

AN INFORMATION-THEORETIC ANALYSIS OF CONTROL AND FILTERING
LIMITATIONS BY THE I-MMSE RELATIONSHIPS

BY

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DISSERTATION

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Abstract

Fundamental limitations in control and filtering, as the trade-offs between the pursuit of performance and the constraints in the design, have been providing guidelines to the practitioners for the design of controllers and filters in a broad range of applications. Some classical control and filtering trade-offs include Bode's sensitivity integral or the waterbed effect in control and the minimum mean-square error in estimation. By modeling control and filtering systems as noisy communication channels, information-theoretic metrics, such as total information (rate), data transmission rate, and differential entropy (rate) difference, can be also utilized to characterize and quantify the fundamental limitations. Previously, complex analysis and the information-theoretic method based on (differential) entropy are two primal tools for calculating and studying these trade-off metrics. However, both methods can only handle a specific type of systems and are not general enough. In this dissertation, by resorting to the I-MMSE (mutual information—minimum mean-square error) relationships, a comprehensive and estimation-based information-theoretic method is proposed to uniformly calculate and analyze total information (rate), i.e., the mutual information (rate) between transmitted message and channel output, or equivalently, the directed information (rate) from channel input to output, as a trade-off metric that captures the fundamental limitations of numerous control and filtering systems in continuous-time, discrete-time, and infinite-dimensional scenarios. In the continuous-time scenario, with the aid of continuous-time I-MMSE relationships and optimal filtering techniques, we relate total information (rate) to the noise sensitivity trade-offs, (average) performance cost, and lowest achievable mean-square estimation error of various continuous-time control and filtering systems in LTI (linear time-invariant), LTV (linear time-varying), and nonlinear forms. In the discrete-time scenario, we first derive the I-MMSE relationships of discrete-time additive white Gaussian channels with and without feedback. With these relationships and optimal

filtering techniques, we then revisit and extend the trade-off properties and interpretations of total information (rate) in discrete-time LTI, LTV, and nonlinear control and filtering systems. Lastly, based on the I-MMSE relationships in abstract Wiener space, a direct and estimation-based scheme is proposed to calculate the total information (rate) of infinite-dimensional control and filtering systems under additive arbitrary Gaussian noise. By applying this scheme to the finite-dimensional systems, we not only recover some well-established fundamental limitations in the control and filtering systems subject to white Gaussian noise, but provide a direct scheme to calculate and study the total information (rate) in some complicated and rarely investigated systems under colored Gaussian noise.

To My Parents and Prof. Naira Hovakimyan!

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List of Abbreviations

ARE	Algebraic Riccati Equation.
AWGN	Additive White Gaussian Noise.
CT	Continuous-Time.
DT	Discrete-Time.
I-MMSE	Mutual Information – Minimum Mean-Square Error.
LQG	Linear-Quadratic-Gaussian.
LQR	Linear-Quadratic Regulator.
LTI	Linear Time-Invariant.
LTV	Linear Time-Varying.
MMSE	Minimum Mean-Square Error.
MS	Mean-Square.
RDE	Riccati Differential Equation.
SNR	Signal-to-Noise Ratio.

Chapter 1

Introduction

Fundamental limitations of control and filtering have been providing guidelines to the practitioners for the design of controllers and filters in a broad range of applications. In general, these limitations are rooted in the physical limitations of systems and appear as the trade-offs between the pursuit of performance and the constraints in the design. Some recognized limitations and trade-off metrics include but are not limited to the Bode-type integrals that characterize the sensitivity trade-offs in noise attenuation [1–3], data rate constraints [4, 5], total and directed information rates [6–8], optimum control performance [9–11], and minimum estimation error [12–14].

Complex analysis and information theory are two primal paradigms for characterizing and investigating these limitations and trade-offs (see [15] for a review), although their weaknesses are also very discernible. The complex analysis method and associated trade-offs, which describe the systems by their input-output relationships or transfer functions, are convenient for studying the linear time-invariant (LTI) systems and capturing the frequency-domain trade-offs, such as Bode’s integral and its variants [2], but can hardly be applied to the time-varying or nonlinear systems. By contrast, the traditional information-theoretic method and trade-offs, which model the systems as noisy communication channels and characterizing the fundamental limitations with information-theoretic quantities, are favorable for analyzing the discrete-time time-varying or nonlinear systems and capturing the time-domain or channel-orientated trade-offs, such as data and information rates [4, 6], time-averaged performance cost [11], and minimum estimation error [16], but are cumbersome and indirect when dealing with the continuous-time systems and trade-offs [7].

To cope with these challenges and search for a trade-off metric as well as an analysis method

that uniformly apply to different control and filtering systems, in this dissertation, by leveraging the I-MMSE (mutual information—minimum mean-square error) relationships, we put forward a universal and direct information-theoretic method to investigate total information (rate) as a trade-off metric characterizing and quantifying the fundamental limitations of various discrete-time, continuous-time, finite-dimensional, and infinite-dimensional control and filtering systems with both linear and nonlinear components. The remaining sections in this chapter can be read separately, as each of them corresponds to an independent introduction or literature review section for the following chapters.

1.1 Fundamental Limitations of Continuous-Time Systems

In [Chapter 2](#), by resorting to Duncan’s theorem or the continuous-time I-MMSE relationships, we conduct an information-theoretic analysis of the fundamental limitations in continuous-time control and filtering systems. The study of fundamental limitations or trade-offs of continuous-time systems can be traced back to the seminal work of Hendrik W. Bode, who proved that for continuous-time open-loop stable systems, the integral of logarithmic sensitivity function over the frequency range must equal zero [\[1\]](#). Freudenberg and Looze later showed that for open-loop unstable systems, this integral equals the sum of open-loop unstable poles [\[17\]](#). Bode’s integral and its variants not only characterize a control trade-off of noise attenuation on the frequency range [\[18, 19\]](#), but quantify the optimum cost of cheap control problem over the time horizon [\[9, 20\]](#). As the dual problem of control, estimation and its performance limits also attracted a great amount of attention. Goodwin and Seron proposed a class of integral constraints, analogous to the control Bode-type integrals, for the filtering, prediction and smoothing problems of continuous-time LTI systems [\[21\]](#). Braslavsky et al. revealed the connection between Bode-type integrals and the lowest achievable mean-square (MS) estimation error in the continuous-time LTI filtering systems under vanishing process and measurement noise [\[12\]](#). However, due to the lack of a powerful mathematical tool other than complex analysis, most of these classical results were confined to the LTI scenario (see [\[2\]](#) for a compilation), with only a few exceptions trying to explore the fundamental trade-offs for time-varying and nonlinear systems [\[22–24\]](#).

As information theory was introduced to characterize the fundamental trade-offs, control and

filtering systems (with more time-varying and nonlinear elements) were modeled and analyzed as communication channels and brought to the attention of researchers. Nair and Evans proved that a discrete-time LTI plant can be stabilized by quantized control if and only if the available data rate exceeds the instability rate of the plant, i.e., sum of logarithmic unstable poles [25]. Zang and Iglesias in [26] and Martins and Dahleh in [27] respectively used the input-output entropy rate difference and total information rate as an information-theoretic interpretation of discrete-time Bode's integral, and similar to the data rate constraint, these quasi-forms of Bode's integral are bounded below by the instability rate of LTI plant. Fundamental limitations of networked control systems [8, 28], quantized control systems [4, 5, 29], LQR/LQG (linear-quadratic-Gaussian) control systems [10, 11, 30], stochastic switched systems [31], contractive nonlinear systems [32], and decentralized systems [33] have also been investigated via information-theoretic approaches, wherein the open-loop unstable poles and non-minimum phase zeros serve as a keystone in quantifying the control trade-offs. Information theory was introduced to interpret the filtering limits in the 1970's, when entropy and mutual information were utilized to investigate the performance limits of filters [34–36]. More recently, the limits of entropy reduction and maximum relative entropy were used to design and analyze the Kalman filter in [37, 38], and rate-distortion function was utilized to describe the filtering trade-offs and synthesize a filter in [13, 14]. Nevertheless, most of the aforementioned analyses only considered the discrete-time setting with LTI plant and relied on the Kolmogorov-Bode formula [32, 39], Szegő limit theorem [40, 41], and their variants [24, 42], accompanied by the assumptions of stationary and Gaussian auto-regressive signals, to connect the entropy rate differences with either a mutual/directed information rate or a quasi-form of Bode's integral. More importantly, entropy (rate), a pivotal tool in capturing the fundamental limitations of discrete-time systems, is undefined for the continuous-time processes over the infinite-dimensional distributions (see the comments in [26, 43] or Remark 2.4 for a technical explanation). Therefore, although information theory is a powerful tool in analyzing discrete-time systems, it seems inadequate for continuous-time systems at first glance.

To study the fundamental continuous-time trade-offs with information-theoretic tools, a feasible scheme is to discretize the original continuous-time system to a discrete-time system via sampling and then apply the established tools and methodology to the approximate system. By making this detour, Li and Hovakimyan proposed a Bode-like integral for the continuous-time LTI control systems

with causal stabilizing controllers [7]. Fang et al. studied the power gain from the disturbance signal to the error signal [8], and Wan et al. investigated the complementary sensitivity trade-offs in continuous-time LTI systems [44]. However, the sampling-based scheme is essentially a compromise when no direct information-theoretic approach is available to the continuous-time problems; the sampling process would inevitably require multiple regularity assumptions on, for example, the time partitions and the boundedness of functions or processes [45, 46], and an auxiliary time-delay has to be imposed for avoiding the causality paradox [31, 47]. Another feasible scheme is to embed the input-output relations into the complex Hardy spaces, such as the H_2 and H_∞ spaces. Braslavsky et al. in [48] and Rojas et al. in [49] cast the signal-to-noise ratio (SNR) constrained channel into a classical H_2 optimization framework and analyze the channel capacity constraints for both continuous- and discrete-time control systems. Unfortunately, noticing that it has complex analysis and analytical functions at its core, this scheme falls short when dealing with non-LTI systems and non-stationary signals. Moreover, with the aid of Duncan’s theorem, Tanaka put forward a semidefinite representation of the rate-distortion function to describe the trade-offs in continuous-time LTI filtering system [50]. It is worth noting that each of the preceding information-theoretic trade-offs and methods only provides an ad-hoc interpretation for a specific type of control or filtering problem, and few results consider the non-LTI control or filtering systems. To answer these challenges, a unified framework and a direct information-theoretic method are put forward in Chapter 2 to calculate and analyze the trade-off metrics of continuous-time control and filtering systems.

1.2 Fundamental Limitations of Discrete-Time Systems

In Chapter 3, we extend the I-MMSE relationships to the discrete-time additive white Gaussian channels with and without feedback and perform an information-theoretic analysis of the fundamental limitations in discrete-time control and filtering systems. Total or directed information rate has been identified as an important information-theoretic metric related to the sensitivity trade-offs or Bode-type integrals [7, 8, 27, 28], average performance cost [11, 16], and minimum estimation error [14], of both discrete- and continuous-time control and filtering systems. Conventionally, (differential) entropy rate, Kolmogorov-Bode formula [32, 39], and their variants are the primal tools

and paradigm for studying and calculating this information rate. Following this paradigm, Zang and Iglesias first showed that the entropy rate difference between the input and output of the control channel serves as an information-theoretic interpretation of Bode’s integral in discrete-time LTI control systems [26]. By utilizing the identity between total information and differential entropy, e.g., (3.15) in Proposition 3.7, Martins and Dahleh later derived an entropy inequality and an information rate inequality that define the fundamental constraints for the plant’s instability rate, a total information rate, in discrete-time control systems; in particular, when the systems are stationary LTI, a Bode-like integral constraint can be restored from their inequalities [27]. Directed information (rate), which equals total information (rate) as (3.4) and (3.7) in Chapter 3 show, was used to capture the fundamental trade-off between communication rate and performance cost of quantized LQR/LQG problems in [11]. For the filtering problem, total information was utilized to formulate the rate-distortion function and design the filter with trade-off criterion in [10, 13]. Similar properties and results were also found for the total and directed information rates in continuous-time control and filtering systems [7, 16, 50].

Apart from the conventional information-theoretic method, the I-MMSE relationship [51], i.e., the relationship between mutual (or total) information in information theory and (predicted, causal, or non-causal) minimum mean-square estimation error in estimation theory, provides an alternative and estimation-based approach to calculate and analyze total information (rate) as a control and filtering trade-off metric. This alternative approach has been utilized to study the control and filtering trade-offs of continuous-time systems, in which the differential entropy rate is not well-defined and cumbersome to use. By using Duncan’s theorem, an identity between total or mutual information and the causal MMSE of input in continuous-time channel, total information rate was studied as a trade-off metric of continuous-time control and filtering systems in [16], and rate-distortion function (or mutual information) was exploited to design an optimal Kalman-Bucy filter with trade-off concern in [50, 52]. However, few existing papers discussed or invoked the I-MMSE relationship when calculating or analyzing the total and directed information (rates) in discrete-time control and filtering systems. One reason, as above mentioned, is that for the discrete-time channels and systems, we already have some mature and convenient tools, such as entropy rate and Kolmogorov-Bode formula, to conduct a pure information-theoretic analysis. More importantly, although the continuous-time I-MMSE relationships, e.g., Duncan’s theorem, have been introduced and developed

for decades [53, 54], their discrete-time counterparts have rarely been scrutinized before. Only Bucy and Guo et al. briefly commented on the I-MMSE relationship of the discrete-time non-feedback channel in [51, 55], while the I-MMSE relationship of the discrete-time channel with feedback, which is essential for analyzing the discrete-time control systems, has been due for a long time. In Chapter 3, we fill this gap by deriving the I-MMSE relationship of discrete-time AWGN channels with and without feedback, which is then leveraged to calculate and analyze total information (rate) as a trade-off metric in discrete-time control and filtering systems.

1.3 Fundamental Limitations in Wiener Spaces

In Chapter 4, by resorting to the I-MMSE relationships in Wiener spaces, we calculate the total information (rate) and characterize the fundamental limitations of infinite-dimensional control and filtering systems. By applying these results to the finite-dimensional systems, we then revisit and explore the fundamental limitations of some common but rarely investigated control and filtering systems of finite dimensions. Differential entropy and I-MMSE relationships are two primal tools for calculating and studying total information (rate). The former method resorts to the classical information-theoretic quantities and relationships, and is mature for analyzing the discrete-time control and filtering systems of finite dimensions. While the I-MMSE method was relatively less discussed, some recent studies in [16, 56] indicate that it has the potential to be applied to a broader class of control and filtering systems, e.g., discrete- and continuous-time systems of both finite- and infinite-dimensions.

The I-MMSE relationships generally refer to the collection of relations between total (or mutual) information in information theory and (predicted, causal or non-causal) minimum mean-square error (MMSE) in estimation theory. The early development of I-MMSE relationships can be traced back to the 1970's when Duncan and Kadota et al. established an identity between total information and causal MMSE of channel input in continuous-time additive white Gaussian noise (AWGN) channels with and without feedback [53, 54]. The progresses of measure transformation technique in stochastic calculus and likelihood ratio calculation in signal processing and statistics also expedited some sequential buildups of the I-MMSE relationships [57–60]. More recently, Guo et al. proved that the derivative of mutual information with respect to (w.r.t.) the signal-to-noise ratio (SNR)

equals half the non-causal MMSE of channel input in the continuous-time AWGN channel without feedback [51], which is known as the differential I-MMSE relationship and has been extended to the AWGN channels with feedback and memory in [61] and abstract Wiener space in [62]. The extension of identity between total information and causal MMSE to abstract Wiener space was contributed by Mayer-Wolf and Zakai in [63]. Nevertheless, as pointed out in [61], the I-MMSE relationships of colored Gaussian channels on either finite- or infinite-dimensional spaces were rarely discussed or examined before.

The I-MMSE relationships provide a comprehensive toolbox to calculate and analyze total information (rates) in different control and filtering systems, which can be modeled into communication channels with and without feedback. With the aid of Duncan’s theorem in [53, 54], total information (rate) was utilized to capture and imitate some fundamental trade-off properties and quantities, such as Bode’s sensitivity integral and lowest achievable MS estimation error, in continuous-time control and filtering systems subject to standard Brownian motion noise and various (e.g., LTI, LTV, and nonlinear) plant, controller, and filter models in [16]. After we extended the I-MMSE relationships and Duncan’s theorem to discrete-time AWGN channels with and without feedback, [56] complemented the results in [16] by analyzing the discrete-time counterpart control and filtering systems and trade-offs. Moreover, Duncan’s theorem was also leveraged in [50] to design an optimal filter subject to a trade-off criterion. However, all the aforementioned and most existing information-theoretic analyses of control and filtering limitations are confined to the finite-dimensional systems under AWGN or standard Brownian motion noise. Few efforts were devoted to the information-theoretic characterization and analysis of control and filtering limitations in the infinite-dimensional systems with general (colored) Gaussian noise. To dismantle these challenges, in Chapter 4, a direct scheme based on the I-MMSE relationships in abstract Wiener spaces is provided for calculating the total information (rate) and characterizing the fundamental limitations of infinite-dimensional control and filtering systems under arbitrary Gaussian noise.

1.4 Contributions, Structure and Miscellaneous

This dissertation consists of three main chapters, in which the I-MMSE relationships are utilized to characterize and calculate the fundamental limitations of control and filtering systems in finite-

dimensional continuous-time (Chapter 2), finite-dimensional discrete-time (Chapter 3), and abstract infinite-dimensional (Chapter 4) scenarios, respectively. Each of these chapters is sourced from an independent research article published by the author. Chapter 2 is mainly based on [16]; Chapter 3 is based on [56], and Chapter 4 is based on [64]. In addition to these three research articles, [19, 44, 65, 66] published by the author during his Ph.D. study, are also closely related to the topic discussed in this dissertation. The main contributions and contents of this dissertation in the following chapters can be summarized as follows:

Chapter 2 lifts the information-theoretic analysis of fundamental control and filtering limitations to continuous function spaces by using Duncan’s theorem or the continuous-time I-MMSE relationships. Continuous-time control and filtering systems are respectively modeled as additive Gaussian channels with and without feedback, and the total information rate is identified as a control and filtering trade-off metric and directly calculated from the minimum mean-square estimation error of channel input. Inequality constraints for the trade-off metric are first derived in a general nonlinear setting and then applied to capture the fundamental limitations of various control and filtering systems subject to linear and nonlinear plants. For the linear systems, we show that total information rate has similar properties as some established trade-offs, such as Bode-type integrals and lowest achievable mean square (MS) estimation error. For the nonlinear systems, we provide a direct method to compute the total information rate and its lower bound by the Stratonovich-Kushner equation.

Chapter 3 extends the I-MMSE relationships to the discrete-time additive white Gaussian channels with and without feedback for the first time, and proposes a new paradigm to estimate and analyze total information rate as a control and filtering trade-off metric. With the proposed framework, we then analyze and extend the trade-off properties of total information rate for a variety of discrete-time control and filtering systems, e.g., LTI, LTV, and nonlinear, and provide an alternative approach to investigate total information rate via optimal estimation.

Chapter 4 extends the notion of total information (rate) as a trade-off metric characterizing the fundamental limitations in control and filtering to the systems of infinite dimensions. With the aid of I-MMSE relationships in Wiener spaces, we propose a direct and estimation-based scheme for calculating and analyzing the total information (rate) of abstract or infinite-dimensional control and filtering systems under arbitrary Gaussian noise. By applying this scheme to finite-dimensional

systems, we then provide several examples to show that the proposed method can not only recover some well-established control and filtering trade-off metrics, but provide a guideline to characterize and analyze the control and filtering trade-offs for some complicated and rarely discussed systems.

[Chapter 5](#) concludes this dissertation and discusses some future research topics.

The structure of this dissertation and interconnections between chapters and sections are outlined as follows.

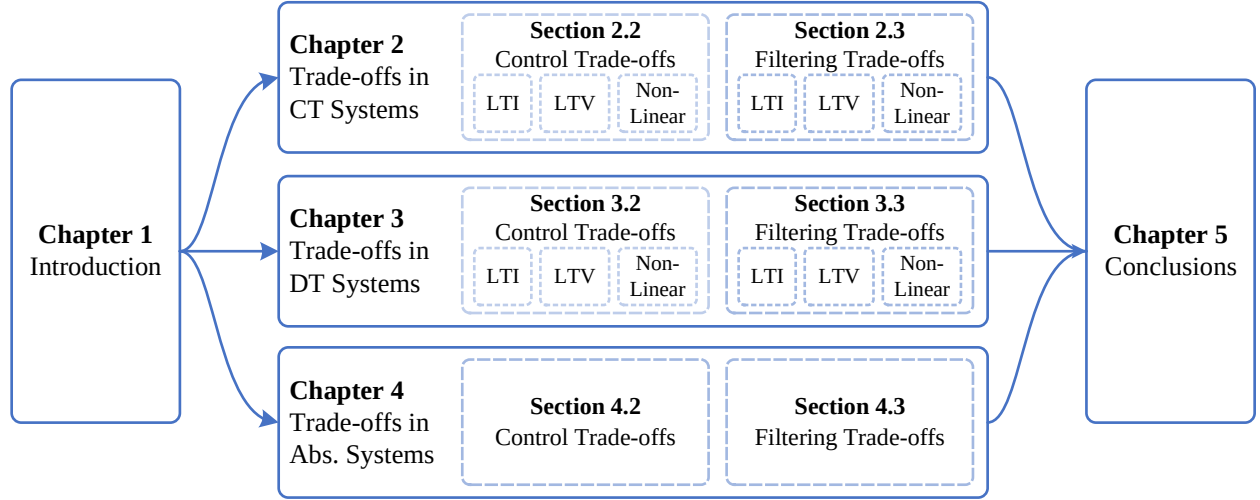


Figure 1.1: Structure of dissertation. CT, DT, and Abs. are respectively the abbreviations for continuous-time, discrete-time, and abstract.

It is worth noting that since this dissertation is a compilation of three independent research articles, to avoid the abuse of notation and ambiguity, the definitions of notations, abbreviations and indices are normally exclusive and self-contained within each chapter. Meanwhile, to give our readers a more intuitive view on the novelty and improvements made in this dissertation, the following table summarizes and compares this dissertation or [16, 56, 64] with existing literature from the perspective of whether their trade-off quantities and associated methods can or have been applied to characterize the fundamental limitations of a) both control and filtering systems, b) continuous- and discrete-time systems, c) linear time-varying and nonlinear systems, d) systems with non-Gaussian and non-stationary signals, e) infinite-dimensional systems, and f) frequency- or time-domain.

Table 1.1: Some control and filtering trade-offs and their application scenarios.

Trade-off quantity	Methods	Ctrl. & Filtering	Cont. & Disc.	LTV & Nonlinear	Non-Gaussian & Non-stationary	Inf.-Dim.	Freq. or Time
Bode's integrals [1, 2, 21, 67, 68]	Complex analysis using transfer func.	✓	✓				Freq.
Bode-like integrals [6, 7, 27] Entropy rate diff. [26, 32]	Info.-theoretic meth. using entropy rate	Ctrl	✓	✓			Time
Data/info. rate [4, 5, 11, 28]	Info.-theoretic meth. using entropy rate	Ctrl	✓	✓			Time
Total info. rate [10, 16, 56, 64]	Info.-theoretic meth. using I-MMSE	✓	✓	✓	✓	✓	Time

✓: May feasible in theory, but cumbersome or has not been studied.
Blank: Most are infeasible in theory; some has not been studied.

Chapter 2

Fundamental Limitations of Continuous-Time Control and Filtering Systems

To answer the challenges raised in [Section 1.1](#), a unified framework based on continuous-time additive white Gaussian channel model is proposed to characterize the fundamental limitations and trade-offs of continuous-time control and filtering systems, and a direct information-theoretic method relying on Duncan's theorem or continuous-time I-MMSE relationships is put forward to calculate and analyze these trade-off metrics. The main contents and contributions of this chapter can be summarized as follows:

- (C1) A unified information-theoretic framework is proposed to model and capture the fundamental limitations of continuous-time control and filtering systems. A continuous-time Gaussian channel with or without feedback is used as a carrier to model the control and filtering systems ([Sections 2.1.3 to 2.1.5](#)). Total information rate is identified as a trade-off metric in a general nonlinear setting ([Sections 2.2.1 and 2.3.1](#)) and then applied to capture the specific fundamental limitations of continuous-time control and filtering systems subject to LTI, LTV, and nonlinear plants ([Sections 2.2.2 to 2.2.4 and Sections 2.3.2 to 2.3.4](#)).
- (C2) A direct information-theoretic paradigm is developed to compute and analyze the total information rate. By invoking Duncan's theorem or the I-MMSE relationships, we lift the computation and

analysis of total information rate to infinite-dimensional function spaces (see [Propositions 2.15](#) and [2.30](#) for a justification). In a general control and filtering setting, we show that i) total information (rate) can be directly computed from the causal MMSE of control input in control ([Theorem 2.14](#)) or the causal MMSE of noise-free output in filtering ([Theorem 2.29](#)), and ii) total information rate is sandwiched between the feedback capacity and instability rate of plant in control ([Theorem 2.17](#)) or the channel capacity and lowest achievable MS estimation error in filtering ([Theorem 2.31](#)). Applying these results to different control and filtering systems, we then establish an equality between total information rate and the sum of open-loop unstable poles in the LTI scenario ([Propositions 2.19](#) and [2.34](#)), and an inequality between total information rate and the sum of spectral values in the LTV scenario ([Propositions 2.24](#) and [2.37](#)), which suggest that total information rate serves a similar role as some established trade-off metrics, such as the Bode-type integrals, average entropy cost function, and minimum MS estimation error, in linear control and filtering systems. By using the Stratonovich-Kushner equation in nonlinear filtering, a direct method to calculate the total information rate in nonlinear control and filtering systems is proposed ([Propositions 2.27](#) and [2.38](#)). No restriction is imposed on the stationarity and distributions of signals and initial states, and the auxiliary time-delay is not needed.

The rest of this chapter is organized as follows. Information theory on continuous function spaces, continuous-time additive Gaussian channel model, and continuous-time control and filtering systems are successively introduced and modeled in [Section 2.1](#). Fundamental limitations of continuous-time control and filtering systems are investigated in [Section 2.2](#) and [Section 2.3](#), respectively, and [Section 2.4](#) concludes this chapter.

Notations: $\log(\cdot)$ denotes the logarithm in base e , where e denotes Euler's number. When A is a square matrix, $\text{tr}(A)$ and $\det(A)$ respectively denote the trace and determinant of matrix A . For a function $f(\tau)$, $f \in \mathbb{C}^r[0, t]$ indicates that $f(\cdot)$ is continuously r th order differentiable at $\tau \in [0, t]$, and $f \in \mathcal{L}^p([0, t])$ implies that the p -th power of $|f(\tau)|$ is Lebesgue integrable on $[0, t]$, i.e., $\int_0^t |f(\tau)|^p d\tau < \infty$. $\mathbb{C}^0[0, t]$ and $\mathcal{L}^2([0, t])$ will be intermittently invoked in this chapter, both of which are the function spaces of infinite dimensions. For two continuous-time random processes (or signals), x_0^t and $y_0^t \in \mathcal{L}^2([0, t])$, we let inner product $\langle x, y \rangle = \langle x, y \rangle_{(t, \omega)} := \int_0^t x(\tau)^\top y(\tau) d\tau$, and norm $\|x\| = \|x\|_{(t, \omega)} := \sqrt{\langle x, x \rangle}$, whose rigorous definitions can be found in [\[60, 69, 70\]](#).

2.1 Preliminaries and Problem Formulation

In this section, continuous-time signals and their associated image measures are defined and employed to reinterpret some information-theoretic metrics and properties. Subsequently, an additive white Gaussian channel model, and continuous-time control and filtering systems are respectively introduced and modeled.

2.1.1 Probability in Function Spaces

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes a sample space, σ -algebra (or -field) $\mathcal{F}$ represents the collection of all subsets on Ω , and $\mathbb{P}$ is a probability measure. When the time parameter $t \in \mathbb{R}^+$, the n -dimensional continuous-time signals (or random processes) on time interval $[0, t]$ are denoted as x_0^t or $x_0^t(\omega) : \Omega \rightarrow \mathcal{R}(\mathbb{R}^+, \mathbb{R}^n)$, where $\mathcal{R}(\mathbb{R}^+, \mathbb{R}^n)$ or $\mathbb{R}^T$, with T denoting the whole or part of the positive real line, represents the space of all real functions from $\mathbb{R}^+$ to $\mathbb{R}^n$. Random variable x_τ , $x(\tau)$ or $x(\tau, \omega) : \Omega \rightarrow \mathbb{R}^n$ implies the value of x_0^t observed or sampled at time $\tau \in [0, t]$. Discrete-time random processes $(x_{t_1}, x_{t_2}, \dots, x_{t_k})$, which consist of finite random variables, are defined on the finite-dimensional spaces, i.e., joint distribution of $x_{t_1}, \dots, x_{t_k}$, and continuous-time random processes $x_0^t(\omega)$, with infinitesimal time steps, are defined on the infinite-dimensional spaces. Meanwhile, we use $\mathcal{B}(\cdot)$ or $\mathcal{B}$ to denote the Borel σ -algebra (or -field) of a random process or variable, and for a collection of random processes $x_0^s, 0 \leq s \leq t$, filtration $\mathcal{F}_t^x$ is defined by $\mathcal{F}_t^x := \mathcal{B}(x_0^s, 0 \leq s \leq t) \subset \mathcal{F}$.

To specify a probability measure on the continuous function space $\mathcal{R}(\mathbb{R}^+, \mathbb{R}^n)$, let μ_x denote the image (or push-forward) measure associated with random process x_0^t such that $\mu_x(A) := \mathbb{P}(x_0^t \in A)$, where Borel set $A \subset \mathcal{R}(\mathbb{R}^+, \mathbb{R}^n)$, $x_0^t : \tau \rightarrow x_0^t(\omega)$, $\tau \in [0, t]$, and $\omega \in \Omega$. If x_0^t is a standard Brownian motion (or Wiener process), μ_x is called Wiener measure. For two probability measures μ_x and μ_y , the absolute continuity of μ_x w.r.t. (with respect to) μ_y is denoted by $\mu_x \ll \mu_y$. More explanations of probability measures on infinite-dimensional spaces can be found in [71–73].

2.1.2 Information Theory for Continuous-Time Systems

With the probability measures on continuous function spaces, we now define the information-theoretic metrics and their related properties for continuous-time signals. Compared with the statistical

notions in [74, 75], the following measure-theoretic notions from [76, 77] are more general and cover more varieties of signals, e.g., signals without probability density functions. Divergence, also known as Kullback-Leibler (KL) divergence and relative entropy, is a function quantifying the dissimilarity between two probability measures.

Definition 2.1 (Divergence). *For two probability measures, μ_x and μ_y , the divergence $D(\mu_x\|\mu_y)$ from μ_x to μ_y is defined by*

$$D(\mu_x\|\mu_y) := \int \left(\log \frac{d\mu_x}{d\mu_y} \right) d\mu_x = \mathbb{E}_{\mu_x} \left[\log \frac{d\mu_x}{d\mu_y} \right],$$

where $\mu_x \ll \mu_y$. In particular, if μ_x and μ_y have densities p_x and p_y , then $D(\mu_x\|\mu_y) := \int \log[p_x(x)/p_y(x)]p_x(x)dx = \mathbb{E}_{p_x}[\log p_x(x)/p_y(x)]$.

When $\mu_x \not\ll \mu_y$, we let $D(\mu_x\|\mu_y) = \infty$ by convention. Differential entropy is a function that measures the amount of randomness in a random vector.

Definition 2.2 (Differential Entropy). *For a random vector x subject to probability measure μ_x , its differential entropy $h(x)$ is defined by*

$$h(x) := -D(\mu_x\|\mu_{\text{Leb}}) = \int \log \left(\frac{d\mu_{\text{Leb}}}{d\mu_x} \right) d\mu_x, \quad (2.1)$$

where μ_{Leb} indicates the Lebesgue measure. In particular, if μ_x has a density p_x , $h(x) := \int \log[1/p_x(x)]p_x(x)dx = \mathbb{E}_{p_x}[\log(1/p_x(x))]$.

The value of $h(x)$ in Definition 2.2 can be positive, negative, $\pm\infty$, or undefined. When μ_x is not absolutely continuous w.r.t. μ_{Leb} , i.e., $\mu_x \not\ll \mu_{\text{Leb}}$, we conventionally let $h(x) = -\infty$. Mutual information is a measure of dependence between two random processes (or vectors), and mutual information rate quantifies the reliable data transmission rate of the communication channel.

Definition 2.3 (Mutual Information & Mutual Information Rate). *Mutual information $I(x_0^t; y_0^t)$ between random processes x_0^t and y_0^t , constrained by probability measures μ_x and μ_y , is defined by*

$$I(x_0^t; y_0^t) := \int \log \frac{d\mu_{xy}}{d(\mu_x \times \mu_y)} d\mu_{xy},$$

where $d\mu_{xy}/d(\mu_x \times \mu_y)$ is the Radon-Nikodym derivative. Mutual information rate $\bar{I}(x; y)$ is then defined by

$$\bar{I}(x; y) := \lim_{t \rightarrow \infty} \frac{1}{t} \cdot I(x_0^t; y_0^t),$$

when the limit exists.¹

Mutual information $I(x_0^t; y_0^t)$ is finite, if $\mu_{xy} \ll \mu_x \times \mu_y$.

Remark 2.4. *There is no differential entropy (rate) for continuous-time random process x_0^t , since the function space $\mathcal{R}(\mathbb{R}^+, \mathbb{R}^n)$, as the domain of image measures μ_x , is of infinite dimensions, while it is well-known that there is no Lebesgue measure defined on infinite-dimensional spaces. Hence, for a continuous-time random process x_0^t , we cannot define its differential entropy (rate) according to Definition 2.2. This motivates to investigate the continuous-time control and filtering trade-off by resorting to the divergence (rate) and mutual information (rate), in which we can replace μ_{Leb} in (2.1) with more general measures, such as Wiener and Gaussian measures, that are well-defined on infinite-dimensional spaces.*

We then introduce some information-theoretic properties based on the metrics defined above. Unless otherwise stressed, the random objects x , y and z that appeared in the following properties can be any random variables or processes that are valid and compatible with Definitions 2.1 to 2.3.

Property 2.5 (Chain Rule). *Given random vectors (or processes) x , y and z , the mutual information $I(x; y, z)$ satisfies*

$$I(x; y, z) = I(x; z) + I(x; y|z),$$

where $I(x; y|z)$ is the conditional mutual information of random vectors (or processes) x and y given z .

Property 2.6 (Data Processing Inequality). *Conditional mutual information $I(x; y|z)$ satisfies*

$$I(x; y|z) \geq I(x; g(y)|z),$$

where equality holds, if $g(\cdot)$ is injective.

¹ This definition is aligned with the definitions of mutual information rate and directed information rate adopted in [47, 78, 79]. When x_0^t and y_0^t are stationary Gaussian or jointly stationary, the limit always exists. When the signals are nonstationary, some useful discussions and calculation results have been provided in [80–82]. Throughout this chapter, we assume and only consider the scenario when this limit exists. Alternatively, we can also define the mutual information rate as $\bar{I}(x; y) := \limsup_{t \rightarrow \infty} t^{-1} I(x_0^t; y_0^t)$, which follows the discrete-time definition in [6] and is allowed to be infinity.

Property 2.7 (Symmetry & Non-negativity). *Given random vectors $x, y \in \mathbb{R}^n$, we have*

$$I(x; y) = I(y; x) = h(x) - h(x|y) = h(y) - h(y|x) \geq 0.$$

Property 2.8 (Invariance of Entropy). *Given random vectors $x, y \in \mathbb{R}^n$, we have*

$$h(x|z) = h(x + g(z)|z),$$

where $g(\cdot)$ is a function.

Property 2.9 (Maximum Entropy). *For a random vector $x \in \mathbb{R}^n$ with covariance matrix $\Sigma_x = \mathbb{E}[xx^\top]$, the differential entropy $h(x)$ is upper bounded by*

$$h(x) \leq h(\bar{x}) = \frac{1}{2} \log[(2\pi e)^n \det(\Sigma_x)],$$

where $\bar{x}$ is a Gaussian vector with the same covariance as x , and equality holds if x is Gaussian.

2.1.3 Continuous-Time Gaussian Channel Model

We now introduce the continuous-time Gaussian channel model, which—dating back to Shannon’s seminal work [83]—is repurposed to facilitate our investigation of control and filtering problems. Consider the following Brownian motion formulation of additive white Gaussian channel with feedback [45, 54]

$$y(t) = \int_0^t \phi(\tau, y_0^\tau, m) d\tau + w(t), \quad (2.2)$$

where $y(t)$ is the channel output; transmitted message m is a random variable or process such that $m \in \mathcal{B}_m$; $\phi(s, y_0^\tau, m)$ is the channel input process and coding function of m ; y_0^τ is the sample path of $y(s)$ on $s \in [0, \tau]$, and channel noise $w(t)$, independent from message m , is a standard Brownian motion. To describe the Gaussian channel without feedback, which can be seen as a special case of the feedback channel, we only need to replace the input process $\phi(\tau, y_0^\tau, m)$ with $\phi(\tau, m)$. Throughout this chapter, we also invoke (2.2) in the differential form, which is $dy_t = \phi(t, y_0^t, m)dt + dw_t$. Some mild assumptions and constraints are imposed on the Gaussian channel (2.2):

(A1) Random process y_0^t is measurable on $\mathcal{B}(\mathbb{R}^+) \times \mathcal{B}_m \times \mathcal{F}_t^w$. Deterministic function $\phi(\tau, y_0^\tau, m)$ is measurable on $\mathcal{F}_\tau^y \times \mathcal{B}_m$ and is defined for all adapted processes y_0^τ .

(A2) There exists a set $B_m \in \mathcal{B}_m$ with $\mathbb{P}(B_m) = 1$ such that for any adapted process y_0^t and $m \in B_m$, if $\int_0^t y^\top(\tau)y(\tau)d\tau < \infty$, then $\int_0^t \phi^\top(\tau, y_0^\tau, m)\phi(\tau, y_0^\tau, m)d\tau < \infty$. Meanwhile, the expectation

$$\mathbb{E}[\int_0^t \phi^\top(\tau, y_0^\tau, m) \phi(\tau, y_0^\tau, m) d\tau] < \infty.$$

(A3) There exists a unique solution y_0^t to the stochastic functional (2.2).

Remark 2.10. (A1) and (A3) are fundamental and commonly posited in the investigations of stochastic control and filtering, e.g., [60, 61, 84]. (A2) is often adopted as a finite-energy constraint and can be automatically fulfilled when signals are restricted in $\mathcal{L}^2([0, t])$ space.

Since entropy (rate), which is pivotal for calculating and analyzing total information (rate) $I(y_0^t; m)$ and will be investigated later as a fundamental limitation of control and filtering, is undefined for continuous-time signals, in this chapter we resort to the following Duncan's theorem from [53, 54] to directly compute $I(y_0^t; m)$ from the causal MMSE of the channel input.

Lemma 2.11. For the Gaussian channel (2.2) subject to (A1)-(A3), the mutual information $I(y_0^t; m)$ between the channel output y_0^t and message m , is given by

$$I(y_0^t; m) = \frac{1}{2} \mathbb{E}[\|\phi\|^2 - \|\hat{\phi}\|^2] = \frac{1}{2} \mathbb{E}[\|\phi - \hat{\phi}\|^2],$$

where $\hat{\phi}(\tau, y_0^\tau, m) = \mathbb{E}[\phi(\tau, y_0^\tau, m) | y_0^\tau]$.²

Additionally, we can also express the total information $I(y_0^t; m)$ in terms of divergence difference by using the following lemma from [63].

Lemma 2.12. For the Gaussian channel (2.2) subject to (A1)-(A3), we have the following identities:

$$\begin{aligned} D(\mu_{y|m} \| \mu_w) &= \frac{1}{2} \mathbb{E}[\|\phi\|^2] \\ D(\mu_y \| \mu_w) &= \frac{1}{2} \mathbb{E}[\|\hat{\phi}\|^2], \end{aligned}$$

where μ_w denotes the Wiener measure induced by w_0^t in (2.2), and $\hat{\phi}(\tau, y_0^\tau, m) = \mathbb{E}[\phi(\tau, y_0^\tau, m) | y_0^\tau]$.

Remark 2.13. Duncan's theorem or Lemma 2.11 not only connects information theory with estimation, but provides a direct approach to compute mutual (or total) information on continuous function spaces, as the divergences or expected norms in Lemma 2.12 are now defined w.r.t. Wiener measure instead of the Lebesgue measure in differential entropy. Meanwhile, Lemmas 2.11 and 2.12

² For a function $f(\tau, y_0^\tau, m) \in \mathcal{L}^2([0, t])$, $\|f\|^2 = \|f\|_{(t, \omega)}^2 := \int_0^t f^\top(\tau, y_0^\tau, m) f(\tau, y_0^\tau, m) d\tau$. The notation of conditional expectation $\mathbb{E}[f(\tau, y_0^\tau, m) | y_0^\tau]$ is the same as $\mathbb{E}[f(\tau, y_0^\tau, m) | \mathcal{F}_\tau^y]$ in the σ -algebra language.

also apply to the Gaussian channel without feedback and always hold regardless of the stationarity and distributions of the channel input and transmitted message.

In addition to total information $I(y_0^t; m)$, directed information $I(\phi_0^t \rightarrow y_0^t)$, as a generalization of mutual information $I(\phi_0^t; y_0^t)$ to random objects obeying causal relations [85], can also be used to measure the information flow from ϕ_0^t to y_0^t and capture the fundamental limitations of control and filtering systems [5, 11, 50]. For the Gaussian channel with or without feedback as (2.2), total information and directed information satisfy [47]

$$I(\phi_0^t \rightarrow y_0^t) = I(y_0^t; m).$$

For the non-feedback channel, we also have $I(\phi_0^t \rightarrow y_0^t) = I(\phi_0^t; y_0^t)$. Directed information rate $\bar{I}(\phi_0^t \rightarrow y_0^t) := \lim_{t \rightarrow \infty} I(\phi_0^t \rightarrow y_0^t)/t$ is similarly defined as the mutual information rate in Definition 2.3. However, since $I(\phi_0^t \rightarrow y_0^t)$ cannot be decomposed when message m consists of multiple sources, for the convenience of later analysis, we will stick with the total information (rate) in this chapter. In the following, control and filtering systems are respectively modeled as an additive Gaussian channel described by (2.2).

2.1.4 Continuous-Time Control Systems

We first consider the following continuous-time control system:

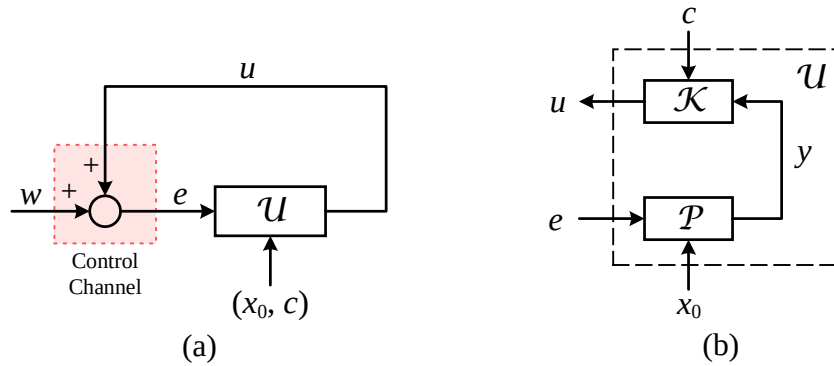


Figure 2.1: Control configuration. (a) Block diagram of control systems. (b) Lumped control process $\mathcal{U}$.

where $w_{(i)} \in \mathbb{C}^0[0, t]$ is a Brownian motion representing noise; $u_{(i)} \in \mathcal{L}^2([0, t])$ stands for the control signal; $e_{(i)}$ represents the error signal determined by $w_{(i)}$ and $u_{(i)}$, and subscript (i) or (ij) denotes

the i -th or ij -th component in a vector or matrix. The lumped control mapping $\mathcal{U}$ in Figure 2.1(a) can be decomposed into a plant model $\mathcal{P}$ and a control mapping $\mathcal{K}$, as Figure 2.1(b) shows, in which $y_{(i)}$ is the output of plant model $\mathcal{P}$, and $c_{(i)}$, independent from x_0 and w_0^t , denotes the command signal or exogenous noise of controller $\mathcal{K}$. The notation y_0^t throughout this chapter has slightly different definitions in control and filtering, which will be explicitly stated before the usage. We use the following Itô process and physical observation model to describe plant $\mathcal{P}$, which not only is general enough to cover most of real-world systems but automatically fulfills (A1)-(A3):

$$\begin{aligned} dx_t &= f(t, x_t)dt + b(t, x_t)de_t \\ y_t &= h(t, x_t), \end{aligned} \tag{2.3}$$

where $x_t \in \mathbb{R}^n$ is the internal state of $\mathcal{P}$; $f(t, x_t)$ and $b(t, x_t)$ are adapted processes, i.e., $\mathcal{F}_t$ -measurable, such that $\int_0^t |f_{(i)}(\tau, y_\tau)|d\tau < \infty$ and $\int_0^t b_{(ij)}^2(\tau, y_\tau)d\tau < \infty$, and $h(t, x_t)$ is a Lebesgue measurable function. The control mapping $\mathcal{K}$ is realized by a square-integrable function

$$u_t = g(t, y_0^t, c_0^t) \tag{2.4}$$

with memory, or a square-integrable function $u_t = g(t, y_t, c_t)$ without memory. We can then collectively formulate the lumped control mapping $\mathcal{U}$ or control channel in Figure 2.1 as

$$de_t = u(t, e_0^t, c_0^t)dt + dw_t, \tag{2.5}$$

where the lumped control process or control signal $u_t = u(t, e_0^t, c_0^t) = u_x(t, x_0^t)$ can be formulated as a function subject to the arguments (e_0^t, c_0^t, x_0) or x_0^t , depending on the statistics of interest. To align with the expression of Gaussian channel in (2.2), we can rewrite (2.5) in an integral form as $e(t) = \int_0^t u(\tau, e_0^\tau, c_0^\tau, x_0)d\tau + w(t)$, where $x_0 \in \mathbb{R}^n$ denotes the initial states of plant $\mathcal{P}$, and the following stability condition is imposed on the closed-loop system.

(A4) The feedback system realized by (2.3)-(2.5) is internally mean-square stable, i.e., $\sup_{t \geq 0} \mathbb{E}[x^\top(t) \cdot x(t)] < \infty$.

2.1.5 Continuous-Time Filtering Systems

To investigate the fundamental limitations of filtering, consider the filtering setup as follows

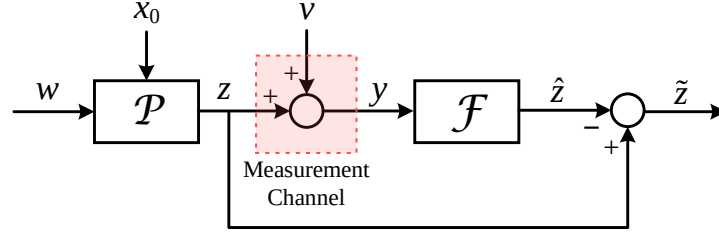


Figure 2.2: Filtering configuration.

where w represents the process input or noise; $\mathcal{P}$ denotes the plant with or without feedback; $x_0 \in \mathbb{R}^n$ is the initial state of $\mathcal{P}$; $z_{(i)} \in \mathcal{L}^2([0, t])$ is the noise-free output to be estimated; v denotes the measurement noise; y implies the measured output; $\mathcal{F}$ represents the filtering process; $\hat{z}$ stands for the estimate of z , and $\tilde{z} = z - \hat{z}$ is the estimation error. We use the following Itô process and mathematical observation model to describe the plant $\mathcal{P}$ and measurement channel in Figure 2.2:

$$\begin{aligned} dx_t &= f(t, x_t)dt + b(t, x_t)[u(t, x_t)dt + \Sigma_{\bar{w}}^{1/2}(t)d\bar{w}_t] \\ dy_t &= z_x(t, x_t^t)dt + \Sigma_v^{1/2}(t)dv_t, \end{aligned} \tag{2.6}$$

where $x_t \in \mathbb{R}^n$ is the internal state of plant $\mathcal{P}$; $b(t, x_t)$ denotes the control matrix; $u(t, x_t)$ is the control signal if plant $\mathcal{P}$ is closed-loop; noise-free output or measurement process $z_t = z_x(t, x_0^t) = z_{\bar{w}}(t, \bar{w}_0^t, x_0)$ is Lebesgue measurable and can be expressed as a function subject to arguments x_0^t or $(\bar{w}_0^t, x_0)$ depending on the statistics of interest; $d\bar{w}_t$ and dv_t are vectors of Brownian components with zero drift and unit-variance rate, and positive semi-definite matrices, $\Sigma_{\bar{w}}$ and Σ_v , respectively denote the covariance matrices of $d\bar{w}_t$ and dv_t . Without loss of generality, in this chapter we mainly consider the scenario when $\Sigma_v = I$ and $\Sigma_{\bar{w}} = \varepsilon I \geq 0$, which allows us to adjust the level of process noise in later analysis, and the filtering model (2.6) can therefore be simplified as

$$\begin{aligned} dx_t &= f(t, x_t)dt + b(t, x_t)[u(t, x_t)dt + dw_t] \\ dy_t &= z(t, w_0^t, x_0)dt + dv_t, \end{aligned} \tag{2.7}$$

where $dw_t = \Sigma_{\bar{w}}^{1/2}d\bar{w}_t = \sqrt{\varepsilon}Id\bar{w}_t$ is the process input, and $z_t = z(t, w_0^t, x_0) = z_{\bar{w}}(t, \bar{w}_0^t, x_0)$ is the

noise-free output. To align with the expression of Gaussian channel in (2.2), we can rewrite the observation formula in (2.7) as $y_t = \int_0^t z(\tau, w_0^\tau, x_0) d\tau + v_t$. Realization and formulas of the filtering process $\mathcal{F}$ can be varied depending on the design characteristics, e.g., MMSE, maximum a posteriori (MAP), minimum conditional inaccuracy. In this chapter, we mainly focus on the scenario and filters, such as Kalman-Bucy filter, that minimize MS estimation error. More detailed interpretations and formulas of these optimal filters will be given later. Up until now, we have modeled the continuous-time control and filtering systems as an additive Gaussian channel with or without feedback in (2.5) and (2.7), where fundamental limitations of control and filtering will be respectively defined and analyzed in the following sections.

2.2 Fundamental Limitations of Continuous-Time Control Systems

In this section, the total information rate $\bar{I}(e; x_0, c)$ or directed information rate $\bar{I}(u \rightarrow e)$, which quantifies the reliable data transmission rate of the control channel in Figure 2.1, is identified as a control trade-off metric in the general setting depicted by (2.3)-(2.5), and then applied to capture the fundamental limitations of control systems subject to LTI, LTV, and nonlinear plants, respectively.

2.2.1 General Control Trade-offs

Based on Duncan's theorem, the following theorem provides a direct method to calculate $I(e_0^t; x_0, c_0^t)$ from the causal MS estimation error of control input.

Theorem 2.14. *For the closed-loop system, described by (2.3)-(2.5) and Figure 2.1, the total information $I(e_0^t; x_0, c_0^t)$ satisfies*

$$I(e_0^t; x_0, c_0^t) = \frac{1}{2} \mathbb{E} [\|u - \hat{u}\|^2] = \frac{1}{2} (\mathbb{E}[\|u\|^2] - \mathbb{E}[\|\hat{u}\|^2]), \quad (2.8)$$

where $\hat{u}(\tau, e_0^\tau, c_0^\tau, x_0) = \mathbb{E}[u(\tau, e_0^\tau, c_0^\tau, x_0) | e_0^\tau]$, and the total information rate is given by

$$\bar{I}(e; x_0, c) = \lim_{t \rightarrow \infty} \frac{1}{t} I(e_0^t; x_0, c_0^t). \quad (2.9)$$

Proof. Since the control channel (2.5) is a noisy communication channel with feedback, by applying the feedback version of Lemma 2.11 to (2.5), we obtain the identities in (2.8). The total information

rate $\bar{I}(e; x_0, c)$ in (2.9) follows Definition 2.3, and the existence of $\bar{I}(e; x_0, c)$ is guaranteed by the boundedness of $\mathbb{E}[\|u\|^2]$ in (2.8) and (A2). $\square$

By using Lemma 2.12 and the chain rule of mutual information, we can connect $I(e_0^t; x_0, c_0^t)$ with $I(e_0^t; x_0)$ or the instability rate of plant $\mathcal{P}$, and express $I(e_0^t; x_0, c_0^t)$ as a difference between the divergences w.r.t. Wiener measure.

Proposition 2.15. *For the closed-loop system subject to (2.3)-(2.5) and Figure 2.1, the total information $I(e_0^t; x_0, c_0^t)$ satisfies*

$$I(e_0^t; x_0, c_0^t) = I(e_0^t; x_0) + I(e_0^t; c_0^t | x_0), \quad (2.10)$$

and can be expressed as a divergence difference:

$$I(e_0^t; x_0, c_0^t) = D(\mu_{e|(x_0, c)} \| \mu_w) - D(\mu_e \| \mu_w), \quad (2.11)$$

where $D(\mu_{e|(x_0, c)} \| \mu_w) = (1/2) \cdot \mathbb{E}[\|u\|^2]$ and $D(\mu_e \| \mu_w) = (1/2) \cdot \mathbb{E}[\|\hat{u}\|^2]$.

Proof. Equation (2.10) can be proved by applying the chain rule of mutual information in Property 2.5 to $I(e_0^t; x_0, c_0^t)$. Equation (2.11) can be straightforwardly verified by replacing the divergences and mutual information with their definition forms in Definition 2.1 and $I(x_0^t; y_0^t) := \mathbb{E}_{(x, y)}[\log(d\mu_{xy}/d(\mu_x \times \mu_y))] = \mathbb{E}_y[\log(d\mu_{y|x}/d\mu_y)]$ in Definition 2.3. Meanwhile, by applying Lemma 2.12 to (2.5), we have $D(\mu_{e|(x_0, c)} \| \mu_w) = (1/2) \cdot \mathbb{E}[\|u\|^2]$ and $D(\mu_e \| \mu_w) = (1/2) \cdot \mathbb{E}[\|\hat{u}\|^2]$. $\square$

Remark 2.16. *Compared with the core equation (11) in [27] and (9.117) in [74], which express the total information as entropy differences in discrete-time setting, equations (2.8) and (2.11), which resort to the MS estimation error and divergence difference, provide the continuous-time counterparts to calculate the total information (rate) without any discretization or approximation.*

Before we present the trade-off properties of $\bar{I}(e; x_0, c)$, the definitions of feedback capacity and channel capacity are given below [47, 79, 86]. For the feedback channel described by (2.5) and Figure 2.1, we define the (information) feedback capacity³ $\mathcal{C}_f := \sup_{(x_0, c)} \bar{I}(e; x_0, c)$ as the supremum of total information rate over all admissible messages, i.e., initial state x_0 and command

³ In this chapter, we adopt the mutual information version of feedback capacity. The definition of (operational) feedback capacity from a coding-theorem perspective can be found in [47, 87].

signal c_0^t . For the same channel without feedback, when the control input is subject to the power constraint, $\mathbb{E}[u_\tau^\top u_\tau] \leq \rho(\tau), \forall \tau \in [0, t]$, we define the *channel capacity* $\mathcal{C} := \sup_{\mathbb{E}[u_\tau^\top u_\tau] \leq \rho(\tau)} \bar{I}(e; u)$ as the supremum of input-output mutual information rate $\bar{I}(e; u)$. The following theorem shows that the total information rate must be sandwiched between the feedback capacity $\mathcal{C}_f$ and mutual information rate $\bar{I}(e; x_0)$.

Theorem 2.17. *For the control system described by (2.3)-(2.5) and Figure 2.1, the total information rate $\bar{I}(e; x_0, c)$ is bounded by*

$$\mathcal{C}_f \geq \bar{I}(e; x_0, c) \geq \bar{I}(e; x_0), \quad (2.12)$$

where $\bar{I}(e; x_0, c) = \bar{I}(e; x_0)$ if e_0^t and c_0^t are conditionally independent given x_0 , i.e., $c_0^t - x_0 - e_0^t$ forms a Markov chain. When the control input u_0^t is further constrained by $\mathbb{E}[u_\tau^\top u_\tau] \leq \rho(\tau), \forall \tau \in [0, t]$, the total information rate $\bar{I}(e; x_0, c)$ satisfies

$$\mathcal{C}_f \geq \bar{I}(e; x_0, c) + \bar{D}(\mu_e \| \mu_w) \geq \bar{I}(e; x_0) + \bar{D}(\mu_e \| \mu_w), \quad (2.13)$$

where $\mathcal{C}_f = \mathcal{C} = (2t)^{-1} \int_0^t \rho(\tau) d\tau$, and the divergence rate $\bar{D}(\mu_e \| \mu_w) := \lim_{t \rightarrow \infty} D(\mu_e \| \mu_w)/t$.

Proof. The first inequality in (2.12) can be readily obtained from the definition of feedback capacity $\mathcal{C}_f$. By the chain rule and non-negativity of mutual information in Properties 2.5 and 2.7, we have $I(e_0^t; x_0, c_0^t) \geq I(e_0^t; x_0)$, which implies the second inequality in (2.12). Meanwhile, according to the chain rule, $\bar{I}(e; x_0, c) = \bar{I}(e; x_0)$ holds when $I(e_0^t; c_0^t | x_0) = 0$, which requires e_0^t and c_0^t be conditionally independent given x_0 , i.e., $c_0^t - x_0 - e_0^t$ forms a Markov chain.

When we impose a power constraint on the control input, since $D(\mu_{e|(x_0, c)} \| \mu_w) = (1/2) \cdot \mathbb{E}[\|u\|^2]$ by Proposition 2.15, the feedback and channel capacities of (2.5) then satisfy ([79, Theorem 6.4.1] and [45, Section 4]):

$$\mathcal{C}_f = \frac{\int_0^t \rho(\tau) d\tau}{2t} \geq \lim_{t \rightarrow \infty} \frac{\mathbb{E}[\|u\|^2]}{2t} = \bar{D}(\mu_{e|(x_0, c)} \| \mu_w), \quad (2.14)$$

where $\mathcal{C}_f = \mathcal{C}$; Fubini's theorem is invoked to swap the order of integration and expectation when deriving the inequality in (2.14), and divergence rate is defined by $\bar{D}(\mu_{e|(x_0, c)} \| \mu_w) := \lim_{t \rightarrow \infty} D(\mu_{e|(x_0, c)} \| \mu_w)/t$. Dividing both sides of (2.11) by t , taking the limit as $t \rightarrow \infty$, and substituting (2.14) into the result give the first inequality in (2.13), and the second inequality in (2.13) follows

from (2.12). □

An immediate or information-theoretic disclosure from Theorem 2.17 is that for an internally mean-square stable feedback channel depicted by (2.3)-(2.5) and Figure 2.1, the coding scheme (or encoder/decoder) must be designed such that the feedback capacity $\mathcal{C}_f$ is large enough to compensate the cost of transmitting message, gauged by $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$, and the cost of attenuating channel noise, quantified by the divergence rate $\bar{D}(\mu_e \| \mu_w)$. Meanwhile, to achieve the channel capacity $\mathcal{C} = \mathcal{C}_f = (2t)^{-1} \int_0^t \rho(\tau) d\tau$, we must have $\bar{D}(\mu_e \| \mu_w) = 0$ or $D(\mu_e \| \mu_w) = o(t)$ as $t \rightarrow \infty$ in (2.13).

When we treat the control system (2.3)-(2.4) as a feedback communication channel and the stabilizing control law as a coding scheme of the feedback channel, as Section 2.1.4 introduces, a control-theoretic interpretation of Theorem 2.17 can then be obtained. To synthesize a mean-square stabilizing controller for the system described by (2.3)-(2.4) and Figure 2.1, the (supremum of) ‘control’ rate $\bar{I}(e; x_0, c)$ (or $\mathcal{C}_f$) achieved by the controller must be large enough to i) offset the instability rate of plant $\bar{I}(e; x_0)$ or its minimum if $\bar{I}(e; x_0)$ is also impacted by the control mapping, and ii) counteract the divergence rate $\bar{D}(\mu_e \| \mu_w)$ induced by the control noise w . The first requirement is a fundamental limitation for the controller to stabilize the continuous-time plant model in (2.3), as we will later show that the instability rate $\bar{I}(e; x_0)$, similar to the Bode-type integrals and data rate, is equal or bounded below by the sum of open-loop unstable poles in the LTI scenario, and analogous to the entropy cost function and LTV Bode’s integral, is bounded below by the sum of spectral values of the antistable component in the LTV scenario. The second requirement is a fundamental constraint for the controller to attenuate the external disturbance. When $\bar{D}(\mu_e \| \mu_w) = 0$, all control effort is allocated to stabilize the plant. Thus far, we have interpreted the trade-off properties of total information rate $\bar{I}(e; x_0, c)$ in Theorem 2.17 and provided a direct method to compute $\bar{I}(e; x_0, c)$ in Theorem 2.14. A vanilla example in Appendix A provides an intuitive preview for the calculation and analysis of total information rate as a metric characterizing Bode’s fundamental limitation in an LTI scalar system. More rigorous treatments are given in the following subsections, in which Theorem 2.14 is utilized to compute and analyze the information rates, $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$, in Theorem 2.17 with various control systems subject to LTI, LTV, and nonlinear plants.

Remark 2.18. *Theorems 2.14, 2.17, and Proposition 2.15 can be applied to all continuous-time*

control systems depicted in [Section 2.1.4](#) despite the linearity of plant, stationarity and distributions of the input, output and commanded signals, and the distribution of initial state. In the absence of command signal c_0^t , total information rate $\bar{I}(e; x_0, c)$ coincides with $\bar{I}(e; x_0)$, and [Theorem 2.14](#) and [Proposition 2.15](#) persist with the second inequalities in (2.12) and (2.13) degenerating to equalities.

2.2.2 Control Trade-offs in LTI Systems

We now narrow down our scope to the LTI systems described as follows

$$\begin{aligned} dx_t &= Ax_t dt + Bde_t \\ y_t &= Cx_t, \end{aligned} \tag{2.15}$$

where A , B and C are time-invariant matrices of proper dimensions. When the control mapping $\mathcal{K}$ in [Figure 2.1](#) is linear, such as the state feedback, output feedback, and observer-based feedback controllers, we use (2.15) to represent the augmented/cascade dynamics of the plant $\mathcal{P}$ and control mapping $\mathcal{K}$ such that the control signal can be expressed as

$$u_t = Kx_t, \tag{2.16}$$

where K is the control gain, and x_t is the states of the augmented dynamics $\mathcal{U}$ in (2.15). By applying [Theorem 2.14](#) to the control channel depicted by (2.5), (2.15), and (2.16), an equality between the information rate $\bar{I}(e; x_0)$ in [Theorem 2.17](#) and the sum of open-loop unstable poles is established for the first time in continuous-time LTI control systems.

Proposition 2.19. *When the closed-loop system, subject to the augmented LTI plant (2.15) and linear controller (2.16) as in [Figure 2.1](#), is internally mean-square stable, the mutual information rate is given by*

$$\bar{I}(e; x_0) = \sum_i \lambda_i^+(A), \tag{2.17}$$

where x_0 denotes the initial states in (2.15), and $\lambda_i^+(A)$ are the eigenvalues of matrix A with positive real parts, i.e., open-loop unstable poles of $\mathcal{U}$.

Proof. By applying [Theorem 2.14](#) to the LTI control system described by (2.15) and (2.16), we

have

$$\bar{I}(e; x_0) = \lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E}[\|u - \hat{u}\|^2],$$

where $u_\tau = Kx_\tau$ and $\hat{u}_\tau = \mathbb{E}[u_\tau | e_0^\tau]$. Equation (2.17) can be then verified by showing that $\lim_{t \rightarrow \infty} (1/2t) \cdot \mathbb{E}[\|u - \hat{u}\|^2] = \sum_i \lambda_i^+(A)$. Substituting the state feedback control law (2.16) into the control channel (2.5) and lumped/cascade LTI model (2.15), we obtain the following linear filtering problem with correlated noise

$$\begin{aligned} dx_t &= \bar{A}x_t dt + Bdw_t \\ de_t &= Kx_t dt + dw_t, \end{aligned} \tag{2.18}$$

where $\bar{A} = A + BK$. Define the estimation error covariance $P_t = \mathbb{E}[(x_t - \hat{x}_t)(x_t - \hat{x}_t)^\top]$, where $\hat{x}_t = \mathbb{E}[x_t | e_0^t]$. By Theorem 10.3 in [60], we can solve the error covariance P_t of optimal filtering problem (2.18) from the following Riccati differential equation (RDE)

$$\dot{P}_t = \bar{A}P_t + P_t\bar{A}^\top + BB^\top - (B + P_tK^\top)(B + P_tK^\top)^\top,$$

or equivalently

$$\dot{P}_t = AP_t + P_tA^\top - P_tK^\top KP_t. \tag{2.19}$$

When (A, K) is detectable, i.e., there exists a matrix $\bar{B}$ such that $A + \bar{B}K$ is Hurwitz, error covariance matrix P_t in (2.19) converges to P as $t \rightarrow \infty$, i.e., $P = \lim_{t \rightarrow \infty} P_t$, by [88]. Since $\dot{P}_t$ determines the boundedness of $\lim_{t \rightarrow \infty} (1/2t) \cdot \mathbb{E}[\|u - \hat{u}\|^2]$, we will preserve $\dot{P}_t$ in (2.19), instead of letting $\dot{P}(t) = 0$ and converting the RDE into an algebraic Riccati equation (ARE), at this point.

To derive $\mathbb{E}[\|u - \hat{u}\|^2]$, we notice that RDE (2.19) also governs the following optimal filtering problem

$$\begin{aligned} dx_t &= Ax_t dt \\ de_t &= Kx_t dt + dw_t, \end{aligned} \tag{2.20}$$

which is equivalent to the system (2.18) without control input and process noise. When the matrix A in (2.20) does not have an eigenvalue on the imaginary axis, we can bring the system (2.20) into

the following modal form

$$\begin{aligned} \begin{bmatrix} dx_s(t) \\ dx_u(t) \end{bmatrix} &= \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix} dt \\ de(t) &= \begin{bmatrix} K_s & K_u \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix} dt + dw(t), \end{aligned} \quad (2.21)$$

where A_s is Hurwitz, and A_u is antistable with eigenvalues identical to the unstable poles of (2.15). Let $\bar{K} = [K_s, K_u]$ and $\bar{x}(t) = [x_s^\top(t), x_u^\top(t)]^\top$. By Lemma 2 in [12] or Lemma 2.35 in Section 2.3.2, when the lumped system (2.15) is both stabilizable and detectable, for any initial condition $\bar{P}(t_0) \geq 0$, the asymptotic estimation error covariance matrix $\bar{P}(t) = \mathbb{E}[(\bar{x}_t - \hat{\bar{x}}_t)(\bar{x}_t - \hat{\bar{x}}_t)^\top] = \lim_{\tau \rightarrow \infty} \bar{P}(\tau)$ of the filtering problem (2.21) takes the form of $\bar{P}(t) = \text{diag}\{0, P_u(t)\} = \lim_{\tau \rightarrow \infty} \text{diag}\{0, P_u(\tau)\}$, where $P_u(t) = \mathbb{E}[(x_u(t) - \hat{x}_u(t))(x_u(t) - \hat{x}_u(t))^\top]$ is the positive definite solution of the following RDE

$$\dot{P}_u(t) = A_u P_u(t) + P_u(t) A_u^\top - P_u(t) K_u^\top K_u P_u(t). \quad (2.22)$$

Consequently, we have $u_t = \bar{K} \bar{x}(t)$, $\text{tr}(\bar{P}_t) = \text{tr}[P_u(t)]$, $\text{tr}(\bar{K}^\top \bar{P}_t \bar{K}) = \text{tr}[K_u^\top P_u(t) K_u]$, and the following MS estimation errors

$$\text{tr}[P_u(t)] = \mathbb{E}[(\bar{x}_t - \hat{\bar{x}}_t)^\top (\bar{x}_t - \hat{\bar{x}}_t)], \quad (2.23a)$$

$$\text{tr}[K_u^\top P_u(t) K_u] = \mathbb{E}[(u_t - \hat{u}_t)^\top (u_t - \hat{u}_t)]. \quad (2.23b)$$

Integrating both sides of (2.23b) from $\tau = 0$ to t gives

$$\int_0^t \text{tr}[K_u^\top P_u(\tau) K_u] d\tau = \mathbb{E} \left[\int_0^t (u_\tau - \hat{u}_\tau)^\top (u_\tau - \hat{u}_\tau) d\tau \right] = \mathbb{E}[\|u - \hat{u}\|^2], \quad (2.24)$$

where we invoked the Fubini's theorem to swap the order of integration and expectation in the first equation. Pre- and post-multiplying (2.22) by $P_u^{-1/2}(\tau)$ and substituting the result into (2.24), we have

$$\int_0^t \text{tr}[K_u^\top P_u(\tau) K_u] d\tau = \int_0^t \text{tr}[P_u^{1/2}(\tau) K_u^\top K_u P_u^{1/2}(\tau)] d\tau = \int_0^t \text{tr}(\Lambda_\tau) d\tau, \quad (2.25)$$

where $\Lambda_\tau = P_u^{-1/2}(\tau)A_u P_u^{1/2}(\tau) + P_u^{1/2}(\tau)A_u^\top P_u^{-1/2}(\tau) - P_u^{-1/2}(\tau)\dot{P}_u(\tau)P_u^{-1/2}(\tau)$. Hence, equating (2.24) and (2.25), we arrive at

$$\begin{aligned}\mathbb{E}[\|u - \hat{u}\|^2] &= \int_0^t \text{tr}(\Lambda_\tau) d\tau = \int_0^t \text{tr}[K_u^\top P_u(\tau)K_u] d\tau \\ &= \int_0^t \text{tr}[A_u + A_u^\top - \dot{P}_u(\tau)P_u^{-1}(\tau)] d\tau \\ &= 2t \cdot \text{tr}(A_u) - \int_0^t \text{tr}[\dot{P}_u(\tau)P_u^{-1}(\tau)] d\tau.\end{aligned}\tag{2.26}$$

Since (A, K) in (2.18) is detectable, the time-invariant matrix pair (A_u, K_u) in (2.21) is also detectable, and therefore, $P_u(t)$ in RDE (2.22) converges as $t \rightarrow \infty$, i.e., $P_u = \lim_{t \rightarrow \infty} P_u(t)$ and $\lim_{t \rightarrow \infty} \dot{P}_u(t) = 0$. Dividing both sides of (2.26) by $2t$ and in the limit of $t \rightarrow \infty$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E}[\|u - \hat{u}\|^2] = \text{tr}(A_u) - \lim_{t \rightarrow \infty} \frac{\int_0^t \text{tr}[\dot{P}_u(\tau)P_u^{-1}(\tau)] d\tau}{2t},\tag{2.27}$$

where $\text{tr}(A_u)$ equals the sum of eigenvalues in A_u , i.e., $\text{tr}(A_u) = \sum_i \lambda_i(A_u)$. By using the L'Hôpital's rule and the convergence of $\lim_{t \rightarrow \infty} P_u(t)$, the last term in (2.27) satisfies

$$\lim_{t \rightarrow \infty} \frac{\int_0^t \text{tr}[\dot{P}_u(\tau)P_u^{-1}(\tau)] d\tau}{t} = \lim_{t \rightarrow \infty} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)] = 0,$$

which together with (2.27) implies

$$\lim_{t \rightarrow \infty} \frac{1}{2t} \cdot \mathbb{E}[\|u - \hat{u}\|^2] = \sum_i \lambda_i(A_u) = \sum_i \lambda_i^+(A).$$

This completes the proof. $\square$

By virtue of Theorem 2.14 or the I-MMES relationships, an equality between total information (or plant's instability) rate and the sum of open-loop unstable poles is established in Proposition 2.19, which shows that in addition to the control trade-off properties interpreted in Theorem 2.17, $\bar{I}(e; x_0)$ also serves as an information-theoretic substitute/interpretation for the classical Bode's integral [1, 17] in the LTI control systems. Previously, in the study of other information-theoretic trade-offs, such as the Bode-like integral in [7], this equality was missing and impossible to attain, e.g., Lemma 4.1 in [7] is proved by using the mean-square stability and maximum entropy. Meanwhile, Proposition 2.19

does not postulate any restriction on the stationarity or distribution of signals and initial states. It is worth noting that same as the classical results of Bode's integral [17, 68],⁴ the right-hand-side (RHS) sum in (2.17) contains the open-loop unstable poles of both the plant $\mathcal{P}$ and control mapping $\mathcal{K}$. When we exclude the unstable poles of control mapping or minimize the RHS sum by choosing a proper control mapping, (2.17) then implies that $\bar{I}(e; x_0)$ must be lower-bounded by the sum of unstable poles in $\mathcal{P}$ or the minimum sum of the unstable poles in $\mathcal{P}$ and $\mathcal{K}$.

For completeness, we also provide a lower bound for the total information and instability rates when the LTI plant (2.15) is stabilized by a causal nonlinear controller (2.4) in the following lemma, which can be implied from Theorem 2.17 and Lemma 4.1 in [7]. Although the exact values of $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$ can still be computed by using Theorem 2.14, since the control mapping $\mathcal{K}$ is now a nonlinear process, we should resort to the nonlinear estimation method, which will be discussed in Section 2.2.4.

Lemma 2.20. *When the closed-loop system, subject to the LTI plant (2.15) and nonlinear control mapping (2.4) as in Figure 2.1, are internally mean-square stable, the information rates $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$ satisfy*

$$\bar{I}(e; x_0, c) \geq \bar{I}(e; x_0) \geq \sum_i \lambda_i^+(A), \quad (2.28)$$

where x_0 denotes the initial states in (2.15), and $\lambda_i^+(A)$ are the eigenvalues of matrix A with positive real parts.

Proposition 2.19 and Lemma 2.20 respectively develop an equality and an inequality between the total information (or plant's instability) rate and the sum of unstable poles. By combining these results with (2.12) and (2.13), an LTI explanation or instance can then be assigned to the control limitation in Theorem 2.17: The total information rate $\bar{I}(e; x_0, c)$ of any causal controller that stabilizes the LTI plant (2.15) must exceed the (minimum) sum of the open-loop unstable poles, which suggests that $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$ serve a similar role as the Bode-type integrals in the continuous-time control systems with LTI plants [1, 7, 17]. While this LTI result seems anticipated and shares some similarities with a few other information-theoretic constraints, e.g., SNR constraint

⁴ Similar to other information-theoretic trade-offs, e.g., Bode-like integral in [7], $\bar{I}(e; x_0)$ or Proposition 2.19 deviates from the classical Bode's integral when the relative degree of lumped/augmented dynamics of (2.15) and (2.16) is less than 2, in which case Bode's integral is usually positive/negative infinity. Nevertheless, when the relative degree of lumped dynamics is greater than or equal to 2, Proposition 2.19 rigorously restores and actually generalizes the equality of Bode's integral.

[48], and data rate constraint of discrete-time control systems [4], in the following subsections, we will show that by using [Theorem 2.14](#), the analysis of $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$ as a control limitation can be easily carried over to the linear time-varying and nonlinear systems, which are challenging for the traditional methods and previously received less attention.

2.2.3 Control Trade-offs in LTV Systems

Consider the LTV plant $\mathcal{P}$ described as follows

$$\begin{aligned} dx_t &= A(t)x_t dt + B(t)de_t \\ y_t &= C(t)x_t, \end{aligned} \tag{2.29}$$

where $A(t)$, $B(t)$ and $C(t)$ are matrix functions of proper dimensions. A brief introduction to the LTV-related notions, such as the Lyapunov exponents, dichotomy spectrum, sensitivity operator, and inner/outer factorization, is given in [Appendix B](#), and more detailed explanations can be found in [\[24, 89\]](#). The following assumptions and constraints are imposed on [\(2.29\)](#).

(A5) The matrix function $A(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times n}$ admits an exponential dichotomy of rank n_s , and the antistable component has dichotomy spectrum

$$[\underline{\lambda}_1, \bar{\lambda}_1] \cup [\underline{\lambda}_2, \bar{\lambda}_2] \cup \cdots \cup [\underline{\lambda}_l, \bar{\lambda}_l] \tag{2.30}$$

with dimensions $n_1, \dots, n_l$, and $\sum_{i=1}^l n_i = n_u$, where $n_s + n_u = n$.

(A6) $A(t) - B(t)C(t)$ is uniformly exponentially stable (UES). Equivalently, the sensitivity operator $S = S_i S_o$ of [\(2.29\)](#), the inner factor S_i , the outer factor S_o and its inverse S_o^{-1} , are all bounded.

(A7) The relative degree of the open-loop system $\mathcal{P}$ is at least 2, i.e., $C(t)B(t) = 0$ for all t .

Remark 2.21. *Similar to the stable/unstable dichotomy of LTI systems, [\(A5\)](#) is a fundamental property of LTV systems that has been extensively studied and adopted in the existing literature [\[90–92\]](#). When [\(2.29\)](#) degenerates to an LTI system, the intervals in [\(2.30\)](#) shrink to the points that coincide with the real parts of the LTI system's unstable poles. [\(A6\)](#) is the prerequisite of any internally stabilizing controller [\[24\]](#). Previous investigations of Bode's integral in LTI systems require the open-loop systems to have a relative degree of at least 2 or $CB = 0$ [\[2\]](#), and [\(A7\)](#) is the*

counterpart constraint in LTV systems.

When $A(t)$ admits an exponential dichotomy, analogous to the modal decomposition on LTI models, we have the following stability preserving state-space transformation that separates the stable and antistable parts of $A(t)$.

Lemma 2.22 ([90, Chapter 5]). *When the matrix function $A(t)$ in (2.29) admits an exponential dichotomy, there exists a bounded matrix function $T(t)$ with bounded inverse such that*

$$\left[\begin{array}{c|c} \Theta(t) & T(t)B(t) \\ \hline C(t)T^{-1}(t) & D(t) \end{array} \right] := \left[\begin{array}{cc|c} A_s(t) & 0 & B_s(t) \\ 0 & A_u(t) & B_u(t) \\ \hline C_s(t) & C_u(t) & D(t) \end{array} \right], \quad (2.31)$$

where $D(t)$ is the feedthrough matrix; $\Theta(t) = [\dot{T}(t) + T(t)A(t)]T^{-1}(t)$; $A_s(t)$ is UES, and $A_u(t)$ is uniformly exponentially antistable (UEA).

By reproducing the relationship between Bode's integral and the average entropy cost function of LTI systems, a time-domain integral $\mathcal{B}$ is proposed as an LTV analogue of Bode's integral [24].

Lemma 2.23. *For the LTV plant (2.29) satisfying (A5)-(A7), the following integral $\mathcal{B}$ defines a control trade-off*

$$\mathcal{B} := \lim_{t \rightarrow \infty} \frac{1}{4t} \int_{-t}^t \text{tr}[\hat{S}_o(\tau, \tau_0)] d\tau = \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t \text{tr}[A_u(\tau)] d\tau,$$

where $\hat{S}_o(\tau, \tau_0) = [B^\top(\tau)X(\tau) - C(\tau)]\Phi_{A-BC}(\tau, \tau_0)B(\tau_0)$ is the kernel of the outer factor S_o with $\tau \geq \tau_0$, $X(\tau) = X^\top(\tau) \geq 0$ being a stabilizing solution to the Riccati differential equation $-\dot{X}(\tau) = A^\top(\tau)X(\tau) + X(\tau)A(\tau) - X(\tau)B(\tau)B^\top(\tau)X(\tau)$, and $\Phi_{A-BC}(\tau, \tau_0)$ standing for the state transition matrix of $A(t) - B(t)C(t)$ from τ_0 to τ . Meanwhile, the control trade-off integral $\mathcal{B}$ satisfies

$$0 < \sum_{i=1}^l n_i \underline{\lambda}_i \leq \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t \text{tr}[A_u(\tau)] d\tau \leq \sum_{i=1}^l n_i \bar{\lambda}_i,$$

where $A_u(\tau)$ is the antistable part of $A(\tau)$, and $\underline{\lambda}_i$ and $\bar{\lambda}_i$, $i = 1, \dots, l$, are the spectral values of $A_u(\tau)$.

When the stabilizing controller $\mathcal{K}$ is linear, we use (2.29) to describe the augmented dynamics of

the plant $\mathcal{P}$ and control mapping $\mathcal{K}$ in Figure 1(b) such that the control signal satisfies

$$u(t) = K(t)x(t), \quad (2.32)$$

where $K(t)$ is the time-varying control gain. By applying Theorem 2.14 to the control system or channel governed by (2.5), (2.29) and (2.32), we obtain the following relationship for $\bar{I}(e; x_0)$ in the LTV scenario.

Proposition 2.24. *When the closed-loop system, subject to the augmented LTV plant (2.29) and linear control mapping (2.32) as in Figure 2.1, is internally mean-square stable, the mutual information rate $\bar{I}(e; x_0)$ satisfies*

$$\begin{aligned} \bar{I}(e; x_0) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \text{tr}[A_u(\tau)] d\tau - \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \text{tr}[\dot{P}_u(\tau) P_u^{-1}(\tau)] d\tau \\ &\geq \sum_{i=1}^l n_i \lambda_i - \limsup_{t \rightarrow \infty} \frac{1}{2} \text{tr}[\dot{P}_u(t) P_u^{-1}(t)], \end{aligned} \quad (2.33)$$

where $A_u(\tau)$ is the antistable component of $A(\tau)$; $P_u(\tau) = \mathbb{E}[(x_u(\tau) - \hat{x}_u(\tau))(x_u(\tau) - \hat{x}_u(\tau))^\top]$ is the estimation covariance matrix of the antistable modes, and $\{\lambda_i\}_{i=1}^l$, with dimensions $n_1, \dots, n_l$, are the spectral values of $A_u(\tau)$.

Proof. By applying Theorem 2.14 to the LTV systems described by (2.29) and (2.32), we have

$$\bar{I}(e; x_0) = \lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E}[\|u - \hat{u}\|^2],$$

where $u_\tau = K_\tau x_\tau$ and $\hat{u}_\tau = \mathbb{E}[u_\tau | e_0^\tau]$. Hence, to prove the equality in (2.33), we only need to show that $(2t)^{-1} \mathbb{E}[\|u - \hat{u}\|^2] = t^{-1} \int_0^t \text{tr}[A_u(\tau)] d\tau - (2t)^{-1} \int_0^t \text{tr}[\dot{P}_u(\tau) P_u^{-1}(\tau)] d\tau$. When computing the estimate $\hat{u}_\tau$, we resort to the following LTV filtering problem, obtained by substituting the control mapping (2.32) into the LTV plant (2.29) and control channel (2.5) respectively:

$$\begin{aligned} dx_t &= \bar{A}(t)x_t dt + B(t)dw_t \\ de_t &= K(t)x_t dt + dw_t, \end{aligned} \quad (2.34)$$

where $\bar{A}(t) = A(t) + B(t)K(t)$. Let the estimation error covariance matrix be $P_t = \mathbb{E}[(x_t - \hat{x}_t)(x_t -$

$\hat{x}_t)^\top]$, where $\hat{x}_t = \mathbb{E}[x_t|e_0^t]$. We can then solve P_t from the following Riccati differential equation or RDE [60, Theorem 10.3]

$$\dot{P}_t = \bar{A}_t P_t + P_t \bar{A}_t^\top + B_t B_t^\top - (B_t + P_t K_t^\top)(B_t + P_t K_t^\top)^\top,$$

or equivalently

$$\dot{P}_t = A_t P_t + P_t A_t^\top - P_t K_t^\top K_t P_t, \quad (2.35)$$

which is also the RDE of (2.34) without the control input:

$$\begin{aligned} dx_t &= A(t)x_t dt \\ de_t &= K(t)x_t dt + dw_t. \end{aligned} \quad (2.36)$$

When the LTV system (2.36) is either exponentially stable or uniformly completely reconstructible [84, 93], given any initial condition $P(t_0) \geq 0$, the solution of (2.35) converges to $P(t)$, i.e., $P(t) = \lim_{\tau \rightarrow \infty} P(\tau)$. By invoking Lemma 2.22, we can transform (2.36) into the following modal form

$$\begin{aligned} \begin{bmatrix} dx_s(t) \\ dx_u(t) \end{bmatrix} &= \begin{bmatrix} A_s(t) & 0 \\ 0 & A_u(t) \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix} dt \\ de_t &= \begin{bmatrix} K_s(t) & K_u(t) \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix} dt + dw_t, \end{aligned} \quad (2.37)$$

where $A_s(t)$ is UES, and $A_u(t)$ is UEA. Let $\bar{K}(t) = K(t)T^{-1}(t) = [K_s(t) \ K_u(t)]$ and $\bar{x}(t) = [x_s^\top(t) \ x_u^\top(t)]^\top$. Based on Lemma 2.35 in Section 2.3.3, when the LTV system (2.36) and thus also (2.37) are uniformly completely reconstructible, the estimation error covariance matrix $\bar{P}(t) = \mathbb{E}[(\bar{x}_t - \hat{\bar{x}}_t)(\bar{x}_t - \hat{\bar{x}}_t)^\top]$ of (2.37) takes the form of $\bar{P}(t) = \text{diag}\{0, P_u(t)\}$, where $P_u(t)$ is the positive definite solution of the following RDE:

$$\dot{P}_u(t) = A_u(t)P_u(t) + P_u(t)A_u^\top(t) - P_u(t)K_u^\top(t)K_u(t)P_u(t). \quad (2.38)$$

Meanwhile, since $u_t = \bar{K}_t \bar{x}_t$, $\text{tr}(\bar{P}_t) = \text{tr}[P_u(t)]$, and $\text{tr}(\bar{K}_t \bar{P}_t \bar{K}_t^\top) = \text{tr}[K_u(t)P_u(t)K_u^\top(t)]$, the MS

estimation errors satisfy

$$\text{tr}[P_u(t)] = \mathbb{E}[(\bar{x}_t - \hat{x}_t)^\top (\bar{x}_t - \hat{x}_t)], \quad (2.39a)$$

$$\text{tr}[K_u(t)P_u(t)K_u^\top(t)] = \mathbb{E}[(u_t - \hat{u}_t)^\top (u_t - \hat{u}_t)]. \quad (2.39b)$$

Integrating both sides of (2.39b) from $\tau = 0$ to t gives

$$\int_0^t \text{tr}[K_u(\tau)P_u(\tau)K_u^\top(\tau)]d\tau = \mathbb{E}[\|u - \hat{u}\|^2], \quad (2.40)$$

where Fubini's theorem is used to exchange the order of integration and expectation. Pre- and post-multiplying (2.38) by $P_u^{-1/2}(\tau)$ and substituting it into (2.40) yields

$$\mathbb{E}[\|u - \hat{u}\|^2] = \int_0^t \text{tr}[P_u^{1/2}(\tau)K_u^\top(\tau)K_u(\tau)P_u^{1/2}(\tau)]d\tau = \int_0^t \text{tr}[2A_u(\tau) - \dot{P}_u(\tau)P_u^{-1}(\tau)]d\tau, \quad (2.41)$$

where the second equality follows $P_u^{1/2}(\tau)K_u^\top(\tau)K_u(\tau)P_u^{1/2}(\tau) = P_u^{-1/2}(\tau)A_u(\tau)P_u^{1/2}(\tau) + P_u^{1/2}(\tau) \cdot A_u^\top(\tau)P_u^{-1/2}(\tau) - P_u^{-1/2}(\tau)\dot{P}_u(\tau)P_u^{-1/2}(\tau)$ obtained from the preceding manipulation on (2.38). From (2.41), we can then derive (2.33) by

$$\begin{aligned} \bar{I}(e; x_0) &= \lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E}[\|u - \hat{u}\|^2] \\ &= \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \text{tr}[2A_u(\tau) - \dot{P}_u(\tau)P_u^{-1}(\tau)]d\tau \\ &\geq \sum_{i=1}^l n_i \underline{\lambda}_i - \limsup_{t \rightarrow \infty} \frac{1}{2} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)], \end{aligned}$$

where the inequality follows from Lemma 2.23, the convergence of $P_u(t) = \lim_{\tau \rightarrow \infty} P_u(\tau)$, and the application of L'Hôpital's rule [94]:

$$\lim_{t \rightarrow \infty} \frac{\int_0^t \text{tr}[\dot{P}_u(\tau)P_u^{-1}(\tau)]d\tau}{t} = \lim_{t \rightarrow \infty} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)] \leq \limsup_{t \rightarrow \infty} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)]. \quad (2.42)$$

Moreover, by comparing (2.33) and (2.43) below, we can further conclude that $\limsup_{t \rightarrow \infty} \text{tr}[\dot{P}_u(t) \cdot P_u^{-1}(t)] \leq 0$ in (2.33). This completes the proof. $\square$

For completeness, we also consider the scenario when the LTV plant (2.29) is stabilized by a

causal nonlinear controller (2.4). Unlike Proposition 2.24, wherein an equality of $\bar{I}(e; x_0)$ is attained by using Theorem 2.14 or Duncan's theorem, the following lemma, based on the mean-square stability and maximum entropy, only provides a lower bound for $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$.

Corollary 2.25. *When the closed-loop system, subject to the LTV plant (2.29) and nonlinear controller (2.4) as in Figure 2.1, are internally mean-square stable, we have*

$$\bar{I}(e; x_0, c) \geq \bar{I}(e; x_0) \geq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \text{tr}[A_u(\tau)] d\tau \geq \sum_{i=1}^l n_i \underline{\lambda}_i, \quad (2.43)$$

where $A_u(\tau)$ is the antistable part of $A(\tau)$, and $\{\underline{\lambda}_i\}_{i=1}^l$, with dimensions $n_1, \dots, n_l$, are the spectral values of the antistable component $A_u(\tau)$.

Proof. The first inequality in (2.43) can be directly obtained by applying Theorem 2.17 to the control channel that consists of the LTV plant (2.29) and nonlinear control mapping (2.4). To prove Corollary 2.25, it remains to verify that $\bar{I}(e; x_0) \geq \lim_{t \rightarrow \infty} (1/t) \int_0^t \text{tr}[A_u(\tau)] d\tau \geq \sum_{i=1}^l n_i \underline{\lambda}_i$. Since the matrix function $A(t)$ in (2.29) possesses an exponential dichotomy, we can decompose the LTV system (2.29) into UES and UEA parts as Lemma 2.22 shows, in which the UEA part can be described by

$$dx_u(t) = A_u(t)x_u(t)dt + B_u(t)de(t), \quad (2.44)$$

where $x_u(t)$ denotes the antistable modes. The solution to (2.44) can be expressed as [95]

$$x_u(t) = \Phi_{A_u}(t, 0)x_u(0) + \int_0^t \Phi_{A_u}(t, \tau)B_u(\tau)de(\tau) = \Phi_{A_u}(t, 0)[x_u(0) + \bar{x}_u(t)], \quad (2.45)$$

where $\Phi_{A_u}(\tau, \tau_0)$ is the state transition matrix of $A_u(t)$ from τ_0 to τ , and $\bar{x}_u(t) = \Phi_{A_u}^{-1}(t, 0) \cdot \int_0^t \Phi_{A_u}(t, \tau)B(\tau)de(\tau)$. Since the closed-loop system is internally mean-square stable, we have

$$\begin{aligned} +\infty &> M > \log \left[\det \left(\mathbb{E} \left[x_u(t)x_u^\top(t) \right] \right) \right] \\ &= 2 \log [\det (\Phi_{A_u}(t, 0))] + \log [\det (\Gamma)] \\ &\stackrel{(a)}{=} 2 \int_0^t \text{tr}[A_u(\tau)] d\tau + \log [\det (\Gamma)], \end{aligned} \quad (2.46)$$

where $\Gamma = \mathbb{E}[(x_u(0) + \bar{x}_u(t))(x_u(0) + \bar{x}_u(t))^\top]$, and step (a) follows from the Liouville's formula.

Meanwhile, the mutual information $I(e_0^t; x_0)$ satisfies

$$\begin{aligned}
I(e_0^t; x_0) &\stackrel{(a)}{\geq} I(e_0^t; x_u(0)) \\
&\stackrel{(b)}{\geq} I(\bar{x}_u(t); x_u(0)) \\
&\stackrel{(c)}{=} h(x_u(0)) - h(x_u(0)|\bar{x}_u(t)) \\
&\stackrel{(d)}{=} h(x_u(0)) - h(x_u(0) + \bar{x}_u(t)|\bar{x}_u(t)) \\
&\stackrel{(e)}{\geq} h(x_u(0)) - h(x_u(0) + \bar{x}_u(t)) \\
&\stackrel{(f)}{\geq} h(x_u(0)) - \frac{1}{2} \log(2\pi e)^k - \frac{1}{2} \log(\det(\Gamma)),
\end{aligned} \tag{2.47}$$

where (a) follows the data processing inequality in [Property 2.6](#) as $x_u(0)$ is a function of x_0 according to (2.29) and (2.31); (b) also follows the data processing inequality as $\bar{x}_u(t)$ is a function of e_0^t in (2.45); (c) uses the identity $I(x; y) = h(y) - h(y|x)$ in [Property 2.7](#); (d) employs the invariance of entropy in [Property 2.8](#); (e) adopts the fact that $I(x; y) = h(y) - h(y|x) \geq 0$ in [Property 2.7](#), and (f) uses the maximum entropy, $h(x) \leq h(\bar{x}) = (1/2) \log[(2\pi e)^n \det(\Sigma_x)]$ where $\bar{x}$ is a Gaussian vector with the same covariance as x . Hence, substituting (2.46) or $-\log(\det(\Gamma)) > -M + 2 \int_0^t \text{tr}(A_u(\tau)) d\tau$ into (2.47) yields

$$I(e_0^t; x_0) \geq h(x_u(0)) - \frac{1}{2} \log(2\pi e)^k - \frac{M}{2} + \int_0^t \text{tr}[A_u(\tau)] d\tau. \tag{2.48}$$

Dividing both sides of (2.48) by t and taking the limit as $t \rightarrow \infty$, we have

$$\begin{aligned}
\bar{I}(e; x_0) &\geq \lim_{t \rightarrow \infty} \left[\frac{h(x_u(0))}{t} - \frac{1}{2t} \log(2\pi e)^n - \frac{M}{2t} + \frac{1}{t} \int_0^t \text{tr}[A_u(\tau)] d\tau \right] \\
&= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \text{tr}[A_u(\tau)] d\tau \geq \sum_{i=1}^l n_i \lambda_i,
\end{aligned}$$

where the last inequality follows [Lemma 2.23](#). This completes the proof. $\square$

Since the lumped input process $\mathcal{U}$ in (2.5) and [Figure 2.1](#) is no longer linear in the setup of [Corollary 2.25](#), for the exact values of $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$ we should resort to [Theorem 2.14](#) and the nonlinear estimation method to be discussed in the following subsection.

[Proposition 2.24](#) and [Corollary 2.25](#) for the first time present an equality and an inequality

between the total information (or instability) rate and the sum of spectral values of the antistable component or the LTV Bode's integral $\mathcal{B}$ in the continuous-time control systems described by (2.29). It is worth noting that similar to the RHS sum in (2.17) and the lower bound of integral $\mathcal{B}$ in Lemma 2.23, the lower bound in (2.33) is determined by the unstable portions of both plant and controller dynamics. Substituting (2.33) and (2.43) into (2.12) and (2.13), we can readily imply that the total information rate attained by any causal controller that stabilizes the LTV plant (2.29) must exceed the (minimum) sum of spectral values. This implication offers an LTV explanation to Theorem 2.17 and is consistent with the preceding interpretations for the LTI and general control systems. More specifically, owing to the relationship between $\bar{I}(e; x_0)$ and $\mathcal{B}$ in (2.33) and (2.43), the total information (or instability) rate serves as a time-domain trade-off, similar to the LTV Bode's integral in [24] and the average entropy cost function of minimum entropy control problem in [96] in the continuous-time control systems subject to the LTV plant (2.29). When the running control cost is smaller than $\bar{I}(e; x_0, c)$ and $\bar{I}(e; x_0)$ on a time interval, the running cost outside this interval must be larger, and vice versa, which is also analogous to the trade-off property of $\bar{I}(e; x_0, c)$ and Bode's integral in the frequency range of LTI systems.

Remark 2.26. *Proposition 2.24 and Corollary 2.25 cover Proposition 2.19 and Lemma 2.20 as a special case. When we apply the former LTV results to the LTI systems described by (2.15), the spectral values in (2.33) and (2.43) will degenerate to the real parts of the open-loop unstable poles in (2.17) and (2.28), and $\text{tr}[\dot{P}_u(t)P_u^{-1}(t)]$ in (2.33) will vanish, since the error covariance matrix $P_u(t)$ converges to a constant matrix P_u , i.e., $\lim_{t \rightarrow \infty} \dot{P}_u(t) = 0$.*

2.2.4 Control Trade-offs in Nonlinear Systems

At the end of this section, we consider the limitation of nonlinear control systems. Substituting the memory-less controller (2.4) into the nonlinear plant (2.3) and channel (2.5), we obtain the following nonlinear filtering problem

$$\begin{aligned} dx_t &= \bar{f}(t, x_t, e_t)dt + b(t, x_t)dw_t \\ de_t &= u(t, e_t)dt + dw_t, \end{aligned} \tag{2.49}$$

where $x_t \in \mathbb{R}^n$ denotes the hidden states; the error signal e_t is observable, and $\bar{f}(t, x_t, e_t) = f(t, x_t) + b(t, x_t)u(t, e_t)$. For conciseness, we first consider the single-input and single-output (SISO) case, i.e., u_t and $e_t \in \mathbb{R}$ in (2.49), and omit the observable commanded signal c_t in (2.4).

To compute the information rate $\bar{I}(e; x_0)$ from (2.49), we use the equality $I(e; x_0) = (1/2) \cdot (\mathbb{E}[\|u\|^2] - \mathbb{E}[\|\hat{u}\|^2])$ from Theorem 2.14, where the first expectation $\mathbb{E}[\|u\|^2]$ is usually observable, and the second expectation $\mathbb{E}[\|\hat{u}\|^2]$ and $\hat{u}(\tau, e_0^\tau, x_0) = \mathbb{E}[u(\tau, e_0^\tau, x_0)|e_0^\tau]$ need to be estimated from (2.49). When $u(t, x_t)$ is not observable, we can follow the scheme in Section 2.3.4 to compute $\bar{I}(e; x_0)$. To implement a nonlinear filter on (2.49), analogous to the stabilizability and detectability conditions for solving ARE and RDE, additional regularity and differentiability constraints are required to guarantee the existence and uniqueness of solution to (2.49) and the pathwise uniqueness of filter equation [60, 97–100]. Let $k(t, x, e)$ denote any of the functions $\bar{f}(t, x_t, e_t)$, $b(t, x_t)$ and $u(t, e_t)$ in (2.49). We require $k(t, x, e)$ be Lipschitz, i.e., $|k(t, x_1, e_1) - k(t, x_2, e_2)|^2 \leq K_1(|x_1 - x_2|^2 + |e_1 - e_2|^2)$ for some constant $K_1 > 0$, and $k^2(t, x, e) \leq K_2(1 + x^2 + e^2)$ for some constant $K_2 > 0$. Meanwhile, the control input $u(t, e_t)$ needs to be at least once differentiable at t and twice differentiable at e_t such that $\sup_{\tau \leq t} \mathbb{E}[u^2(\tau, e_\tau)] < \infty$, $\int_0^t \mathbb{E}[(u'_{e_\tau}(\tau, e_\tau))^2] d\tau < \infty$, and $\int_0^t \mathbb{E}[\mathcal{L}u(\tau, e_\tau)]^2 d\tau < \infty$, where $\mathcal{L}u(\tau, e_\tau) = u'_\tau(\tau, e_\tau) + u'_{e_\tau}(\tau, e_\tau)u(\tau, e_\tau) + (1/2)u''_{e_\tau e_\tau}(\tau, e_\tau)$. When (2.49) fulfills the preceding regularity and differentiability constraints, based on Theorem 2.14 and the Stratonovich-Kushner equation, the following proposition provides an approach to compute the information rate $\bar{I}(e; x_0)$ in nonlinear control systems.

Proposition 2.27. *When the closed-loop systems described by the nonlinear plant and controller (2.49) as in Figure 2.1, are internally mean-square stable, we have*

$$\bar{I}(e; x_0) = \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \mathbb{E}[u^2(\tau, e_0^\tau, x_0)] d\tau - \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \mathbb{E}[\hat{u}^2(\tau, e_0^\tau, x_0)] d\tau, \quad (2.50)$$

where $\hat{u}(\tau, e_0^\tau, x_0) = \mathbb{E}[u(\tau, e_0^\tau, x_0)|e_0^\tau] = \pi_\tau(u)$ satisfies

$$\pi_\tau(u) = \pi_0(u) + \int_0^\tau \pi_s(\mathcal{L}u) ds + \int_0^\tau [\pi_s(u'_{e_s}) + \pi_s(u^2) - \pi_s^2(u)] d\tilde{w}_s; \quad (2.51)$$

$\mathcal{L}u(s, e_s) = u'_s(s, e_s) + u'_{e_s}(s, e_s)u(s, e_s) + (1/2)u''_{e_s e_s}(s, e_s)$, and $\tilde{w}_\tau = \int_0^\tau de_s - \int_0^\tau \pi_s(u) ds$ is a Wiener process w.r.t. $\mathcal{F}_\tau^e$, $\tau \in [0, t]$.

Proof. Applying [Theorem 2.14](#) to the control channel in [\(2.49\)](#), we have

$$\begin{aligned}\bar{I}(e; x_0) &\stackrel{(a)}{=} \lim_{t \rightarrow \infty} \frac{1}{2t} (\mathbb{E}[\|u\|^2] - \mathbb{E}[\|\hat{u}\|^2]) \\ &\stackrel{(b)}{=} \lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E} \left[\int_0^t u^2(\tau, e_0^\tau, x_0) d\tau \right] - \lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E} \left[\int_0^t \hat{u}^2(\tau, e_0^\tau, x_0) d\tau \right],\end{aligned}$$

where (a) is obtained by applying [\(2.8\)](#) to [\(2.49\)](#), dividing the result by t , and taking the limit as $t \rightarrow \infty$; (b) expands the norms inside expectations, and [\(2.50\)](#) is then obtained by using the Fubini's theorem to swap the order of expectation and integration in (b). While the control signal $u(\tau, e_0^\tau, x_0)$ in [\(2.50\)](#) is observable, the optimal estimate $\pi_\tau(u) = \hat{u}(\tau, e_0^\tau, x_0) = \mathbb{E}[u(\tau, e_0^\tau, x_0)|e_0^\tau]$ needs to be estimated from the nonlinear filtering problem [\(2.49\)](#). By applying the one-dimensional Stratonovich-Kushner equation in [\[60, Theorem 8.3\]](#) to [\(2.49\)](#), we obtain the estimate $\hat{u}(\tau, e_0^\tau, x_0)$ in [\(2.51\)](#). $\square$

Due to the various and complex structures of nonlinear plants and systems, the trade-off properties of $\bar{I}(e; x_0)$ in [\(2.50\)](#) as well as its connections to some established control trade-offs, such as the Bode-type integrals and entropy cost function, are not as explicit as the preceding discussions for linear systems. Meanwhile, similar to [Propositions 2.19](#) and [2.24](#), the value of $\bar{I}(e; x_0)$ in [Proposition 2.27](#) is determined by both the nonlinear plant and controller. However, by combining [Proposition 2.27](#) with [Theorem 2.17](#) or [Proposition 2.15](#), we can still glimpse some trade-off features of $\bar{I}(e; x_0)$ in nonlinear control systems. Consistent with the interpretations for linear systems, a straightforward observation from [Proposition 2.27](#) is that the total information rate of any causal controller that stabilizes the nonlinear plant [\(2.49\)](#) must exceed the (minimum) value of $\bar{I}(e; x_0)$ given in [\(2.50\)](#), which can be used as a criterion to examine whether a causal controller is able to stabilize the nonlinear system [\(2.49\)](#). Meanwhile, since $D(\mu_{e|(x_0, c)} \|\mu_w) = (1/2) \cdot \mathbb{E}[\|u\|^2]$ and $D(\mu_{e|(x_0, c)} \|\mu_w) = I(e_0^t; x_0, c_0^t) + D(\mu_e \|\mu_w)$ by [Proposition 2.15](#), the total information rate $\bar{I}(e; x_0, c)$ or $\bar{I}(e; x_0)$ also serves as a time-domain trade-off, similar to the entropy cost function and LTV Bode's integral $\mathcal{B}$, in the nonlinear setting. When the quadratic control cost $\mathbb{E}[u_\tau^\top u_\tau]$ is smaller than the value of $\bar{I}(e; x_0)$ in [\(2.50\)](#) or, more precisely, the value of $\bar{I}(e; x_0, c) + \bar{D}(\mu_e \|\mu_w)$ on a time interval, the control cost outside this interval must be larger, and vice versa. Up until now, we have analyzed and computed total information rate as a fundamental control trade-off in the continuous-time control systems subject to LTI, LTV, and nonlinear plants. In the following section,

total information rate as a fundamental filtering limit will be investigated.

Remark 2.28. *To compute the mutual information rate $\bar{I}(e; x_0)$ of a MIMO control channel, we only need to rewrite (2.49) and (2.51) into a matrix-vector form and replace the one-dimensional filtering equation in Proposition 2.27 by the multi-dimensional Stratonovich-Kushner equation in [60, Theorem 9.3].*

2.3 Fundamental Limitations of Continuous-Time Filtering Systems

For the filtering problem, we first show that total information rate $\bar{I}(y; x_0, w)$ or directed information rate $\bar{I}(z \rightarrow y)$ serves as a filtering limit in the general filtering systems depicted by (2.7) and Figure 2.2, and then apply it to capture the fundamental limitations of the filtering systems subject to LTI, LTV, and nonlinear plants, respectively.

2.3.1 General Filtering Limits

Based on Duncan's theorem, the following theorem provides a direct method to compute the total information (rate) $I(y_0^t; x_0, w_0^t)$ from the MS estimation error of noise-free output $z(t, w_0^t, x_0)$ in (2.7).

Theorem 2.29. *For the filtering systems described by (2.7) and Figure 2.2, the total information $I(y_0^t; x_0, w_0^t)$ satisfies*

$$I(y_0^t; x_0, w_0^t) = \frac{1}{2} \mathbb{E} [\|z - \hat{z}\|^2] = \frac{1}{2} (\mathbb{E}[\|z\|^2] - \mathbb{E}[\|\hat{z}\|^2]), \quad (2.52)$$

where $\hat{z}(\tau, w_0^\tau, x_0) = \mathbb{E}[z(\tau, w_0^\tau, x_0) | y_0^\tau]$, and the total information rate is given by

$$\bar{I}(y; x_0, w) = \lim_{t \rightarrow \infty} \frac{1}{t} I(y_0^t; x_0, w_0^t). \quad (2.53)$$

Proof. Since the observation model in (2.7) is an additive white Gaussian channel without feedback, by applying the non-feedback version of Lemma 2.11 to (2.7), we obtain (2.52). The total information rate (2.53) follows from Definition 2.3. $\square$

[Theorem 2.29](#) shows that regardless of how the filtering process $\mathcal{F}$ is designed, $I(y_0^t; x_0, w_0^t)$ equals half of the time integral of MS estimation error and $\bar{I}(y; x_0, w)$ equals half of the time-averaged or steady-state MS estimation error as $t \rightarrow \infty$. By using the chain rule of mutual information and [Lemma 2.12](#), more insights on $I(y_0^t; x_0, w_0^t)$ can be extracted.

Proposition 2.30. *For the filtering systems described by (2.7) and [Figure 2.2](#), the total information $I(y_0^t; x_0, w_0^t)$ satisfies*

$$I(y_0^t; x_0, w_0^t) = I(y_0^t; x_0) + I(y_0^t; w_0^t | x_0) \quad (2.54)$$

and can be expressed as a difference between divergences

$$I(y_0^t; x_0, w_0^t) = D(\mu_{y|(x_0, w)} \| \mu_v) - D(\mu_y \| \mu_v), \quad (2.55)$$

where μ_v is the Wiener measure induced by the measurement noise v_0^t ; $D(\mu_{y|(x_0, w)} \| \mu_v) = (1/2) \cdot \mathbb{E}[\|z\|^2]$, and $D(\mu_y \| \mu_v) = (1/2) \cdot \mathbb{E}[\|\hat{z}\|^2]$.

Proof. By applying the chain rule of mutual information in [Property 2.5](#) to $I(y_0^t; x_0, w_0^t)$, we can prove (2.54). Equation (2.55) can be verified by replacing the mutual information and divergences with their definition forms in [Definition 2.1](#) and $I(x_0^t; y_0^t) := \mathbb{E}_{(x, y)}[\log(d\mu_{xy}/d(\mu_x \times \mu_y))] = \mathbb{E}_y[\log(d\mu_{y|x}/d\mu_y)]$ in [Definition 2.3](#). Meanwhile, by applying the non-feedback version of [Lemma 2.12](#) to the measurement channel (2.7), we have $D(\mu_{y|(x_0, w)} \| \mu_v) = (1/2) \cdot \mathbb{E}[\|z\|^2]$ and $D(\mu_y \| \mu_v) = (1/2) \cdot \mathbb{E}[\|\hat{z}\|^2]$. This completes the proof. $\square$

[Proposition 2.30](#) verifies that when using [Theorem 2.29](#) or Duncan's theorem to compute $I(y_0^t; x_0, w_0^t)$ or $\bar{I}(y; x_0, w)$, we actually lift the calculation and analysis of total information (rate) to a continuous function space, since the causal MMSE in (2.52) can be equivalently expressed as a divergence rate difference w.r.t. the Wiener measure on infinite-dimensional spaces.

We now consider the filtering trade-off properties of $\bar{I}(y; x_0, w)$ and $\bar{I}(y; x_0)$. Suppose the average power of noise-free output in (2.7) is finite, i.e., $\mathbb{E}[z_\tau^\top z_\tau] \leq \rho(\tau)$, $\forall \tau \in [0, t]$. Define the filtering capacity $\mathcal{C}_f := \sup_{(x_0, w)} \bar{I}(y; x_0, w)$ as the supremum of total information rate, and the channel capacity $\mathcal{C} := \sup_{\mathbb{E}[z_\tau^\top z_\tau] \leq \rho(\tau)} \bar{I}(y; z)$ as the supremum of input-output mutual information rate. The following theorem then reveals some fundamental relationships among the capacities, information rates, and divergence rates in the measurement channel (2.7).

Theorem 2.31. *For the filtering systems described by (2.7) and Figure 2.2, the total information rate $\bar{I}(y; x_0, w)$ is bounded by*

$$\mathcal{C}_f \geq \bar{I}(y; x_0, w) \geq \bar{I}(y; x_0), \quad (2.56)$$

where $\bar{I}(y; x_0, w) = \bar{I}(y; x_0)$ if w_0^t and y_0^t are conditionally independent given x_0 , i.e., $w_0^t - x_0 - y_0^t$ forms a Markov chain. Meanwhile, when the noise-free output is subject to the average power constraint, $\mathbb{E}[z_\tau^\top z_\tau] \leq \rho(\tau)$, $\forall \tau \in [0, t]$, we have

$$\mathcal{C}_f \geq \bar{I}(y; x_0, w) + \bar{D}(\mu_y \| \mu_v) \geq \bar{I}(y; x_0) + \bar{D}(\mu_y \| \mu_v), \quad (2.57)$$

where $\mathcal{C}_f = \mathcal{C} = (2t)^{-1} \int_0^t \rho(\tau) d\tau$, and the divergence rate $\bar{D}(\mu_y \| \mu_v) := \lim_{t \rightarrow \infty} D(\mu_y \| \mu_v)/t$.

Proof. The first inequality in (2.56) follows the definition of filtering capacity. By applying the chain rule of mutual information in Property 2.5 to $I(y_0^t; x_0, w_0^t)$ and using the non-negativity of mutual information in Property 2.7, we can derive the second inequality in (2.56). Meanwhile, by the chain rule, we have $I(y_0^t; x_0, w_0^t) = I(y_0^t; x_0)$ when w_0^t and y_0^t are conditionally independent given x_0 .

To verify (2.57), we first impose an average power constraint on the noise-free output such that $\mathbb{E}[z_\tau^\top z_\tau] \leq \rho(\tau)$, $\forall \tau \in [0, t]$ or $(1/t) \int_0^t \mathbb{E}[z_\tau^\top z_\tau] d\tau \leq \rho$. Since $D(\mu_{y|(x_0, w)} \| \mu_v) = (1/2) \cdot \mathbb{E}[\|z\|^2]$ by Proposition 2.30, based on [79, Theorem 6.4.1], the filtering and channel capacities of (2.7) then satisfy

$$\mathcal{C}_f = \frac{\int_0^t \rho(\tau) d\tau}{2t} \geq \lim_{t \rightarrow \infty} \frac{\mathbb{E}[\|z\|^2]}{2t} = \bar{D}(\mu_{y|(x_0, w)} \| \mu_v), \quad (2.58)$$

where $\mathcal{C}_f = \mathcal{C}$; the inequality follows from the average power constraint and Fubini's theorem, and the divergence rate is given by $\bar{D}(\mu_{y|(x_0, w)} \| \mu_v) := \lim_{t \rightarrow \infty} D(\mu_{y|(x_0, w)} \| \mu_v)/t$. Dividing both sides of (2.55) by t , taking the limit as $t \rightarrow \infty$, and substituting (2.58) into the results give the first inequality in (2.57). The second inequality in (2.57) follows (2.56). $\square$

An information-theoretic implication from Theorem 2.31 is that for the measurement channel in (2.7), the channel (or filtering) capacity $\mathcal{C}$ (or $\mathcal{C}_f$), determined by the plant $\mathcal{P}$, must be large enough to offset the cost of the transmitting message, quantified by $\bar{I}(y; x_0, w)$ or $\bar{I}(y; x_0)$, and the cost of attenuating channel noise, measured by $\bar{D}(\mu_y \| \mu_v)$. Meanwhile, the capacity of the measurement

channel can be achieved when $\bar{D}(\mu_y \parallel \mu_v) = 0$ or $D(\mu_y \parallel \mu_v) = o(t)$ as $t \rightarrow \infty$.

Compared with the preceding control interpretations, the total information rate $\bar{I}(y; x_0, w)$ and mutual information rate $\bar{I}(y; x_0)$ in [Theorem 2.31](#) have a more explicit and consistent filtering-theoretic explanation across different filtering systems. As [Theorem 2.29](#) shows, $\bar{I}(y; x_0, w)$ is half of the time-averaged or steady-state MS estimation error, and $\bar{I}(y; x_0) = \lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w)$ is half of the lowest achievable MS estimation error in the absence of process noise. When the covariance of process noise or the intensity of corresponding white noise tends to zero, $\Sigma_{\bar{w}} = \varepsilon I \geq 0$ with $\varepsilon \rightarrow 0$, we have

$$\bar{I}(y; x_0, w) \geq \lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w) = \bar{I}(y; x_0), \quad (2.59)$$

which can be verified by [Proposition 2.30](#), [Theorem 2.31](#), and the fact that $\lim_{\varepsilon \rightarrow 0} I(y_0^t; w_0^t | x_0) = 0$ as the lumped process input $w_0^t \rightarrow 0$ and becomes deterministic. In practice, when the process noise exists and the filtering process $\mathcal{F}$ is not optimal, the practical MS estimation error must be larger than the minimum errors gauged by $\bar{I}(y; x_0, w)$ and $\bar{I}(y; x_0)$. Hence, a filtering-theoretic disclosure of [Theorem 2.31](#) is that to effectively transmit a message or distinguish the noise-free output from the measured output in (2.7), the channel capacity $\mathcal{C}$ (or the power of noise-free output) must be larger than $\bar{I}(y; x_0, w)$ and $\bar{I}(y; x_0)$ (or the minimum MS estimation error). In the following subsections, the lowest achievable MS estimation errors, or $\bar{I}(y; x_0, w)$ and $\bar{I}(y; x_0)$ in [Theorem 2.31](#), will be respectively investigated in the filtering systems subject to LTI, LTV, and nonlinear plants.

Remark 2.32. *Theorems 2.29, 2.31 and Proposition 2.30 apply to all filtering problems depicted by (2.7) and Figure 2.2, regardless of the linearity of the plant and the stationarity or distributions of the signals and initial states. Moreover, when the process noise w_0^t is excluded or completely vanishes, the total information rate $\bar{I}(y; x_0, w)$ coincides with $\bar{I}(y; x_0)$, and Theorem 2.29 and Proposition 2.30 persist, while the second inequalities in (2.56) and (2.57) degenerate to equalities.*

2.3.2 Filtering Limits in LTI Systems

To capture the filtering limitations of LTI plants, we consider the following plant and observation models

$$dx_t = Ax_t dt + Bdw_t \quad (2.60)$$

$$dy_t = Cx_t dt + \Sigma_v^{1/2} dv_t,$$

where x_t is the vector of the hidden states; y_t is observable; A, B and C are time-invariant matrices of proper dimensions; w_t is the lumped process noise such that $dw_t = \Sigma_{\bar{w}}^{1/2} d\bar{w}_t$; independent standard Brownian motions $\bar{w}_t$ and v_t respectively denote the process and measurement noise with covariance functions $\Sigma_{\bar{w}}$ and Σ_v , and the noise-free output $z_t = Cx_t$. Let $\hat{x}_t$ and $\hat{z}_t = C\hat{x}_t$ be the estimates of states x_t and output z_t , and the estimation error $\tilde{z}_t = z_t - \hat{z}_t = C(x_t - \hat{x}_t)$. The following optimal linear or Kalman-Bucy filter then gives an optimal MS estimation of x_t and z_t by conditioning on the measured output y_0^t [60, 93]:

$$d\hat{x}_t = A\hat{x}_t dt + PC^\top \Sigma_v^{-1} [dy_t - C\hat{x}_t dt], \quad (2.61)$$

where the optimal estimates are given by $\hat{x}_t = \mathbb{E}[x_t | y_0^t]$ and $\hat{z}_t = \mathbb{E}[Cx_t | y_0^t]$, and the positive semi-definite matrix $P_t = \mathbb{E}[(x_t - \hat{x}_t)(x_t - \hat{x}_t)^\top]$ denotes the estimation error covariance. When the LTI system (2.60) is either i) asymptotically stable or ii) both stabilizable and detectable, given any initial condition $P(t_0) \geq 0$, the covariance matrix P_t converges as $t \rightarrow \infty$, i.e., $P = \lim_{t \rightarrow \infty} P_t$, which can be solved from the ARE [84]:

$$0 = AP + PA^\top + B\Sigma_{\bar{w}}B^\top - PC^\top \Sigma_v^{-1} CP. \quad (2.62)$$

The following lemma from [12] then shows the minimum MS estimation error achieved by the optimal filter (2.61) when the covariance matrices $\Sigma_v = I$ and $\Sigma_{\bar{w}} = \varepsilon I \rightarrow 0$ in (2.60).

Lemma 2.33. *For the filtering systems described by the stabilizable and detectable LTI plant (2.60) and the optimal linear filter (2.61) as in Figure 2.2, the lowest achievable MS error in the estimation of noise-free output z_t is*

$$\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \mathbb{E}[\tilde{z}_t^\top \tilde{z}_t] = 2 \sum_i \lambda_i^+(A),$$

where $\tilde{z}_t = z_t - \hat{z}_t$; $\hat{z}_t = \mathbb{E}[z_t | y_0^t]$, and $\lambda_i^+(A)$ indicates the eigenvalues of matrix A with positive real parts.

Lemma 2.33 tells that the lowest achievable MS estimation error, achieved by the optimal filter

(2.61) in the idealist setting without process noise, equals twice the sum of the unstable poles in (2.60).

For the filtering systems subject to LTI plant (2.60), irrespective of how the filtering process $\mathcal{F}$ is designed, the following proposition develops an equality between the mutual information rate $\bar{I}(y; x_0)$ in Theorem 2.31 and the sum of open-loop unstable poles in (2.60).

Proposition 2.34. *For the filtering system subject to the stabilizable and detectable LTI plant (2.60) as in Figure 2.2, we have*

$$\bar{I}(y; x_0) = \sum_i \lambda_i^+(A), \quad (2.63)$$

where $\lambda_i^+(A)$ denotes the eigenvalues of matrix A with positive real parts.

Proof. Since $\bar{I}(y; x_0) = \lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w)$ by (2.59), in order to prove Proposition 2.34, we only need to show that $\lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w) = \sum_i \lambda_i^+(A)$. By applying Theorem 2.29 to the LTI system (2.60) under vanishing process noise or $\Sigma_{\bar{w}} = \varepsilon I \rightarrow 0$, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w) &\stackrel{(a)}{=} \lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E}[\|\tilde{z}\|^2] \\ &\stackrel{(b)}{=} \lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E} \left[\int_0^t \tilde{z}_\tau^\top \tilde{z}_\tau d\tau \right] \\ &\stackrel{(c)}{=} \lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \mathbb{E} \left[\tilde{z}_\tau^\top \tilde{z}_\tau \right] d\tau \\ &\stackrel{(d)}{=} \lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{2} \mathbb{E} \left[\tilde{z}_t^\top \tilde{z}_t \right] \stackrel{(e)}{=} \sum_i \lambda_i^+(A), \end{aligned}$$

where (a) is obtained by substituting (2.60) into Theorem 2.29 and taking the limit as $t \rightarrow \infty$; in step (b) we expand the norm $\mathbb{E}[\|\tilde{z}\|^2]$; Fubini's theorem is employed to swap the order of integration and expectation in step (c); L'Hôpital's rule is invoked to derive (d), and step (e) follows from Lemma 2.33. This completes the proof. $\square$

Proposition 2.34 shows that when the plant model $\mathcal{P}$ in Figure 2.2 is an LTI model as (2.60), the total information rate $\bar{I}(y; x_0, w)$ and mutual information rate $\bar{I}(y; x_0)$ are bounded below or equal to the sum of open-loop unstable poles in (2.60). Meanwhile, along with the relationships between $\bar{I}(y; x_0, w)$, $\bar{I}(y; x_0)$ and the minimum MS estimation error, we can imply that for the LTI filtering systems, $\bar{I}(y; x_0, w)$ and $\bar{I}(y; x_0)$ serve as an information-theoretic interpretation or replacement for

the lowest achievable MS estimation error (in the absence of process noise, [12]), and the (minimum) MS estimation error of the filtering system (2.60) must be larger than twice the sum of the unstable poles. Analogous to the trade-off properties of Bode's integral in the frequency range and the entropy cost function in the time domain, $\bar{I}(y; x_0, w)$, $\bar{I}(y; x_0)$ or the lowest achievable MS estimation error serves as a filtering trade-off in the time domain. For the LTI filtering system (2.60), if the MS estimation error of z on a time interval is smaller than twice the sum of the unstable poles, $\bar{I}(y; x_0, w)$ or $\bar{I}(y; x_0)$, the MS estimation error outside this interval must be larger, and vice versa.

2.3.3 Filtering Limits in LTV Systems

We then consider the LTV filtering systems with the following plant $\mathcal{P}$ and observation model

$$\begin{aligned} dx_t &= A(t)x_t dt + B(t)dw_t, \\ dy_t &= C(t)x_t dt + \Sigma_v^{1/2}(t)dv_t, \end{aligned} \tag{2.64}$$

where x_t denotes the hidden states; the measured output y_t is observable; $A(t)$, $B(t)$ and $C(t)$ are time-varying matrix functions of proper dimensions; w_t is the lumped process noise that satisfies $dw_t = \Sigma_w^{1/2}(t)d\bar{w}_t$; the independent standard Brownian motions $\bar{w}_t$ and v_t are respectively the process and measurement noise with covariance functions $\Sigma_w(t)$ and $\Sigma_v(t)$, and the definitions of the noise-free output z_t , the estimate $\hat{z}_t$, and the estimation error $\tilde{z}_t$ can be implied from (2.60). The optimal MS estimates $\hat{x}_t = \mathbb{E}[x_t|y_0^t]$ and $\hat{z}_t = C\hat{x}_t$ can be calculated from (2.64) by resorting to the following Kalman-Bucy filter [60, 93]:

$$d\hat{x}_t = A(t)\hat{x}_t dt + P(t)C^\top(t)\Sigma_v^{-1}(t)[dy_t - C(t)\hat{x}_t dt], \tag{2.65}$$

where $P(t) = \mathbb{E}[(x_t - \hat{x}_t)(x_t - \hat{x}_t)^\top] = \mathbb{E}[\tilde{x}_t\tilde{x}_t^\top]$ denotes the estimation error covariance. When the LTV system (2.64) is either i) exponentially stable or ii) both uniformly completely stabilizable and uniformly completely reconstructible, for any initial condition $P(t_0) \geq 0$ the covariance function $P(\tau)$ converges to $P_t = \lim_{\tau \rightarrow \infty} P_\tau$, which can be solved from the following RDE [84]:

$$\dot{P}_t = A_t P_t + P_t A_t^\top + B_t \Sigma_w(t) B_t^\top - P_t C_t^\top \Sigma_v^{-1}(t) C_t P_t.$$

When the process input vanishes, i.e., $\Sigma_{\bar{w}}(t) = \varepsilon I \rightarrow 0$, the optimal filter (2.65) then gives the lowest achievable MS estimation error under vanishing process noise. Since w_0^t and v_0^t in (2.64) are uncorrelated and zero-mean white Gaussian, the MS estimation error attained by (2.65) is the optimal and minimum solution for all feasible filters [101]. To reveal more details about this estimation error, we decompose the original LTV system (2.64) into an UES part with states $x_s(t)$ and an UEA part with states $x_u(t)$ as (2.31) in Lemma 2.22. The lowest achievable MS estimation error of the transformed system satisfies the following lemma.

Lemma 2.35. *When the LTV filtering system (2.64) under vanishing process noise is uniformly completely stabilizable and detectable, the estimation error covariance, achieved by applying the optimal filter (2.65) to the modal-decomposed system of (2.64), satisfies*

$$\lim_{\varepsilon \rightarrow 0} \lim_{\tau \rightarrow \infty} \mathbb{E} \left\{ \begin{bmatrix} \tilde{x}_s(\tau) \\ \tilde{x}_u(\tau) \end{bmatrix} \begin{bmatrix} \tilde{x}_s(\tau) \\ \tilde{x}_u(\tau) \end{bmatrix}^\top \right\} = \begin{bmatrix} 0 & 0 \\ 0 & P_u(t) \end{bmatrix}, \quad (2.66)$$

where $\tilde{x}_s(\tau) = x_s(\tau) - \hat{x}_s(\tau)$ and $\tilde{x}_u(\tau) = x_u(\tau) - \hat{x}_u(\tau)$ are the estimation errors, and the lowest achievable MS estimation error under vanishing noise is

$$\lim_{\varepsilon \rightarrow 0} \lim_{\tau \rightarrow \infty} \mathbb{E} \left\{ \begin{bmatrix} \tilde{x}_s(\tau) \\ \tilde{x}_u(\tau) \end{bmatrix} \begin{bmatrix} \tilde{x}_s(\tau) \\ \tilde{x}_u(\tau) \end{bmatrix}^\top \right\} = \text{tr}[P_u(t)], \quad (2.67)$$

where $P_u(t)$ can be solved from the following RDE

$$\dot{P}_u(t) = A_u(t)P_u(t) + P_u(t)A_u^\top(t) - P_u(t)C_u^\top(t)C_u(t)P_u(t). \quad (2.68)$$

Proof. When the matrix function $A(t)$ in (2.64) admits an exponential dichotomy, by Lemma 2.22, we can decompose (2.64) into the following modal form

$$\begin{aligned} \begin{bmatrix} dx_s(t) \\ dx_u(t) \end{bmatrix} &= \begin{bmatrix} A_s(t) & 0 \\ 0 & A_u(t) \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix} dt + \begin{bmatrix} B_s(t) \\ B_u(t) \end{bmatrix} dw(t) \\ dy(t) &= \begin{bmatrix} C_s(t) & C_u(t) \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix} dt + dv(t). \end{aligned} \quad (2.69)$$

Let $\bar{x}(t) = [x_s^\top(t), x_u^\top(t)]^\top$ be the state of modal-decomposed systems, and $\tilde{x}(t) = [x_s^\top(t) - \hat{x}_s^\top(t), x_u^\top(t) - \hat{x}_u^\top(t)]^\top$ be the estimation error. When (2.31) or (2.64) is uniformly completely stabilizable and detectable, we have $\lim_{\tau \rightarrow \infty} \mathbb{E}[\tilde{x}(\tau)\tilde{x}^\top(\tau)] = P_t$ and $\lim_{\tau \rightarrow \infty} \mathbb{E}[\tilde{x}^\top(\tau)\tilde{x}(\tau)] = \text{tr}(P_t)$, where P_t can be solved from the following RDE

$$\begin{aligned} \dot{P}_t = & \begin{bmatrix} A_s(t) & 0 \\ 0 & A_u(t) \end{bmatrix} P_t + \varepsilon^2 \begin{bmatrix} B_s(t)B_s^\top(t) & B_s(t)B_u^\top(t) \\ B_u(t)B_s^\top(t) & B_u(t)B_u^\top(t) \end{bmatrix} + P_t \begin{bmatrix} A_s^\top(t) & 0 \\ 0 & A_u^\top(t) \end{bmatrix} \\ & - P_t \begin{bmatrix} C_s^\top(t)C_u(t) & C_s^\top(t)C_u(t) \\ C_u^\top(t)C_s(t) & C_u^\top(t)C_u(t) \end{bmatrix} P_t. \end{aligned} \quad (2.70)$$

For the singularly perturbed equation (2.70), we seek its solution $P_t = P_t^\top$ in the form of power series in ε [12, 102, 103]:

$$P(t) = \begin{bmatrix} \varepsilon^2 P_s(t) + O(\varepsilon^3) & \varepsilon^2 P_0(t) + O(\varepsilon^3) \\ * & P_u(t) + O(\varepsilon) \end{bmatrix}. \quad (2.71)$$

Substituting the leading terms of (2.71) into the RDE (2.70), $P_s(t)$, $P_0(t)$ and $P_u(t)$ in (2.71) then satisfy

$$\varepsilon^2 \dot{P}_s(t) = \varepsilon^2 [A_s(t)P_s(t) + P_s(t)A_s^\top(t) + B_s(t)B_s^\top(t)] + O(\varepsilon^4) \quad (2.72a)$$

$$\begin{aligned} \varepsilon^2 \dot{P}_0(t) = & \varepsilon^2 [A_s(t)P_0(t) + P_0(t)A_u^\top(t) + B_s(t)B_u^\top(t) - P_s(t)C_s^\top(t)C_u(t)P_u(t) \\ & - P_0(t)C_u^\top(t)C_u(t)P_u(t)] + O(\varepsilon^4) \end{aligned} \quad (2.72b)$$

$$\dot{P}_u(t) = A_u(t)P_u(t) + P_u(t)A_u^\top(t) - P_u(t)C_u^\top(t)C_u(t)P_u(t) + O(\varepsilon^2) \quad (2.72c)$$

After setting $\varepsilon = 0$ in (2.71) and (2.72), we find that the only nonzero element in $P(t)$ is $P_u(t)$, as (2.66) shows, and $P_u(t)$ is the positive definite solution of (2.68). This completes the proof. $\square$

Remark 2.36. Since $P_u(t)$ is the estimation error covariance of the antistable subsystem, Lemma 2.35 reveals that under the vanishing process noise, only the antistable modes contribute to the MS estimation error. An LTI version of Lemma 2.35 is given and used to prove Lemma 2.33 in [12].

Applying Theorem 2.29 and Lemma 2.35 to the LTV filtering system (2.64) under vanishing

process noise, we can obtain the following proposition that reveals some equality and inequality relationships for the total information rate or minimum MS estimation error in [Theorem 2.31](#), regardless of how the filtering process $\mathcal{F}$ is designed.

Proposition 2.37. *For the filtering system described by the LTV plant (2.64) under vanishing process noise, as in [Figure 2.2](#), the mutual information rate $\bar{I}(y; x_0)$ satisfies*

$$\begin{aligned} \bar{I}(y; x_0) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \text{tr}[A_u(\tau)] d\tau - \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \text{tr}[\dot{P}_u(\tau) P_u^{-1}(\tau)] d\tau \\ &\geq \sum_{i=1}^l n_i \underline{\lambda}_i - \limsup_{t \rightarrow \infty} \frac{1}{2} \text{tr}[\dot{P}_u(t) P_u^{-1}(t)] \\ \text{or } &\leq \sum_{i=1}^l n_i \bar{\lambda}_i - \liminf_{t \rightarrow \infty} \frac{1}{2} \text{tr}[\dot{P}_u(t) P_u^{-1}(t)], \end{aligned} \quad (2.73)$$

where $A_u(t)$ is the antistable part of $A(t)$; $\underline{\lambda}_i$ and $\bar{\lambda}_i$, $i = 1, \dots, l$, are the spectral values of $A_u(t)$, and the estimation error covariance matrix $P_u(t)$ satisfies (2.68).

Proof. When the lumped process noise vanishes or the covariance function $\Sigma_{\bar{w}} = \varepsilon I \rightarrow 0$, by [Lemma 2.35](#), the lowest achievable MS estimation error of (2.64) satisfies

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \lim_{\tau \rightarrow \infty} \mathbb{E}[\tilde{z}_\tau^\top \tilde{z}_\tau] &\stackrel{(a)}{=} \lim_{\varepsilon \rightarrow 0} \lim_{\tau \rightarrow \infty} \mathbb{E} \left\{ \begin{bmatrix} \tilde{z}_s(\tau) \\ \tilde{z}_u(\tau) \end{bmatrix}^\top \begin{bmatrix} \tilde{z}_s(\tau) \\ \tilde{z}_u(\tau) \end{bmatrix} \right\} \\ &\stackrel{(b)}{=} \text{tr}[P_u(t) C_u^\top(t) C_u(t)] \\ &\stackrel{(c)}{=} \text{tr}[P_u^{1/2}(t) C_u^\top(t) C_u(t) P_u^{1/2}(t)] \\ &\stackrel{(d)}{=} \text{tr}(\Psi_t), \end{aligned} \quad (2.74)$$

where (a) follows the identity $\tilde{z}_\tau = C_\tau \tilde{x}_\tau = [C_s(\tau) \ C_u(\tau)][\tilde{x}_s^\top(\tau) \ \tilde{x}_u^\top(\tau)]^\top$ from (2.31); when the LTV system (2.64) is uniformly completely stabilizable and detectable, (b) follows from [Lemma 2.35](#), $\tilde{z}_u(t) = C_u(t) \tilde{x}_u(t)$, $P_u(t) = \mathbb{E}[\tilde{x}_u(t) \tilde{x}_u^\top(t)]$, and the trace property $\text{tr}(MN) = \text{tr}(NM)$ for matrices M and N of proper dimensions; (c) utilizes the positive definiteness of covariance $P_u(t)$ and the same trace property in (b), and $\Psi_t = P_u^{-1/2}(t) A_u(t) P_u^{1/2}(t) + P_u^{1/2}(t) A_u^\top(t) P_u^{-1/2}(t) - \dot{P}_u(t) P_u^{-1}(t)$ in (d), which can be derived from (2.68).

By applying [Theorem 2.29](#) to the measurement channel (2.64) under vanishing process noise, we

can derive (2.73) and the following relationships for $\lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w) = \bar{I}(y; x_0)$

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w) &\stackrel{(a)}{=} \lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \mathbb{E}[\tilde{z}_\tau^\top \tilde{z}_\tau] d\tau \\
&\stackrel{(b)}{=} \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \text{tr}(\Psi_\tau) d\tau \\
&\stackrel{(c)}{=} \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \text{tr}[2A_u(\tau) - \dot{P}_u(\tau)P_u^{-1}(\tau)] d\tau \\
&\stackrel{(d)}{\geq} \sum_{i=1}^l n_i \underline{\lambda}_i - \limsup_{t \rightarrow \infty} \frac{1}{2} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)] \\
\text{or } &\stackrel{(e)}{\leq} \sum_{i=1}^l n_i \bar{\lambda}_i - \liminf_{t \rightarrow \infty} \frac{1}{2} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)],
\end{aligned} \tag{2.75}$$

where (a) utilizes Theorem 2.29 and Fubini's theorem to exchange the order of integration and expectation; (b) owes to (2.74); (c) implies the equality in (2.73), and when (2.64) fulfills (A5)-(A7), (d) and (e) follow Lemma 2.23 and the application of L'Hôpital's rule

$$\liminf_{t \rightarrow \infty} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)] \leq \lim_{t \rightarrow \infty} \frac{\int_0^t \text{tr}[\dot{P}_u(\tau)P_u^{-1}(\tau)] d\tau}{t} = \lim_{t \rightarrow \infty} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)] \leq \limsup_{t \rightarrow \infty} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)].$$

Compared with the proof of Proposition 2.34, we notice that in the LTI scenario, the spectral values in (2.75) coincide with the real parts of the open-loop unstable poles in (2.60), i.e., $\text{tr}[A_u(t)] = \sum_i \lambda_i^+(A)$, and $\lim_{t \rightarrow \infty} \text{tr}[\dot{P}_u(t)P_u^{-1}(t)] = 0$ as the RDE (2.68) degenerates to a steady-state ARE when (2.60) is stabilizable and detectable [12, 88]. This completes the proof. $\square$

Proposition 2.37 provides an equality and a lower bound for $\bar{I}(y; x_0, w)$ and $\bar{I}(y; x_0)$ in the LTV filtering (2.64) regardless of the filter $\mathcal{F}$ design, and an upper bound for the (worst) MS estimation error of the Kalman-Bucy filter (2.65) under vanishing process noise. Combining Proposition 2.37 with Theorem 2.31, we can imply that for the filtering systems subject to the LTV plant (2.64), the total information rate $\lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w)$ or mutual information rate $\bar{I}(y; x_0)$ serves a similar role as (or equals half) the lowest achievable MS estimation error $\lim_{\varepsilon \rightarrow 0} \mathbb{E}[\|z - \hat{z}\|^2]$ for all feasible filters in the presence of process noise. This LTV filtering limit, according to (2.73), is determined by the spectral values of the antistable component $A_u(t)$ and the steady-state error covariance $P_u(t)$ of the optimal filter (2.65), and shares the same time-domain properties as the LTI trade-off: when the estimation error is smaller than this filtering limit on a time interval, the error outside this interval

must be larger. Meanwhile, the LTV result and derivation presented in [Proposition 2.37](#) cover the LTI result in [Proposition 2.34](#) as a special case, which can be explained by the same arguments in [Remark 2.26](#).

2.3.4 Filtering Limits in Nonlinear Systems

In the end, we consider the nonlinear filtering problem described as follows

$$\begin{aligned} dx_t &= \bar{f}(t, x_t)dt + \bar{b}(t, x_t)d\bar{w}_t \\ dy_t &= z(t, x_t)dt + \Sigma_v^{1/2}(t)dv_t, \end{aligned} \tag{2.76}$$

where x_t and y_t are the hidden and observable states; $\bar{f}(t, x_t) = f(t, x_t) + b(t, x_t)u(t, x_t)$ is the deterministic drift term with passive dynamics $f(t, x_t)$, control matrix $b(t, x_t)$, and control signal $u(t, x_t)$; $\bar{b}(t, x_t) = b(t, x_t)\Sigma_{\bar{w}}^{1/2}(t)$ denotes the lumped control matrix, and the process noise $\bar{w}_t$ and measurement noise v_t are two independent standard Brownian motions with covariance functions $\Sigma_{\bar{w}}$ and $\Sigma_v = I$. The noise-free output $z(t, w_0^t, x_0)$ in (2.7) is rewritten as the $z_x(t, x_t)$ or $z(t, x_t)$ in (2.76), which satisfies $z_t = z(t, w_0^t, x_0) = z_x(t, x_t) = z(t, x_t)$ and implies that $I(y_0^t; x_0, w_0^t) = I(y_0^t; x_0^t)$ by applying [Theorem 2.29](#) to (2.7) and (2.76), respectively. For conciseness, only the SISO measurement channel with z_t and $y_t \in \mathbb{R}$ is considered, and the MIMO extension has been discussed in [Remark 2.28](#).

To compute the total information rate $\bar{I}(y; x_0, w)$, or equivalently $\bar{I}(z \rightarrow y)$ and $\bar{I}(z; y)$, from (2.76) we resort to the identity $I(y_0^t; x_0, w_0^t) = (1/2)\mathbb{E}[\|z - \hat{z}\|^2]$ in [Theorem 2.29](#). However, neither the noise-free output $z(t, x_t)$, nor the optimal estimate $\hat{z}(t, x_t)$ can be directly measured in (2.76). A feasible approach to derive $\mathbb{E}[\|z - \hat{z}\|^2]$ is to adopt the following identities from [60, Chapter 10 & 13]

$$\mathbb{E}[(z - \hat{z})^2] = \mathbb{E}[(z - \hat{z})^2 | y_0^\tau] = \mathbb{E}[z^2(\tau, x_\tau) | y_0^\tau] - (\mathbb{E}[z(\tau, x_\tau) | y_0^\tau])^2, \tag{2.77}$$

where both $\mathbb{E}[z(\tau, x_\tau) | y_0^\tau]$ and $\mathbb{E}[z^2(\tau, x_\tau) | y_0^\tau]$ can be estimated from solving the nonlinear filtering problem (2.76). Meanwhile, additional regularity and differentiability constraints are required to guarantee the existence and uniqueness of the solution to (2.76) and the pathwise uniqueness of the filter equation. Let $k(t, x)$ denote any of the functions $\bar{f}(t, x_t)$, $\bar{b}(t, x_t)$, and $z_x(t, x_0^t)$ in (2.76). We require $k(t, x)$ be Lipschitz such that $|k(t, x_1) - k(t, x_2)|^2 \leq K_1(|x_1 - x_2|^2)$ for some constant $K_1 > 0$, and $k^2(t, x) \leq K_2(1 + x^2)$ for some constant $K_2 > 0$. The estimated function $g(t, x_t)$, which denotes

any of the functions $z(t, x_t)$ and $z^2(t, x_t)$ in (2.77), is required to be at least once differentiable at t and twice differentiable at x_t such that $\sup_{\tau \leq t} \mathbb{E}[g^2(\tau, x_\tau)] < \infty$, $\int_0^t \mathbb{E}[g'_x(\tau, x_\tau)^2] d\tau < \infty$, and $\int_0^t \mathbb{E}[\mathcal{L}g(\tau, x_\tau)]^2 d\tau < \infty$, where $\mathcal{L}g(\tau, x_\tau) = g'_\tau(\tau, x_\tau) + g'_{x_\tau}(\tau, x_\tau)\bar{f}(\tau, x_\tau) + (1/2)g''_{x_\tau x_\tau}(\tau, x_\tau)\bar{b}^2(t, x_t)$. When (2.76) fulfills the preceding regularity and differentiability conditions, by resorting to Theorem 2.29 and the Stratonovich-Kushner equation, we can calculate the total information rate $\bar{I}(y; x_0, w)$ by the following proposition.

Proposition 2.38. *For the nonlinear filtering systems described by (2.76) as in Figure 2.2, the total information rate $\bar{I}(y; x_0, w)$ satisfies*

$$\bar{I}(y; x_0, w) = \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t [\mathbb{E}[z^2(\tau, x_\tau)|y_0^\tau] - (\mathbb{E}[z(\tau, x_\tau)|y_0^\tau])^2] d\tau, \quad (2.78)$$

where $\pi_\tau(g) = \mathbb{E}[g(\tau, x_\tau)|y_0^\tau]$ satisfies

$$\pi_\tau(g) = \pi_0(g) + \int_0^\tau \pi_s(\mathcal{L}g) ds + \int_0^\tau [\pi_s(g'_{x_s}) + \pi_s(z \cdot g) - \pi_s(z)\pi_s(g)] d\tilde{w}_s; \quad (2.79)$$

$g(\tau, x_\tau)$ denotes any of the functions $z(\tau, x_\tau)$ and $z^2(\tau, x_\tau)$ in (2.78); $\mathcal{L}g(s, x_s) = g'_s(s, x_s) + g'_{x_s}(s, x_s) \cdot \bar{f}(s, x_s) + g''_{x_s x_s}(s, x_s) \bar{b}^2(s, x_s)$, and $\tilde{w}_\tau = \int_0^\tau dy_s - \int_0^\tau \pi_s(z) ds$ is a Wiener process w.r.t. $\mathcal{F}_\tau^y$, $\tau \in [0, t]$.

Proof. By applying Theorem 2.29 to the measurement channel in (2.76), we have

$$\begin{aligned} \bar{I}(y; x_0, w) &\stackrel{(a)}{=} \lim_{t \rightarrow \infty} \frac{1}{2t} \mathbb{E}[\|z - \hat{z}\|^2] \\ &\stackrel{(b)}{=} \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t \mathbb{E}[(z(\tau, x_\tau) - \hat{z}(\tau, x_\tau))^2] d\tau \\ &\stackrel{(c)}{=} \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t [\mathbb{E}[z^2(\tau, x_\tau)|y_0^\tau] - (\mathbb{E}[z(\tau, x_\tau)|y_0^\tau])^2] d\tau, \end{aligned}$$

where (a) is obtained by applying (2.52) to the measurement channel in (2.76), dividing the result by t and taking the limit as $t \rightarrow \infty$; (b) invokes the Fubini's theorem to swap the order of integration and expectation; (c) is the application of (2.77), and the conditional expectations $\pi_\tau(z) = \mathbb{E}[z(\tau, x_\tau)|y_0^\tau]$ and $\pi_\tau(z^2) = \mathbb{E}[z^2(\tau, x_\tau)|y_0^\tau]$ need to be estimated from solving the nonlinear filtering problem (2.76). By applying the one-dimensional Stratonovich-Kushner equation from [60, Theorem 8.3] to (2.76), we obtain the optimal estimate (2.79). $\square$

Proposition 2.38 provides a direct approach to compute the total information rate $\bar{I}(y; x_0, w)$ in

the nonlinear filtering system (2.76). By letting the covariance function $\Sigma_{\bar{w}} = \varepsilon I \rightarrow 0$ in (2.76) or $\bar{b}(t, x_t) \rightarrow 0$ in (2.79), Proposition 2.38 can also be used to calculate the mutual information rate $\bar{I}(y; x_0)$ or $\lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w)$ when the process noise in (2.76) vanishes. Consistent with the interpretations of filtering trade-off metrics in general and linear filtering systems, for nonlinear filtering system (2.76) the total information rate $\bar{I}(y; x_0, w)$ serves a similar role as (or equals half) the MMSE in the presence of process noise, and the mutual information rate $\bar{I}(y; x_0)$ or $\lim_{\varepsilon \rightarrow 0} \bar{I}(y; x_0, w)$ equals half the lowest achievable MS estimation error of (2.76) in the absence of process noise, irrespective of the filter design. Meanwhile, the filtering limit and optimal estimates in Proposition 2.38 cover the LTI and LTV results in Propositions 2.34 and 2.37 as special cases, as we can easily restore the linear optimal or Kalman-Bucy filter when applying (2.79) to a linear filtering system (2.60) or (2.64).

2.4 Conclusion and Discussion

In this chapter, a unified framework based on the additive Gaussian channel model and a direct information-theoretic method relying on Duncan's theorem were proposed to capture and analyze the fundamental limitations of continuous-time control and filtering systems. Control and filtering systems were respectively modeled as an additive white Gaussian channel with or without feedback. Total information rate, directly derived from the causal MS estimation error of the channel input, was identified as an information-theoretic trade-off in the general control and filtering setting, and then applied to characterize the fundamental limitations of continuous-time control and filtering systems subject to LTI, LTV, and nonlinear systems, respectively.

An immediate generalization of the results presented in this chapter is the extension of the trade-off metrics and analysis method to discrete-time or abstract control and filtering systems, which requires a discrete-time or abstract version of Lemma 2.11, and will be discussed later in Chapters 3 and 4. Meanwhile, since the information-theoretic trade-offs in Sections 2.2.2 and 2.3.2 are related to the unstable poles of LTI plants, it is then natural to discuss whether we can define the complementary information-theoretic trade-offs that relate to the non-minimum phase zeros of LTI systems. Some heuristic work on this complementary or dual extension is given in [9, 12, 44, 104, 105]. Lastly, it is also worthwhile to apply or generalize the results in this chapter to the

networked control and filtering systems, wherein the control and filtering systems with multiple sensors, explicit encoders/decoders, and quantizers are considered.

Chapter 3

Fundamental Limitations of Discrete-Time Control and Filtering Systems

In this chapter, we fill the theoretical gap raised in [Section 1.2](#) by deriving the I-MMSE relationship of discrete-time white Gaussian channels with and without feedback. With this relationship, discrete-time control and filtering systems are respectively modeled into Gaussian channels with and without feedback, and total information (rate) is then analyzed as a control and filtering trade-off metric and estimated by using the optimal filtering techniques, which together provide a discrete-time counterpart result and complement our previous analysis of continuous-time systems in [Chapter 2](#) and [\[16\]](#). The main contributions of this chapter can be summarized as follows:

(C1) We derive the discrete-time I-MMSE relationship, which shows that total information of discrete-time white Gaussian channel with or without feedback is sandwiched between the sums of causal and predicted MMSEs of channel input ([Theorem 3.4](#)). This relationship supplements the non-feedback result in [\[51, Theorem 9\]](#) and serves as a discrete-time counterpart of Duncan's theorem in [\[54, Theorem 1\]](#).

(C2) By resorting to the discrete-time I-MMSE relationship, fundamental constraints of total information rate, $\bar{I}(E; C, X_0)$ or $\bar{I}(E; X_0)$, are first derived in a general nonlinear control setup ([Theorems 3.6 and 3.8](#)), and then utilized to characterize the control trade-offs of various systems.

By applying the I-MMSE relationship to discrete-time LTI control systems, a more concise proof is presented to show that $\bar{I}(E; X_0)$ equals the sum of logarithmic unstable poles and serves a similar role as the established Bode-type integrals (Proposition 3.10 and Corollary 3.11). For LTV control systems, by using the I-MMSE relationship, we show that $\bar{I}(E; X_0)$, determined by the antistable dynamics of plant, is identical to a time-domain Bode's integral, which quantifies the average cost function (Proposition 3.15 and Corollary 3.16). For nonlinear control systems, in which $\bar{I}(E; X_0)$ can be interpreted as a limit of data transmission rate or average cost function, we provide an optimal-filtering-based approach to estimate $\bar{I}(E; X_0)$ (Proposition 3.18).

(C3) By using the discrete-time I-MMSE relationship, we prove that total information rate $\bar{I}(Y; X_0)$ (or $\bar{I}(Y; X_0, W)$) serves as the lowest achievable (or a lower bound of average) predicted MMSE of noise-free output in a general nonlinear filtering setup (Theorems 3.20 and 3.22). Subsequently, similar to the individual control results in (C2), we show that total information rate $\bar{I}(Y; X_0)$, determined by the unstable/antistable dynamics of plant, equals the sum of logarithmic unstable poles in LTI filtering systems (Proposition 3.23), and is identical to a time-domain Bode's integral in LTV filtering systems (Proposition 3.25). For nonlinear filtering systems, we propose a nonlinear-filtering-based approach to estimate $\bar{I}(Y; X_0, W)$ (Section 3.3.4).

The remainder of this chapter is organized as follows. Section 3.1 introduces the preliminaries of information theory, and derives the discrete-time I-MMSE relationship. Sections 3.2 and 3.3 respectively investigate total information rates in different control and filtering systems, and Section 3.4 draws the conclusions.

Notations: For a discrete-time random variable X_i , x_i denotes a sample or value of X_i . For a discrete-time random process, X_0^n or $X_{0:n}$, from $i = 0$ to n , x_0^n stands for a sample path or value of X_0^n . The norm of X_0^n is defined by $\|X_0^n\| = (\sum_{i=0}^n X_i^\top X_i)^{1/2}$. x_0^t stands for a continuous-time random process from $\tau = 0$ to t . For a complex number c , $|c|$ denotes its modulus. e is Euler's number. For a matrix M , $\det M$ or $|M|$ denotes its determinant. $\mu_i(M)$ denote the singular values of M and are ordered as $\bar{\mu}(M) := \mu_1(M) \geq \mu_2(M) \geq \cdots \geq \mu_r(M) =: \underline{\mu}(M)$. $\|M\|$ denotes the norm of M and equals $\bar{\mu}(M)$. Limit inferior or $\liminf$ is abbreviated as $\underline{\lim}$, and limit superior or $\limsup$ is shortened as $\overline{\lim}$.

3.1 Preliminaries and Problem Formulation

In this section, the definitions and preliminary results of some discrete-time information-theoretic metrics and discrete-time additive Gaussian channels are given, and the I-MMSE relationship of discrete-time white Gaussian channel with feedback is then derived for the first time.

3.1.1 Information Theory for Discrete-Time Systems

In this chapter, information-theoretic metrics, e.g., differential entropy and mutual information, are defined and considered with their measure-theoretic notions [76, 77], which apply to the discrete-time processes/signals without probability density function (pdf), and thus are more general than the traditional statistical notions [74]. Differential entropy (rate) measures the amount (or time density) of randomness in a random vector or process.

Definition 3.1 (Differential Entropy). *For a discrete-time random process X_0^n with image (or push-forward) measure μ_X , its differential entropy $h(X_0^n)$ is defined by*

$$h(X_0^n) := \int_{\mathcal{X}} \log \left(\frac{d\mu_{\text{Leb}}}{d\mu_X} \right) d\mu_X,$$

where $\mathcal{X}$ denotes the Borel σ -algebra of X_0^n , and μ_{Leb} stands for the Lebesgue measure. In particular, if X_0^n has pdf p_X , then $h(X_0^n) := \int_{\mathcal{X}} \log[1/p_X(X_0^n)] p_X(X_0^n) dX_0^n = \mathbb{E}_{p_X}[\log(1/p_X(X_0^n))]$. Differential entropy rate of X_0^n is defined by

$$\bar{h}(X) := \overline{\lim}_{n \rightarrow \infty} h(X_0^n)/(n+1).$$

If X_0^n is stationary, $\bar{h}(X) := \lim_{n \rightarrow \infty} h(X_0^n)/(n+1)$.

Mutual information (rate) quantifies the dependency between two random objects and reliable data transmission (rate) of communication channel.

Definition 3.2 (Mutual Information). *For discrete-time random processes X_0^n and Y_0^n with image measures μ_X and μ_Y , the mutual information between X_0^n and Y_0^n is defined by*

$$I(X_0^n; Y_0^n) := \int_{\mathcal{X} \times \mathcal{Y}} \log \frac{d\mu_{XY}}{d(\mu_X \times \mu_Y)} d\mu_{XY},$$

where $\mathcal{X}$ and $\mathcal{Y}$ respectively denote the Borel σ -algebras of X_0^n and Y_0^n , and $d\mu_{XY}/d(\mu_X \times \mu_Y)$ is the Radon-Nikodym derivative between joint measure μ_{XY} and product measure $\mu_X \times \mu_Y$. In particular, if X_0^n and Y_0^n have pdfs p_X and p_Y , then $I(X_0^n; Y_0^n) := \int_{\mathcal{X} \times \mathcal{Y}} \log[p_{X \times Y}(X_0^n, Y_0^n)/(p_X(X_0^n) \times p_Y(Y_0^n))] dX_0^n dY_0^n$. Mutual information rate between X_0^n and Y_0^n is defined by¹

$$\bar{I}(X; Y) := \overline{\lim}_{n \rightarrow \infty} I(X_0^n; Y_0^n)/(n+1).$$

If (X_0^n, Y_0^n) are jointly stationary, then $\bar{I}(X; Y) := \lim_{n \rightarrow \infty} I(X_0^n; Y_0^n)/(n+1)$.

Definitions of properties for differential entropy and mutual information, such as invariance and maximum of differential entropy, data processing inequality, chain rule, symmetry, and non-negativity of mutual information, are not provided in this chapter. Interested readers are referred to [Section 2.1.2](#) and [\[7, 8, 44, 74, 76\]](#) for their rigorous definitions and proofs.

3.1.2 Discrete-Time Gaussian Channel Model

Consider a discrete-time additive white Gaussian channel with or without feedback described by [Figure 3.1](#):

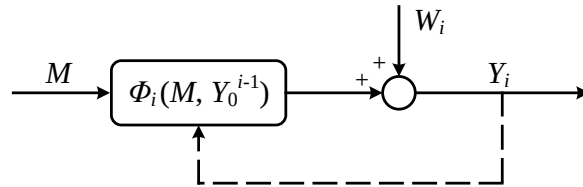


Figure 3.1: Discrete-time additive Gaussian channel.

and the following difference equation

$$Y_i = \Phi_i(M, Y_0^{i-1}) + W_i, \quad (3.1)$$

where Y_i and Y_0^{i-1} respectively denote the channel output at time i and the sample path of channel output from 0 to $i-1$; the random variable or process M is the transmitted message; $\Phi_i(M, Y_0^{i-1})$ stands for the channel input process or coding function of M , and W_i is a white Gaussian noise independent from M . To describe a non-feedback channel, we replace the input

¹ The definitions of entropy rate and mutual information rate adopted in this chapter are aligned with those in [\[6, 27\]](#).

function $\Phi_i(M, Y_0^{i-1})$ in (3.1) with $\Phi_i(M)$. Compared with the analysis of continuous-time channels performed on the infinite-dimensional spaces in Chapter 2 and [16, 54, 79], the convenience of analyzing the discrete-time channel (3.1) is that we can start from a finite-dimensional or -horizon problem with $i = 0, 1, \dots, n$ and then generalize the results to infinite-horizon by taking the limit $n \rightarrow \infty$ [106].

Before we introduce the discrete-time I-MMSE relationship, some estimation performance metrics of channel (3.1) are defined as follows. Let the prior estimate of the input process be $\hat{\Phi}_i^- := \mathbb{E}[\Phi_i | Y_0^{i-1}]$. The one-step prediction MMSE of input process Φ_i is then defined by

$$\text{pmmse}(\Phi_i) := \mathbb{E}[(\Phi_i - \hat{\Phi}_i^-)^\top (\Phi_i - \hat{\Phi}_i^-)]. \quad (3.2)$$

Let the posterior estimate of the input process be $\hat{\Phi}_i := \mathbb{E}[\Phi_i | Y_0^i]$. The causal (filtering) MMSE of Φ_i is then denoted by

$$\text{cmmse}(\Phi_i) := \mathbb{E}[(\Phi_i - \hat{\Phi}_i)^\top (\Phi_i - \hat{\Phi}_i)]. \quad (3.3)$$

The following continuous-time I-MMSE relationship, which is also known as the Duncan's theorem [53], will be used to derive its discrete-time counterpart.

Lemma 3.3. *For a continuous-time additive Gaussian channel $dy_t = \sqrt{\text{snr}} \phi(t, m, y_0^t) dt + dw_t$ with transmitted message m , signal-to-noise ratio $\text{snr} > 0$, channel input process $\phi_t(m, y_0^t)$ or $\phi_t(m)$ satisfying $\mathbb{E}[\phi_t^\top \phi_t] < \infty$, channel output y_t , and channel noise w_t – a standard Brownian motion, the directed information $I(\phi_0^t \rightarrow y_0^t)$ and total information $I(m; y_0^t)$ satisfy*

$$I(\phi_0^t \rightarrow y_0^t) = I(m; y_0^t) = \frac{\text{snr}}{2} \int_0^t \text{cmmse}(\phi_\tau, \text{snr}) d\tau, \quad (3.4)$$

where $\text{cmmse}(\phi_\tau, \text{snr}) := \mathbb{E}[(\phi_\tau - \hat{\phi}_\tau)^\top (\phi_\tau - \hat{\phi}_\tau)]$ denotes the continuous-time causal MMSE, and $\hat{\phi}_\tau := \mathbb{E}[\phi_\tau | y_0^\tau]$ is the posterior estimate of channel input.

The definition of continuous-time directed information, a generalization of input-output mutual information, e.g., $I(\phi_0^t; y_0^t)$, to random objects obeying causal or feedback relation, and the proof of first equality in (3.4) are given in [47]. It is worth noting that total information is favored in this chapter, since it grants more flexibility and convenience in our derivations than directed

information. The second equality in (3.4) is the continuous-time I-MMSE relationship or Duncan's theorem, whose proof is available in [51, 54]. Based on Lemma 3.3, the following theorem presents the discrete-time I-MMSE relationship in Gaussian channel (3.1).

Theorem 3.4. *For the discrete-time additive white Gaussian channel with or without feedback described by (3.1) and Figure 3.1, when the input process satisfies $\mathbb{E}[\Phi_i^\top \Phi_i] < \infty$, $i = 0, \dots, n$, the total information $I(M; Y_0^n)$ in (3.1) satisfies*

$$\frac{1}{2} \sum_{i=0}^n \text{cmmse}(\Phi_i) \leq I(M; Y_0^n) \leq \frac{1}{2} \sum_{i=0}^n \text{pmmse}(\Phi_i). \quad (3.5)$$

Proof. When (3.1) represents a non-feedback channel, inequalities (3.5) can be proved by following Theorem 9 in [51] and the identities $I(\Phi_0^n; Y_0^n) = I(M; Y_0^n) = I(\Phi_0^n \rightarrow Y_0^n)$ of non-feedback Gaussian channel presented in [85, 107], where the discrete-time directed information is defined by $I(\Phi_0^n \rightarrow Y_0^n) := \sum_{i=0}^n I(\Phi_0^i; Y_i | Y_0^{i-1})$. However, it is not trivial to prove (3.5) for the feedback Gaussian channel, as several fundamental paradigms, relationships, such as the SNR-incremental channel and differential I-MMSE relationship used in [51], and information-theoretic quantities, e.g., input-output mutual information $I(\Phi_0^n; Y_0^n)$, are either invalid or not well-defined in the presence of feedback.

In the following, motivated by [106], we are to prove inequalities (3.5) for the discrete-time feedback channel by using the continuous-time I-MMSE relationship or Duncan's theorem in Lemma 3.3. Consider the following continuous-time channel with piecewise-constant message and input as an equivalent of the discrete-time channel (3.1)

$$dy_t = \sqrt{\text{snr}} \phi_t(m, y_0^t) dt + dw_t, \quad t \in [0, \infty). \quad (3.6)$$

To align with the expression of (3.1), we assume $\text{snr} = 1$ in (3.6) by default. In the time interval $t \in (i-1, i]$, or more precisely $t \in (t_{i-1}, t_i]$, the piecewise-constant message m is equal to the value of random variable/vector M at time i , and the input to the continuous-time channel (3.6) equals the value of random vector/process Φ_i at time i , i.e., $\phi_t(m, y_0^t) = \Phi_i(M, Y_0^{i-1})$ if $t \in (i-1, i]$. At time step $t = i$, we further have the equality $w_t = W_i$ and thus $y_t = Y_i$ between (3.1) and (3.6). Consequently, the following relationships between the total information in (3.1) and (3.6) can be

established

$$I(M; Y_0^n) = I(M; y_0^n) = I(m; y_0^n), \quad (3.7)$$

where the first equality follows the fact that Y_0^n , or samples of continuous-time process y_0^n at natural numbers, are sufficient statistics for message M ; the second equality follows the fact that discrete-time message M and piecewise-constant message m have the same randomness, and $I(M; Y_0^n)$ also equals the directed information $I(\Phi_0^n \rightarrow Y_0^n)$ by [107].

As the causal MMSEs of the discrete-time channel (3.1) and continuous-time channel (3.6) are identical at time $t = i$, we denote them collectively by $\text{cmmse}(\Phi_i, \text{snr})$ in the following. Meanwhile, when $t \in (i - 1, i]$, since the filtration generated by y_0^i or Y_0^i contains more information about ϕ_i or Φ_i than the filtration generated by y_0^t , which in turn contains more information about ϕ_i or Φ_i than y_0^{i-1} or Y_0^{i-1} , we have

$$\text{cmmse}(\Phi_i, \text{snr}) \leq \text{cmmse}(\phi_t, \text{snr}) \leq \text{pmmse}(\Phi_i, \text{snr}), \quad (3.8)$$

where $\text{pmmse}(\Phi_i, \text{snr})$ and $\text{cmmse}(\Phi_i, \text{snr})$ follow the definitions in (3.2) and (3.3) when $\text{snr} = 1$, and the continuous-time causal MMSE $\text{cmmse}(\phi_t, \text{snr})$ was defined in (3.4). Letting $\text{snr} = 1$ in (3.6), and integrating (3.8) over t from 0 to n , we can obtain the inequalities in (3.5), which follow the facts that $I(M; Y_0^n) = (\text{snr}/2) \int_0^n \text{cmmse}(\phi_t, \text{snr}) dt$ by Lemma 3.3 and (3.7), $\int_0^n \text{cmmse}(\Phi_i, \text{snr}) dt = \sum_{i=0}^n \text{cmmse}(\Phi_i, \text{snr})$, and $\int_0^n \text{pmmse}(\Phi_i, \text{snr}) dt = \sum_{i=0}^n \text{pmmse}(\Phi_i, \text{snr})$. This completes the proof. $\square$

Theorem 3.4 presents an I-MMSE relationship for the discrete-time Gaussian channel with or without feedback in (3.1), which not only serves as a discrete-time counterpart of the Duncan's theorem in Lemma 3.3, but extends the I-MMSE relationship in [51, Theorem 9] to feedback scenario. Compared with Duncan's theorem and the differential entropy representation (or decomposition) of mutual information, both of which are equality constraints, Theorem 3.4 provides a sandwich-type estimate/constraint for $I(M; Y_0^n)$, $I(\Phi_0^n \rightarrow Y_0^n)$, and $I(\Phi_0^n; Y_0^n)$. It is worth noting that for discrete-time Gaussian channel (3.1), there exists a differential I-MMSE relationship, an equality, between the directed/total information and the sum of non-causal (smoothing) MMSEs, $\text{nmmse}(\Phi_i) = \mathbb{E}[(\Phi_i - \hat{\Phi}_i^+)^T (\Phi_i - \hat{\Phi}_i^+)]$ with $i = 0, \dots, n$ and smoothed estimate $\hat{\Phi}_i^+ := \mathbb{E}[\Phi_i | Y_0^n]$, and a correctional

term [61]. Nevertheless, since the differential relationship is not as intuitive and convenient as (3.5), and the non-casual MMSE is not always available to calculate in practice, we will not adopt the differential relationship in this chapter. In the following sections, by utilizing the I-MMSE relationship in Theorem 3.4, we will interpret how total and directed information (rates) serve as a trade-off metric/quantity characterizing some fundamental limitations of control and filtering systems.

3.2 Fundamental Limitations of Discrete-Time Control Systems

To capture the control limitations, the general discrete-time control system is first modeled as an additive white Gaussian channel with feedback. By utilizing the discrete-time I-MMSE relationship and optimal estimation theory, we then use total information rate and its sandwich bounds to characterize the fundamental control trade-offs of various control systems.

3.2.1 General Control Systems and Trade-offs

Consider the general discrete-time feedback control system illustrated by the diagrams in Figure 3.2:

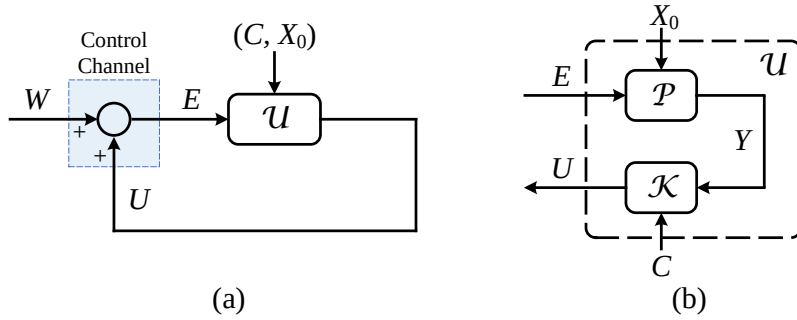


Figure 3.2: Configuration of a general control system. (a) Block diagram of control system. (b) Block diagram of control input process $\mathcal{U}$.

where $\mathcal{U}$ stands for the control input function or the coding function of “message” (C, X_0) , which can be further decomposed into a plant model $\mathcal{P}$ and a control mapping $\mathcal{K}$; channel noise W is additive white Gaussian or memory-less; E represents the error signal; U denotes the control input; Y and X_0 are respectively the measured output and initial states of $\mathcal{P}$, and C is the command signal

or control noise of $\mathcal{K}$.² We use the following stochastic difference equation to describe the plant $\mathcal{P}$:

$$X_{i+1} = f_i(X_i) + b_i(X_i)E_i \quad (3.9)$$

$$Y_i = h_i(X_i),$$

where X_i , E_i , and Y_i are the internal states of plant $\mathcal{P}$, error signal, and measured output at time i , respectively. The control mapping $\mathcal{K}$ can be realized by either the following causal stabilizing function with memory

$$U_i = g_i(Y_0^i, C_0^i) \quad (3.10)$$

or the memory-less function $U_i = g_i(Y_i, C_i)$. With (3.9) and (3.10), the control channel in Figure 3.2 and error signal E_i in (3.9) can be described by

$$E_i = U_i(E_0^{i-1}, C_0^i, X_0) + W_i, \quad (3.11)$$

where the input function is $U_i(E_0^{i-1}, C_0^i, X_0) = g_i(Y_0^i, C_0^i)$, and the channel noise is $W_i \sim \mathcal{N}(0, I)$. Meanwhile, we postulate the following assumption on the control channel depicted by (3.9)-(3.11).

(A1) The control system described by (3.9)-(3.11) is internally mean-square stable, i.e., $\sup_i \mathbb{E}[X_i^\top X_i] < \infty$, and the average power of control input U_i is finite, i.e., $\mathbb{E}[U_i^\top U_i] < \infty, \forall i$.

Remark 3.5. *The stability condition in (A1) is fundamental and commonly posited in the analysis of stochastic control systems [7, 27]. The average power constraint in (A1) is a preliminary condition for investigating the I-MMSE relationships, since it guarantees the absolute continuity or equivalence between the image measures for calculating total and mutual information [79, 106].*

By applying Theorem 3.4 to (3.11), we obtain a MMSE-based sandwich bound for the total information (rate) $I(E_0^n; C_0^n, X_0)$ in control channel (3.11), which depending on the particular model $\mathcal{U}$ generating U_i , and similar to some established control trade-off metrics, such as the Bode(-like) integrals and average entropy cost, characterizes several fundamental limitations of different control systems.

² We follow the general control system setup in [44, 108] when the channel noise W is added to the control signal U . By switching the roles of $\mathcal{P}$ and $\mathcal{K}$ in Figure 3.2, we can also analyze the case when the noise W is imposed on the measured output Y . The analysis procedures and results of these two scenarios are symmetric, while the only difference is in the notations.

Theorem 3.6. For the discrete-time control systems described by (3.9)-(3.11) and Figure 3.2, the total information $I(E_0^n; C_0^n, X_0)$ in (3.11) is bounded between

$$\frac{1}{2} \sum_{i=0}^n \text{cmmse}(U_i) \leq I(E_0^n; C_0^n, X_0) \leq \frac{1}{2} \sum_{i=0}^n \text{pmmse}(U_i), \quad (3.12)$$

and the total information rate $\bar{I}(E; C, X_0)$ satisfies

$$\lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{cmmse}(U_i)}{2(n+1)} \leq \bar{I}(E; C, X_0) \leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(U_i)}{2(n+1)}. \quad (3.13)$$

In particular, when (E_0^n, C_0^n) are jointly stationary, we have $\lim_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(U_i) \leq \bar{I}(E; C, X_0) \leq \lim_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{pmmse}(U_i)$.

Proof. Inequalities (3.12) can be directly proved by applying Theorem 3.4 to the discrete-time control channel (3.11), wherein (C_0^i, X_0) corresponds to the message M in (3.1). Inequalities (3.13) can be derived from (3.12) and the definition of mutual information rate in Definition 3.2. $\square$

By invoking the fundamental identities and properties of differential entropy and mutual information, we have the following equality constraints and decomposition for $I(E_0^n; C_0^n, X_0)$.

Proposition 3.7. For the discrete-time control systems in (3.9)-(3.11) and Figure 3.2, the total information $I(E_0^n; C_0^n, X_0)$ satisfies

$$I(E_0^n; C_0^n, X_0) = I(E_0^n; X_0) + I(E_0^n; C_0^n | X_0), \quad (3.14)$$

and

$$I(E_0^n; C_0^n, X_0) = h(E_0^n) - h(W_0^n). \quad (3.15)$$

Proof. Equation (3.14) can be verified by applying the chain rule of mutual information to $I(E_0^n; C_0^n, X_0)$. Equation (3.15) can be proved by

$$\begin{aligned} I(E_0^n; C_0^n, X_0) &\stackrel{(a)}{=} h(E_0^n) - \sum_{i=0}^n h(E_i | E_0^{i-1}, C_0^n, X_0) \\ &\stackrel{(b)}{=} h(E_0^n) - \sum_{i=0}^n h(W_i | E_0^{i-1}, C_0^i, X_0) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(c)}{=} h(E_0^n) - \sum_{i=0}^n h(W_i | W_0^{i-1}, C_0^i, X_0) \\
&\stackrel{(d)}{=} h(E_0^n) - \sum_{i=0}^n h(W_i | W_0^{i-1}),
\end{aligned}$$

where (a) follows the identity between mutual information and differential entropy, and the chain rule of differential entropy; (b) relies on (3.11), in which U_i is a function of (E_0^{i-1}, C_0^i, X_0) ; (c) uses the fact that (E_0^{i-1}, C_0^i, X_0) is a function of (W_0^{i-1}, C_0^i, X_0) , and (d) implies (3.15) by the chain rule of differential entropy. $\square$

The identities and representation in Proposition 3.7, developed from the pure and traditional information-theoretic relations, have been a pivotal tool for studying the total information (rate) as a performance trade-off metric of various control systems in [7, 33, 44]. For example, the important entropy inequality in [27, Theorem 4.2], i.e., $h(E_0^n) \geq I(E_0^n; X_0) + h(W_0^n)$, can be easily restored from (3.14) and (3.15). On the contrary, based on the I-MMSE relationship, Theorem 3.6 establishes an inequality constraint and sandwich-type bounds for the total information (rate), which provide an alternative approach for estimating and analyzing this control trade-off metric and will be studied in more detail in the rest of this section.

By combining Theorem 3.6 and Proposition 3.7, we are able to extend the current understanding and bounds for the total information rate $\bar{I}(E; C, X_0)$. Before we show this extension, the notations of feedback capacity $\mathcal{C}_f$ and channel capacity $\mathcal{C}$ are introduced. For the control channel (3.11), we define the (information) feedback capacity³ $\mathcal{C}_f := \sup_{(C, X_0)} \bar{I}(E; C, X_0)$ as the supremum of total information rate over all admissible (message) sets of (C, X_0) ; for the same channel without feedback, when the power of input signal is limited by $\mathbb{E}[U_i^\top U_i] \leq \rho_i, \forall i$, the channel capacity $\mathcal{C} := \sup_{\mathbb{E}[U_i^\top U_i] \leq \rho_i} \bar{I}(U; E)$ is defined as the supremum of input-output mutual information rate. The following theorem provides an extended sandwich bound on $\bar{I}(E; C, X_0)$.

Theorem 3.8. *For the discrete-time control systems described by (3.9)-(3.11) and Figure 3.2, the total information rate $\bar{I}(E; C, X_0)$ in (3.11) is bounded between*

$$\text{CLB} \leq \bar{I}(E; C, X_0) \leq \mathcal{C}_f \leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(U_i)}{2(n+1)}, \quad (3.16)$$

³ In this chapter, we adopt the mutual information (rate) version of feedback capacity as in [79, 86], as opposed to the (operational) feedback capacity, or the supremum of data rate, used in [47, 87].

where the (control) lower bound $\text{CLB} = \max\{\bar{I}(E; X_0), \overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(U_i)\}$. When $\mathbb{E}[U_i^\top U_i] \leq \rho_i$, $\forall i$, and Σ_{U_i} denotes the covariance of U_i , the feedback capacity $\mathcal{C}_f = \mathcal{C} = \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[U_i^\top U_i] \leq \rho_i} [2(n+1)]^{-1} \sum_{i=0}^n \log(|\Sigma_{U_i} + I|)$.

Proof. We first derive the control lower bound CLB in (3.16). By the non-negativity of conditional mutual information $I(E_0^n; C_0^n | X_0)$ in (3.14), we can imply that $\bar{I}(E; X_0)$ is a lower bound of $\bar{I}(E; C, X_0)$ in CLB. Meanwhile, from inequalities (3.13), we can tell that $\overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \cdot \sum_{i=0}^n \text{cmmse}(U_i)$ is also a lower bound for $\bar{I}(E; C, X_0)$. As for the upper bounds, since $\mathcal{C}_f$ is the lowest upper bound for $\bar{I}(E; C, X_0)$, by invoking (3.13) we have $\bar{I}(E; C, X_0) \leq \mathcal{C}_f \leq \overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{pmmse}(U_i)$ in (3.16).

It then remains to show that under the power constraint, $\mathbb{E}[U_i^\top U_i] \leq \rho_i, \forall i$, the feedback capacity $\mathcal{C}_f$ equals $\overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[U_i^\top U_i] \leq \rho_i} [2(n+1)]^{-1} \sum_{i=0}^n \log(|\Sigma_{U_i} + I|)$. Without loss of generality, suppose that E_i and $W_i \in \mathbb{R}^s, \forall i$, in the control channel (3.11). We first show that $\mathcal{C} \leq \mathcal{C}_f \leq \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[U_i^\top U_i] \leq \rho_i} [2(n+1)]^{-1} \sum_{i=0}^n \log(|\Sigma_{U_i} + I|)$, where the first inequality $\mathcal{C} \leq \mathcal{C}_f$ can be implied from their definitions and the identity $I(E_0^n; C_0^n, X_0) = I(E_0^n; U_0^n)$ when (3.11) is a channel without feedback, and the second equality can be proved by using the facts that $\mathcal{C}_f$ is the lowest upper bound for $\bar{I}(E; C, X_0)$ and

$$\begin{aligned} \bar{I}(E; C, X_0) &\stackrel{(a)}{=} \bar{h}(E) - \bar{h}(W) \leq \mathcal{C}_f \\ &\stackrel{(b)}{\leq} \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[U_i^\top U_i] \leq \rho_i} \frac{\sum_{i=0}^n \log[(2\pi e)^s |\Sigma_{E_i}|]}{2(n+1)} - \frac{\log[(2\pi e)^s]}{2} \\ &\stackrel{(c)}{\leq} \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[U_i^\top U_i] \leq \rho_i} \frac{1}{2(n+1)} \sum_{i=0}^n \log(|\Sigma_{U_i} + I|) \\ &\stackrel{(d)}{=} \overline{\lim}_{n \rightarrow \infty} \frac{1}{2(n+1)} \sum_{i=0}^n \log(1 + \rho_i), \end{aligned} \quad (3.17)$$

where (a) follows from (3.15) and Definitions 3.1 and 3.2; (b) invokes the property $h(E_0^n) \leq \sum_{i=0}^n h(E_i)$ with equality attained if and only if $\{E_i\}_{i=0}^n$ are mutually independent, maximum entropy condition in Property 2.9, and $W_i \sim \mathcal{N}(0, I)$; (c) uses the fact that $\Sigma_{E_i} = \Sigma_{U_i} + \Sigma_{W_i}$, where $\Sigma_{W_i} = I$, and (d) can be obtained when (3.11) is one-dimensional, i.e., $s = 1$. To prove the other direction, consider a special scenario when (3.11) is a Gaussian channel without feedback and $U_i \sim \mathcal{N}(0, \Sigma_{U_i})$ are mutually independent for $i = 0$ to n . Since the channel capacity under this

special setting satisfies

$$\mathcal{C} \geq \bar{I}(E; U) = \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[U_i^\top U_i] \leq \rho_i} [2(n+1)]^{-1} \sum_{i=0}^n \log(|\Sigma_{U_i} + I|), \quad (3.18)$$

by combining (3.17) and (3.18), we prove the last statement in Theorem 3.8. This completes the proof. $\square$

A more comprehensive description of $\bar{I}(E; C, X_0)$, as well as the relation between the MMSE-based bounds in Theorem 3.6 and the established bounds in Proposition 3.7, are revealed in Theorems 3.8. Depending on the plant $\mathcal{P}$ and controller $\mathcal{K}$ in Figure 3.2, total information rate $\bar{I}(E; C, X_0)$ and plant's instability rate $\bar{I}(E; X_0)$ in (3.16) capture different control trade-offs. For example, as later subsections demonstrate, when $\mathcal{P}$ is linear, $\bar{I}(E; X_0)$ serves as an information-theoretic interpretation of the famous Bode-type integrals and average risk-sensitive cost function, and $\bar{I}(E; C, X_0)$ is closely related to the data-rate constraint and rate-distortion trade-offs considered in [4, 10]. Typically, in the presence of channel noise W_0^n and “message” (C_0^n, X_0) , the smaller (or larger) the MMSEs and other bounds in (3.16) are, the milder (or rougher) the fundamental control limitations tend to be. In the subsequent subsections, based on Theorem 3.8, $\bar{I}(E; C, X_0)$ and $\bar{I}(E; X_0)$ are scrutinized as the control trade-off metrics for some particular control systems, in which explicit expression and also the order of bounds in (3.16) are available to derive.

Remark 3.9. *Theorems 3.6, 3.8, and Proposition 3.7 also hold for the control channel (3.9)-(3.11) without command signal C_0^n , where the total information rate $\bar{I}(E; C, X_0)$ coincides with the instability rate $\bar{I}(E; X_0)$ in (3.14) and (3.16).*

3.2.2 Control Trade-offs in LTI Systems

When both the plant model $\mathcal{P}$ and control mapping $\mathcal{K}$ in Figure 3.2 are LTI, we use the following equations to describe the dynamics of $\mathcal{P}$ or the augmented/lumped dynamics of $\mathcal{P}$ and $\mathcal{K}$:

$$\begin{aligned} X_{i+1} &= AX_i + BE_i \\ Y_i &= HX_i, \end{aligned} \quad (3.19)$$

such that the control input signal can be expressed as

$$U_i = GX_i, \quad (3.20)$$

where A, B, H and G are time-invariant matrices of proper dimensions, and X_i is the internal states of $\mathcal{P}$ or augmented states of $\mathcal{P}$ and $\mathcal{K}$. Equations (3.19) and (3.20) are general enough to depict the scenarios when $\mathcal{K}$ is an LTI controller, e.g., linear state feedback, output feedback, and observer-based feedback. Consequently, the control channel can be described by

$$E_i = U_i + W_i = GX_i + W_i, \quad (3.21)$$

where $W_i \sim \mathcal{N}(0, I)$ denotes a white Gaussian noise. By resorting to [Theorem 3.8](#), we have the following result on the total information rate $\bar{I}(E; X_0)$ of the LTI control system and the channel depicted in (3.19)-(3.21).

Proposition 3.10. *When the discrete-time LTI control system described by (3.19)-(3.21) and [Figure 3.2](#) is internally mean-square stable, the total information rate $\bar{I}(E; X_0)$ is bounded between*

$$\overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{cmmse}(U_i)}{2(n+1)} \leq \bar{I}(E; X_0) \leq \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(U_i)}{2(n+1)}, \quad (3.22)$$

where $\bar{I}(E; X_0) = \sum_j \log |\lambda_j^+(A)|$ with $\lambda_j^+(A)$ denoting the eigenvalues of matrix A with positive real parts, i.e., open-loop unstable poles of $\mathcal{U}$ in [Figure 3.2](#).

Proof. A sketch or an overview of the proof of [Proposition 3.10](#) is first given. The two inequalities in (3.22) can be directly obtained by applying [Theorem 3.6](#) or [3.8](#) to the LTI control channel (3.21). It remains to prove that $\bar{I}(E; X_0) = \sum_j \log |\lambda_j^+(A)|$, which is an important identity/evidence showing that $\bar{I}(E; X_0)$ serves as an information-theoretic interpretation and a substitute to the recognized Bode's integral of discrete-time LTI control systems [\[109\]](#). We can derive this equality by separately proving the statements i) $\bar{I}(E; X_0) \geq \sum_j \log |\lambda_j^+(A)|$, and ii) $\bar{I}(E; X_0) \leq \sum_j \log |\lambda_j^+(A)|$. Statement i) is well-known and has been proved by using the internally mean-square stability and maximum entropy conditions in [\[6, Lemma 4.1\]](#). However, to the authors' knowledge, statement ii) has not been proved in any existing literature yet. Surprisingly, even without proving statement ii), equality

$\bar{I}(E; X_0) = \sum_j \log |\lambda_j^+(A)|$ was incidentally derived by [10, Corollary 1] during the minimization of directed information (rate) $I(U_0^n \rightarrow E_0^n) = I(E_0^n; X_0)$ subject to a LQG performance constraint. In this proof, we will prove statement ii) by using the tools from information and estimation theories, which along with the proof of statement i), provides a more straightforward and concise approach to derive the identity $\bar{I}(E; X_0) = \sum_j \log |\lambda_j^+(A)|$.

In order to calculate the estimates and estimation error (covariance) of control input $U_i = GX_i$, consider the following filtering problem originated from (3.19)-(3.21)

$$\begin{aligned} X_{i+1} &= (A + BG)X_i + BW_i \\ E_i &= GX_i + W_i, \end{aligned} \tag{3.23}$$

where E_i is observable, and X_i is the vector of the hidden states to estimate. By applying the LTI Kalman filter to (3.23), and letting the prior estimate be $\hat{X}_i^- := \mathbb{E}[X_i | E_0^{i-1}]$, the prior error covariance $P_i^- := \mathbb{E}[(X_i - \hat{X}_i^-)(X_i - \hat{X}_i^-)^\top]$ satisfies

$$P_i^- = AP_{i-1}^-(GP_{i-1}^-G^\top + I)^{-1}A^\top. \tag{3.24}$$

Defining the posterior estimate $\hat{X}_i = \mathbb{E}[X_i | E_0^i]$, the posterior error covariance $P_i := \mathbb{E}[(X_i - \hat{X}_i)(X_i - \hat{X}_i)^\top]$ obeys

$$P_i = P_i^- - P_i^-G^\top(GP_i^-G^\top + I)^{-1}GP_i^-. \tag{3.25}$$

The evolution of error covariance matrices in (3.24) and (3.25) is equivalent to the filtering result of the following system without control input and noise⁴

$$\begin{aligned} X_{i+1} &= AX_i, \\ E_i &= GX_i + W_i. \end{aligned} \tag{3.26}$$

When (3.23) is detectable, we can partition (3.23) and (3.26) into stable and unstable parts by modal decomposition. By the LTI version of Lemma 3.24, the asymptotic prior error covariance

⁴ The filtering equivalence only exists for the prior/posterior error covariance matrices of (3.23) and (3.26), while the prior/posterior estimates of these two systems are not identical.

$\lim_{i \rightarrow \infty} P_i^-$ in (3.24) takes the form of

$$\lim_{i \rightarrow \infty} P_i^- = \begin{bmatrix} 0 & 0 \\ 0 & P_u^- \end{bmatrix}, \quad (3.27)$$

where $P_u^- := \lim_{i \rightarrow \infty} \mathbb{E}[(X_u(i) - \hat{X}_u^-(i))(X_u(i) - \hat{X}_u^-(i))^\top]$, with unstable modes $X_u(i)$ and their prior estimates $\hat{X}_u^-(i)$, is the asymptotic prior error covariance of the unstable part and satisfies the steady-state ARE (algebraic Riccati equation)

$$P_u^- = A_u P_u^- (G_u P_u^- G_u^\top + I)^{-1} A_u^\top, \quad (3.28)$$

with A_u and G_u being the unstable parts of A and G .

With the filtering result above, we now prove statement i) and show that $\bar{I}(E; X_0) \geq \sum_j \log |\lambda_j^+(A)| = (1/2) \log \det(G_u P_u^- G_u^\top + I)$, where the inequality is proved by resorting to [6, Lemma 4.1], and the equality can be obtained by taking $\log \det(\cdot)$ on both sides of (3.28). To prove statement ii) and show that $\bar{I}(E; X_0) \leq (1/2) \log \det(G_u P_u^- G_u^\top + I) = \sum_j \log |\lambda_j^+(A)|$, consider the following manipulation on $I(E_0^n; X_0)$

$$\begin{aligned} I(E_0^n; X_0) &\stackrel{(a)}{=} h(E_0^n) - h(W_0^n) \\ &\stackrel{(b)}{=} h(G(X_0^n - \hat{X}_{0:n}^-) + W_0^n) - h(W_0^n) \\ &\stackrel{(c)}{\leq} \frac{1}{2} \sum_{i=0}^n \log[(2\pi e)^s |G P_i^- G^\top + I|] - \frac{n}{2} \log(2\pi e)^s \\ &\stackrel{(d)}{=} n \sum_j \log |\lambda_j^+(A_u)|, \end{aligned} \quad (3.29)$$

where (a) invokes (3.15) in Proposition 3.7; (b) follows from (3.21) and equality/property $h(GX_0^n + W_0^n) = h(G(X_0^n - \hat{X}_{0:n}^-) + W_0^n)$, since $\hat{X}_{0:n}^- = \{\mathbb{E}[X_0^i | E_0^{i-1}]\}_{i=0}^n$ is deterministic; (c) follows property $h(X_0^n) \leq \sum_{i=0}^n h(X_i)$ and the maximum entropy condition (Property 2.9) when E_i and $W_i \in \mathbb{R}^s$ in (3.21), and (d) relies on (3.27) and (3.28). Dividing both sides of (3.29) by n and taking the limit superior as $n \rightarrow \infty$, we have $\bar{I}(E; X_0) \leq \sum_j \log |\lambda_j^+(A)|$. Combining statements i) and ii), we then prove the identity $\bar{I}(E; X_0) = \sum_j \log |\lambda_j^+(A)|$ in Proposition 3.10. More interesting observations on

the lower and upper bounds of (3.22) can also be implied from (3.24) and (3.25).⁵ $\square$

Proposition 3.10 presents a sandwich bound and an equality condition for the total information rate or plant instability rate $\bar{I}(E; X_0)$ subject to (3.19)-(3.21). The equality, $\bar{I}(E; X_0) = \sum_j \log |\lambda_j^+(A)|$, indicates that total information rate serves as an information-theoretic interpretation/substitute to the Bode(-like) integral [17, 27], which is a recognized sensitivity trade-off metric equaling or bounded below by $\sum_j \log |\lambda_j^+(A)|$. The newly derived MMSE-based bounds in (3.22) supplement this information-theoretic trade-off metric by offering a sandwich estimate and an alternative lower bound to $\bar{I}(E; X_0)$.

For completeness, we also consider the scenario when the LTI plant (3.19) is stabilized by the nonlinear control mapping

$$U_i = g_i(Y_0^i), \quad (3.30)$$

which induces the following control channel

$$E_i = U_i(E_0^{i-1}, X_0) + W_i. \quad (3.31)$$

Since the control mapping and channel are now nonlinear, the MMSE-based bounds should be calculated by nonlinear filtering technique, which will be discussed later in Section 3.2.4. By applying Theorem 3.8 to (3.31), we have the following result on the total information rate $\bar{I}(E; X_0)$.

Corollary 3.11. *When the discrete-time LTI plant (3.19) under the nonlinear control mapping (3.30), as depicted in Figure 3.2, is internally mean-square stable, the total information rate $\bar{I}(E; X_0)$ in (3.31) is bounded between*

$$\text{CLB}_{\text{LTI}} \leq \bar{I}(E; X_0) \leq \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(U_i)}{2(n+1)}, \quad (3.32)$$

where $\text{CLB}_{\text{LTI}} = \max\{\sum_j \log |\lambda_j^+(A)|, \overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(U_i)\}$, with $\lambda_j^+(A)$ denoting the eigenvalues of matrix A with positive real parts.

⁵ For example, **a)** $2 \sum_j \log |\lambda_j^+(A)| = \sum_k \log |1 + \eta_k|$ with $\{\eta_k\}$ being the eigenvalues of $G_u P_u^- G_u^\top$, which is obtained by taking $\log \det(\cdot)$ on both sides of (3.28); **b)** $\sum_j \log |\lambda_j^+(A)| \geq \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \text{cmmse}(U_i) / [2(n+1)]$, which is derived by pre- and post-multiplying both sides of (3.25) by G and $G^\top$, taking the trace operator $\text{tr}(\cdot)$, invoking inequality $x/(1+x) \leq \log(1+x)$ when $x > 0$ and result a), and taking average over $i = 0, \dots, n$ as $n \rightarrow \infty$, and **c)** $\sum_j \log |\lambda_j^+(A)| \leq \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \text{pmmse}(U_i) / [2(n+1)]$, which is attained by taking $\log \det(\cdot)$ on both sides of (3.24), applying the relation $\log \det(I + G P_i^- G^\top) = -2 \log(\mathbb{E}[\exp[-(U_i - \hat{U}_i^-)^\top (U_i - \hat{U}_i^-) / 2]]) \leq \text{pmmse}(U_i)$ from [96, Lemma 3.1] and Jensen's inequality, and taking average over $i = 0, \dots, n$ as $n \rightarrow \infty$. Observations b) and c) along with the identity $\bar{I}(E; X_0) = \sum_j \log |\lambda_j^+(A)|$ reaffirm (3.22) in Proposition 3.10.

Proof. Applying [Theorem 3.6](#) to the control channel (3.31), we obtain the MMSE-based lower and upper bounds in (3.32). By applying [6, Lemma 4.1] or the nonlinear time-varying result in [Corollary 3.16](#) to the control systems depicted in (3.19), (3.30), and (3.31), we can tell that $\sum_j \log |\lambda_j^+(A)|$ is also a lower bound of $\bar{I}(E; X_0)$. $\square$

Under the same setting of [Corollary 3.11](#), if all the signals in [Figure 3.2](#) are further stationary Gaussian, [27] has shown that the information rate $\bar{I}(E; X_0)$ is identical to a Bode-like integral, which is defined by the power spectral density ratio between E_0^n and W_0^n , lower-bounded by $\sum_j \log |\lambda_j^+(A)|$, and – similar to the classical Bode’s integral in [110] – captures the fundamental noise sensitivity trade-off property. With the aid of I-MMSE relationship in [Theorem 3.8](#), [Proposition 3.10](#) and [Corollary 3.11](#) supplement the previous findings in [27] by showing that i) $\bar{I}(E; X_0)$ equals $\sum_j \log |\lambda_j^+(A)|$ when the control mapping $\mathcal{K}$ is also linear, and ii) an alternative and MMSE-based sandwich bound for $\bar{I}(E; X_0)$, in case the established bounds, $\sum_j \log |\lambda_j^+(A)|$ and $\mathcal{C}_f$, are loose or difficult to calculate.

3.2.3 Control Trade-offs in LTV Systems

When the plant model $\mathcal{P}$ in [Figure 3.2](#) is LTV, we use the following equations to describe the dynamics of $\mathcal{P}$ (or the augmented dynamics of $\mathcal{P}$ and $\mathcal{K}$ when $\mathcal{K}$ is linear and U_i satisfies (3.34)):

$$\begin{aligned} X_{i+1} &= A_i X_i + B_i E_i \\ Y_i &= H_i X_i, \end{aligned} \tag{3.33}$$

where A_i , B_i and H_i are time-varying matrices of proper dimensions. Definitions of some notations related to the LTV system (3.33), such as uniformly exponentially (anti)stable, exponential dichotomy, weighted shift operator and its spectrum, are briefly interpreted in [Appendix C](#) and references therein. Meanwhile, we postulate the following assumptions on (3.33).

(A2) The sequence $\{A_i\}$ admits an exponential dichotomy of rank m_u , and the spectrum of the antistable part satisfies $\cup_{j=1}^l \{\lambda \in \mathbb{C} : \underline{\kappa}_j \leq |\lambda| \leq \bar{\kappa}_j\}$ with multiplicities $m_1, \dots, m_l$, where $\underline{\kappa}_j \geq 0$ and $\sum_{j=1}^l m_j = m_u$.

(A3) The sequence $\{A_i - B_i H_i\}$ is uniformly exponentially stable (UES).

Remark 3.12. (A2) can be regarded as an LTV counterpart of the stable/unstable dichotomy in LTI systems and is extensively postulated and studied in the literature on LTV systems [91, 92, 111, 112]. When (3.33) is regular, e.g., time-invariant or periodic, the annuli in (A2) shrink to circles, and the modulus of the spectrum in (A2) coincide with the discrete-time Lyapunov exponents or the modulus of unstable poles. (A3) guarantees the boundedness of sensitivity operator, a state-space representation of the sensitivity function, which is also a necessary requirement of any internally stabilizing controller [112].

The exponential dichotomy in (A2) allows us to define a stability preserving state space transformation, analogous to the modal decomposition of LTI systems, that separates the stable and antistable parts of $\{A_i\}$.

Lemma 3.13 (See [111]). *The sequence $\{A_i\}$ in (3.33) possesses an exponential dichotomy if and only if there exist a bounded sequence of matrices $\{T_i\}$ with bounded inverse such that*

$$\left[\begin{array}{c|c} T_{i+1}A_iT_i^{-1} & T_{i+1}B_i \\ \hline H_iT_i^{-1} & D_i \end{array} \right] := \left[\begin{array}{cc|c} A_s(i) & 0 & B_s(i) \\ 0 & A_u(i) & B_u(i) \\ \hline H_s(i) & H_u(i) & D(i) \end{array} \right],$$

where D_i denotes the direct feedthrough matrix, and $\{A_s(i)\}$ and $\{A_u(i)\}$ respectively stand for the stable and antistable parts of $\{A_i\}$.

By using the relationship between entropy and risk-sensitive cost functions in [96, 113], we define an LTV Bode's integral $\mathcal{B}$, which gives a time-domain characterization of the sensitivity property of (3.33).

Lemma 3.14 (See [112]). *For the discrete-time LTV system (3.33) fulfilling (A2) and (A3), the time-domain Bode's integral $\mathcal{B}$ satisfies⁶*

$$\mathcal{B} = \overline{\lim}_{n \rightarrow \infty} \frac{\log \det[\Phi_{A_u}(n, 0)]}{n} \geq \sum_{j=1}^l m_j \log \kappa_j \geq m_u \log \beta,$$

⁶ Rigorously, the LTV Bode's integral is defined as $\mathcal{B} := \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det(E_n^\top S^\top S E_n)$ in [112], where E_n denotes the embedding operator of X_0^n and S is the sensitivity operator of (3.33). Meanwhile, there exist an identity between $\mathcal{B}$ and the average risk-sensitive cost function $\mathcal{B} = \overline{\lim}_{n \rightarrow \infty} -[2(n+1)]^{-1} \log(\mathbb{E}[\exp(\|W_0^n\|/2 - \|E_0^n\|/2)])$, where W_0^n and E_0^n are the noise and error signals in Figure 3.2, and identities $\mathcal{B} = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \sum_{i=0}^n \log \det R_i = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det[\Phi_{A_u}(n+1, 0)]$, where block-diagonal operator $R_i = I + B_u(i)^\top X_u(i+1)B_u(i)$ with $X_u(i)$ and $B_u(i)$ being the antistable parts of X_i and B_i .

where $\Phi_{A_u}(n, 0) = \prod_{i=0}^{n-1} A_u(i)$ stands for the discrete-time state transition matrix of $\{A_u(i)\}$ from $i = 0$ to n , and α and $\beta > 1$ are positive constants, independent from n , such that $\underline{\mu}(\Phi_{A_u}(n, 0)) \geq \alpha\beta^n$.

In particular, if the sequence $\{A_i\}$ in (3.33) is open-loop stable, the LTV Bode's integral reduces to $\mathcal{B} = 0$. When the sequence $\{A_u(i)\}$ of unstable part is regular, by Remark 3.12, the LTV Bode's integral takes the form $\mathcal{B} = \sum_{j=1}^l m_j \log \underline{\kappa}_j = \sum_{j=1}^l m_j \log \bar{\kappa}_j$.

With the assumptions and properties above, we now turn to investigate the control trade-offs in discrete-time LTV systems. Similar to the LTI study in Section 3.2.2, we first consider the scenario when the control mapping $\mathcal{K}$ in Figure 3.2 is linear, and the control input signal satisfies

$$U_i = G_i X_i, \quad (3.34)$$

where G_i is a time-varying feedback gain, and X_i denotes the internal states of $\mathcal{P}$ or the augmented states of $\mathcal{P}$ and $\mathcal{K}$. Subsequently, the control channel subject to the LTV plant (3.33) and controller (3.34) becomes

$$E_i = U_i + W_i = G_i X_i + W_i. \quad (3.35)$$

By applying Theorem 3.8 to (3.33)-(3.35), we have the following result on the total information rate $\bar{I}(E; X_0)$ of LTV systems.

Proposition 3.15. *When the discrete-time LTV system subject to (3.33)-(3.35), assumptions (A2) and (A3), as Figure 3.2 shows, is internally mean-square stable, the total information rate $\bar{I}(E; X_0)$ in (3.35) is bounded between*

$$\text{CLB}_{\text{LTV}} \leq \bar{I}(E; X_0) \leq \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(U_i)}{2(n+1)}, \quad (3.36)$$

where $\bar{I}(E; X_0) = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det[\Phi_{A_u}(n+1, 0)] = \mathcal{B}$, and $\text{CLB}_{\text{LTV}} = \max\{\sum_{j=1}^l m_j \log \underline{\kappa}_j, \overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(U_i)\}$.

Proof. By applying Theorem 3.8 to the LTV control system and channel depicted by (3.33)-(3.35), we can obtain the MMSE-based sandwich bounds in (3.36). Hence, it remains to show that $\bar{I}(E; X_0) = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det \Phi_{A_u}(n+1, 0) \geq \sum_{j=1}^l m_j \log \underline{\kappa}_j$, where the inequality can be directly implied from Lemma 3.14, and the equality can be verified by respectively proving the

statements i) $\bar{I}(E; X_0) \geq \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det \Phi_{A_u}(n+1, 0)$, and ii) $\bar{I}(E; X_0) \leq \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det \Phi_{A_u}(n+1, 0)$.

To calculate the prior/posterior estimates and estimation error (covariance) of the control input $U_i = G_i X_i$, consider the following LTV filtering problem originated from (3.33)-(3.35)

$$\begin{aligned} X_{i+1} &= (A_i + B_i G_i) X_i + B_i W_i \\ E_i &= G_i X_i + W_i, \end{aligned} \tag{3.37}$$

where E_i is observable, and X_i is the vector of the hidden states to be estimated. By applying the LTV Kalman filter, [114, Sec. 2.7.2] or [84, Sec. 6.5], to (3.37), and defining the state prior estimate $\hat{X}_i^- := \mathbb{E}[X_i | E_0^{i-1}]$, the prior error covariance $P_i^- := \mathbb{E}[(X_i - \hat{X}_i^-)(X_i - \hat{X}_i^-)^\top]$ of the filtering problem (3.37) satisfies

$$P_i^- = A_{i-1} P_{i-1}^- (G_{i-1} P_{i-1}^- G_{i-1}^\top + I)^{-1} A_{i-1}^\top. \tag{3.38}$$

Let the posterior estimate be $\hat{X}_i := \mathbb{E}[X_i | E_0^i]$. The posterior error covariance $P_i := \mathbb{E}[(X_i - \hat{X}_i)(X_i - \hat{X}_i)^\top]$ of (3.37) follows

$$P_i = P_i^- - P_i^- G_i^\top (G_i P_i^- G_i^\top + I)^{-1} G_i P_i^-. \tag{3.39}$$

The evolution of error covariance matrices (not state estimates) of the filtering problem (3.37) in (3.38) and (3.39) is identical to the filtering result of the following LTV system without control input and process noise

$$\begin{aligned} X_{i+1} &= A_i X_i, \\ E_i &= G_i X_i + W_i. \end{aligned} \tag{3.40}$$

Since (3.40) is subject to assumption (A2), we can partition (3.40) into stable and antistable parts, as Lemma 3.13 shows. When (3.37) or (3.40) is either a) uniformly completely reconstructible or b) exponentially stable, by Lemma 3.24, the asymptotic (or steady-state) prior error covariance $\lim_{i' \rightarrow \infty} P_{i'}^-$ in (3.38) takes the form of

$$\lim_{i' \rightarrow \infty} P_{i'}^- = \begin{bmatrix} 0 & 0 \\ 0 & \bar{P}_u^-(i) \end{bmatrix}, \tag{3.41}$$

where the asymptotic prior error covariance of antistable part $\bar{P}_u^-(i) := \lim_{i' \rightarrow \infty} \mathbb{E}[(X_u(i') - \hat{X}_u^-(i')) \cdot (X_u(i') - \hat{X}_u^-(i'))^\top]$, with antistable modes $X_u(t)$ and their prior estimates $\hat{X}_u^-(i)$, satisfies the following RDE (Riccati difference equation)

$$\bar{P}_u^-(i+1) = A_u(i)\bar{P}_u^-(i)[G_u(i)\bar{P}_u^-(i)G_u^\top(i) + I]^{-1}A_u^\top(i) \quad (3.42)$$

with $A_u(i)$ and $G_u(i)$ being the antistable parts of $A(i)$ and $G(i)$.

With the filtering result ahead, we now prove statement i) and show that $\bar{I}(E; X_0) \geq \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det \Phi_{A_u}(n+1, 0) = \overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \log \det [G_u(i)\bar{P}_u^-(i)G_u^\top(i) + I]$. The former inequality will be verified later in the proof of [Corollary 3.16](#) by resorting to the mean-square stability and maximum entropy conditions. To show the latter equality, take $\log \det(\cdot)$ on both sides of (3.42), which gives

$$\log \det A_u(i) = \frac{1}{2} \left[\log \det [G_u(i)\bar{P}_u^-(i)G_u^\top(i) + I] + \log \det \bar{P}_u^-(i+1) - \log \det \bar{P}_u^-(i) \right]. \quad (3.43)$$

Summing up (3.43) from $i = 0$ to n , dividing the result by $n+1$, and taking the limit superior as $n \rightarrow \infty$, we then prove statement i). To verify statement ii) and show that $\bar{I}(E; X_0) \leq \overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \log \det [G_u(i)\bar{P}_u^-(i)G_u^\top(i) + I] = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det \Phi_{A_u}(n+1, 0)$, consider the following manipulation on $I(E_0^n; X_0)$:

$$\begin{aligned} I(E_0^n; X_0) &\stackrel{(a)}{=} h(E_0^n) - h(W_0^n) \\ &\stackrel{(b)}{=} h(G_0^n(X_0^n - \hat{X}_{0:n}^-) + W_0^n) - h(W_0^n) \\ &\stackrel{(c)}{\leq} \frac{1}{2} \sum_{i=0}^n \log[(2\pi e)^s |G_i P_i^- G_i^\top + I|] - \frac{n}{2} \log(2\pi e)^s \\ &\stackrel{(d)}{=} \log \det \Phi_{A_u}(n+1, 0) + \frac{1}{2} \log \det \bar{P}_u^-(0) - \frac{1}{2} \log \det \bar{P}_u^-(n+1), \end{aligned} \quad (3.44)$$

where (a) uses (3.15) in [Proposition 3.7](#); (b) follows from (3.35) and the identity/property $h(G_0^n X_0^n + W_0^n) = h(G_0^n(X_0^n - \hat{X}_{0:n}^-) + W_0^n)$, since $\hat{X}_{0:n}^- := \{\mathbb{E}[X_0^i | E_0^{i-1}]\}_{i=0}^n$ is deterministic; (c) adopts the differential entropy property $h(X_0^n) \leq \sum_{i=0}^n h(X_i)$, with equality attained if and only if $\{X_i\}_{i=0}^n$ are mutually independent, and the maximum entropy condition ([Property 2.9](#)) when E_i and $W_i \in \mathbb{R}^s$ in (3.35), and (d) invokes (3.41) and (3.43) in the limit of $i \rightarrow \infty$. By dividing both sides of (3.44) by

$n + 1$ and taking the limit superior as $n \rightarrow \infty$, statement ii) is verified. Combining the proofs of statements i) and ii), we verify the identity $\bar{I}(E; X_0) = \overline{\lim}_{n \rightarrow \infty} (n + 1)^{-1} \log \det[\Phi_{A_u}(n + 1, 0)]$ in [Proposition 3.15](#). This completes the proof. $\square$

The equality in [Proposition 3.15](#) reveals that for the LTV systems governed by (3.33)-(3.35), total information rate $\bar{I}(E; X_0)$ is identical to the LTV Bode's integral $\mathcal{B}$ in [Lemma 3.14](#). This equality can also be regarded as an LTV extension of the LTI identity, $\bar{I}(E; X_0) = \sum_j \log |\lambda_j^+(A)|$, in [Proposition 3.10](#), a technical explanation of which is given in [Remarks 3.12](#) and [3.17](#). Meanwhile, our proofs of [Proposition 3.10](#) and [3.15](#), based on information and estimation theories, and applicable to both LTI and LTV control systems, are more concise and straightforward than the existing optimization-based method [10], which has not been verified in the LTV scenario yet. Similar to the extra findings⁵ in the proof of [Proposition 3.10](#), more interesting observations on the MMSE-based bounds in (3.36) can be extracted from the propagation formulae of error covariance matrices in (3.38) and (3.39).⁷

For completeness, we briefly discuss the scenario when the LTV plant (3.33) is stabilized by the nonlinear controller (3.30), which can be collectively described by the control channel (3.31), and is no longer an LTV system due to the non-linearity in (3.30). By applying [Theorem 3.8](#) to the control system depicted by (3.30), (3.31) and (3.33), we have the following finding.

Corollary 3.16. *When the discrete-time LTV plant (3.33) under nonlinear control mapping (3.30) in [Figure 3.2](#) is internally mean-square stable, the total information rate $\bar{I}(E; X_0)$ satisfies*

$$\text{CLB}_{\text{LTV}} \leq \bar{I}(E; X_0) \leq \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(U_i)}{2(n+1)}, \quad (3.45)$$

where $\text{CLB}_{\text{LTV}} = \max\{\sum_{j=1}^l m_j \log \kappa_j, \overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(U_i)\}$.

Proof. The MMSE-based sandwich bounds in (3.45) can be readily derived by applying [Theorem 3.8](#) to the LTV plant (3.33) and control channel (3.31). It then remains to show that

⁷ For example, **a)** $2 \sum_{i=0}^n \sum_j \log |\lambda_j(A_u(i))| + \log \det P_u^-(0) - \log \det P_u^-(n+1) = \sum_{i=0}^n \sum_k (1 + \eta_{i,k})$, where $\{\lambda_j(A_u(i))\}$ denote the eigenvalues of $A_u(i)$, and $\{\eta_{i,k}\}$ are the eigenvalues of $G_u(i)P_u^-(i)G_u^\top(i)$; **b)** $\lim_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(U_i) \leq \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \sum_{i=0}^n \sum_j \log |\lambda_j(A_u(i))|$; **c)** $\lim_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{pmmse}(U_i) \geq \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \sum_{i=0}^n \sum_j \log |\lambda_j(A_u(i))|$, where observations b) and c) along with the identity $\bar{I}(E; X_0) = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det[\Phi_{A_u}(n+1, 0)]$ reaffirm (3.36) in [Proposition 3.15](#). The proofs of observations a)-c) can be inferred from the LTI derivations in [5](#).

$\bar{I}(E; X_0) \geq \sum_{j=1}^l m_j \log \kappa_j$, which, analogous to the constraint of LTV Bode's integral $\mathcal{B}$, indicates the control trade-off property of $\bar{I}(E; X_0)$, and is also an intermediate result invoked in the proof of [Proposition 3.15](#). To prove this inequality, consider the antistable part of (3.33) governed by the following equation

$$X_u(i+1) = A_u(i)X_u(i) + B_u(i)E(i), \quad (3.46)$$

where $A_u(i)$ and $B_u(i)$ follow the definitions in [Lemma 3.13](#). The solution to (3.46), $X_u(n)$ or $X_u(n+1)$, is

$$\begin{aligned} X_u(n) &= \Phi_{A_u}(n, 0)X_u(0) + \sum_{i=0}^{n-1} \Phi_{A_u}(n, i+1)B_u(i)E(i) \\ &= \Phi_{A_u}(n, 0)[X_u(0) + \bar{X}_u(n)], \end{aligned} \quad (3.47)$$

where the state transition matrix is given by $\Phi_{A_u}(n, 0) := \prod_{i=0}^{n-1} A_u(i)$, and $\bar{X}_u(n) := \Phi_{A_u}(n, 0)^{-1} \cdot \sum_{i=0}^{n-1} \Phi_{A_u}(n, i+1)B_u(i)E(i)$. Since the LTV subsystem (3.46), stabilized by the nonlinear control mapping (3.30) or (3.31), is internally mean-square stable, i.e., $\sup_i \mathbb{E}[X_i^\top X_i] < \infty$, substituting (3.47) into the mean-square stability condition gives

$$+\infty > S > \log \det \mathbb{E}[X_u(n)X_u^\top(n)] = 2 \log \det \Phi_{A_u}(n, 0) + \log \det \Gamma_n, \quad (3.48)$$

where $\Gamma_n := \mathbb{E}[(X_u(0) + \bar{X}_u(n))(X_u(0) + \bar{X}_u(n))^\top]$. Moreover, the total information $I(E_0^{n-1}; X_0)$ in (3.31) satisfies

$$\begin{aligned} I(E_0^{n-1}; X_0) &\stackrel{(a)}{\geq} I(\bar{X}_u(n); X_u(0)) \\ &= h(X_u(0)) - h(X_u(0)|\bar{X}_u(n)) \\ &\stackrel{(b)}{=} h(X_u(0)) - h(X_u(0) + \bar{X}_u(n)|\bar{X}_u(n)) \\ &\stackrel{(c)}{\geq} h(X_u(0)) - h(X_u(0) + \bar{X}_u(n)) \\ &\stackrel{(d)}{\geq} h(X_u(0)) - \frac{s_u}{2} \log(2\pi e) - \frac{1}{2} \log \det \Gamma_n, \end{aligned} \quad (3.49)$$

where (a) follows from the fact that $X_u(0)$ is a function of $X(0)$, and $\bar{X}_u(n)$ is a function of E_0^{n-1} ; (b) uses the differential entropy property $h(X|Y) = h(X + f(Y)|Y)$; (c) can be explained by the

non-negativity of mutual information, since $I(X; Y) = h(X) - h(X|Y) \geq 0$, and (d) assumes that $X_u \in \mathbb{R}^{s_u}$ and resorts to the maximum condition of differential entropy. Substituting inequality (3.48) into (3.49) gives

$$I(E_0^{n-1}; X_0) \geq h(X_u(0)) - (s_u/2) \log(2\pi e) - S/2 + \log \det \Phi_{A_u}(n, 0). \quad (3.50)$$

Dividing both sides of (3.50) by n and taking the limit superior as $n \rightarrow \infty$, we have

$$\bar{I}(E; X_0) \geq \overline{\lim}_{n \rightarrow \infty} \frac{\log \det \Phi_{A_u}(n, 0)}{n} \geq \sum_{j=1}^l m_j \ln \kappa_j \geq m_u \ln \beta.$$

This completes the proof. $\square$

Since the control mapping (3.30) is nonlinear, the estimates of control action, $\text{cmmse}(U_i)$ and $\text{pmmse}(U_i)$, in (3.45) can no longer be calculated from the linear Kalman filters adopted in Proposition 3.10 or 3.15. Instead, the nonlinear filtering method, to be discussed in the following subsection, should be used.

When the plant $\mathcal{P}$ in Figure 3.2 is LTV, Proposition 3.15 and Corollary 3.16 connect the total information rate $\bar{I}(E; X_0)$ with the LTV Bode's integral $\mathcal{B}$ and the sum of spectral values $\sum_{j=1}^l m_j \log \kappa_j$. This connection reveals that analogous to the LTV Bode's integral and average risk-sensitive cost function, total information (rate) captures the time-domain control trade-off properties of LTV system (3.33), i.e., when the running risk-sensitive or entropy cost is less (or greater) than $\bar{I}(E; X_0)$ on a time interval, the cost outside this interval must be greater (or less). Meanwhile, with the aid of the I-MMSE relationship, the MMSE-based sandwich bounds for this control trade-off metric are also derived, which not only provide a sandwich estimate to $\bar{I}(E; X_0)$, but complement the preceding bounds, such as $\mathcal{C}_f$ in Theorem 3.8 and $\sum_{j=1}^l m_j \log \kappa_j$ in Lemma 3.14.

Remark 3.17. *The LTV derivations and results presented in Proposition 3.15 and Corollary 3.16 cover the LTI ones in Proposition 3.10 and Corollary 3.11 as a special case. If we apply the LTV results of this section to an LTI system as (3.19), the spectral values κ_j in the CLB_{LTV} of (3.36) and (3.45) will coincide with the modulus of unstable poles $|\lambda_j^+(A)|$ in the CLB_{LTI} of (3.22) and (3.32), and the time-varying estimation error covariance matrices in (3.38)-(3.44) will degenerate to a time-invariant or steady-state matrix and be canceled. Similar relationship also exists when we*

investigate the total information rates of continuous-time LTI and LTV control systems [16].

3.2.4 Control Trade-offs in Nonlinear Systems

When both $\mathcal{P}$ and $\mathcal{K}$ are nonlinear in Figure 3.2, consider the following nonlinear plant dynamics

$$\begin{aligned} X_{i+1} &= f_i(X_i) + b_i(X_i)E_i, \\ Y_i &= h_i(X_i), \end{aligned} \tag{3.51}$$

stabilized by the nonlinear control mapping

$$U_i = g_i(Y_i). \tag{3.52}$$

The control channel and error signal E_i in Figure 3.2 then satisfy

$$E_i = U_i(E_0^{i-1}, X_0) + W_i = U_i(X_{i-1}) + W_i, \tag{3.53}$$

where W_i is a white Gaussian process noise, and notation U_i is abused here to denote proper functions that satisfy $g(Y_i) = U_i(E_0^{i-1}, X_0) = U_i(X_{i-1})$. By applying Theorem 3.8 and results from nonlinear estimation theory to the control system and channel depicted in (3.51)-(3.53), the MMSE-based sandwich bounds of $\bar{I}(E; X_0)$ and their calculation scheme are given in the following proposition and proof.

Proposition 3.18. *When the discrete-time control system in Figure 3.2, subject to nonlinear plant (3.51) and control mapping (3.52), is internally mean-square stable, the total information rate $\bar{I}(E; X_0)$ in (3.53) is bounded by*

$$\overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{cmmse}(U_i)}{2(n+1)} \leq \bar{I}(E; X_0) \leq \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(U_i)}{2(n+1)}. \tag{3.54}$$

Proof. The MMSE-based bounds and inequality in (3.54) can be obtained by applying Theorem 3.8 to control channel (3.53). However, since both the plant and controller are nonlinear, we can no longer resort to the preceding linear Kalman filters when calculating the estimates and estimation errors of control input U_i in (3.54). Assume the control input signal U_i in (3.52) and (3.53) is

observable. Otherwise, see [Section 3.3.4](#) for the alternative estimation and calculation scheme. In order to compute the prior estimate $\hat{U}_i^- := \mathbb{E}[U_i(X_i)|E_0^{i-1}]$ in $\text{pmmse}(U_i)$ and the posterior estimate $\hat{U}_i := \mathbb{E}[U_i(X_i)|E_0^i]$ in $\text{cmmse}(U_i)$, consider the following nonlinear filtering problem with correlated noise and originated from (3.51)-(3.53):

$$\begin{aligned} X_{i+1} &= \bar{f}_i(X_i) + b_i(X_i)W_i \\ E_i &= U_i(X_i) + W_i, \end{aligned} \tag{3.55}$$

where X_i denotes the hidden internal states of plant $\mathcal{P}$; the error signal E_i is observable, and $\bar{f}_i(X_i) := f_i(X_i) + b_i(X_i)U_i(X_i)$. To implement the nonlinear filter and estimate the predicted and posterior densities from (3.55), we first decouple the correlated noise by transforming (3.55) into the following filtering problem with independent noise [\[115, 116\]](#):

$$\begin{aligned} X_{i+1} &= F_i(X_i, E_i) \\ E_i &= U_i(X_i) + W_i, \end{aligned} \tag{3.56}$$

where $F_i(X_i, E_i) := \bar{f}_i(X_i) + b_i(X_i)[E_i - U_i(X_i)]$. Since (X_i, E_i) in (3.56) form a Markov chain, by letting $\pi_{t|s}(x) := \mathbb{P}(X_t = x|E_0^s = e_0^s)$, we can calculate the prior density $\pi_{i+1|i}(x_{i+1}) = \mathbb{P}(X_{i+1} = x_{i+1}|E_0^i = e_0^i)$ by the time update

$$\pi_{i+1|i}(x_{i+1}) = \int_{\mathcal{X}_i} p(x_{i+1}|x_i, e_0^i) \pi_i(x_i) dx_i, \tag{3.57}$$

where $p(x_{i+1}|x_i, e_0^i) := \mathbb{P}(X_{i+1} = x_{i+1}|X_i = x_i, E_0^i = e_0^i)$ is the state transition probability of (3.56), and $\mathcal{X}_i$ denotes the state space of X_i . With the measurement e_{i+1} , we can compute the posterior density $\pi_{i+1}(x_{i+1}) := \mathbb{P}(X_{i+1} = x_{i+1}|E_0^{i+1} = e_0^{i+1})$ from the measurement update

$$\pi_{i+1}(x_{i+1}) = \frac{\pi_{i+1|i}(x_{i+1})p(e_{i+1}|x_{i+1})}{\int_{\mathcal{X}_{i+1}} \pi_{i+1|i}(x_{i+1})p(e_{i+1}|x_{i+1})dx_{i+1}}, \tag{3.58}$$

where $p(e_{i+1}|x_{i+1}) := \mathbb{P}(E_{i+1} = e_{i+1}|X_{i+1} = x_{i+1})$ is the conditional distribution of observation e_{i+1} given the state x_{i+1} . By utilizing the density functions (3.57) and (3.58), and control mapping (3.52), we can then calculate the estimates of control input $\hat{U}_i^-$ and $\hat{U}_i$, and thus the estimation errors $\text{cmmse}(U_i)$ and $\text{pmmse}(U_i)$ in (3.54). This completes the proof. $\square$

It is worth noting that density functions (3.57) and (3.58) can be approximated by the sampling methods, such as MCMC, which are arbitrarily accurate but tend to be slow and computationally intensive. In practice, by postulating some proper assumptions, e.g., Gaussian distribution assumption in [117], we can simplify and expedite this approximation process with the aid of some sub-optimal filters, such as the extended Kalman filter and particle filter [118]. Meanwhile, due to the variety and complexity of nonlinear models in (3.51)-(3.53), the control trade-off property of $\bar{I}(E; X_0)$ in Proposition 3.18 is not as explicit as the linear cases discussed in the preceding subsections. Nevertheless, since the I-MMSE relationship and inequalities (3.16) still hold for nonlinear control systems, we can interpret the total information rate $\bar{I}(E; X_0)$ in (3.53) as i) a performance limit related to the average control quadratic cost, similar to the LTV interpretation, or ii) a data transmission limit (or instability rate) of the nonlinear control channel and system, as we explained in Section 3.2.1.

3.3 Fundamental Limitations of Discrete-Time Filtering Systems

To characterize the filtering limitations, we first model the general discrete-time filtering system into an additive white Gaussian channel without feedback. With the aid of discrete-time I-MMSE relationship and optimal estimation theory, total information rate and its sandwich bounds are used to quantify the performance limits of different filtering systems.

3.3.1 General Filtering Systems and Limits

Consider the general discrete-time filtering system depicted in Figure 3.3:

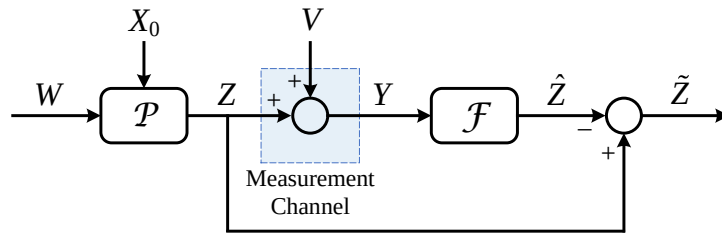


Figure 3.3: Configuration of a general filtering system.

where $\mathcal{P}$ and $\mathcal{F}$ denote the plant model and filtering process, respectively; (W, X_0) is regarded as

the transmitted message of measurement channel with W being a white Gaussian process noise, and X_0 being the initial states of $\mathcal{P}$; V is a white Gaussian measurement noise; W , V , and X_0 are mutually independent; Z stands for the noise-free output to estimate; Y is the measured output; $\hat{Z}$ represents the estimate of Z , and the estimation error $\tilde{Z} := Z - \hat{Z}$. The open- or closed-loop plant $\mathcal{P}$ in Figure 3.3 is described by the following nonlinear model

$$\begin{aligned} X_{i+1} &= f_i(X_i) + b_i(X_i)W_i \\ Z_i &= h_i(X_i), \end{aligned} \tag{3.59}$$

where X_i and Z_i are the internal states and noise-free output of $\mathcal{P}$ at time i ; process noise $W_i \sim \mathcal{N}(0, \varepsilon^2 I)$, and $f_i(\cdot)$, $b_i(\cdot)$, and $h_i(\cdot)$ are measurable functions. Since the noise-free output Z_i is a function of message (W_0^{i-1}, X_0) by (3.59), the measurement channel in Figure 3.3 is governed by

$$Y_i = Z_i(W_0^{i-1}, X_0) + V_i, \tag{3.60}$$

where the channel input is $Z_i(W_0^{i-1}, X_0) = Z_i = h_i(X_i)$, and the measurement noise is $V_i \sim \mathcal{N}(0, I)$. By combining the first equation of (3.59) and (3.60), a discrete-time filtering problem can be established, in which X_i is the vector of the hidden states to be estimated, and Y_i denotes the observable output. The following assumption is then imposed on this filtering problem.

(A4) The noise-free output Z_i is of bounded power, i.e., $\mathbb{E}[Z_i^\top Z_i] < \infty, \forall i$.

Remark 3.19. (A4) is a prerequisite for applying the I-MMSE relationship or Theorem 3.4 to the measurement channel (3.60), since it guarantees that Z_i belongs to the reproducing kernel Hilbert space induced by V_i , which is a necessary (and sufficient, when Z_i is Gaussian) condition for the boundedness of total information $I(Y_0^n; W_0^{n-1}, X_0)$ and input-output mutual information $I(Y_0^n; Z_0^n)$ in (3.60). Meanwhile, no restriction is imposed on the distribution of Z_i , i.e., Z_i can be non-Gaussian.

By applying Theorem 3.4 to the filtering problem developed from (3.59) and (3.60), the following theorem shows that total information rate $\bar{I}(Y; W, X_0)$ is a trade-off metric quantifying the filtering limits of noise-free output.

Theorem 3.20. *For the discrete-time filtering systems subject to (3.59), (3.60) and Figure 3.3, the total information $I(Y_0^n; W_0^{n-1}, X_0)$ in (3.60) is bounded between*

$$\frac{1}{2} \sum_{i=0}^n \text{cmmse}(Z_i) \leq I(Y_0^n; W_0^{n-1}, X_0) \leq \frac{1}{2} \sum_{i=0}^n \text{pmmse}(Z_i), \quad (3.61)$$

and the total information rate $\bar{I}(Y; W, X_0)$ satisfies:

$$\lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{cmmse}(Z_i)}{2(n+1)} \leq \bar{I}(Y; W, X_0) \leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(Z_i)}{2(n+1)}. \quad (3.62)$$

In particular, if signals Y_0^n and W_0^n are jointly stationary, we have $\lim_{n \rightarrow \infty} \sum_{i=0}^n \text{cmmse}(Z_i)/[2(n+1)] \leq \bar{I}(Y; W, X_0) \leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \text{pmmse}(Z_i)/[2(n+1)]$.

Proof. Inequalities (3.61) are obtained by applying the I-MMSE relationship in Theorem 3.4 to the measurement channel (3.60), and inequalities (3.62) are derived by the definition of mutual information rate in Definition 3.2. $\square$

Theorem 3.20 shows that for the general discrete-time filtering systems, the total information (rate) serves as a lower bound for the (average) predicted MMSE of noise-free output Z_i and an upper bound for the (average) causal MMSE. By using the fundamental properties of differential entropy and mutual information, we have the following inequality constraints and decomposition for $I(Y_0^n; W_0^{n-1}, X_0)$.

Proposition 3.21. *For the discrete-time filtering systems subject to (3.59), (3.60) and Figure 3.3, the total information $I(Y_0^n; W_0^n, X_0)$ satisfies*

$$I(Y_0^n; W_0^{n-1}, X_0) = I(Y_0^n; X_0) + I(Y_0^n; W_0^{n-1} | X_0), \quad (3.63)$$

and

$$I(Y_0^n; W_0^{n-1}, X_0) = h(Y_0^n) - h(V_0^n). \quad (3.64)$$

Proof. Equation (3.63) is obtained by applying the chain rule of mutual information to $I(Y_0^n; W_0^{n-1}, X_0)$.

Equation (3.64) can be verified by

$$I(Y_0^n; W_0^{n-1}, X_0) = h(Y_0^n) - h(Y_0^n | W_0^{n-1}, X_0)$$

$$\begin{aligned}
&= h(Y_0^n) - \sum_{i=0}^n h(V_i | Y_0^{i-1}, W_0^{i-1}, X_0) \\
&= h(Y_0^n) - \sum_{i=0}^n h(V_i | V_0^{i-1}),
\end{aligned}$$

which follows the similar arguments as the proof of (3.15). $\square$

The analysis of total information (rate) as a filtering trade-off metric in [Theorem 3.20](#) and [Proposition 3.21](#) is symmetric and can be seen as the filtering counterpart to the control trade-off analysis in [Theorem 3.6](#) and [Proposition 3.7](#).

By incorporating the bounds and constraints in [Theorem 3.20](#) and [Proposition 3.21](#) together, a comprehensive description of the total information rate as a filtering limit metric can be obtained. Before showing this result, we define the filtering capacity $\mathcal{C}_f := \sup_{(W, X_0)} \bar{I}(Y; W, X_0)$ as the supremum of total information rate over all admissible sets of (W, X_0) in the measurement channel (3.60). When the power of noise-free output is limited by $\mathbb{E}[Z_i^\top Z_i] \leq \rho_i, \forall i$, the channel capacity $\mathcal{C} := \sup_{\mathbb{E}[Z_i^\top Z_i] \leq \rho_i} \bar{I}(Y; Z)$ is defined as the supremum of input-output mutual information rate. The following theorem then provides a full description on the total information rate $\bar{I}(Y; W, X_0)$ of general nonlinear filtering systems.

Theorem 3.22. *For the discrete-time filtering systems subject to (3.59), (3.60) and [Figure 3.3](#), the total information rate $\bar{I}(Y; W, X_0)$ is bounded between*

$$\text{FLB} \leq \bar{I}(Y; W, X_0) \leq \mathcal{C}_f \leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(Z_i)}{2(n+1)}, \quad (3.65)$$

where the filtering lower bound is $\text{FLB} = \max\{\overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(Z_i), \bar{I}(Y; X_0)\}$. When $\mathbb{E}[Z_i^\top Z_i] \leq \rho_i, \forall i$, and Σ_{Z_i} denotes the covariance of Z_i , the filtering capacity is given by $\mathcal{C}_f = \mathcal{C} = \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[Z_i^\top Z_i] \leq \rho_i} [2(n+1)]^{-1} \sum_{i=0}^n \log(|\Sigma_{Z_i} + I|)$. When the process noise W_i in (3.59) vanishes, i.e., $\varepsilon \rightarrow 0$, the total information rate $\bar{I}(Y; X_0) = \lim_{\varepsilon \rightarrow 0} \bar{I}(Y; W, X_0)$ in (3.60) satisfies

$$\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{cmmse}(Z_i)}{2(n+1)} \leq \bar{I}(Y; X_0) \leq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(Z_i)}{2(n+1)}. \quad (3.66)$$

Proof. We first derive the lower and upper bounds of $\bar{I}(Y; W, X_0)$ in (3.65). From (3.62), we can tell that $\overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(Z_i)$ and $\overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{pmmse}(Z_i)$ are respectively

a lower bound and an upper bound for $\bar{I}(Y; W, X_0)$. By (3.63) and the non-negativity of mutual information, or $I(Y_0^n; W_0^{n-1}|X_0) \geq 0$, we see that $\bar{I}(Y; X_0)$ is also a lower bound of $\bar{I}(Y; W, X_0)$. Meanwhile, according to the definition of filtering capacity, $\mathcal{C}_f$ should be the lowest upper bound for $\bar{I}(Y; W, X_0)$.

We then consider the scenario when the noise-free output Z_i is subject to the power constraint, i.e., $\mathbb{E}[Z_i^\top Z_i] < \rho_i$, $\forall i$. Without loss of generality, suppose Y_i and $V_i \in \mathbb{R}^s$ in (3.60). To verify $\mathcal{C}_f = \mathcal{C} = \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[Z_i^\top Z_i] \leq \rho_i} [2(n+1)]^{-1} \sum_{i=0}^n \log(|\Sigma_{Z_i} + I|)$ in Theorem 3.22, we first show that $\mathcal{C} \leq \mathcal{C}_f = \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[Z_i^\top Z_i] \leq \rho_i} [2(n+1)]^{-1} \sum_{i=0}^n \log(|\Sigma_{Z_i} + I|)$, where the inequality can be implied by the definition of $\mathcal{C}$ and $\mathcal{C}_f$ and the identity between total information and input-output mutual information in (3.60), i.e., $I(Y_0^n; W_0^{n-1}, X_0) = I(Y_0^n; Z_0^n)$, and the equality follows

$$\begin{aligned} \bar{I}(Y; W, X_0) &\stackrel{(a)}{=} \bar{h}(Y) - \bar{h}(V) \leq \mathcal{C}_f \\ &\stackrel{(b)}{\leq} \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[Z_i^\top Z_i] \leq \rho_i} \frac{\sum_{i=0}^n \log[(2\pi e)^s |\Sigma_{Y_i}|]}{2(n+1)} - \frac{\log[(2\pi e)^s]}{2} \\ &\stackrel{(c)}{\leq} \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[Z_i^\top Z_i] \leq \rho_i} \frac{1}{2(n+1)} \sum_{i=0}^n \log(|\Sigma_{Z_i} + I|) \\ &\stackrel{(d)}{=} \overline{\lim}_{n \rightarrow \infty} \frac{1}{2(n+1)} \sum_{i=0}^n \log(1 + \rho_i), \end{aligned} \tag{3.67}$$

where (a) follows from (3.64) and Definitions 3.1 and 3.2; (b) uses the property $h(Y_0^n) \leq \sum_{i=0}^n h(Y_i)$ with equality attained if and only if $\{Y_i\}_{i=0}^n$ are mutually independent, maximum entropy condition in Property 2.9, and $V_i \sim \mathcal{N}(0, I)$; (c) utilizes the fact that $\Sigma_{Y_i} = \Sigma_{Z_i} + \Sigma_{V_i}$, where $\Sigma_{V_i} = I$, and (d) holds when (3.60) is one-dimensional, i.e., $s = 1$. To show the other direction, consider a special scenario when the noise-free outputs $Z_i \sim \mathcal{N}(0, \Sigma_{Z_i})$ for $i = 0, \dots, n$ in the non-feedback channel (3.60) are mutually independent. The channel capacity $\mathcal{C}$ of this special scenario satisfies

$$\mathcal{C} \geq \bar{I}(Y; Z) = \overline{\lim}_{n \rightarrow \infty} \sup_{\mathbb{E}[Z_i^\top Z_i] \leq \rho_i} \frac{1}{2(n+1)} \sum_{i=0}^n \log(|\Sigma_{Z_i} + I|). \tag{3.68}$$

Combining (3.67) and (3.68), we prove the equations subject to the finite-power constraint in Theorem 3.22.

Lastly, when the process noise $W_i \sim \mathcal{N}(0, \varepsilon^2 I)$ vanishes, i.e., $\varepsilon \rightarrow 0$ in (3.59), the total information

rate $\bar{I}(Y; W, X_0)$ in (3.62) satisfies

$$\bar{I}(Y; W, X_0) \geq \lim_{\varepsilon \rightarrow 0} \bar{I}(Y; W, X_0) = \bar{I}(Y; X_0), \quad (3.69)$$

which follows the decomposition in (3.63) and the fact that when $W_0^n \rightarrow 0$ and becomes deterministic, $\lim_{\varepsilon \rightarrow 0} I(Y_0^n; W_0^n | X_0) = 0$. Taking the limit $\varepsilon \rightarrow 0$ in (3.62) and substituting (3.69) into the result, we obtain (3.66). $\square$

In addition to the filtering trade-off properties characterized in Theorem 3.20 and Proposition 3.21, Theorem 3.22 and inequalities (3.66) reveal that for the general discrete-time filtering systems under vanishing process noise, total information (rate) serves as a lower bound for the (average) *lowest achievable* predicted MMSE of noise-free output despite the design of filtering process $\mathcal{F}$. In other words, no matter how well the filter $\mathcal{F}$ in Figure 3.3 is designed, the average predicted MMSE of Z_i cannot be lower than twice $\bar{I}(Y; X_0)$ in practice (or with the presence of process noise). Meanwhile, we will later show that when there exists an analytic or explicit relationship between the predicted and causal MMSEs, $\bar{I}(Y; X_0)$ can also be used to define a lower bound for the (average) lowest achievable causal MMSE. In the following subsections, with the aid of Theorem 3.22, $\bar{I}(Y; X_0)$ is investigated as a filtering trade-off metric for some specific filtering systems, in which the closed-form expression of $\bar{I}(Y; X_0)$ can be derived.

3.3.2 Filtering Limits in LTI Systems

We now narrow down our scope to the filtering system depicted by the following LTI plant $\mathcal{P}$ and measurement model

$$\begin{aligned} X_{i+1} &= AX_i + BW_i \\ Y_i &= Z_i + V_i = HX_i + V_i, \end{aligned} \quad (3.70)$$

where A , B , and H are time-invariant matrices of proper dimensions, and process noise $W_i \sim \mathcal{N}(0, \varepsilon^2 I)$ and measurement noise $V_i \sim \mathcal{N}(0, I)$ are mutually independent. Since W_i and V_i in (3.70) are uncorrelated and zero-mean white Gaussian, the optimal MS estimates and minimum MS estimation errors of the filtering problem (3.70) can be calculated by the LTI Kalman filter [101].

By applying [Theorem 3.22](#) and LTI Kalman filter to the filtering problem (3.70) under vanishing process noise, the following proposition interprets the filtering trade-off property of $\bar{I}(Y; X_0)$.

Proposition 3.23. *For the filtering system subject to stabilizable and detectable LTI plant (3.70) and vanishing process noise ($\varepsilon \rightarrow 0$) in [Figure 3.3](#), the total information rate $\bar{I}(Y; X_0)$ in (3.70) satisfies*

$$\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{cmmse}(Z_i)}{2(n+1)} \leq \bar{I}(Y; X_0) \leq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(Z_i)}{2(n+1)}, \quad (3.71)$$

where $\bar{I}(Y; X_0) = \sum_j \log |\lambda_j^+(A)|$, with $\lambda_j^+(A)$ denoting the eigenvalues of matrix A with positive real parts, i.e., open-loop unstable poles of $\mathcal{P}$ in [Figure 3.3](#).

Proof. Since [Proposition 3.23](#) can be virtually proved by replacing the feedback control channel in the proof of [Proposition 3.10](#) or [Proposition 3.15](#) with the non-feedback measurement channel in (3.70), in this proof we only show the key results and skip the overlapped details. To calculate the optimal output estimates, $\hat{Z}_i^- := \mathbb{E}[Z_i|Y_0^{i-1}] = \mathbb{E}[HX_i|Y_0^{i-1}]$ and $\hat{Z}_i := \mathbb{E}[Z_i|Y_0^i] = \mathbb{E}[HX_i|Y_0^i]$, and the predicted/causal MMSEs of Z_i in (3.71), we can apply the LTI Kalman filter to (3.70). Define the state prior estimate $\hat{X}_i^- := \mathbb{E}[X_i|Y_0^{i-1}]$ and state posterior estimate $\hat{X}_i := \mathbb{E}[X_i|Y_0^i]$. We can calculate $\hat{Z}_i^-$, $\hat{X}_i^-$, and $\text{pmmse}(Z_i)$ from the prediction step of Kalman filter

$$\begin{aligned} \hat{X}_i^- &= A\hat{X}_{i-1} \\ P_i^- &= AP_{i-1}^-(HP_{i-1}^-H^\top + I)^{-1}A^\top + \varepsilon^2 BB^\top, \end{aligned} \quad (3.72)$$

where the state prior error covariance is $P_i^- := \mathbb{E}[(X_i - \hat{X}_i^-)(X_i - \hat{X}_i^-)^\top]$. In the meantime, $\hat{Z}_i$, $\hat{X}_i$, and $\text{cmmse}(Z_i)$ can be calculated from the correction step of Kalman filter

$$\begin{aligned} \hat{X}_i &= \hat{X}_i^- + K_i(Y_i - H\hat{X}_i^-) \\ P_i &= P_i^- - P_i^- H^\top (HP_i^- H^\top + I)^{-1} HP_i^-, \end{aligned} \quad (3.73)$$

where the Kalman gain is given by $K_i := P_i^- H^\top (I + HP_i^- H^\top)^{-1}$, and the state posterior error covariance is $P_i := \mathbb{E}[(X_i - \hat{X}_i)(X_i - \hat{X}_i)^\top]$.

Inequalities (3.71) are obtained by applying [Theorem 3.22](#) to the measurement channel in (3.70). We then prove $\bar{I}(Y; X_0) = \sum_j \log |\lambda_j^+(A)|$ by respectively showing that i) $\bar{I}(Y; X_0) \geq \sum_j \log |\lambda_j^+(A)|$, and ii) $\bar{I}(Y; X_0) \leq \sum_j \log |\lambda_j^+(A)|$. Statement i) can be obtained by using (A4) and mimicking the proof of [Corollary 3.16](#). Meanwhile, in the limits of $i \rightarrow \infty$ and $\varepsilon \rightarrow 0$, by taking $\log \det(\cdot)$ on both

sides of (3.72) and applying the LTI version of Lemma 3.24 to the prior error covariance matrices in (3.72), we have

$$\lim_{\varepsilon \rightarrow 0} \lim_{i \rightarrow \infty} \log \det(HP_i^- H^\top + I) = \log \det(H_u P_u^- H_u^\top + I) = 2 \sum_j \log |\lambda_j^+(A)|, \quad (3.74)$$

where P_u^- and H_u are the unstable parts of $\lim_{i \rightarrow \infty} P_i^-$ and H , and $\lambda_j^+(A)$ denote the eigenvalues of A_u , which is the unstable part of A . Supposing Y_i and $V_i \in \mathbb{R}^s$ in (3.70), and following (3.64), (3.74) and the similar arguments for (3.29), we have

$$\begin{aligned} I(Y_0^n; X_0) &= h(Y_0^n) - h(V_0^n) \\ &= h(H(X_0^n - \hat{X}_{0:n}^-) + V_0^n) - h(V_0^n) \\ &\leq \frac{1}{2} \sum_{i=0}^n \log[(2\pi e)^s |HP_i^- H^\top + I|] - \frac{n}{2} \log(2\pi e)^s \\ &\stackrel{\varepsilon \rightarrow 0}{\underset{n \rightarrow \infty}{\rightleftharpoons}} n \sum_j \log |\lambda_j^+(A)|. \end{aligned} \quad (3.75)$$

Statement ii) and thus Proposition 3.23 can be proved by dividing both sides of (3.75) by n and taking the limit superior (or limit for the stationary case) as $n \rightarrow \infty$. This completes the proof. $\square$

Proposition 3.23 shows that the total information rate $\bar{I}(Y; X_0)$ serves as a time-domain trade-off metric for the filtering systems subject to the LTI plant (3.70), i.e., no matter how well the filtering mapping $\mathcal{F}$ in Figure 3.3 is designed in practice (or with the presence of process noise), if the predicted MMSE of Z_i is lower than $2\bar{I}(Y; X_0)$ in a time interval, the estimation error outside this interval must be larger, and vice versa. Since $\bar{I}(Y; X_0) = \sum_j \log |\lambda_j^+(A)|$, the lowest achievable predicted MMSE of Z_i is determined by the unstable parts of LTI plant (3.70). If W_i and V_i in (3.70) are not uncorrelated or zero-mean white Gaussian, $\sum_j \log |\lambda_j^+(A)|$ then only defines the filtering limit of all linear filters. Meanwhile, by utilizing the analytic relation between the predicted and causal MMSEs, a lower bound for the lowest achievable causal MMSE of Z_i can also be defined by $\bar{I}(Y; X_0)$.⁸

⁸ Pre- and post-multiplying (3.73) by H and $H^\top$, and using the Woodbury matrix identity, we have $HP_i^- H^\top = HP_i^- H^\top (I + HP_i^- H^\top)^{-1}$. Taking $\text{tr}(\cdot)$ on both sides of this equation, and using Lemma 3.24 in the limits of $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$, we have $\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sum_{i=0}^n \text{cmmse}(Z_i)/(n+1) = 2 \sum_k \eta_k/(1+\eta_k)$, where $\{\eta_k\}$ denotes the eigenvalues of $H_u P_u^- H_u^\top$ in (3.74). When $\{\eta_k\}$ are known, we can directly calculate the lowest achievable casual MMSE by $2 \sum_k \eta_k/(1+\eta_k)$. When $\bar{I}(Y; X_0)$ is known, we can estimate a lower bound for the lowest achievable causal MMSE by solving the minimization problem $\min \sum_k \eta_k/(1+\eta_k)$, s.t. $\sum_k \eta_k \geq 2\bar{I}(Y; X_0)$, and $\eta_k > 0, \forall k$.

3.3.3 Filtering Limits in LTV Systems

We then consider the filtering systems subject to the following LTV plant and measurement model

$$\begin{aligned} X_{i+1} &= A_i X_i + B_i W_i \\ Y_i &= Z_i + V_i = H_i X_i + V_i, \end{aligned} \tag{3.76}$$

where A_i , B_i , H_i are time-varying matrices of proper dimensions and satisfy (A2) and (A3), and process noise $W_i \sim \mathcal{N}(0, \varepsilon^2 I)$ and measurement noise $V_i \sim \mathcal{N}(0, I)$ are mutually independent. Since W_i and V_i in (3.76) are uncorrelated and zero-mean white Gaussian, we can use the LTV Kalman filter to infer the optimal MS estimates and minimum MS estimation errors of filtering problem (3.76). The following lemma shows that when the process noise W_i in (3.76) vanishes, the state prior and posterior error covariance matrices of filtering problem (3.76) are determined by the antistable dynamics.

Lemma 3.24. *For the uniformly completely reconstructible and stabilizable LTV system (3.76) subject to vanishing process noise, i.e., $\varepsilon \rightarrow 0$, the asymptotic prior error covariance of its modal-decomposed system satisfies*

$$\lim_{\varepsilon \rightarrow 0} \lim_{i' \rightarrow \infty} \mathbb{E} \left\{ \begin{bmatrix} \tilde{X}_s^-(i') \\ \tilde{X}_u^-(i') \end{bmatrix} \begin{bmatrix} \tilde{X}_s^-(i') \\ \tilde{X}_u^-(i') \end{bmatrix}^\top \right\} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{P}_u^-(i) \end{bmatrix}, \tag{3.77}$$

where $\tilde{X}_s^-(i') := X_s(i') - \hat{X}_s^-(i')$ and $\tilde{X}_u^-(i') := X_u(i') - \hat{X}_u^-(i')$ respectively denote the prior errors of stable and antistable modes, and $\bar{P}_u^-(i) := \lim_{\varepsilon \rightarrow 0} \lim_{i' \rightarrow \infty} \mathbb{E}[\tilde{X}_u^-(i') \tilde{X}_u^-(i')^\top]$ is the asymptotic prior error covariance of antistable mode that satisfies $\bar{P}_u^-(i+1) = A_u(i) \bar{P}_u^-(i) [H_u(i) \cdot \bar{P}_u^-(i) H_u^\top(i) + I]^{-1} A_u^\top(i)$, with A_u and H_u being the antistable parts of A and H . The asymptotic posterior error covariance of the modal-decomposed system satisfies

$$\lim_{\varepsilon \rightarrow 0} \lim_{i' \rightarrow \infty} \mathbb{E} \left\{ \begin{bmatrix} \tilde{X}_s(i') \\ \tilde{X}_u(i') \end{bmatrix} \begin{bmatrix} \tilde{X}_s(i') \\ \tilde{X}_u(i') \end{bmatrix}^\top \right\} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{P}_u(i) \end{bmatrix}, \tag{3.78}$$

where $\tilde{X}_s(i') := X_s(i') - \hat{X}_s(i')$ and $\tilde{X}_u(i') := X_u(i') - \hat{X}_u(i')$ are respectively the posterior errors of stable and antistable modes, and $\bar{P}_u(i) := \lim_{\varepsilon \rightarrow 0} \lim_{i' \rightarrow \infty} \mathbb{E}[\tilde{X}_u(i') \cdot \tilde{X}_u(i')^\top]$ is the asymp-

otic posterior error covariance of anti-stable mode and satisfies $\bar{P}_u(i) = \bar{P}_u^-(i) - \bar{P}_u^-(i)H_u^\top(i) \cdot [H_u(i)P_u^-(i)H_u^\top(i) + I]^{-1}H_u(i)P_u^-(i)$.

Proof. For the discrete-time filtering system (3.76) satisfying (A2) and (A3), we can decompose it into stable and antistable parts by Lemma 3.13:

$$\begin{aligned} \begin{bmatrix} X_s(i'+1) \\ X_u(i'+1) \end{bmatrix} &= \begin{bmatrix} A_s(i') & 0 \\ 0 & A_u(i') \end{bmatrix} \begin{bmatrix} X_s(i') \\ X_u(i') \end{bmatrix} + \begin{bmatrix} B_s(i') \\ B_u(i') \end{bmatrix} W(i'), \\ Y(i') &= \begin{bmatrix} H_s(i') & H_u(i') \end{bmatrix} \begin{bmatrix} X_s(i') \\ X_u(i') \end{bmatrix} + V(i'), \end{aligned} \quad (3.79)$$

where $X_s(i')$ and $X_u(i')$ respectively denote the stable and antistable modes. Since the original system (3.76) is uniformly completely stabilizable and reconstructible, the transformed system and its stable/antistable subsystems in (3.79) are also stabilizable and reconstructible. Hence, in the limit of $i' \rightarrow \infty$, the asymptotic state prior error covariance $\bar{P}_i^- := \lim_{i' \rightarrow \infty} \mathbb{E}\{\tilde{X}_s^-(i')^\top, \tilde{X}_u^-(i')^\top\}^\top [\tilde{X}_s^-(i')^\top, \tilde{X}_u^-(i')^\top]$ of filtering problem (3.79) satisfies [84, Theorem 6.45]

$$\bar{P}_{i+1}^- = \bar{A}_i \bar{P}_i^- (\bar{H}_i \bar{P}_i^- \bar{H}_i^\top + I)^{-1} \bar{A}_i^\top + \varepsilon^2 \bar{B}_i \bar{B}_i^\top, \quad (3.80)$$

where $\bar{A}_i = \text{diag}\{A_s(i), A_u(i)\}$, $\bar{B}_i = [B_s^\top(i), B_u^\top(i)]^\top$, and $\bar{H}_i = [H_s(i), H_u(i)]$. When the process noise W_i vanishes in (3.76) and (3.79), or in the limit of $\varepsilon \rightarrow 0$, (3.80) becomes a singularly perturbed equation, and we seek its solution $\bar{P}_i^-$ in the form of power series in ε [12, 103]:

$$\bar{P}_i^- = \begin{bmatrix} \varepsilon^2 \bar{P}_s^-(i) + O(\varepsilon^3) & \varepsilon^2 \bar{P}_0^-(i) + O(\varepsilon^3) \\ * & \bar{P}_u^-(i) + O(\varepsilon) \end{bmatrix}. \quad (3.81)$$

By letting

$$(\bar{H}_i \bar{P}_i^- \bar{H}_i^\top + I)^{-1} =: \begin{bmatrix} F_s(i) & F_0(i) \\ F_0(i) & F_u(i) \end{bmatrix}, \quad (3.82)$$

and substituting (3.81) and (3.82) into (3.80), we have

$$\varepsilon^2 \bar{P}_s^-(i+1) = \varepsilon^2 [A_s(i) \bar{P}_s^-(i) F_s(i) A_s^\top(i) + A_s(i) \bar{P}_0^-(i) F_0(i) A_s^\top(i) + B_s(i) B_s^\top(i)] + O(\varepsilon^3) \quad (3.83a)$$

$$\varepsilon^2 \bar{P}_0(i+1) = \varepsilon^2 [A_s(i) \bar{P}_s^-(i) F_0(i) A_u^\top(i) + A_s(i) P_0^-(i) F_u(i) A_u^\top(i) + B_s(i) B_u^\top(i)] + O(\varepsilon^3) \quad (3.83b)$$

$$\bar{P}_u^-(i+1) = \varepsilon^2 [A_u(i) \bar{P}_0^-(i) F_0(i) A_u^\top(i) + B_u(i) B_u^\top(i)] + A_u(i) \bar{P}_u^-(i) F_u(i) A_u^\top(i) + O(\varepsilon^3). \quad (3.83c)$$

After setting $\varepsilon = 0$ in (3.81) and (3.83), the only non-zero element in $\bar{P}_i^-$ is $\bar{P}_u^-(i)$. Since $F_u(i) = [H_u(i) \bar{P}_u^-(i) H_u^\top(i) + I]^{-1}$ in (3.83c), we can then imply the asymptotic prior error covariance in (3.77). Substituting the asymptotic prior error covariance (3.77) into the correction step of LTV Kalman filter (3.86), we obtain the asymptotic posterior error covariance in (3.78). This completes the proof. $\square$

Lemma 3.24 also holds for the discrete-time LTI filtering systems and has been previously invoked in the proofs of **Propositions 3.10**, **3.15** and **3.23**. By applying **Theorem 3.22** and **Lemma 3.24** to the LTV filtering problem and measurement channel (3.76), we then have the following result on $\bar{I}(Y; X_0)$.

Proposition 3.25. *For the discrete-time filtering systems depicted by **Figure 3.3** and uniformly completely stabilizable and reconstructible LTV plant (3.76) under vanishing process noise, the total information rate $\bar{I}(Y; X_0)$ in (3.76) satisfies*

$$\text{FLB}_{\text{LTV}} \leq \bar{I}(Y; X_0) \leq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \frac{\text{pmmse}(Z_i)}{2(n+1)}, \quad (3.84)$$

where $\bar{I}(Y; X_0) = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det[\Phi_{A_u}(n+1, 0)] = \mathcal{B}$, and the filtering lower bound is $\text{FLB}_{\text{LTV}} = \max\{\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} [2(n+1)]^{-1} \sum_{i=0}^n \text{cmmse}(Z_i), \sum_{j=1}^l m_j \log \kappa_j\}$.

Proof. Since **Proposition 3.25** and this proof can be seen as the filtering and non-feedback counterpart of **Proposition 3.15** and its proof, in the following, we only show the key steps while skip the overlapped details. As the process and measurement noises in (3.76) are uncorrelated and zero-mean white Gaussian, we can use the LTV Kalman filter to calculate the optimal estimates of Z_i , $\hat{Z}_i^- := \mathbb{E}[Z_i | Y_0^{i-1}] = \mathbb{E}[H_i X_i | Y_0^{i-1}]$ and $\hat{Z}_i := \mathbb{E}[Z_i | Y_0^i] = \mathbb{E}[H_i X_i | Y_0^i]$, and the predicted/causal MMSEs of Z_i in (3.84). Let the state prior estimate be $\hat{X}_i^- := \mathbb{E}[X_i | Y_0^{i-1}]$ and the state posterior estimate be $\hat{X}_i := \mathbb{E}[X_i | Y_0^i]$. We can then calculate $\hat{Z}_i^-$, $\hat{X}_i^-$, $\text{pmmse}(Z_i)$ in (3.84), and the state prior error covariance $P_i^- := \mathbb{E}[(X_i - \hat{X}_i^-)(X_i - \hat{X}_i^-)^\top]$ from the prediction step of Kalman filter

$$\begin{aligned}
\hat{X}_{i+1}^- &= A_i \hat{X}_i, \\
P_{i+1}^- &= A_i P_i^- (H_i P_i^- H_i^\top + I)^{-1} A_i^\top + \varepsilon^2 B_i B_i^\top.
\end{aligned} \tag{3.85}$$

Meanwhile, $\hat{Z}_i$, $\hat{X}_i$, $\text{cmmse}(Z_i)$ in (3.84), and state posterior error covariance $P_i := \mathbb{E}[(X_i - \hat{X}_i)(X_i - \hat{X}_i)^\top]$ can be calculated from the correction step of Kalman filter

$$\begin{aligned}
\hat{X}_i &= \hat{X}_i^- + K_i (Y_i - H_i \hat{X}_i^-), \\
P_i &= P_i^- - P_i^- H_i^\top (H_i P_i^- H_i^\top + I)^{-1} H_i P_i^-,
\end{aligned} \tag{3.86}$$

where $K_i := P_i^- H_i^\top (H_i P_i^- H_i^\top + I)^{-1}$ is the Kalman gain.

With the above estimates, we now prove the results in Proposition 3.25. The MMSE-based sandwich bounds in (3.84) can be derived by applying Theorem 3.22 to the measurement channel (3.76). Meanwhile, once we show that $\bar{I}(Y; X_0) = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det[\Phi_{A_u}(n+1, 0)]$, we can tell that $\sum_{j=1}^l m_j \log \kappa_j$ is also a lower bound of $\bar{I}(Y; X_0)$ by Lemma 3.14. Hence, to prove Proposition 3.25, it remains to verify that $\bar{I}(Y; X_0) = \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det[\Phi_{A_u}(n+1, 0)]$, which is equivalent to showing that i) $\bar{I}(Y; X_0) \geq \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det[\Phi_{A_u}(n+1, 0)]$, and ii) $\bar{I}(Y; X_0) \leq \overline{\lim}_{n \rightarrow \infty} (n+1)^{-1} \log \det[\Phi_{A_u}(n+1, 0)]$. Statement i) can be proved by using the finite-power constraint (A4) and applying (the proof of) Corollary 3.16 to (3.76). Taking $\log \det(\cdot)$ on both sides of (3.85), summing up the result from $i = 0$ to n , and applying Lemma 3.24 to P_i^- , as $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$, we have

$$\overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^n \log \det \Psi(i) = 2 \overline{\lim}_{n \rightarrow \infty} \log \det \Phi_{A_u}(n+1, 0) + \log \det \bar{P}_u^-(0) - \log \det \bar{P}_u^-(n+1), \tag{3.87}$$

where $\Psi(i) := H_u(i) \bar{P}_u^-(i) H_u^\top(i) + I$; the asymptotic prior error covariance of antistable mode $\bar{P}_u^-(i)$ is defined in (3.77), and $H_u(i)$ denotes the antistable part of H_i in (3.76). Suppose Y_i and $V_i \in \mathbb{R}^s$ in (3.76). By using (3.64) and (3.87), and following the similar derivations of (3.44) and (3.75), we have

$$\begin{aligned}
I(Y_0^n; X_0) &= h(Y_0^n) - h(V_0^n) \\
&= h(H(X_0^n - \hat{X}_{0:n}^-) + V_0^n) - h(V_0^n) \\
&\leq \frac{1}{2} \sum_{i=0}^n \log[(2\pi e)^s |H_i P_i^- H_i^\top + I|] - \frac{n}{2} \log(2\pi e)^s
\end{aligned} \tag{3.88}$$

$$\stackrel{\varepsilon \rightarrow 0}{\underset{n \rightarrow \infty}{\equiv}} \log \det \Phi_{A_u}(n+1, 0) + \frac{1}{2} \log \det \bar{P}_u^-(0) - \frac{1}{2} \log \det \bar{P}_u^-(n+1),$$

where the last equality follows [Lemma 3.24](#) and (3.87). Statement ii) and hence [Proposition 3.25](#) are proved by dividing (3.88) by $n+1$ and taking the limit superior (or limit in the stationary case) as $n \rightarrow \infty$. This completes the proof. $\square$

[Proposition 3.25](#) reveals that when the plant $\mathcal{P}$ and measurement channel in [Figure 3.3](#) are governed by (3.76), the total information rate $\bar{I}(Y; X_0)$ serves as a lower bound for the lowest achievable predicted MMSE of Z_i despite the design of filtering process $\mathcal{F}$. Specifically, with process noise W_i or $\varepsilon^2 > 0$ in (3.76), no matter how well the filtering process $\mathcal{F}$ is designed, the actual average predicted MMSE of Z_i must be larger than $2\bar{I}(Y; X_0)$, which is determined by the antistable dynamics of (3.76). When the process and measurement noises are not uncorrelated or zero-mean white Gaussian, $\bar{I}(Y; X_0)$ only captures the lower bounds of linear filters. Similar to the discussion⁸ following [Proposition 3.23](#), with the analytic relation between predicted and causal MMSEs, $\bar{I}(Y; X_0)$ can also be used to define a lower bound for the lowest achievable causal MMSE of Z_i in (3.84).

3.3.4 Filtering Limits in Nonlinear Systems

Lastly, we consider the nonlinear filtering system subject to nonlinear plant (3.59) and measurement model (3.60). According to [Theorem 3.22](#), in the presence of process noise or $\varepsilon^2 > 0$ in (3.59), the information rate $\bar{I}(Y; W, X_0)$, satisfying inequalities (3.65), is a lower bound for the time-averaged or stationary predicted MMSE of Z_i , which is sensitive to the noise level ε . When the process noise in (3.59) vanishes or $\varepsilon \rightarrow 0$, the information rate $\bar{I}(Y; X_0) = \lim_{\varepsilon \rightarrow 0} \bar{I}(Y; W, X_0)$, satisfying inequalities (3.66), defines a lower bound for the lowest achievable predicted MMSE regardless of the design of filtering process $\mathcal{F}$.

Hence, it only remains to solve how to calculate the MMSE-based bounds or estimate the information rates, $\bar{I}(Y; W, X_0)$ and $\bar{I}(Y; X_0)$, in [Theorem 3.22](#), especially, when the noise-free output or channel input Z_i is unobservable, compared with the observable control input signal considered in [Section 3.2.4](#). When Y_i and Z_i in (3.59) and (3.60) are Gaussian or normal, using the theorem

on normal correlation [60, Chapter 13], we can compute the MMSEs of Z_i by

$$\mathbb{E}[(Z_i - \hat{Z}_i)^\top (Z_i - \hat{Z}_i)] = \mathbb{E}[(Z_i - \hat{Z}_i)^\top (Z_i - \hat{Z}_i) | Y_0^i] = \mathbb{E}[Z_i^\top Z_i | Y_0^i] - \mathbb{E}[Z_i | Y_0^i]^\top \mathbb{E}[Z_i | Y_0^i], \quad (3.89)$$

which also holds when we replace the posterior estimate $\hat{Z}_i$ and Y_0^i by the prior estimate $\hat{Z}_i^-$ and Y_0^{i-1} . To calculate the expectations in (3.89), we need to figure out the prior density $\pi_{i|i-1}(x_i) := \mathbb{P}(X_i = x_i | Y_0^{i-1} = y_0^{i-1})$ and posterior density $\pi_i(x_i) := \mathbb{P}(X_i = x_i | Y_0^i = y_0^i)$. Since X_i in (3.59) is Markovian, we can calculate the prior density by the time update step

$$\pi_{i|i-1}(x_i) = \int_{\mathcal{X}_{i-1}} p(x_i | x_{i-1}) \pi_{i-1}(x_{i-1}) dx_{i-1}, \quad (3.90)$$

where $p(x_i | x_{i-1}) := \mathbb{P}(X_i = x_i | X_{i-1} = x_{i-1})$ is the state transition probability of (3.59), and $\mathcal{X}_{i-1}$ denotes the state space of X_{i-1} . With the output measurement y_{i+1} , we can calculate the posterior density by the measurement update step

$$\pi_i(x_i) = \frac{\pi_{i|i-1}(x_i) p(y_i | x_i)}{\int_{\mathcal{X}_i} \pi_{i|i-1}(x_i) p(y_i | x_i) dx_i}, \quad (3.91)$$

where $p(y_i | x_i)$ denotes the conditional distribution of observation y_i given state x_i . With the prior and posterior densities in (3.90) and (3.91), we can evaluate $\text{pmmse}(Z_i)$ and $\text{cmmse}(Z_i)$, and hence sandwich estimate $\bar{I}(Y; W, X_0)$ and $\bar{I}(Y; X_0)$ by (3.89) and Theorem 3.22.

Similar to the challenges of nonlinear control scenario in Section 3.2.4, accurate computation of densities in (3.90) and (3.91) is slow and computationally intensive. By adopting some proper assumptions, e.g., Gaussian distribution assumption, and replacing the optimal estimator with the sub-optimal filters, e.g., extended Kalman filter, we can simplify and expedite the process of estimating $\pi_{i|i-1}(x_i)$, $\pi_i(x_i)$, and the related MMSEs. Due to the variety and complexity of nonlinear dynamical models, information rates, $\bar{I}(Y; W, X_0)$ and $\bar{I}(Y; X_0)$, are not directly tied to (the antistable dynamics of) nonlinear plant (3.59). However, the above analysis and Theorem 3.22 indicate that even for the complex nonlinear filtering systems, $\bar{I}(Y; W, X_0)$ or $\bar{I}(Y; X_0)$ is still a lower bound for the (lowest achievable) predicted MMSE of Z_i . Similar to the linear filtering cases, when the relationship between predicted and causal MMSEs is explicit, it is possible to define a lower bound for the (lowest achievable) causal MMSE of Z_i by using $\bar{I}(Y; W, X_0)$ or $\bar{I}(Y; X_0)$.

3.4 Conclusion and Discussion

In this chapter, we studied total information rate as a control and filtering trade-off metric by using the I-MMSE relationships. We first derived and extended the I-MMSE relationships to the discrete-time additive white Gaussian channels with and without feedback. For the control scenario, we first showed that total information rate is sandwiched between the average causal and predicted MMSEs of control input, and serves a similar role as the Bode-type integrals and average risk-sensitive cost function. Specifically, when the plant and controller are respectively LTI, LTV, and nonlinear, based on the optimal filtering theory, we calculated and estimated the information rates and their MMSE-based bounds. For the filtering problem, we showed that total information rate is bounded between the stationary causal and predicted MMSEs of noise-free output, and defines a lower bound for the (lowest achievable) predicted (and possibly causal) MMSEs of noise-free output. When the plant generating output is respectively LTI, LTV, and nonlinear, we provided an optimal-filtering-based approach to calculate and estimate the information rates and their MMSE-based bounds. Some promising buildups of this chapter include extending the I-MMSE relationships and trade-off analysis to the control and filtering systems subject to colored and non-Gaussian noises.

Chapter 4

Characterization of Control and Filtering Limitations in Wiener Spaces

To dismantle the challenges raised in [Section 1.3](#), in this chapter, we provide a direct scheme, based on the I-MMSE relationships in abstract Wiener space, for calculating the total information (rate) and characterizing the fundamental limitations of (spatially and temporally) infinite-dimensional control and filtering systems under arbitrary Gaussian noise. After relating total information (rate) to the MS estimation error of channel input under the noise-induced RKHS norm, we first propose an estimation-based method to calculate the total information (rates) in the abstract or infinite-dimensional control and filtering systems. With several examples and applications in the finite-dimensional systems, we then show that the proposed calculation scheme not only recovers some acknowledged trade-off results in the control and filtering systems subject to AWGN or standard Brownian motion noise, but provides a direct method to calculate and analyze the total information (rates) of control and filtering systems subject to colored Gaussian noise, such as Ornstein-Uhlenbeck process and Gauss-Markov process with factorizable covariance.

The remainder of this chapter is organized as follows. [Section 4.1](#) introduces the abstract Gaussian channel and I-MMSE relationship. [Sections 4.2](#) and [4.3](#) characterize the control and filtering limitations, respectively, and [Section 4.4](#) draws the conclusion.

Notation: We use bold letters, e.g., $\mathbf{m}$ and $\mathbf{w}$, to denote the random processes and vectors of indefinite (finite or infinite) dimensions. For a normed space $\mathcal{S}$, we use $\mathcal{S}^*$ to denote its dual space,

the collection of all bounded linear functionals on $\mathcal{S}$. $\mathcal{L}^2([0, T]; \mathbb{R}^p)$ or $\mathcal{L}^2([0, T])$ denotes the Hilbert space of all square-integrable functions defined over $[0, T]$ and taking values from $\mathbb{R}^p$. $\mathbb{C}^n([0, T]; \mathbb{R}^{\mathbb{N}})$ or $\mathbb{C}^n([0, T])$ is the space of n times continuously differentiable functions defined over $[0, T]$ and taking values from $\mathbb{R}^p$. $\mathbb{C}_a([0, T])$ denotes the space of absolutely continuous function on $[0, T]$. $\|\cdot\|_{\mathcal{H}}$ stands for the norm associated with RKHS $\mathcal{H}$. $\mu_1 \ll \mu_2$ indicates that measure μ_1 is absolutely continuous with measure μ_2 . $\sigma\{\cdot\}$ represents the Borel σ -algebra or -field of random vectors or processes.

4.1 Preliminaries and Problem Formulation

In this section, we formulate the additive Gaussian channel model in abstract Wiener space, and the I-MMSE relationship of abstract Gaussian channel is then introduced.

4.1.1 Abstract Gaussian Channel

We first give an intuitive view on the additive Gaussian channel. Consider the following additive Gaussian channel with or without feedback,

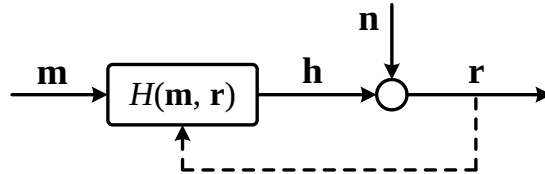


Figure 4.1: Abstract Gaussian channel with or without feedback.

where random vector (or process) $\mathbf{m}$ denotes the transmitted message; $H(\cdot, \cdot)$ is the channel input function or coding function of message $\mathbf{m}$; $\mathbf{h}$ stands for the channel input signal; $\mathbf{n}$ represents the channel noise, and $\mathbf{r}$ denotes the channel output signal. All signals in Figure 4.1 are allowed to be (spatially) infinite-dimensional.

Analytically, we formulate the abstract additive Gaussian channel with or without feedback as

$$\begin{cases} \mathbf{r} = \mathbf{h} + \mathbf{n} \\ \mathbf{h} = H(\mathbf{m}, \mathbf{r}) \quad \text{or} \quad H(\mathbf{m}), \end{cases} \quad (4.1)$$

where $H(\mathbf{m}, \mathbf{r})$ and $H(\mathbf{m})$ are respectively the input functions with and without feedback. To provide a concrete example, we can consider (4.1) as $r_t = h_t + n_t$, where the time variable $t \in [0, T]$; n_t with $t \in [0, T]$ is a standard Brownian motion representing noise; h_t depends on the transmitted message m_t for $t \in [0, T]$ and past channel output r_τ with $\tau \in [0, t]$ via feedback, i.e., $h_t = H(m_0^T, r_0^t)$. Meanwhile, the following abstract settings are imposed on (4.1):

S1) Gaussian channel (4.1) is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in \mathcal{I} \text{ or } [0, T]\}, \mathbb{P})$ with sample space Ω , σ -algebra $\mathcal{F}$, complete and right-continuous filtration $\{\mathcal{F}_t, t \in \mathcal{I}\}$, and probability measure $\mathbb{P}$.

S2) Transmitted message $\mathbf{m}$ is a random vector or process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in a Polish space¹ M , and inducing an image or push-forward measure $\mu_{\mathbf{m}}$.

S3) Channel noise $\mathbf{n} : \Omega \rightarrow \mathcal{B}$, is a centered non-degenerate Gaussian random vector, independent of $\mathbf{m}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in a separable Banach space $\mathcal{B}$, and inducing an image measure $\mu_{\mathbf{n}}$ and a separable associated reproducing kernel Hilbert space (RKHS) $\mathcal{H}$.² The non-degeneracy of $\mathbf{n}$ implies that $\mathcal{H}$ is densely embedded in $\mathcal{B}$, i.e., $(\mathcal{B}, \mathcal{H}, \mu_{\mathbf{n}})$ is an abstract Wiener space and $\mathcal{B}^* \subset \mathcal{H} \subset \mathcal{B}$.

S4) We define a time structure on $(\mathcal{B}, \mathcal{H}, \mu_{\mathbf{n}})$ in the form of a continuous and strictly increasing resolution of the identity $\{\pi_t, t \in \mathcal{I} \text{ or } [0, T]\}$ of $\mathcal{H}$, such that $\pi_t \mathcal{B}^* \subset \mathcal{B}^*$ and $\delta_{\mathbf{n}}(\pi_t l) := {}_{\mathcal{B}}\langle \mathbf{n}, \pi_t l \rangle_{\mathcal{B}^*}$ is $\mathcal{F}_t$ -adapted for all $t \in \mathcal{I}$ and $l \in \mathcal{B}^*$.³

S5) Channel input function $H(\cdot, \cdot) : M \times \mathcal{B} \rightarrow \mathcal{H}$ is a jointly measurable mapping and π -nonanticipative in its second variable. A pair of $\mathcal{F}$ -adapted random vectors $\mathbf{h} \in \mathcal{L}^2(\mathbb{P}; \mathcal{H})$ and $\mathcal{B}$ -valued $\mathbf{r}$ satisfy (4.1) almost surely with respect to probability measure $\mathbb{P}$ (or $\mathbb{P}$ -a.s.).

More detailed introduction on the definitions and setups of abstract Wiener space, RKHS, and abstract Gaussian channel can be found in [59, 62, 63, 119] and references therein.

Remark 4.1. The time parameter set $\mathcal{I}$ in S1) and S4) can be quite general [57]. For example, $\mathcal{I}$

¹ A complete and separable metric space.

² $\mathcal{H}$ is also called the Cameron-Martin space associated with $(\mathcal{B}, \mu_{\mathbf{n}})$.

³ In this chapter, we also use π and $\mathcal{F}$ to denote $\{\pi_t, t \in \mathcal{I} \text{ or } [0, T]\}$ and $\{\mathcal{F}_t, t \in \mathcal{I} \text{ or } [0, T]\}$. Given a generic sub-filtration $\{\mathcal{G}_t, t \in \mathcal{I}\}$ of $\{\mathcal{F}_t, t \in \mathcal{I}\}$, a $\mathcal{B}$ -valued random variable or vector $\mathbf{v}$ is $(\pi, \mathcal{G})$ -adapted if ${}_{\mathcal{B}}\langle \mathbf{v}, \pi_t l \rangle_{\mathcal{B}^*}$ is $\mathcal{G}_t$ -measurable $\forall t \in \mathcal{I}$ and $l \in \mathcal{B}^*$. A mapping $f : \mathcal{B} \rightarrow \mathcal{B}$ is π -nonanticipative if $f(\mathbf{v})$ is $(\pi, \mathcal{F}^{\mathbf{v}})$ -adapted for any $\mathcal{B}$ -valued $\mathbf{v}$, where $\mathcal{F}^{\mathbf{v}} := \sigma\{{}_{\mathcal{B}}\langle \mathbf{v}, \pi_t l \rangle_{\mathcal{B}^*}, \forall t \in \mathcal{I}, l \in \mathcal{B}^*\}$. A $\mathcal{B}$ -valued random vector $\mathbf{v}$ is a $\mathcal{G}$ -abstract Wiener process if for all $l \in \mathcal{B}^*$, $\delta_{\mathbf{v}}(\pi \cdot l) := {}_{\mathcal{B}}\langle \mathbf{v}, \pi \cdot l \rangle_{\mathcal{B}^*}$ is a zero-mean, continuous and Gaussian $\mathcal{G}$ -martingale with quadratic variation $\langle \delta_{\mathbf{v}}(\pi \cdot l) \rangle_t = \|\pi_t l\|_{\mathcal{H}}^2$. A vanilla example or candidate of $\mathbf{v}$ and l is the noise $\mathbf{n}$ and input $\mathbf{h}$ in (4.1).

can be a subset of the real line or a finite set of points on $[0, T]$. For easiness, in this chapter, we only consider $\mathcal{I}$ as an interval of the real line (or continuous-time parameter set), e.g., $[0, 1]$, $[0, T]$ or $[t_1, t_2]$.

With the abstract Gaussian channel defined in Figure 4.1 and (4.1), we now introduce total information or the mutual information between $\mathbf{m}$ and $\mathbf{r}$, which measures the dependency between message and channel output and will be later used to characterize the fundamental control and filtering limitations.

Definition 4.2 (Total Information). *The total information of the abstract Gaussian channel (4.1) is defined by*

$$I(\mathbf{m}; \mathbf{r}) = \mathbb{E} \left[\log \frac{d\mu_{\mathbf{m} \times \mathbf{r}}}{d(\mu_{\mathbf{m}} \times \mu_{\mathbf{r}})} \right],$$

if $\mu_{\mathbf{m} \times \mathbf{r}} \ll \mu_{\mathbf{m}} \times \mu_{\mathbf{r}}$, $\mathbb{P}$ -a.s.; otherwise, $I(\mathbf{m}; \mathbf{r}) = \infty$.

Compared with the input-output mutual information $I(\mathbf{h}; \mathbf{r})$, which does not preserve the causal relationship between $\mathbf{h}$ and $\mathbf{r}$ in the feedback scenario, the total information $I(\mathbf{m}; \mathbf{r})$ is well-defined for both the feedback and non-feedback channels defined in (4.1). Meanwhile, when (4.1) represents a continuous-time additive white Gaussian channel of finite dimensions, we further have $I(\mathbf{m}; \mathbf{r}) = I(\mathbf{h} \rightarrow \mathbf{r}) = I(\mathbf{h}; \mathbf{r})$, where $I(\mathbf{h} \rightarrow \mathbf{r})$ denotes the directed information from input $\mathbf{h}$ to output $\mathbf{r}$, and the second equality only holds when the channel is without feedback [47].

4.1.2 Abstract I-MMSE Relationship

The following lemma, which is proved by using the generalized Girsanov theorem for abstract Wiener space and process in [63], establishes an equality relationship between the total (or mutual) information and causal estimation error in abstract Gaussian channel (4.1).

Lemma 4.3. *For the abstract Gaussian channel with or without feedback depicted in (4.1), and under the settings S1) to S5), the total information $I(\mathbf{m}; \mathbf{r})$ satisfies*

$$I(\mathbf{m}; \mathbf{r}) = \frac{1}{2} \mathbb{E} \|\mathbf{h} - \hat{\mathbf{h}}\|_{\mathcal{H}}^2, \quad (4.2)$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the RKHS norm associated with noise $\mathbf{n}$, and $\hat{\mathbf{h}} := \mathbb{E}[\mathbf{h} | \mathcal{F}^{\mathbf{r}}]$ is the dual predictable

projection or causal estimate of $\mathbf{h}$ w.r.t. the filtration $\mathcal{F}^\tau$.⁴

The celebrated Duncan's theorem in [53, 54], i.e., the I-MMSE relationship for continuous-time additive white Gaussian channel of finite dimensions, can be recovered from Lemma 4.3, if we let the channel noise $\mathbf{n} \in \mathbb{C}^0([0, T]; \mathbb{R}^N)$ in (4.1) be a finite-dimensional standard Brownian motion B_t taking values in the Banach space of real-valued and continuous functions on $[0, T]$. Total information (rate) has been utilized to capture the fundamental limitations of various finite-dimensional control and filtering systems subject to AWGN, e.g., [7, 16, 27, 56]. In the following, by using the abstract I-MMSE relationship in Lemma 4.3, we are to calculate this important trade-off metric for also the infinite-dimensional control and filtering systems subject to arbitrary Gaussian noise.

Remark 4.4. Lemma 1 and its proof in [63] assume that the time parameter set $\mathcal{I}$ is continuous, i.e., an interval of the real line. To derive the abstract I-MMSE relationship for discrete-time Gaussian channel, one can extend the discrete-time I-MMSE relationship and its proof in [56] to the abstract Wiener space and Gaussian channel subject to the discrete-time parameter set $\mathcal{I}$, e.g., a finite set of points on $[0, T]$.

4.2 Control Trade-offs in Wiener Spaces

To characterize total information (rate) as a control trade-off metric, consider the general feedback control system depicted in the following diagram:

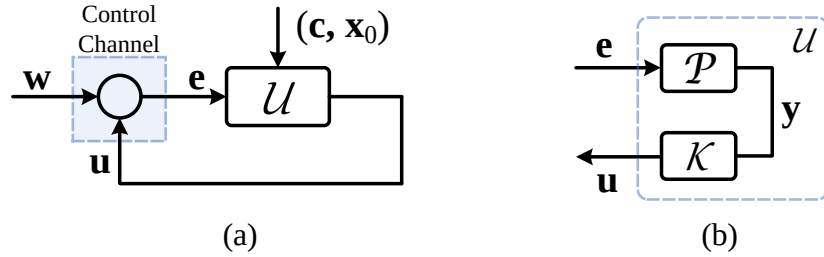


Figure 4.2: Feedback control system.

where $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{e}$ are the control signal, external disturbance and error signal of the control system, respectively, and $\mathcal{U}$ denotes the control process or coding function of ‘message’ $(\mathbf{c}, \mathbf{x}_0)$, which, as Figure 4.2 (b) shows, can be further decomposed into a plant $\mathcal{P}$ and a controller $\mathcal{K}$ with $\mathbf{x}_0$ being

⁴ The filtration $\mathcal{F}^\tau$ satisfies $\mathcal{F}^\tau := \sigma\{\langle \mathbf{r}, \pi_\tau l \rangle_{\mathcal{B}^*}, l \in \mathcal{B}^*, \tau \in [0, t]\} = \sigma\{\langle \mathbf{h}, \pi_\tau l \rangle_{\mathcal{H}} + \delta_{\mathbf{n}}(\pi_\tau l), l \in \mathcal{B}^*, \tau \in [0, t]\}$.

the initial state of $\mathcal{P}$ and $\mathbf{c}$ being the command signal or control noise of $\mathcal{K}$. When all signals in Figure 4.2 are of infinite dimensions, we can model this feedback control system into an abstract Gaussian channel with feedback as follows:

$$\begin{cases} \mathbf{e} = \mathbf{u} + \mathbf{w} \\ \mathbf{u} = U(\mathbf{e}, \mathbf{c}, \mathbf{x}_0), \end{cases} \quad (4.3)$$

where $U(\cdot)$ denotes the input function or realization of control process $\mathcal{U}$, and signals and function in (4.3) fulfill the same settings S1) to S5) for (4.1). Typically, the plant model $\mathcal{P}$ can be individually described by the following equations

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) + b(\mathbf{x})\mathbf{e} \\ \mathbf{y} &= h(\mathbf{x}), \end{aligned} \quad (4.4)$$

where $\mathbf{x}$ denotes the internal states of $\mathcal{P}$, and $f(\cdot)$, $b(\cdot)$ and $h(\cdot)$ are the passive dynamics, control and output functions of $\mathcal{P}$. Compared with the previous analyses of finite-dimensional control systems under AWGN, the infinite-dimensional control systems subject to arbitrary Gaussian noise modeled in (4.3) and (4.4) are significantly more general.

By applying Lemma 4.3 to the abstract control system/channel depicted in (4.3), we have the following result on the total information $I(\mathbf{e}; \mathbf{c}, \mathbf{x}_0)$ of feedback control systems.

Theorem 4.5. *For the abstract control systems depicted in Figure 4.2 and (4.3), and under the settings S1) to S5), the total information $I(\mathbf{e}; \mathbf{c}, \mathbf{x}_0)$ of control channel (4.3) satisfies*

$$I(\mathbf{e}; \mathbf{c}, \mathbf{x}_0) = \frac{1}{2} \mathbb{E} \|\mathbf{u} - \hat{\mathbf{u}}\|_{\mathcal{H}_{\mathbf{w}}}^2, \quad (4.5)$$

where $\|\cdot\|_{\mathcal{H}_{\mathbf{w}}}$ is the RKHS norm associated with external disturbance $\mathbf{w}$, and $\hat{\mathbf{u}} := \mathbb{E}[\mathbf{u}|\mathcal{F}^{\mathbf{e}}]$ stands for the dual predictable projection or causal estimate of $\mathbf{u}$ w.r.t. the filtration $\mathcal{F}^{\mathbf{e}}$.

The proof of Theorem 4.5 is direct. When the time parameter set $\mathcal{I}$ is $[0, T]$, we define total information rate as the limit of time-averaged mutual information, i.e., $\bar{I}(\mathbf{e}; \mathbf{c}, \mathbf{x}_0) := \lim_{T \rightarrow \infty} (1/T) I(\mathbf{e}; \mathbf{c}, \mathbf{x}_0)$,

if the limit exists. Hence, by (4.5), we have

$$\bar{I}(\mathbf{e}; \mathbf{c}, \mathbf{x}_0) = \lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} \|\mathbf{u} - \hat{\mathbf{u}}\|_{\mathcal{H}_w}^2. \quad (4.6)$$

The calculation of projection $\hat{\mathbf{u}}$ or estimation error $\mathbf{u} - \hat{\mathbf{u}}$ in (4.5) and (4.6) demands the usage of infinite-dimensional optimal filtering [120–123], which will not be covered in this chapter. In the following, we are to show how total information (rate) and Theorem 4.5 can be leveraged to characterize the fundamental limitations of several common finite-dimensional control systems by i) recovering some well-established control trade-offs and results of finite-dimensional systems, e.g., Bode’s sensitivity integral [2], from Theorem 4.5, and ii) calculating and analyzing total information (rate) for some important but rarely investigated finite-dimensional control systems, such as the systems subject to colored Gaussian noise. In the first example, we consider a classical scenario when the external disturbance or control channel noise is a standard Brownian motion. By using Theorem 4.5, we show that total information (rate) equals the (average) minimum mean-square estimation error of control input and serves as an information-theoretic interpretation to the Bode-type integrals.

Example 4.6. *Consider the following finite-dimensional additive white Gaussian control channel or system:*

$$e_t = \int_0^t u_\tau(e_0^\tau, c_0^T, x_0) d\tau + w_t, \quad (4.7)$$

where the time parameter $t \in [0, T]$; the external disturbance or control channel noise w_t is a p -dimensional standard Brownian motion B_t ; (differential) control input function satisfies $u_\tau \in \mathcal{L}^2([0, t]; \mathbb{R}^p)$ or $\mathbb{E}[\int_0^t u_\tau^\top u_\tau d\tau] < \infty$, and e_t stands for the error signal or control channel output.⁵ Since the (covariance) kernel function of w_t is $K_1(s, t) = \text{Cov}(w_s, w_t) = \min(s, t)$, the RKHS associated with kernel $K_1(s, t)$ or the Cameron-Martin space associated with $(\mathbb{C}^0([0, T]; \mathbb{R}^p), \mu_w)$ is a Hilbert sub-subspace of $\mathbb{C}^0([0, T])$ that consists of absolutely continuous functions with square-integrable time derivatives [119], i.e.,

$$\mathcal{H}_1 := \{\varphi(\cdot) \mid \varphi(0) = 0, \varphi \in \mathbb{C}_a([0, T]), \dot{\varphi} \in \mathcal{L}^2([0, T])\},$$

⁵ To model (4.7) in the classical Wiener space or $\mathbb{C}^0([0, T]; \mathbb{R}^p)$, we let $\mathbf{e} = e_t$, $\mathbf{u} = \int_0^t u_\tau(e_0^\tau, c_0^T, x_0) d\tau$, and $\mathbf{w} = w_t$ in (4.3). Meanwhile, the control channel (4.7) can be equivalently expressed as $de_t = u_t(e_0^t, c_0^T, x_0) dt + dw_t$, where dw_t/dt represents a p -dimensional white Gaussian noise.

which belongs to the class of Sobolev spaces. The associated norm (or inner product) of $\mathcal{H}_1$ is

$$\|\varphi\|_{\mathcal{H}_1}^2 := \int_0^T \dot{\varphi}(t)^\top \dot{\varphi}(t) dt. \quad (4.8)$$

The triplet $(\mathbb{C}^0([0, T]), \mathcal{H}_1, \mu_w)$ is also known as the classical Wiener space. Hence, when the control systems and channel (4.7) are subject to standard Brownian motion noise, by Theorem 4.5 and (4.8), the total information in (4.7) satisfies

$$I(e_0^T; c_0^T, x_0) = \frac{1}{2} \mathbb{E} \left[\int_0^T (u_t - \hat{u}_t)^\top (u_t - \hat{u}_t) dt \right], \quad (4.9)$$

where $\hat{u}_t := \mathbb{E}[u_t | \mathcal{F}_t^e] = \mathbb{E}[u_t | e_0^t]$ can be calculated from (4.7) by the optimal filtering techniques, such as Kalman-Bucy filter and Stratonovich-Kushner equation. Identity (4.9) is a pivotal relation for investigating total information (rate) as a control trade-off metric in [16] and was previously derived by using Duncan's theorem. Depending on the stationarity of signals and particular forms of plant (4.4) and control mapping (4.7), total information (rate) can be used to characterize the fundamental limitations of different control systems. More detailed explanations and instances about $I(e_0^T; c_0^T, x_0)$ and $\bar{I}(e; c, x_0)$ under white Gaussian or standard Brownian motion noise can be found in Appendix D or Chapter 2. $\square$

By implementing Theorem 4.5 in the finite-dimensional control systems under standard Brownian motion noise (4.7), Example 4.6 along with Appendix D provides a new paradigm, based on the RKHS norm of input estimation error, to calculate and investigate total information (rate) as a control trade-off metric. Compared with the preceding works using Duncan's theorem and finite-dimensional I-MMSE relationships [16, 53, 86], a major improvement of the new paradigm and Theorem 4.5 is that they can also be directly applied to the control systems and channels subject to more general types of Gaussian noise, e.g., Ornstein-Uhlenbeck process, Gauss-Markov process, and other colored Gaussian processes.

To compute the total or mutual information (rate) of colored Gaussian channels by using Duncan's theorem, an alternative in the past is to preprocess the signals and whiten the colored noise by some techniques, such as orthogonal transformation [79] and canonical representation [58, 124], which represent the channel noise by Wiener process, but obscure the (causal) relationships of

signals in the original channels. Now, with the aid of [Lemma 4.3](#) and [Theorem 4.5](#), we can directly calculate the total information (rate) by expanding the associated RKHS norm as in (4.8) and (4.9). In the following, more examples of control systems and channels under colored Gaussian noise are provided, in which total information (rate) cannot be directly calculated by using Duncan's theorem or conventional I-MMSE relationships.

Example 4.7. *Consider the same additive Gaussian channel model in (4.7), except that the control channel noise w_t is now a p -dimensional Ornstein-Uhlenbeck or O-U process satisfying*

$$dw_t = -w_t dt + dB_t, \quad (4.10)$$

where B_t stands for a p -dimensional standard Brownian motion. The (covariance) kernel of w_t in (4.10) is $K_2(s, t) = \text{Cov}(w_s, w_t) = (1/2)\exp(-|t - s|)$. Hence, the corresponding RKHS $\mathcal{H}_2$ is a Hilbert sub-space of $\mathbb{C}^0([0, T])$ consisting of absolutely continuous and square-integrable functions with square-integrable time derivatives [119], i.e.,

$$\mathcal{H}_2 := \{\varphi(\cdot) \mid \varphi \in \mathbb{C}_a([0, T]), \varphi \text{ and } \dot{\varphi} \in \mathcal{L}^2([0, T])\},$$

which belongs to the class of Sobolev spaces. The associated norm (or inner product) of $\mathcal{H}_2$ is

$$\|\varphi\|_{\mathcal{H}_2}^2 := \int_0^T \left[\dot{\varphi}(t)^\top \dot{\varphi}(t) + \varphi(t)^\top \varphi(t) \right] dt. \quad (4.11)$$

Therefore, when the control system and channel are governed by (4.7) with O-U process noise (4.10), by [Theorem 4.5](#) and (4.11), the total information in (4.7) satisfies

$$I(e_0^T; c_0^T, x_0) = \frac{1}{2} \mathbb{E} \left[\int_0^T (u_t - \hat{u}_t)^\top (u_t - \hat{u}_t) dt \right] + \frac{1}{2} \mathbb{E} \left[\int_0^T \int_0^t (u_\tau - \hat{u}_\tau)^\top d\tau \int_0^t (u_\tau - \hat{u}_\tau) d\tau dt \right], \quad (4.12)$$

where $\hat{u}_t := \mathbb{E}[u_t | \mathcal{F}_t^c] = \mathbb{E}[u_t | e_0^t]$. Since the control channel noise w_t in (4.7) is an O-U process, the calculation of estimates $\hat{u}_t$ and $\hat{u}_\tau$ in (4.12) demands estimation methods specific to the O-U process noise, e.g., [125, 126]. □

In the following, we consider an example when the external disturbance in (4.3) is a finite-dimensional Gauss-Markov process, which covers the disturbances in Examples 4.6 and 4.7 as special cases.

Example 4.8. *In addition to the control model (4.7), we can also consider the following finite-dimensional realization of (4.3),*

$$e_t = u_t(e_0^t, c_0^T, x_0) + w_t, \quad (4.13)$$

where the time parameter $t \in [0, T]$; e_t and $u_t(\cdot)$ respectively denote the error signal and continuous control input function, and the external disturbance w_t is a Gauss-Markov process with factorizable (covariance) kernel $K_3(s, t) = \text{Cov}(w_s, w_t) = a(s)b(t)$.⁶ Supposing $0 \leq s \leq t \leq T$, the associated RKHS $\mathcal{H}_3$ is a Hilbert space consisting of functions such that their time derivatives are defined almost everywhere (a.e.) and are square-integrable, i.e.

$$\mathcal{H}_3 = \{\varphi(\cdot) \mid d/dt[\varphi(t)/b(t)] \text{ exists a.e. and } \in \mathcal{L}^2([0, T])\}.$$

The associated norm of $K_3(s, t)$ and $\mathcal{H}_3$ is

$$\|\varphi\|_{\mathcal{H}_3}^2 = \int_0^T \left[\frac{d}{dt} \left\{ \frac{\varphi(t)}{b(t)} \right\} \right]^2 \left[\frac{d}{dt} \left\{ \frac{a(t)}{b(t)} \right\} \right]^{-1} dt. \quad (4.14)$$

Hence, by applying Theorem 4.5 and (4.14) to the control system or channel (4.13) under Gauss-Markov process noise, the total information of (4.13) satisfies

$$I(e_0^T; c_0^T, x_0) = \frac{1}{2} \mathbb{E} \left[\int_0^T \left[\frac{d}{dt} \left\{ \frac{u_t - \hat{u}_t}{b(t)} \right\} \right]^2 \left[\frac{d}{dt} \left\{ \frac{a(t)}{b(t)} \right\} \right]^{-1} dt \right], \quad (4.15)$$

where, depending on the specific forms of external disturbance w_t , the estimate $\hat{u}_t := \mathbb{E}[u_t | \mathcal{F}_t^e] = \mathbb{E}[u_t | e_0^t]$ can be calculated from (4.13) by corresponding optimal filters, e.g., [127, 128]. $\square$

With the aid of abstract I-MMSE relationship and Theorem 4.5, Examples 4.7 and 4.8 provide a direct approach to calculate the total information (rate) of control systems or channels subject to O-U process and Gauss-Markov process disturbances. Thanks to the generality of RKHS norm in (4.5), this approach can also be extended to calculate the total information (rates) of control systems

⁶ This (covariance) kernel covers the kernels in Examples 4.6 and 4.7 as special cases. For example, if $s \leq t$, we can restore $K_1(s, t) = \min(s, t)$ in Example 4.6 by letting $a(s) = s$ and $b(t) = 1$, and $K_2(s, t) = (1/2) \exp(-|t - s|)$ in Example 4.7 by letting $a(s) = \exp(s)$ and $b(t) = (1/2) \exp(-t)$.

subject to other Gaussian disturbances [119], such as the processes with triangular and Dirichlet (covariance) kernels, and processes that induce various Sobolev and Hardy spaces. Although the total information (rates) presented in Examples 4.7 and 4.8 share some common trade-off properties, e.g., an upper bound of plant's instability rate $\bar{I}(e; x_0)$, with the total information (rate) discussed in Example 4.6 and Appendix D, to comprehensively understand and interpret the trade-off properties of total information (rates) under colored Gaussian noise and reveal their connections to the internal features and limitations of physical plants, such as the unstable poles of linear plant in Appendix D, a more thorough study, similar to [16], Chapters 2 and 3, but beyond the scope and length of this chapter, needs to be carried out on the basis of Theorem 4.5 and Examples 4.7 and 4.8. It is also worth noting that as (4.12) and (4.15) shows, when external disturbance w_t is colored, the total information (rate) can no longer be expressed as a simple function of the MMSE of control input, and the estimates $\hat{u}_t$ in (4.5), (4.12), and (4.15) should not be calculated by the common filters for white Gaussian noise.

4.3 Filtering Trade-offs in Wiener Spaces

To investigate total information (rate) as a filtering trade-off metric, consider the general estimation system depicted in the following diagram,

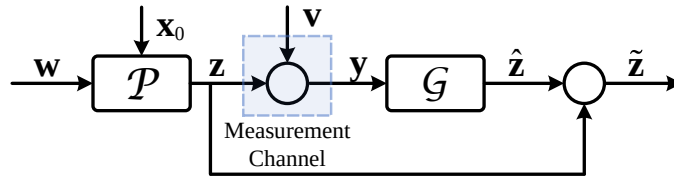


Figure 4.3: Abstract filtering systems.

where process noise $\mathbf{w}$ and initial states $\mathbf{x}_0$, mutually independent, can be regarded as the message of measurement channel; channel input $\mathbf{z}$, generated by plant $\mathcal{P}$, is the noise-free output to be estimated; channel noise $\mathbf{v}$, independent from $\mathbf{w}$ and $\mathbf{x}_0$, is the measurement noise; $\mathbf{y}$ is the measured output; $\mathcal{G}$ stands for the filtering process; $\hat{\mathbf{z}}$ is an estimate of $\mathbf{z}$, and $\tilde{\mathbf{z}} := \mathbf{z} - \hat{\mathbf{z}}$ is the estimation error. Similar to the modeling of feedback control system in Section 4.2, when all signals in Figure 4.3 are infinite-dimensional, we can describe this filtering system and measurement channel by an abstract

Gaussian channel without feedback

$$\begin{cases} \mathbf{y} = \mathbf{z} + \mathbf{v} \\ \mathbf{z} = Z(\mathbf{w}, \mathbf{x}_0), \end{cases} \quad (4.16)$$

where $Z(\cdot)$ denotes the coding function of message $(\mathbf{w}, \mathbf{x}_0)$ or realization of plant $\mathcal{P}$, and all signals and the function in (4.16) obey the same settings S1) to S5) for (4.1). Specifically, we can realize the plant $\mathcal{P}$ and input function $Z(\cdot)$ by

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) + \mathbf{w} \\ \mathbf{z} &= h(\mathbf{x}), \end{aligned} \quad (4.17)$$

where $\mathbf{x}$ denotes the internal states of $\mathcal{P}$, $f(\cdot)$ is the open- or closed-loop dynamics function, and $h(\cdot)$ is the measurement function that satisfies $\mathbf{z} = h(\mathbf{x}) = Z(\mathbf{w}, \mathbf{x}_0)$. As the noise $\mathbf{v}$ in (4.16) can be arbitrary Gaussian, the measurement channel modeled by (4.16), (4.17) and Figure 4.3 is general enough to represent a wide variety of filtering systems, such as the common system subject to AWGN.

By applying Lemma 4.3 to the abstract measurement channel (4.16), we have the following result on the total information $I(\mathbf{y}; \mathbf{w}, \mathbf{x}_0)$ of abstract filtering systems.

Theorem 4.9. *For the abstract filtering systems described by Figure 4.3 and (4.16) satisfying settings S1) to S5), the total information $I(\mathbf{y}; \mathbf{w}, \mathbf{x}_0)$ of measurement channel (4.16) satisfies*

$$I(\mathbf{y}; \mathbf{w}, \mathbf{x}_0) = \frac{1}{2} \mathbb{E} \|\mathbf{z} - \hat{\mathbf{z}}\|_{\mathcal{H}_{\mathbf{v}}}^2, \quad (4.18)$$

where $\|\cdot\|_{\mathcal{H}_{\mathbf{v}}}$ denotes the RKHS norm associated with measurement noise $\mathbf{v}$, and $\hat{\mathbf{z}} := \mathbb{E}[\mathbf{z} | \mathcal{F}^{\mathbf{y}}]$ is the dual predictable projection or causal estimate of $\mathbf{z}$ w.r.t. the filtration $\mathcal{F}^{\mathbf{y}}$.

The proof of Theorem 4.9 is direct. When the time parameter set $\mathcal{I}$ is $[0, T]$, total information rate is defined as the limit of time-averaged total information, i.e., $\bar{I}(\mathbf{y}; \mathbf{w}, \mathbf{x}_0) := \lim_{T \rightarrow \infty} (1/T) I(\mathbf{y}; \mathbf{w}, \mathbf{x}_0)$, if the limit exists. Hence, by (4.18), we have

$$\bar{I}(\mathbf{y}; \mathbf{w}, \mathbf{x}_0) = \lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} \|\mathbf{z} - \hat{\mathbf{z}}\|_{\mathcal{H}_{\mathbf{v}}}^2. \quad (4.19)$$

Similar to the control scenario in [Section 4.2](#), the calculation of projection $\hat{\mathbf{z}}$ or estimation error $\mathbf{z} - \hat{\mathbf{z}}$ in (4.18) and (4.19) demands the infinite-dimensional optimal filters.

A few examples are presented below to interpret how total information (rate) and [Theorem 4.9](#) can be utilized to capture the fundamental limitations in finite-dimensional filtering systems. The first example invokes [Theorem 4.9](#) to restore an important filtering limit, lowest achievable MS estimation error, in the systems subject to Wiener process noise.

Example 4.10. *Consider the following finite-dimensional filtering system and measurement channel*

$$y_t = \int_0^t z_\tau(w_0^T, x_0) d\tau + v_t, \quad (4.20)$$

where the time parameter $t \in [0, T]$; the measurement noise v_t is a p -dimensional standard Brownian motion B_t ; the noise-free output satisfies $z_\tau \in \mathcal{L}^2([0, t]; \mathbb{R}^p)$ or $\mathbb{E}[\int_0^t z_\tau^\top z_\tau d\tau] < \infty$, and y_t denotes the measured output.⁵ The Cameron-Martin space and RKHS norm induced by v_t in (4.20) are identical to the space $\mathcal{H}_1$ and norm $\|\cdot\|_{\mathcal{H}_1}$ discussed in [Example 4.6](#). Hence, when the filtering system is subject to standard Brownian motion noise, the total or input-output mutual information in (4.20) satisfies

$$I(y_0^T; w_0^T, x_0) = \frac{1}{2} \mathbb{E} \left[\int_0^T (z_t - \hat{z}_t)^\top (z_t - \hat{z}_t) dt \right], \quad (4.21)$$

where $I(y_0^T; w_0^T, x_0) = I(y_0^T; z_0^T)$ by [\[47\]](#), and $\hat{z}_t := \mathbb{E}[z_t | \mathcal{F}_t^y] = \mathbb{E}[z_t | y_0^t]$ can be calculated by some proper optimal filtering techniques, e.g., Kalman-Bucy filter and Stratonovich-Kushner equation, depending on the linearity of plant (4.17). Equation (4.21) shows that the total information in (4.20) is half the MMSE of noise-free output. When the process noise $\mathbf{w}$ vanishes in (4.17), total information rate $\lim_{\mathbf{w} \rightarrow 0} \bar{I}(y; w, x_0)$ then defines a lower bound for the MS estimation errors of all filters. More detailed explanation on the trade-off properties of $I(y_0^T; w_0^T, x_0)$ and $\bar{I}(y; w, x_0)$ is provided in [Appendix E](#). $\square$

With the following two examples, we show that [Theorem 4.9](#) provides a feasible approach to calculate the total information (rate) and characterize the fundamental limitations for some rarely discussed filtering systems and measurement channels subject to colored Gaussian noises, in which Duncan's theorem fails to hold.

Example 4.11. *Consider the same filtering problem and additive Gaussian measurement channel*

as (4.20) in [Example 4.10](#), except that the measurement noise v_t in (4.20) is now a p -dimensional O - U process obeying

$$dv_t = -v_t dt + dB_t, \quad (4.22)$$

where B_t is a p -dimensional standard Brownian motion. The Cameron-Martin space and RKHS norm induced by the O - U process noise v_t in (4.20) and (4.22) are identical to the space $\mathcal{H}_2$ and norm $\|\cdot\|_{\mathcal{H}_2}$ in [Example 4.7](#). Hence, when the filtering system (4.20) is subject to the O - U process noise (4.22), the total information $I(y_0^T; w_0^T, x_0)$ in (4.20) satisfies

$$I(y_0^T; w_0^T, x_0) = \frac{1}{2} \mathbb{E} \left[\int_0^T (z_t - \hat{z}_t)^\top (z_t - \hat{z}_t) dt \right] + \frac{1}{2} \mathbb{E} \left[\int_0^T \int_0^t (z_\tau - \hat{z}_\tau)^\top d\tau \int_0^t (z_\tau - \hat{z}_\tau) d\tau dt \right], \quad (4.23)$$

where $\hat{z}_t := \mathbb{E}[z_t | \mathcal{F}_t^y] = \mathbb{E}[z_t | y_0^t]$ needs to be calculated from (4.17) and (4.20) by the optimal filtering techniques designed for O - U process noise. $\square$

The following example considers the scenario when the measurement noise is a p -dimensional Gauss-Markov process that covers the measurement noises in [Examples 4.10](#) and [4.11](#) as special cases.

Example 4.12. Apart from the measurement model (4.20), we can also realize (4.16) with the following finite-dimensional model

$$y_t = z_t(w_0^T, x_0) + v_t, \quad (4.24)$$

where the time parameter $t \in [0, T]$; y_t and $z_t(\cdot)$ respectively denote the measured output and noise-free output function, and measurement noise v_t is a p -dimensional Gauss-Markov process with factorizable (covariance) kernel $K_3(s, t) = \text{Cov}(w_s, w_t) = a(s)b(t)$. The Cameron-Martin space and RKHS norm associated with the Gauss-Markov process noise in (4.24) are identical to the space $\mathcal{H}_3$ and norm $\|\cdot\|_{\mathcal{H}_3}$ in [Example 4.8](#). Therefore, by deploying [Theorem 4.9](#) and (4.14) on the filtering system (4.24), the total information in (4.24) satisfies

$$I(y_0^T; w_0^T, x_0) = \frac{1}{2} \mathbb{E} \left[\int_0^T \left[\frac{d}{dt} \left\{ \frac{z_t - \hat{z}_t}{b(t)} \right\} \right]^2 \left[\frac{d}{dt} \left\{ \frac{a(t)}{b(t)} \right\} \right]^{-1} dt \right], \quad (4.25)$$

where depending on the specific forms of measurement noise v_t in (4.24), $\hat{z}_t := \mathbb{E}[z_t | \mathcal{F}_t^y] = \mathbb{E}[z_t | y_0^t]$ can be calculated from (4.17) and (4.24) by a proper optimal filter. $\square$

Examples 4.11 and 4.12 indicate that when the measurement noise is colored Gaussian, the total information (rate) of measurement channel is no longer half the MMSE of noise-free output as (4.21) or Duncan's theorem tells. By following the same scheme in Examples 4.10 to 4.12, we can also calculate the total information (rate) of filtering systems subject to other Gaussian noise. It is worth noting that depending on the specific types of plant model, filtering process and measurement noise, proper filters should be utilized to calculate the estimates in (4.18), (4.21), (4.23), and (4.25). To fully understand and interpret the filtering trade-off properties of total information (rate) in Examples 4.11 and 4.12, a more thorough and systematic study, similar to the trade-off analysis of filtering systems subject to AWGN in [16], should be conducted on the basis of (4.23) and (4.25).

4.4 Conclusion and Discussion

In this chapter, by the I-MMSE relationships, we proposed a direct and estimation-based method to calculate the total information (rates), which characterize the fundamental limitations of infinite-dimensional control and filtering systems subject to arbitrary Gaussian noise. Examples and applications to the finite-dimensional systems were presented, which showed that the proposed method not only recovers some well-established control and filtering trade-offs, but provides a feasible approach to calculate and analyze the total information (rate) of some complicated control and filtering systems. Some promising followups of this chapter include i) more detailed and systematic investigations on the control and filtering trade-off properties of total information (rates) in Examples 4.7, 4.8, 4.11 and 4.12, and ii) extending the abstract I-MMSE relationship in Lemma 4.3 and the characterization of fundamental limitations to infinite-dimensional discrete-time control and filtering systems.

Chapter 5

Conclusions

In this dissertation, by resorting to the I-MMSE relationships and optimal filtering techniques, we proposed a comprehensive and estimation-based paradigm to calculate and investigate total information (rate) as a trade-off metric characterizing the fundamental limitations of various control and filtering systems in continuous-time, discrete-time, and infinite-dimensional systems. For the continuous-time scenario discussed in [Chapter 2](#), with the aid of Duncan's theorem or continuous-time I-MMSE relationships, we related total information (rate) to the fundamental limitations, such as noise sensitivity trade-offs and average performance cost in control and lowest achievable mean-square estimation error in the filtering of continuous-time control and filtering systems in LTI, LTV, and nonlinear forms. For the discrete-time scenario covered by [Chapter 3](#), we extended the I-MMSE relationships to discrete-time white Gaussian channels with and without feedback. With these relationships and optimal filtering techniques, we then revisited and extended the trade-off properties and interpretations of total information (rate) in discrete-time LTI, LTV, and nonlinear control and filtering systems. For the infinite-dimensional scenario investigated in [Chapter 4](#), based on the I-MMSE relationships in abstract Wiener space, a direct and estimation-based method was proposed to calculate and study the total information (rate) in abstract or infinite-dimensional control and filtering systems subject to arbitrary Gaussian noise. By applying this method to the finite-dimensional systems, we not only recovered some well-established trade-off metrics and properties in the control and filtering systems subject to white Gaussian noise, but provided a direct alternative to calculate the total information (rate) in some complicated systems under colored Gaussian noise.

Many future research problems can be investigated based on this dissertation. We have provided some open problems in the conclusion and discussion sections at the end of each chapter. Among these problems, the author believes that the following topics are immediate and of special interest:

- Relate total information (rate) to the fundamental limitations of control and filtering systems subject to colored Gaussian noise. In [Chapters 2](#) and [3](#), by using the I-MMSE relationships and optimal filtering techniques, we related total information (rate) to some fundamental limitations of control and filtering systems with AWGN. For example, when investigating the continuous-time LTI control and filtering systems under AWGN in [Chapter 2](#), we related total information rate to the sum of unstable poles, and hence, Bode's integral and lowest achievable mean-square estimation error. Based on the examples presented in [Chapter 4](#), similar results are expected to be derived in the control and filtering systems subject to colored Gaussian noise.
- Extend the abstract I-MMSE relationship in [Lemma 4.3](#) and subsequent characterization of control and filtering limitations to the discrete-time parameter set. Although the abstract Gaussian channel model [\(4.1\)](#) allows the time parameter set $\mathcal{I}$ to be a finite set of points, the derivation of abstract I-MMSE relationship or [Lemma 4.3](#) in [\[63\]](#) only considers the scenario when $\mathcal{I}$ is an interval of the real line. A possible approach to derive the abstract I-MMSE relationship under the discrete-time parameter set is to generalize the proof of discrete-time I-MMSE relationships in [Chapter 3](#) to the abstract Wiener space.

Appendix A

Control Trade-off in a Scalar System

To give a concrete example on the control trade-off and analysis method introduced in [Section 2.2.1](#), we consider the following open-loop scalar system

$$dy_t = \alpha \cdot y_t dt + de_t, \quad (\text{A.1})$$

where $\alpha > 0$ implies that [\(A.1\)](#) possesses an open-loop unstable pole, and $\alpha \leq 0$ stands for a stable pole. To stabilize the closed-loop system, as [Figure 2.1](#) shows, we adopt the control mapping

$$u_t = -k \cdot y_t, \quad (\text{A.2})$$

where $k > 0$ is the control gain. The lumped control mapping or control channel [\(2.5\)](#) then satisfies

$$de_t = u_t dt + dw_t = -k \cdot y_t dt + dw_t, \quad (\text{A.3})$$

where w_t is a standard Brownian motion representing noise. By substituting [\(A.2\)](#) and [\(A.3\)](#) into [\(A.1\)](#), we can then use the following Ornstein-Uhlenbeck process to describe the output signal y_0^t

$$dy_t = (\alpha - k) \cdot y_t dt + dw_t = \bar{\alpha} \cdot y_t dt + dw_t, \quad (\text{A.4})$$

which has a solution

$$y_t = y_0 \cdot \exp(\bar{\alpha}t) + \int_0^t \exp[\bar{\alpha}(t-s)] dw_s. \quad (\text{A.5})$$

When the closed-loop pole $\bar{\alpha} = \alpha - k \leq 0$, the closed-loop system (A.4) is stable, and the solution (A.5) implies a convergent or stable output signal.

Theorem 2.14 provides two alternative approaches to calculate $I(e_0^t; x_0) = (1/2) \cdot (\mathbb{E}[\|u\|^2] - \mathbb{E}[\|\hat{u}\|^2]) = (1/2) \cdot \mathbb{E}[\|u - \hat{u}\|^2]$, which require the explicit expressions for $\mathbb{E}[\|u\|^2]$, $\mathbb{E}[\|\hat{u}\|^2]$, and $\mathbb{E}[\|u - \hat{u}\|^2]$. Assuming that the initial condition y_0 in (A.5) is normal $y_0 \sim \mathcal{N}(0, \sigma_y^2)$ and substituting (A.5) into (A.2), we have

$$\mathbb{E}[\|u\|^2] = \mathbb{E}\left[\int_0^t u^2(\tau) d\tau\right] = k^2 \cdot \left[\exp(2\bar{\alpha} \cdot t) \cdot \sigma_y^2 - \frac{1}{2\bar{\alpha}} (1 - \exp(2\bar{\alpha} \cdot t))\right].$$

By applying the Kalman-Bucy filter in [60, Theorem 10.3] to the filtering problem described by (A.3) and (A.4) or $du_t = \bar{\alpha} \cdot u_t dt - k \cdot dw_t$, we can then compute $\mathbb{E}[\|\hat{u}\|^2]$ by solving the following filter equation

$$d\hat{u}_t = \bar{\alpha} \cdot \hat{u}_t dt + [-k + p(t)] \cdot [de_t - \hat{u}_t dt],$$

or $\mathbb{E}[\|u - \hat{u}\|^2]$ by solving the following error covariance equation

$$\dot{p}(t) = -p^2(t) + 2\alpha \cdot p(t), \tag{A.6}$$

where $\hat{u}_t = \mathbb{E}[u_t | e_0^t] = -k \cdot \hat{y}_t = -k \cdot \mathbb{E}[y_t | e_0^t]$, and $p(t) = \mathbb{E}[(u(t) - \hat{u}(t))^2]$ is the estimation error covariance. Since the solution to (A.6) is more explicit, we use $I(e_0^t; x_0) = (1/2) \cdot \mathbb{E}[\|u - \hat{u}\|^2]$ to compute $I(e_0^t; x_0)$ and $\bar{I}(e; x_0)$. The solution to (A.6) is

$$p(t) = \frac{2\alpha \cdot p(0) \cdot \exp(2\alpha t)}{p(0) \cdot \exp(2\alpha t) - p(0) + 2\alpha}, \tag{A.7}$$

where, by letting $t \rightarrow \infty$, we compute the asymptotic error covariance $\lim_{t \rightarrow \infty} p(t) = 2\alpha$ when $\alpha > 0$, and $\lim_{t \rightarrow \infty} p(t) = 0$ when $\alpha \leq 0$. With the estimation error covariance given in (A.7), we compute the mutual information $I(e_0^t; x_0)$ in (A.1)-(A.3) via

$$I(e_0^t; x_0) = \frac{1}{2} \mathbb{E}[\|u - \hat{u}\|^2] = \frac{1}{2} \int_0^t p(\tau) d\tau = \frac{1}{2} [\log(p(0) \exp(2\alpha t) - p(0) + 2\alpha) - \log(2\alpha)],$$

where the Fubini's theorem is used to swap the order of integration and expectation. Consequently,

the mutual information rate

$$\bar{I}(e; x_0) = \lim_{t \rightarrow \infty} \frac{1}{2t} \int_0^t p(\tau) d\tau = \begin{cases} \alpha & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha \leq 0, \end{cases} \quad (\text{A.8})$$

where the limit is evaluated by using the L'Hôpital's rule and the limit of (A.7). Since $\bar{I}(e; x_0)$ equals the (sum of) open-loop unstable poles, as (A.8) shows, we can imply that the mutual information rate $\bar{I}(e; x_0)$ is identical to the famous Bode's integral [17], which serves as a control trade-off characterizing the disturbance attenuation ability of scalar control system (A.1)-(A.3). Meanwhile, with Proposition 2.15 and preceding results, we can also compute $D(\mu_{e|x_0} \parallel \mu_w) = (1/2) \cdot \mathbb{E}[\|u\|^2]$ and $D(\mu_e \parallel \mu_w) = (1/2) \cdot \mathbb{E}[\|\hat{u}\|^2] = I(e_0^t; x_0) - (1/2) \cdot \mathbb{E}[\|u\|^2]$. Rigorous analysis on the control trade-offs $I(e_0^t; x_0)$ and $\bar{I}(e; x_0)$ is performed in Section 2.2, and more examples in LTI, LTV and nonlinear control systems can be found by mimicking the steps in this example.

Appendix B

Stability and Sensitivity of Continuous-Time LTV Systems

We start with the following n -dimensional homogeneous LTV system

$$\dot{x}_t = A(t)x_t. \quad (\text{B.1})$$

Let $\{x^{(i)}(t), 1 \leq i \leq n\}$ be n linearly independent solutions to (B.1). We then define the *fundamental solution matrix* $X(t) = [x^{(1)}(t), \dots, x^{(n)}(t)]$, which satisfies $\dot{X}(t) = A(t)X(t)$, and the state transition matrix $\Phi_A(t, t_0) = X(t)X^{-1}(t_0)$. When $\{p_i\}_{i=1}^n$ is an orthonormal basis in $\mathbb{R}^n$, the *characteristic numbers* are defined by

$$\lambda_i(p_i) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|X(t)p_i\|.$$

When the sum of the characteristics numbers $\sum_{i=1}^n \lambda_i(p_i)$ is minimized, the orthonormal basis $\{p_i\}_{i=1}^n$ is called *normal*, and the λ_i 's are called the *Lyapunov exponents*. The characteristic numbers and Lyapunov exponents in LTV systems serve the same role as the real parts of the eigenvalues of A in LTI systems. Definitions of *uniform exponential stability* and *antistability* are given as follows.

Definition B.1 (Uniformly Exponentially Stable). *The matrix function $A(t)$ is uniformly exponentially stable (UES) if there exist positive constants γ and δ such that*

$$\|\Phi_A(t, \tau)\| \leq \gamma e^{-\delta(t-\tau)}$$

for all $t \geq \tau$.

Definition B.2 (Uniformly Exponentially Antistable). *The matrix function $A(t)$ is uniformly exponentially antistable (UEA), if there exist positive constants γ and δ such that*

$$\|\Phi_A(t, \tau)\| \leq \gamma e^{-\delta(t-\tau)}$$

for all $t \leq \tau$.

The LTV system (B.1) is said to possess an *exponential dichotomy*, if there exists a projection P , and positive constants γ and δ such that

$$\begin{aligned} \|X(t)PX(\tau)^{-1}\| &\leq \gamma e^{-\delta(t-\tau)}, & \forall t \geq \tau, \\ \|X(t)(I-P)X(\tau)^{-1}\| &\leq \gamma e^{-\delta(t-\tau)}, & \forall t \leq \tau. \end{aligned}$$

If $\text{rank}(P) = n_s$, exponential dichotomy implies that n_s fundamental solutions are UES, whereas $n_u = n - n_s$ are UEA. The *dichotomy spectrum* of LTV system (B.1) is the set of real values λ for which the translated systems $\dot{x}(t) = [A(t) - \lambda I]x(t)$ fail to have a dichotomy. In general, the dichotomy spectrum is a collection of compact non-overlapping intervals

$$\mathcal{S}_{\text{dich}} = \cup_{i=1}^m [\underline{\lambda}_i, \bar{\lambda}_i],$$

where $m \leq n$ and $\underline{\lambda}_1 \leq \bar{\lambda}_1 < \underline{\lambda}_2 \leq \bar{\lambda}_2 < \dots < \underline{\lambda}_m \leq \bar{\lambda}_m$. It is straightforward to verify that when $\lambda_0 < \underline{\lambda}_1$ all trajectories of $\dot{x}(t) = [A(t) - \lambda_0 I]x(t)$ are unbounded, and when $\lambda_{m+1} > \bar{\lambda}_m$ all trajectories of $\dot{x}(t) = [A(t) - \lambda_{m+1} I]x(t)$ are bounded. When (B.1) is a LTI system or the matrix function $A(t)$ in (B.1) is periodic, each of these intervals $[\underline{\lambda}_i, \bar{\lambda}_i]$ is a point equivalent to a Lyapunov exponent λ_i . The dichotomy is then known as a *point spectrum*, and the system (B.1) is said to be *regular*.

For the LTV system Σ_L defined in (2.29), its sensitivity operator S (or sensitivity system Σ_S) has the following state-space representation

$$\Sigma_S = \left[\begin{array}{c|c} A(t) - B(t)C(t) & B(t) \\ \hline -C(t) & I \end{array} \right],$$

which serves the same role as the (state-space representation of) *sensitivity function* $S(s) = 1/[1 + L(s)]$ in LTI systems [68]. When $A(t)$ admits an exponential dichotomy and the pair $(A(t), B(t))$ is stabilizable, the sensitivity operator has an inner/outer factorization $S = S_i S_o$, where the state-space representations of these two factors are respectively

$$\Sigma_{S_i} = \left[\begin{array}{c|c} A(t) - B(t)B(t)^\top X(t) & B(t) \\ \hline -B(t)^\top X(t) & I \end{array} \right],$$

and

$$\Sigma_{S_o} = \left[\begin{array}{c|c} A(t) - B(t)C(t) & B(t) \\ \hline B(t)^\top X(t) - C(t) & I \end{array} \right].$$

Define the kernel of the outer factor as

$$\hat{S}_o(t, \tau) = [B(t)^\top X(t) - C(t)]\Phi_{A-BC}(t, \tau)B(\tau),$$

for all $t \geq \tau$, and then the following integral

$$\mathcal{B} := \lim_{t \rightarrow \infty} \frac{1}{4t} \int_{-t}^t \text{tr}[\hat{S}_o(\tau, \tau)] d\tau$$

is an analogue of Bode's sensitivity integral for continuous-time LTV systems [24, 68]. More comprehensive introductions on the stability and sensitivity of LTV systems refer to [24, 89, 90, 92] and references therein.

Appendix C

Stability and Sensitivity of Discrete-Time LTV Systems

Consider the following m -dimensional discrete-time homogeneous LTV system

$$X_{i+1} = A_i X_i, \tag{C.1}$$

where $X_i \in \mathbb{R}^m$, $\{A_i\}$ is a sequence of $m \times m$ matrices, and $i = k, k+1, \dots, k+n$. Let $\Phi(k+n, k) := \prod_{i=k}^{k+n-1} A_i$ be the discrete-time state transition matrix of (C.1) from k to $k+n$. We say that the sequence $\{A_i\}$ is *uniformly exponentially stable* (UES), if there exist positive constants α and $\beta < 1$, independent of k and n , such that $\|\Phi_A(k+n, k)\| = \bar{\mu}(\Phi_A(k+n, k)) < \alpha\beta^n$, and $\{A_i\}$ is *uniformly exponentially antistable* (UEA) if there exist positive constants α and $\beta > 1$, independent of k and n , such that $\underline{\mu}(\Phi_A(k+n, k)) > \alpha\beta^n$.

The LTV system (C.1) is said to possess an *exponential dichotomy* if it can be decomposed into stable and antistable parts. Let $\{P_k\}$ be a bounded sequence of projections in $\mathbb{R}^m$ such that the rank of P_k is constant. We say that $\{P_k\}$ is a *dichotomy* for $\{A_i\}$ if the commutativity condition $A_k P_k = P_{k+1} A_k$ is satisfied for all k and there exist positive constants α and $\beta > 1$ such that $\|\Phi_A(k+n, k)P_k x\| > \alpha\beta^{n+1}\|P_k x\|$ and $\|\Phi_A(k+n, k)(I - P_k)x\| < (\alpha\beta^l)^{-1}\|(I - P_k)x\|$ for any x .

Define the weighted shift operator $\mathcal{S}_A$ by $(\mathcal{S}_A X)_i := A_{i-1} X_{i-1}$, and let $\sigma(\mathcal{S}_A)$ denote the spectrum of the operator. The sequence $\{A_k\}$ is stable (or antistable) if and only if $\sigma(\mathcal{S}_A) \in \mathbb{D}$ (or $\sigma(\mathcal{S}_A) \in \mathbb{C} \setminus \mathbb{D}$), where $\mathbb{D}$ denotes the unit disk, and $\mathbb{C}$ stands for the entire complex plane. The

spectrum of the weighted shift is contained in r concentric annuli:

$$\sigma(\mathcal{S}_A) = \cup_{j=1}^r \{\lambda \in \mathbb{C} : \underline{\kappa}_j \leq |\lambda| \leq \bar{\kappa}_j\},$$

where $0 \leq \underline{\kappa}_1 \leq \bar{\kappa}_1 < \underline{\kappa}_2 < \cdots < \underline{\kappa}_r < \bar{\kappa}_r < \infty$ and $1 \leq r \leq m$. Exponential dichotomy is equivalent to the existence of an integer $q \in \{1, \dots, r-1\}$ such that either $\bar{\kappa}_q < 1 < \underline{\kappa}_{q+1}$ or $\bar{\kappa}_r < 1$. More detailed definitions and explanations on the terminology related to the discrete-time LTV systems can be found in [112, 129] and references therein.

Appendix D

Extension of Example 4.6

In this appendix, we show some calculations and explanations of the total information (rate) in (4.9), when the plant (4.4) and control mapping u_τ in (4.7) take different forms, e.g., LTI, LTV and nonlinear. More detailed examples and interpretations on the control trade-off properties of total information (rate) can be found in the Chapter 2 of this dissertation, [7, 16, 27, 44, 56] and references therein. We first consider an LTI scenario when the control signal u_τ in (4.7) is generated by the LTI plant with state dynamics $dx_\tau = Ax_\tau d\tau + Bde_\tau$ and observation model $y_\tau = Cx_\tau$, and control mapping $u_\tau = Gx_\tau$, where A, B, C and G are time-invariant matrices of proper dimensions. By using Kalman-Bucy filter to calculate $\hat{u}_t$ in (4.9), and regardless of the stationarity of signals in (4.7), we have the following identity on total information rate

$$\bar{I}(e; c, x_0) = \sum_i \lambda_i^+(A), \quad (\text{D.1})$$

where $\lambda_i^+(A)$ denote the eigenvalues of matrix A with positive real parts, and $\bar{I}(e; c, x_0)$ degenerates to $\bar{I}(e; x_0)$ in the absence of command signal c_0^t . Since the classical Bode's sensitivity integral also equals $\sum_i \lambda_i^+(A)$ by [2], equation (D.1) indicates that total information rate is analogous to the Bode's integral for LTI control systems. We then consider an LTV scenario when the control signal u_τ in (4.7) is generated by the LTV plant with state dynamics $dx_\tau = A_\tau x_\tau d\tau + B_\tau de_\tau$ and measurement model $y_\tau = C_\tau x_\tau$, and control mapping $u_\tau = G_\tau x_\tau$, where A_τ, B_τ, C_τ and G_τ are time-varying matrices of proper dimensions. Using the time-varying Kalman-Bucy filter to calculate

$\hat{u}_t$ in (4.9), we have the following result on the total information rate

$$\bar{I}(e; c, x_0) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{tr}[A_u(\tau)] d\tau - \lim_{T \rightarrow \infty} \frac{1}{2T} \int_0^T \text{tr}[\dot{P}_u(\tau) P_u^{-1}(\tau)] d\tau, \quad (\text{D.2})$$

where $A_u(\tau)$ denotes the antistable component of A_τ , and $P_u(\tau)$ is the estimation covariance matrix of antistable modes. Since the first integral in (D.2) equals the time-domain Bode's integral defined in [24], total information rate defines and captures the trade-off property of average entropy cost functions for LTV control systems. When u_τ in (4.7) is generated by nonlinear plant and control mapping, the total information rate and estimate $\hat{u}_\tau$ in (4.9) can be calculated and analyzed via the Stratonovich-Kushner equation as in [16].

In particular, when the plant (4.4) and control mapping in (4.7) are both LTI, and all signals in (4.7) and Figure 4.2 are (wide sense) stationary Gaussian with valid power spectral densities $\phi_x(\omega)$, i.e., the Fourier transforms of (auto-)covariance functions $\text{Cov}[x(t), x(t + \tau)]$, in addition to using Kalman-Bucy filter, we can also calculate the estimate and estimation error in (4.9) by using the causal Wiener filter or orthogonality principle, which connects total information and MMSE with the power spectral densities of signals and Bode-like integral in (D.3). Meanwhile, we can define a Bode-like integral $\mathcal{B}$ as an information-theoretic interpretation or substitute to the classical Bode's integral [7]:

$$\mathcal{B} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \left[\frac{\phi_e(\omega)}{\phi_w(\omega)} \right] d\omega \geq \sum_i \lambda_i^+(A), \quad (\text{D.3})$$

which can be related to the total information rate via (D.1).

Appendix E

Extension of Example 4.10

In this appendix, we provide some further interpretations on the filtering trade-off properties of total information (rate) in [Example 4.10](#) by showing the calculation results of [\(4.21\)](#), when the plant model [\(4.17\)](#) or input function z_τ in [\(4.20\)](#) take different forms, e.g., LTI, LTV and nonlinear. More detailed derivations and discussion can be found in [\[16\]](#). We first consider the scenario when signals in [Figure 4.3](#) are not necessarily stationary. When the process noise w_t in [\(4.17\)](#) vanishes, i.e., covariance $\Sigma_w \rightarrow 0$, total information rate $\bar{I}(y; x_0) = \lim_{\Sigma_w \rightarrow 0} \bar{I}(y; w, x_0)$ serves as a lower bound for the MS estimation error of noise-free output z_t , i.e., in the presence of process noise or $\Sigma_w \not\rightarrow 0$, no matter how well the filtering process $\mathcal{G}$ in [Figure 4.3](#) is designed, the time-averaged MS estimation error of z_t cannot be smaller than $2\bar{I}(y; x_0)$. Similar to the control trade-off analyses in [Appendix D](#), when the plant model [\(4.17\)](#) and filtering process $\mathcal{G}$ are linear, by using Kalman-Bucy filter and [\(4.21\)](#), we can show that the value of $\bar{I}(y; x_0)$ is determined by the antistable dynamics of plant. When [\(4.17\)](#) or $\mathcal{G}$ is nonlinear, we can calculate information rate $\bar{I}(y; x_0)$ by resorting to the nonlinear filtering or Stratonovich-Kushner equation.

An interesting and distinct result of this filtering trade-off can be derived when all signals in [\(4.20\)](#) and [Figure 4.3](#) are stationary Gaussian with valid power spectral densities. Applying the orthogonality principle and Karhunen-Loève expansion to [\(4.21\)](#), we can develop a direct equation between total information rate and the spectral density of noise-free output z_t

$$\bar{I}(y; w, x_0) = \lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} \left[\int_0^T (z_t - \hat{z}_t)^\top (z_t - \hat{z}_t) dt \right] = \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \det(I + \phi_z(\omega)) d\omega, \quad (\text{E.1})$$

where the second equality is a multi-dimensional extension of the celebrated Yovits-Jackson formula. When the plant model (4.17) is LTI, and measurement channel (4.20) is SISO, by [56, Proposition 4.5], the total information rate in (4.21) or (E.1) equals the sum of unstable poles in the limit of $\Sigma_w \rightarrow 0$, and the last integral in (E.1) can be used as an information-theoretic analogue to the classical Bode's integrals in [2]. Unfortunately, due to the correlation between control signal and external disturbance, and the fact that K-L expansion cannot preserve the causality among signals, the counterpart result of (E.1) has not been reported for any feedback control scenario yet.

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