

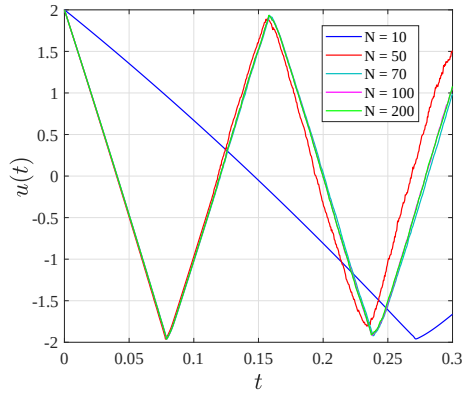


(a) Without filter.

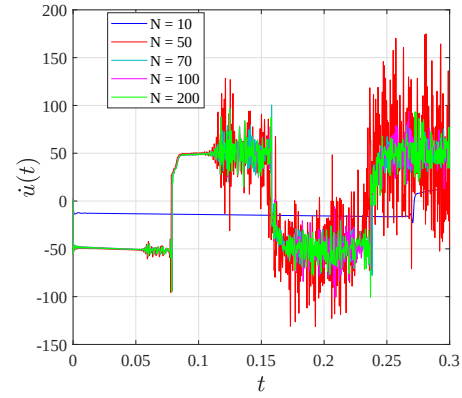


(b) with filter $\beta(\eta) = e^{-700\eta^4}$.

Figure 6.11: Comparison of the energy conservation in the bar problem for different values of N .

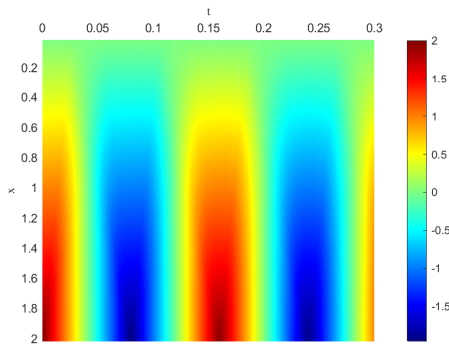


(a) Displacement.

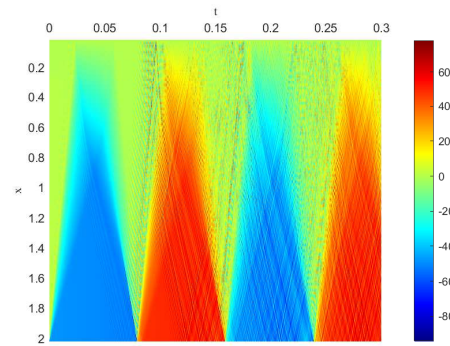


(b) Velocity.

Figure 6.12: Comparison of the solution of the bar problem (endpoint) for different values of N with the filter $\beta(\eta) = e^{-700\eta^4}$.

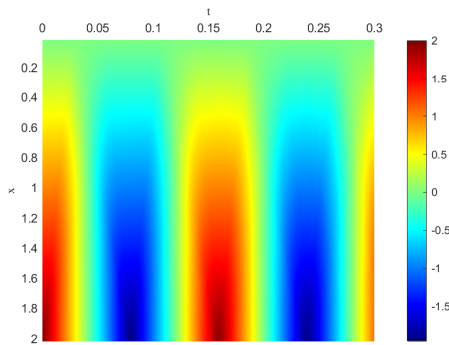


(a) Displacement.

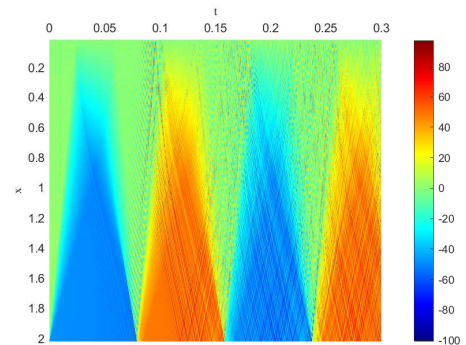


(b) Velocity.

Figure 6.13: Displacement and velocity profiles for the bar problem.



(a) Displacement.



(b) Velocity.

Figure 6.14: Displacement and velocity profiles for the bar problem with the filter $\beta(\eta) = e^{-700\eta^4}$.

Chapter 7

Conclusion and Future Work

In this dissertation, the alternative field equations in elastodynamics were thoroughly considered. The stress framework to solve the IBVP for an electromagneto-elastic material has been established in chapter 3. To that end, using an alternative representation of the equations of motion stemming from [43, 44], with the aid of Lemmas 1 and 2, we generalized the Ignaczak equation of elastodynamics for such materials. For the set of conditions enumerated in Theorem 4, the uniqueness of the solution was secured. The reciprocity theorem, which is of importance for a class of numerical algorithms, was obtained in Theorem 5. Finally, planar problems for a homogeneous transversely isotropic electromagneto-elastic material were formulated within the context of Corollaries 1 and 2 as particular cases of the general 3D problem.

The formulation presented here is an alternative framework through which one can analyze the elastodynamics of electromagneto-elastic materials. This formalism is an appropriate approach particularly when the mechanical boundary conditions are entirely traction-type. For example, on the basis of (3.21), [26] proposed a variational formulation in terms of stress for the case of pure elastodynamics such that the stress boundary conditions are exactly satisfied. This is a significant step in numerical solutions, since the satisfaction of such boundary conditions in the displacement approach is only secured implicitly. Furthermore, as stated in [14], there exist some class of elastic bodies whose constitutive equations are implicit in the nonlinear regime, leading to expressing the infinitesimal strain tensor as a nonlinear function of the Cauchy stress. These types of constitutive equations are not invertible in general [14] and, thus, the stress language may be suitable to successfully deal with the relevant IBVPs (e.g., see [15]). In this regard, the knowledge of linear stress formulation is the cornerstone as there are methods in which the solution

of nonlinear problems is iteratively constructed in terms of linear solutions (e.g., Newton–Raphson method). Also, the fact that there exist several common physical phenomena in the linear and nonlinear regimes demonstrates the importance of the linear solution. However, this approach has its own challenges and needs to be considered in future works. For instance, the stress formulation contains more unknowns compared to the displacement approach, which may increase the cost of computations, particularly in numerical methods associated with spatial and time discretization.

In parallel to [26], on the basis of alternative field equations for electromagneto-elastic materials the convolutional variational principles have been derived and proved rigorously in chapter 4. In Theorem 10, a general convolutional variational form, in which the admissible process is not required to meet any field equations and/or boundary/initial conditions, was derived. The convolutional variational principle corresponding to an admissible process that meets only the strain-displacement relation was formulated in Theorem 11. The result for a more restricted process - namely, a kinematically admissible process - was formulated in Theorem 12. As a corollary of Theorem 12, the convolutional variational principle corresponding to a kinematically admissible displacement-potential was set in Theorem 13. Also, through Theorem 14 and Theorem 15, variational principles in terms of stress was established, respectively, for general problems and traction problems. On the application side and the future work, the results of the present work provide a robust basis for a numerical analysis of electromagneto-elastic materials with general domain geometry and boundary/initial conditions.

In chapter 5, an approach dealing with the governing PDEs of elastodynamics and structural mechanics was suggested. It was shown that, for one step methodology, other than the end point of a given time interval, the method converges to the problem’s solution at all times provided that enough time-wise terms are included. Also, it has been indicated that the ‘endpoint problem’ can be eliminated either by means of (5.63) or the time step version of the algorithm. The main features of the algorithm can be listed as follows:

- although the algorithm is stemmed from in a coupled variational time-space form, there is no space-time discretization.

- the present solution procedure accords with those of time integration methods. That is, the procedure is quite similar to FEM for stationary problems.
- by comparison to other time integration methods, the present algorithm is capable of obtaining the solution for an arbitrary time interval with only one set of matrix calculations provided that a sufficient number of terms is taken. As a result, the algorithm can be employed for both small and large time intervals. Also, in the time step version, in contrast to the classical time integration algorithm, time step length and the solution at a particular time(s) in every time interval can be arbitrarily determined.
- the algorithm successfully captures general forms (5.54). However, because of a certain property of the resulted matrix in (6.50), the algorithm is a computationally suitable choice for elastodynamics problems.
- in contrast to time integration algorithms, the algorithm keeps the history of the solution in the sense that the corresponding solution converges to the response of structure for almost all times.
- by increasing N , the initial conditions and conservation of the energy are satisfied appropriately for any given time interval.
- it is feasible to apply filters to improve the convergence of the method without computational cost since the explicit form of the solution is obtained in this method. These filters may exclude higher modal frequency while the lower ones remain unchanged; a desired property leads to removal of artificial high modes resulted from spatial discretization of the problem.
- higher-order of accuracy can be achieved.

Hence, the new finite element method can be successfully utilized, both practice and future research, as the main solver for analysis of linear elastic materials.

In chapter 6, a novel solution procedure to deal with the nonlinear governing equations of motion of elastic bodies was proposed. The formulation was presented for the infinitesimal strain theory where the nonlinearity comes from constitutive equations, i.e., material's nonlinearity. However,

the method can be straightforwardly extended for the case of large deformation (geometrical nonlinearity). The results show that the method can successfully address the energy conservation issue existing in the trapezoidal method without further enforcing parameters such as Lagrange multipliers. In addition to that, the method is conspicuously versatile to obtain accurate results. The main characteristics of the algorithm are:

- The linear solver, i.e., the convolution FEM, does not require discretization in time, although it is rooted in a coupled space-time variational principle.
- The solution procedure is similar to the classical nonlinear dynamic finite element method where the Newmark average acceleration method (the trapezoidal rule) is combined with the Newton-Raphson method.
- From a computational perspective, the resultant block arrowhead matrix in (6.50) justifies the application of the method for nonlinear elastodynamics.
- The linear solver is suitable for parallel programming as most operations in the convolution-Newton-Raphson solver can be executed independently.
- A higher-order accurate algorithm can be automatically generated when increasing N .
- It is feasible to apply filters to improve the results as the explicit form of the solution is obtained in the linear solver.

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