

ASPECTS OF QUANTUM FIELD THEORY, HOLOGRAPHY AND DE SITTER

BY

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DISSERTATION

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Abstract

This thesis explores several topics in Euclidean gravitational path integral, holography and quantum field theory.

We first examine factorization and continuity properties of fields in the Euclidean gravitational path integral for actions constructed from powers of the Riemann tensor. We find the boundary terms corresponding to the microcanonical ensemble and show that the saddle point approximation to the path integral with a quasilocal energy constraint generally yields a saddle point with discontinuous temperature. Our work shows that the constrained state idea for the Euclidean Schwarzschild-de Sitter (SdS) geometry is robust against at least some types of quantum corrections from heavy fields. As an application of our construction, we compute the entropy of SdS in $D = 4$ using the BTZ method.

We then discuss gravitational back-reaction in the semiclassical treatment of quantum fields near black hole horizons in D dimensions. Motivated by the Cohen-Kaplan-Nelson bound, we construct states that deviate significantly from the Hartle-Hawking state and determine the conditions under which the energy density of these states have large back-reaction on the background geometry. We find the characteristic energy scale $(r_s^2 G_N)^{1/D}$ with r_s the Schwarzschild radius of the black hole. Our analysis suggests that well-controlled effective field theory outside the black hole can only account for an entropy that scales as $A^{\frac{D-2}{2}}$, where A is the area of the black hole horizon in Planck units.

Next, we analyze the problem of bubble nucleation in gravitational context. We consider the setting where a true vacuum bubble nucleates inside a false vacuum region. We approximate the spacetimes in the true and false regions as de Sitter spaces in static slicing, with different cosmological constants. A novel point aspect of our construction is that we allow the lapse degree of freedom on the bubble to remain unfixed. We derive the Israel junction conditions across the bubble and determine the bubble's effective action. We confirm that the bubble's effective action reproduces the junction conditions and use this effective action to analyze bubble nucleation.

Within the framework of Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence we analyze the recently introduced Weyl-Fefferman-Graham (WFG) gauge. In the WFG gauge, both a metric and a Weyl connection are induced at the conformal boundary and conformal geometry is upgraded to Weyl geometry. We present in detail the geometric ingredients of Weyl geometry, such as Weyl-covariant tensors. We study the holographic Weyl anomaly in this context, focusing on its cohomological properties. We compute the Weyl anomaly for $2d, 4d$ -dimensional boundaries and interestingly find the WFG gauge only trivially modifies the Weyl anomaly cocycle in the FG gauge. Motivated by the WFG gauge we then generalize the ambient metric construction of Fefferman-Graham for conformal manifolds to a corresponding construction for Weyl

manifolds, referred to as the Weyl-ambient construction. This is achieved by promoting a conformal manifold into a Weyl manifold by assigning a Weyl connection to the principal $\mathbb{R}_+$ -bundle, realizing a Weyl structure. We then show that the Weyl structure admits a well-defined initial value problem, which determines the Weyl-ambient metric. Through the Weyl-ambient construction, we also investigate Weyl-covariant tensors on the Weyl manifold and explicitly define the extended Weyl-obstruction tensors. The Weyl-ambient construction places the WFG gauge into a solid geometric framework.

In the final chapter of this thesis we investigate a recently discovered connection between certain L -loop conformal ladder graphs and thermal expectation values of a $U(1)$ charge of a free massive complex scalar $\phi(x)$ in $d = 2L + 1$ dimensions. We broaden this correspondence by showing that the free energy itself corresponds to a certain L -loop conformal ladder graph which evaluates a certain four-point function of the singular two-dimensional conformal fishnet model of Kazakov and Olivucci. Thermal partition functions offer a novel new perspective on conformal graphs. As an application of this point of view we reproduce the resummation of an infinite number of loop diagrams through the language of thermal partition functions.

To my Mother.

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Chapter 1

Introduction

Quantum field theory is arguably the most successful theory in human history. It encompasses all known physical phenomena in a beautiful unified framework. On one hand the Standard Model (SM) describes the three fundamental interactions, the electromagnetic force, the weak and strong interactions, together with all the currently known matter content. On the other hand, Einstein’s General theory of relativity (GR) beautifully describes the gravitational interaction as the curvature of spacetime.

Both the Standard model and General relativity have passed numerous tests, which include but are not limited to: The discovery of the Higgs boson at the Large Hadron Collider at CERN [1], [2], the first direct detection of gravitational waves by LIGO [3] and the first direct image of a black hole by the Event Horizon Telescope [4]. Despite the tremendous successes the full story is still missing. The modern perspective regards the Standard Model and GR as an *Effective Field theory* (EFT) (see e.g. [5], [6]). EFT provides a powerful coherent framework to classify the relevant interactions at each scale of interest¹, but as the name suggests, it is not a complete theory. Perhaps the most compelling hints of new physics come from Gravity. As a non renormalizable interaction, new physics are expected to emerge at high energies. The celebrated singularity theorems [8], [9] in classical GR already indicate where such situations arise, with the most significant examples being black hole and Big Bang cosmological singularities.

The search for a UV complete theory of Quantum Gravity is one of the longest-standing open problems in theoretical physics. String theory successfully resolves the UV divergences responsible for the non-renormalizability of gravitational interaction,² but a non-perturbative definition of the theory is lacking. Based on string theoretic constructions, Maldacena discovered the *AdS/CFT correspondence* [11] which is currently our best proposal of a UV complete theory of Quantum Gravity. The duality is a concrete realization of the holographic principle [12], [13] and postulates that Quantum Gravity for asymptotically Anti-de Sitter (AdS_{d+1}) spacetime in $d+1$ dimensions (usually referred to as the “bulk”) is dual to a conformal field theory (CFT_d) in d spacetime dimensions (usually referred to as the “boundary”). Asymptotically AdS spacetimes are solutions to Einstein’s equations with negative cosmological constant ($\Lambda < 0$). This is particularly important for the bulk physics due to the existence of a conformal boundary, which allows one to

¹See e.g. [7] for a very nice book on EFT methods.

²Another noteworthy approach to a UV complete theory of Quantum Gravity is the program of asymptotic safety (see, e.g., [10] for a review).

define diffeomorphism invariant observables by anchoring them at the boundary. According to the duality, diffeomorphism invariant observables in the bulk correspond to gauge invariant observables in the dual CFT.

However, our universe is well modeled by the Λ CDM model with small positive cosmological constant $\Lambda \simeq 1.1 \times 10^{-52} \text{ m}^{-2}$ [14]. Thus an important question is whether the holographic principle can be extended to spacetimes with positive cosmological constant. Although there have been some holographic proposals of de Sitter space [15]–[17], de Sitter holography is still in its early stages compared to AdS/CFT. In this regard, one hopes semiclassical considerations will provide us with important hints that will survive the final formulation. This question is explored in Chapter 2. An important clue comes from the thermodynamic properties of the $D = 4$ Schwarzschild de-Sitter (SdS) solution. As the mass of the black hole in SdS geometry increases, the combined areas of the cosmological and black hole horizons decrease. Relatedly, Banks et al. [16] postulated that local excitations can be viewed as constrained states of horizon degrees of freedom of empty de Sitter space. This idea was successfully implemented in the semiclassical Euclidean path integral [18]. Because SdS possesses two horizons the Euclidean continuation generically introduces at least one conical singularity, which prevents it from being a saddle point of the Euclidean gravitational path integral with Einstein-Hilbert action. Such obstruction is lifted when one imposes an appropriate constraint, and SdS can be thought of as a genuine saddle point of the constrained gravitational path integral. In Chapter 2 we investigate the constrained state idea in the Effective Field Theory context by allowing higher curvature corrections in the gravitational action.

The holographic principle implies a more radical breakdown of effective field theory in the bulk than is typically predicted by conventional EFT reasoning. Holographic entropy bounds [12], [13], inspired by the Bekenstein-Hawking area law for the black hole entropy [19], indicate that EFT significantly overestimates the number of degrees of freedom. In their seminal work [20], Cohen-Kaplan-Nelson (CKN) observed that energetic back-reaction bounds impose stronger constraints on the EFT Hilbert space than entropic bounds. They analyzed hot boxes in flat space and identified the conditions under which the box would collapse into a black hole. Such strong gravitational back-reaction is beyond current theoretic control. In Chapter 3, we develop the analog of the CKN bound for black horizons. Instead of a hot box, we consider a modification of the Hartle-Hawking [21] state in which thermally populated Boulware modes are cooled down to the Boulware vacuum [21], and determine the conditions before the back-reaction to the geometry becomes significant.

The application of the path integral formulation to gravity implies that gravity should exhibit quantum mechanical effects. A quintessential quantum phenomenon, understood since the early days of quantum mechanics (see, e.g., [22] for a brief historical overview), is quantum tunneling. Nearly fifty years ago, Coleman and collaborators developed powerful Euclidean path integral methods to compute vacuum decay rates in a semiclassical expansion [23]–[25]. This phenomenon, which holds significant cosmological importance, continues to drive interesting theoretical work, most importantly the development of new calculational tools [26]–[36]. In Chapter 4, we explore quantum tunneling effects in gravity within the context of false vacuum decay, commonly referred to as bubble nucleation. Specifically, we consider tunneling from a false de Sitter vacuum with cosmological constant Λ_F to a true de Sitter vacuum with Λ_T (see Figure 4.1). The gravitational fields in both the false and true vacuum regions are static but the gravitational back-reaction of the bubble is incorporated.

Chapters 5 and 6 of this thesis are directly related to the AdS/CFT correspondence, focusing mostly in the (classical) bulk side of the correspondence. A key tool in the early developments of AdS/CFT is the

Fefferman-Graham (FG) gauge of the bulk metric. This is a series expansion of the bulk metric of $(d + 1)$ -dimensional asymptotically locally AdS (AlAdS) spacetime in terms of a “radial” bulk coordinate z that moves you deeper into the bulk, where $z = 0$ denotes the location of the conformal boundary.³ It was shown that the FG gauge induces a conformal geometry in the boundary geometry with Weyl transformations induced by a bulk diffeomorphism called the Penrose-Brown-Henneaux (PBH) transformation [37], [38]. The basic idea behind conformal geometry is a manifold equipped with a conformal class of metrics $[g]$, where two metrics belong in the same conformal class if one metric is a smooth positive multiple of the other. A different perspective on conformal geometry was introduced by Weyl [39], whose idea was to make the physical scale a local quantity. The Weyl connection a_i was introduced so that one can transport the physical scale between two points of the manifold. Through Weyl’s idea one can modify the conformal class $[g]$ to a Weyl class $[g, a]$, which is the equivalence class formed by the pairs $(g_{ij}, a_i) \sim (\mathcal{B}(x)^{-2}g_{ij}, a_i - \partial_i \ln \mathcal{B}(x))$. This defines a Weyl manifold $(M, [g, a])$, and the conformal geometry is promoted to Weyl geometry [40]–[42]. The Weyl connection was incorporated into the formalism of AdS/CFT in [43]. By relaxing the FG gauge to the *Weyl-Fefferman-Graham* (WFG) gauge, it was shown [43] that the bulk Levi-Civita connection induces a Weyl connection on the conformal boundary. In Chapter 5 we investigate the geometric aspects of the WFG gauge and its role in AdS/CFT. Among other things we clarify the cohomological properties of the holographic Weyl anomaly in this new gauge. Chapter 6 sets the WFG gauge into a solid geometric framework by introducing the *Weyl-ambient metric*. This is a generalization of the Fefferman-Graham ambient construction [44]. We translate many of the theorems of [44] in this new setting. We also define Weyl-obstruction tensor rigorously through the Weyl-ambient construction.

Finally, Chapter 7 stands somewhat apart from the earlier chapters. In [45] it was observed that the logarithm of the partition function Z_L of a free massive complex scalar $\phi(x)$, twisted by the global $U(1)$ charge along the thermal circle in $d = 2L + 1$ dimensions, is given in terms of a class of single-valued polylogarithms. The latter functions are ubiquitous in multiloop QFT calculations (see e.g. [46] for a recent review), and their intriguing mathematical properties have been discussed in a number of works [47], [48]. The twisting parameter μ corresponds to an imaginary chemical potential for the abelian “charge” operator $\mathcal{Q} = \phi^\dagger \overleftrightarrow{\mathcal{D}}_\tau \phi$, with $\mathcal{D}_\tau = \partial_\tau - i\mu$, which together with $\mathcal{O} = |\phi|^2$ can be viewed as integrable relevant deformations of the massless free theory. From $\ln Z_L$ we can calculate the thermal one-point functions $\langle \mathcal{O} \rangle_L$ and $\langle \mathcal{Q} \rangle_L$ respectively, and it was shown in [45] that $\langle \mathcal{Q} \rangle_L$ is essentially given by the L -loop Davydychev-Usyukina conformal ladder graph [49], [50]. In this work we show that $\ln Z_L$ itself is also given by an L -loop conformal ladder graph which evaluates a certain four-point function of the singular two-dimensional conformal fishnet model of Kazakov and Olivucci [51]. Although the origin of the correspondence is still unknown, it provides a novel new perspective on conformal ladder diagrams. As an application of this new point of view we reproduce the resummation of an infinite number of loop diagrams [52] using the language of thermal partition functions.

³See Eq. (5.53) and Eq. (5.54) for the precise form of the FG metric expansion.

Chapter 2

Path Integral Factorization and the Gravitational Effective Action

Work in this chapter is based on “Path Integral Factorization and the Gravitational Effective Action” [53]. The paper was authored by Patrick Draper (UIUC), Szilard Farkas and Manthos Karydas (UIUC).

2.1 Introduction

A desirable property of functional integrals is “factorizability”: it should be possible to split the path integral defined on a spacetime manifold $\mathcal{M}$ into integrals on subregions of $\mathcal{M}$. For factorization to hold, generally the fields need to respect some continuity properties, so that the classical action functional over subregions is suitably additive.

As a result of these continuity conditions, the factorized form of the path integral on $\mathcal{M}$ is not usually a simple product of subregion path integrals, but rather it is given by the integral of such a product over field data defined on the codimension-1 boundaries between the subregions. Loosely we may think of this decomposition analogously to the insertion of a complete set of states in canonical quantization. Restricting the lower-dimensional path integral so that some data on the interface are fixed corresponds to the insertion of a constraint.

The continuity properties of the fields across a factorization surface follow from the form of the classical action. Consequently there is a close relationship between the factorization properties of the functional integral and the variational problem associated with the semiclassical approximation. Fields that are fixed on the boundary in the classical variational problem are generally required to be continuous across a factorization surface, and thus appear in the lower-dimensional integral over the interface data. Fields that are not included in the lower-dimensional integral over the interface data are not generally continuous. This can have interesting consequences for the semiclassical approximation to factorized path integrals, for fixed factorization surface data or equivalently in the presence of a constraint: the fields at the stationary point may not be continuous.

An example of this phenomenon arises in the Euclidean path integral in 4-dimensional pure Einstein gravity with positive cosmological constant. In the absence of a constraint, there is a completely smooth solution given

by the 4-sphere. However, imposing a spherically symmetric constraint on the Brown-York (BY) quasilocal energy [54], the relevant solution is given by the Euclidean Schwarzschild de-Sitter (SdS) metric [18], [55]. The black hole and cosmological “horizons” can both be made smooth by a suitable choice of Euclidean time periodicity on either side of the surface where the constraint is imposed, but the choices are different, so the metric is not continuous across the surface. The discontinuity may be encoded as a jump in the lapse function in static slicing. The lapse is not part of the field data that needs to be provided in the variational problem with fixed BY energy, and it is not integrated over in the factorized form of the path integral, so it is perfectly acceptable for it to appear with a discontinuity in the semiclassical solution. At the leading semiclassical order, at least, the on-shell action is independent of the radius at which the constraint is imposed [18]. This resolution of the semiclassical interpretation of SdS also provides support to the idea that localized matter should correspond to a constrained state of the underlying quantum theory of de Sitter space [16].¹

One should ask whether discontinuous saddle point geometries obtained in Einstein gravity still make sense beyond the leading semiclassical order. The approach to this question taken here will be to determine whether the factorization properties of more general gravitational Wilsonian effective actions are, in a suitable sense, consistent with the factorization properties of Einstein gravity. Higher-dimension operators can account for fluctuations of short-wavelength degrees of freedom, and so provide a test of whether the continuity conditions inferred from the Einstein-Hilbert action continue to hold beyond the leading order.

At a technical level, we work with “ $f(\text{Riemann})$ ” gravity, where the action density is an arbitrary function of the Riemann tensor, in an auxiliary field formulation [57] where the bulk Lagrangian is second order in derivatives, and we consider “canonical ensemble” and “microcanonical ensemble” versions, where boundary terms are chosen so that the semiclassical variational problem corresponds to fixing the induced boundary metric or fixing the boundary energy and momentum fluxes, respectively. The choice of $f(\text{Riemann})$ gravity could be viewed as one of technical convenience, since we can borrow the auxiliary field formulation of [57]. It is not a completely general effective action, as it omits some terms of higher order in covariant derivatives as well as parity odd operators. However, up to dimension four parity-even operators it is general, and in pure gravity through dimension six it is general up to field redefinitions and total derivatives [58].² To our knowledge there has been limited investigation in the literature of how total derivatives, generated either by integrating out massive fields or by field redefinitions, modify the variational problem needed to implement the semiclassical approximation in effective field theories with boundaries (see, however, [59] and refs therein.) We will not take up this problem here, and in any case it is not important for path integral factorization, where any consistent set of boundary conditions can be considered on a factorization surface. (This is analogous to how we may insert either a complete set of position states or a complete set of momentum states in any expectation value in ordinary quantum mechanics.) Instead, we rewrite the general effective Lagrangian of $f(\text{Riemann})$ type in the auxiliary field formulation of [57], where it is particularly transparent how to implement a consistent Dirichlet-type, or “canonical ensemble,” boundary condition. The auxiliary field formulation has been useful for defining a Brown-York quasi-local stress tensor for curvature squared Lagrangian [60], with sensible results. The auxiliary field formulation is also essential to pass to a Hamiltonian formulation, and so cleanly facilitates the construction of a second set of consistent “microcanonical ensemble”

¹See also [56] and refs therein.

²The dimension four terms R^2 and $R_{ab}R^{ab}$ may be removed by a field redefinition $g_{ab} \rightarrow g_{ab} + bR_{ab} + cRg_{ab}$, modulo a shift in $1/G_N$ proportional to the cosmological constant. The third independent dimension four term may be taken to be the Gauss-Bonnet term, a total derivative. The dimension six terms may be categorized as of $f(\text{Riemann})$ type, total derivatives, and terms proportional to the vacuum Einstein equation. The latter can again be removed by field redefinitions, up to total derivatives and terms of higher dimension.

boundary conditions. In any event, the class of theories we examine is illustrative of a general procedure which can be applied to other effective actions; for example, higher derivatives may be accommodated by expanding the set of auxiliary fields.

Although we do not know the precise generalization of the Euclidean SdS solution in an arbitrary $f(\text{Riemann})$ theory, the presence of two Euclidean horizons at different temperatures implies that a lapse discontinuity is unavoidable. To that end, one of the main results we will obtain is that the microcanonical boundary condition (and associated microcanonical factorization and constraints) can be naturally generalized to theories with higher dimension operators, in such a way that the lapse function is still not part of the boundary data in boundary-value problems or part of the continuous field data in factorization problems. Thus we may infer that a similar “saddle point of a constrained path integral” interpretation will apply to whatever geometry corrects Euclidean SdS in the semiclassical treatment of more general gravitational effective actions, at least ones that may be cast by field redefinitions to the $f(\text{Riemann})$ class. In the simplest case of Einstein gravity plus curvature-squared corrections, SdS remains an exact solution, and we show that the Wald entropy [61], [62] matches the thermodynamic entropy computed using the BTZ method which is well-adapted to the Hamiltonian formalism [63].

The organization of this chapter is as follows. In Section 2.2 we set up the notation and conventions for this chapter. In Sec. 2.3 we give an overview of our approach to path integral factorization. For a given action we explain the importance of the set $\mathcal{B}_i$, which is the set of continuous fields required for the action to satisfy the additivity property. This is most naturally identified in the purely bulk form of the action. We also point out the close connection between the set $\mathcal{B}_i$ and the boundary conditions needed for a well-posed variational problem. In Sec. 2.4 we review the derivation of the canonical action I_{ADM} for $f(\text{Riemann})$ effective Lagrangians, the path integral factorization of which we study in Sec. 2.5. We provide the proof of the Gauss-Codazzi-Ricci equations in Appendix A.1. We define the canonical path integral in Eq. (2.39) and in Appendix A.3 we exhibit the reduction of the canonical path integral to that of General Relativity for a specific function f . We end Section 2.4 with remarks regarding local symmetries in the canonical form of the action. Section 2.5 is split into subsections 2.5.1 and 2.5.2 focused on the canonical and microcanonical ensemble respectively. In subsection 2.5.1 we find the set $\mathcal{B}_i$ for the canonical ensemble by expressing the action in purely bulk form. The purely bulk form of the canonical action facilitates defining the microcanonical ensemble, following [64], in subsection 2.5.2. We also identify the set $\mathcal{B}_i$ for the microcanonical factorization, which, among other fields, contains the surface energy density ϵ_{BY} and surface momentum density J_a , for which explicit expressions are found in Eq. (2.51). Details of the derivation as well as the full form of the purely bulk form of the canonical/microcanonical action are given in Appendices A.2, A.4. In Section 2.6 we turn to applications. We show that SdS in 4 dimensions is a constrained state for the general parity-even effective action with operators up to dimension 4. We calculate the constraints and show that SdS satisfies the equation of motion on either side of the constrained surface. We also compute the entropy from the Euclidean on-shell action and verify that it matches Wald’s formula. The details of these computations are presented in Appendix A.5. Finally, we end with some discussion and outlook in Section 2.7.

2.2 Conventions

In this chapter we consider ambient D -dimensional spacetime $\mathcal{M}$ with metric g_{MN} , cf. Fig. 2.1. The time coordinate is labeled by τ and is timelike/spacelike for $\sigma = -1/+1$. All our expressions up to Section 2.6 will

hold for general signature, except the explicit path integral expressions in Eqs. (2.10), (2.39), and (2.56); it is straightforward to modify them to cover Lorentzian signature. In Section 2.6 we will work with Euclidean signature only, i.e. $\sigma = +1$. Capital Greek letters $\{M, N, K, \Lambda, \dots\}$ label the coordinates x^M of $\mathcal{M}$ and ∇_M is the metric-compatible covariant derivative on $\mathcal{M}$.

Σ_τ denotes codimension-1 hypersurfaces of constant time τ with induced metric $h_{\mu\nu}$. Lowercase Greek letters $\{\mu, \nu, \kappa, \lambda, \dots\}$ (except ρ) label coordinates x^μ of Σ_τ , and D_μ denotes the induced covariant derivative on Σ_τ , defined as

$$D_K T^{M_1 \dots M_p}_{N_1 \dots N_q} := \gamma^{M_1}_{M'_1} \dots \gamma^{M_p}_{M'_p} \gamma^{N'_1}_{N_1} \dots \gamma^{N'_q}_{N_q} \gamma^\Sigma_K \nabla_\Sigma \left(\gamma^{M'_1}_{M''_1} \dots \gamma^{N''_1}_{N'_1} \dots T^{M''_1 \dots M''_p}_{N''_1 \dots N''_q} \right), \quad (2.1)$$

where $\gamma_{MN} = g_{MN} - \sigma n_M n_N$ is the projector on Σ_τ .

$\mathcal{T}$ denotes the codimension-1 factorization surface. It is defined as a surface of constant spatial coordinate $\rho(x^\mu)$. Throughout this work we assume $\mathcal{T}$ intersects Σ_τ normally. When the factorization surface is fixed to a particular $\rho = r$ we denote it as $\mathcal{T}_r$.

$\mathcal{S}_\tau = \Sigma_\tau \cap \mathcal{T}$ denotes codimension-2 surfaces defined as the intersection of Σ_τ and $\mathcal{T}$ with induced metric s_{ab} . Lowercase Latin letters $\{a, b, c, d, \dots\}$ label coordinates x^a of $\mathcal{S}_\tau$ and $^{(2)}D_\mu$ denotes the codimension-2 induced covariant derivative on $\mathcal{S}_\tau$. In our analysis we will assume $\mathcal{S}_\tau$ surfaces are compact and thus drop boundary terms of the form $\int d^{D-2}x \sqrt{s} {}^{(2)}D_a T^a$. For SdS in $D = 4$ the surfaces $\mathcal{S}_\tau$ are 2-spheres.

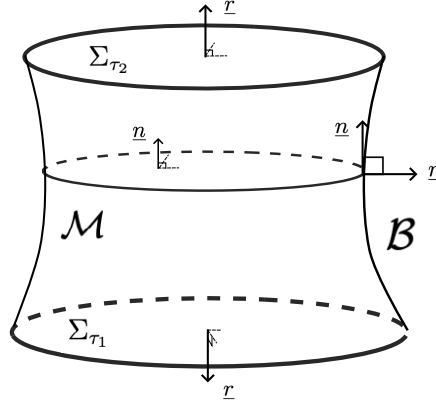


Figure 2.1: The manifold $\mathcal{M}$ with boundary $\partial\mathcal{M} = \Sigma_{\tau_1} \cup \mathcal{B} \cup \Sigma_{\tau_2}$. The unit normal $\underline{n}$ is future pointing. The unit normal $\underline{r}$ at $\partial\mathcal{M}$ is defined as outwards pointing. We assume the Σ_τ surfaces intersect $\mathcal{B}$ normally, i.e. $g_{MN} n^M r^N = 0$. The boundary $\mathcal{B}$ does not have to be connected, but is drawn that way for simplicity. Figure taken from [53].

For the divergence theorem in pseudo-Riemannian manifolds we assume a volume $\mathcal{M}$ with boundary $\partial\mathcal{M} = \cup_i \Sigma_i$ comprised from surfaces Σ_i that are solely spacelike/timelike. We define the outward unit normal r_M to the boundary Σ_i with $r_M r^M = \epsilon_i$ and projector $r_{MN} = g_{MN} - \epsilon_i r_M r_N$. The divergence theorem is then

$$\int_{\mathcal{M}} d^D x \sqrt{|g|} \nabla_M A^M = \int_{\partial\mathcal{M}} d^{D-1} y \epsilon_i \sqrt{|r|} r_M A^M, \quad (2.2)$$

where $|r|$ is the determinant of the induced metric on $\partial\mathcal{M}$. Note that in the case where $\sigma = -1$, the surfaces $\Sigma_{1,2} := \Sigma_{\tau_1, \tau_2}$ of Figure 2.1 are spacelike with $\epsilon_{1,2} = -1$. For the timelike boundary $\Sigma_3 := \mathcal{B}$ we have $\epsilon_3 = +1$. We will assume $\mathcal{B}$ and Σ_τ intersect normally. We will also drop corner terms, namely boundary total derivatives.

2.3 Factorization Methodology

In this section we review some aspects of the factorization properties of gravitational path integrals at leading order in the semiclassical expansion. Restricting to the leading order mainly allows us to be relaxed regarding local functional measures. (For subleading semiclassics there are interesting and subtle issues and ambiguities regarding the path integral measure that will not be addressed here.)

On a manifold $\mathcal{M}$ of fixed topology the canonical gravitational action takes the form

$$I_{\text{ADM}}(\mathcal{M}) = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \left(\mathcal{P}_i \partial_\tau \mathcal{Q}^i \right) - H(\mathcal{Q}^i, \mathcal{P}_i, \lambda_m), \quad (2.3)$$

where λ_m are fields that act as Lagrange multipliers. The corresponding path integral is

$$Z = \int_{\mathcal{M}} \mathcal{D}\mathcal{Q}^i \mathcal{D}\mathcal{P}_i \mathcal{D}\lambda_m e^{-I_{\text{ADM}}(\mathcal{M})}. \quad (2.4)$$

Suppose that $\mathcal{M}$ is partitioned into two regions $\mathcal{M}_1, \mathcal{M}_2$ separated by a surface $\mathcal{T}$ with $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. Then it is natural to try to write a factorized version of the path integral,

$$Z = \int_{\mathcal{T}} \mathcal{D}\mathcal{B}_i Z(\mathcal{M}_1|\mathcal{B}_i) Z(\mathcal{M}_2|\mathcal{B}_i). \quad (2.5)$$

Here the integrand is written as a product of two path integrals over $\mathcal{M}_{1,2}$ with fixed boundary data $\mathcal{B}_i$ at the interface $\mathcal{T}$. In the subscript of the integral in (2.5), $\mathcal{T}$ is a mnemonic indicating that the integration variables are fields defined on the boundary $\mathcal{T}$. The question is which field variables on $\mathcal{T}$ are included in $\mathcal{B}_i$?

To answer this question, it is convenient to write the action in a *purely bulk form*, where any boundary terms (like the Gibbons-Hawking-York term, or the Hamiltonian boundary terms) are rewritten as bulk total derivatives. Suppose that the action has been written in such a form. Then Eq. (2.5) can hold if the path integrals are over fields satisfying the additivity property

$$I_{\text{ADM}}(\mathcal{M}) = I_{\text{ADM}}^{(1)}(\mathcal{M}_1) + I_{\text{ADM}}^{(2)}(\mathcal{M}_2). \quad (2.6)$$

The fields $\mathcal{B}_i$ on $\mathcal{T}$ are given by the restriction to $\mathcal{T}$ of all fields on $\mathcal{M}$ whose continuity across $\mathcal{T}$ is required by the additivity property (2.6). (We use “fields” as a general term that encompasses fundamental fields, their derivatives and linear combinations, etc.) In the purely bulk form of the action, additivity will hold if there are no Dirac delta functions in the action density at $\mathcal{T}$. Products of discontinuous functions are permissible and such configurations can play an important physical role.³

Let us consider a concrete example. We adopt a coordinate system where $\mathcal{T} = \mathcal{T}_r$ lies at a fixed “radial” coordinate $\rho = r$. (Here and elsewhere “radial” will be a general term referring to some spatial coordinate, a level surface of which specifies a factorization surface.) Suppose that in a purely bulk form of the action all

³The simplicity of these conditions motivate but do not require the use of the purely bulk form of the action. If we prefer a form of the action with explicit boundary terms, then the right-hand side of (2.6) includes these terms on either side of $\mathcal{T}$. In this case, the continuity properties required for factorization permit integrable Dirac delta function singularities on $\mathcal{T}$ in the action density on the left-hand side of (2.6), so long as they are captured by the boundary terms on the right-hand side of (2.6).

the terms with radial derivatives have the following form

$$I(\mathcal{M}) = \int_{\mathcal{M}} A \partial_{\rho} B, \quad (2.7)$$

where A, B are fields whose exact identity is not needed for now. It is clear that the additivity property holds as long as B is continuous in ρ at $\mathcal{T}_r$. The field A can be discontinuous in ρ . Thus the field B belongs to the set $\mathcal{B}_i$ mentioned before and must be integrated on $\mathcal{T}_r$ in the factorized form of the path integral (2.5), while the field A is integrated over separately in each path integral appearing in the integrand of (2.5). A standard example of this type is a canonical path integral in quantum mechanics, where ρ is a time coordinate, B is a particle coordinate, and A is its conjugate momentum.

There is also a close relationship between the continuity properties required for the factorization in (2.5) and the boundary conditions required for a well-defined variational problem in each of the subregions $\mathcal{M}_{1,2}$. The functional derivatives of the actions $I^{(1,2)}(\mathcal{M}_{1,2})$ are well defined for variations in which Dirichlet conditions are imposed on the $\mathcal{B}_i$. If we restrict attention to fields for which the additivity property holds, then we can implement the saddle point approximation on each path integral $Z(\mathcal{M}_{1,2}|\mathcal{B}_i)$ separately, with consistent boundary data on $\mathcal{T}_r$.

If the variational problem is well-defined with Dirichlet conditions on the $\mathcal{B}_i$, then we can also apply the saddle approximation to a constrained path integral, namely a path integral where the values of the fields $\mathcal{B}_i$ are held fixed on $\mathcal{T}_r$ and *not* integrated over. This is similar to inserting a projection into a matrix element, or a delta functional into a path integral. The Euclidean SdS solution is precisely a saddle point solution of such constrained path integral when the I_{ADM} is the Einstein-Hilbert (EH) action with microcanonical boundary conditions [18].

In the next sections of this chapter, we will determine the continuity properties of fields required for factorization of the gravitational path integral, with action given by a general functional of the Riemann tensor, and with microcanonical boundary terms. We focus on the phase space path integral which is a natural formulation for discussing the microcanonical ensemble. As an example application of our results, in Section 2.6 we will show that Euclidean SdS is again a saddle point solution of $D = 4$ $f(\text{Riemann})$ gravity theories with microcanonical boundary conditions and operators up to dimension 4, and the on-shell action computes the leading EFT corrections to the entropy.

2.4 Canonical action of $f(\text{Riemann})$ gravity

In this section we review the derivation of the canonical action I_{ADM} for $f(\text{Riemann})$ theories of gravity, originally presented in [57]. The main result is given in Eq. (2.36), which is input to the canonical path integral factorization analysis in Section 2.5. In this section we keep track both Lorentzian/Euclidean $\sigma = -1/+1$ signature of the time coordinate.

$f(\text{Riemann})$ gravity theories have action of the following form:⁴

$$S[g_{MN}] = \frac{1}{2} \int_{\mathcal{M}} d^D x \sqrt{|g|} f(\mathcal{R}^{MN}{}_{K\Lambda}), \quad (2.8)$$

⁴For now we ignore boundary terms in the action. We will introduce them in Eq. (2.10) after we have defined the auxiliary fields needed for the Hamiltonian description.

where $\mathcal{R}^{MN}_{K\Lambda}$ is the Riemann tensor associated with g_{MN} . The Euler-Lagrange equations are

$$\mathcal{R}^{(M}_{K\Lambda P} \frac{\partial f}{\partial \mathcal{R}_{N)K\Lambda P}} - 2\nabla_K \nabla_\Lambda \frac{\partial f}{\partial \mathcal{R}_{K(MN)\Lambda}} - \frac{1}{2} f g^{MN} = 0. \quad (2.9)$$

The action (2.8) generically contains second-order time derivatives which is not suitable for Hamiltonian analysis. To convert the equations of motion (2.9) into equivalent first order differential equations, one can introduce auxiliary fields $\varrho_{MNK\Lambda}, \varphi^{MNK\Lambda}$ that act as Lagrange multipliers. The action and path integral proposed in [57] is ⁵

$$I_{\mathcal{L}} = \frac{1}{2} \int_{\mathcal{M}} d^D x \sqrt{|g|} [f(\varrho^{MN}_{K\Lambda}) + \varphi^{MNK\Lambda} (\mathcal{R}_{MNK\Lambda} - \varrho_{MNK\Lambda})] - \mathcal{S}_{\text{GHY}}, \quad (2.10)$$

$$Z = \int \mathcal{D}g_{MN} \mathcal{D}\varrho^{MN}_{K\Lambda} \mathcal{D}\varphi^{MNK\Lambda} e^{-I_{\mathcal{L}}[g_{MN}, \varrho^{MN}_{K\Lambda}, \varphi^{MNK\Lambda}]}. \quad (2.11)$$

The fields $\varrho_{MNKL}, \varphi^{MNKL}$ are assumed to have the same symmetries as the Riemann tensor. The generalized Gibbons-Hawking-York (GHY) term $\mathcal{S}_{\text{GHY}}$, whose role we will see shortly, is defined as [66] (see also [67])

$$\mathcal{S}_{\text{GHY}} = \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{|r|} \mathcal{K}_{MN} \ominus^{MN}, \quad (2.12)$$

where $|r|$ is the determinant of the induced metric on $\partial\mathcal{M}$ and

$$\mathcal{K}_{MN} := r_M^K r_N^\Sigma \nabla_K r_\Sigma, \quad \ominus^{MN} := -2r^{MA} r^{NB} \varphi_{A\Gamma B\Delta} r^\Gamma r^\Delta. \quad (2.13)$$

Note that $\mathcal{K}_{MN}$ is the extrinsic curvature of $\partial\mathcal{M}$, and by definition r_M is the outward pointing unit normal to $\partial\mathcal{M}$ with $r_M r^M = \epsilon$. The tensor $r_{MN} = g_{MN} - \epsilon r_M r_N$ is the projector (cf. Figure 2.1) on $\mathcal{M}$. For timelike outward normal unit vector $\underline{r}$ we have $\epsilon = -1$ and for spacelike $\epsilon = +1$.

At the classical level the auxiliary action (2.10) and $f(\text{Riemann})$ action in (2.8) are equivalent, resulting in the same equations of motion for g_{MN} . To see this we take the functional variation of the auxiliary action (2.10)

$$\begin{aligned} \delta I_{\mathcal{L}} = & \frac{1}{2} \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[E^{MN} \delta g_{MN} + \left(\frac{\partial f}{\partial \varrho^{MN}_{P\Sigma}} - \varphi^{ABP\Sigma} g_{AM} g_{BN} \right) \delta \varrho^{MN}_{P\Sigma} \right. \\ & \left. + \delta \varphi^{MNP\Sigma} (\mathcal{R}_{MNP\Sigma} - \varrho_{MNP\Sigma}) \right] \\ & + \epsilon \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{|r|} (r_\Sigma (\nabla_P \varphi^{MPN\Sigma}) \delta g_{MN} - r_\Sigma \varphi^{MPN\Sigma} \nabla_P \delta g_{MN}) - \delta \mathcal{S}_{\text{GHY}}, \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} E^{MN} = & \frac{1}{2} g^{MN} f(\varrho^{AB}_{\Gamma\Delta}) + \frac{1}{2} g^{MN} \varphi^{ABP\Sigma} (\mathcal{R}_{ABP\Sigma} - \varrho_{ABP\Sigma}) \\ & - 2\varphi^{(M|ABP} \varrho^{N)}_{ABP} + \varphi^{ABP(M} \mathcal{R}_{ABP}{}^{N)} + 2\nabla_A \nabla_B \varphi^{A(MN)B}. \end{aligned} \quad (2.15)$$

⁵It was noted in [57] that for the exceptional case of Lovelock gravity, the phase space reduction of the auxiliary field formulation does not straightforwardly lead to the Lovelock Hamiltonian found in [65], and to make sense of the reduced Lagrangian theory a more complicated boundary term is needed. We do not consider reductions in this chapter and such issues do not arise.

The equations of motion are

$$E^{MN} = 0, \quad \varphi_{MN}{}^{P\Sigma} = \frac{\partial f}{\partial \varrho^{MN}{}_{P\Sigma}}, \quad \varrho_{MNK\Lambda} = \mathcal{R}_{MNK\Lambda}. \quad (2.16)$$

If we substitute the second and third equations of (2.16) above into $E^{MN} = 0$ we find the f (Riemann) metric equations of motion given in Eq. (2.9). The role of generalized GHY term (2.12) is to make the variational problem of the action (2.10) well posed [66] with the following Dirichlet boundary conditions on $\partial\mathcal{M}$ ⁶

$$\delta r^{MN}|_{\partial\mathcal{M}} = \delta (r_K r_\Lambda \varphi^{MKN\Lambda})|_{\partial\mathcal{M}} = 0. \quad (2.17)$$

The utility of the auxiliary action (2.10) is that it contains only first-order time derivatives, suitable for canonical analysis. We assume a “time” function $\tau : \mathcal{M} \rightarrow \mathbb{R}$ that defines a foliation of spacetime $\mathcal{M}$. The surfaces of constant τ could be timelike ($\sigma = -1$) or spacelike ($\sigma = +1$). The metric is decomposed into an Arnowitt-Deser-Misner (ADM) [68] form adapted to the foliation:

$$ds^2 = \sigma N^2 d\tau^2 + h_{\mu\nu} (dx^\mu + N^\mu d\tau) (dx^\nu + N^\nu d\tau). \quad (2.18)$$

We define the unit normal form to constant time slices⁷

$$n_M = \sigma N \partial_M \tau, \quad N > 0, \quad n_N n^N = \sigma. \quad (2.19)$$

The sign convention for (2.19) is such that the dual unit normal vector $\underline{n}$ points to increasing values of τ , i.e. $n^M \partial_M \tau \geq 0$. The time evolution vector $\underline{\tau} = \frac{\partial}{\partial \tau}$, the normal vector $\underline{n}$ and the shift vector $\underline{N}$ are related by

$$\underline{\tau} = N \underline{n} + \underline{N}. \quad (2.20)$$

The codimension-1 projection tensor γ on constant time surfaces is

$$\gamma_{MN} = g_{MN} - \sigma n_M n_N. \quad (2.21)$$

A very useful property of the projection tensor that we will repeatedly use is that $\gamma^{\mu\nu} = h^{\mu\nu}$ and $\gamma^{MN} = 0$ for $(M, N) \neq (\mu, \nu)$. Here $h^{\mu\nu}$ is the inverse of the induced metric on each time slice Σ_τ . The Gauss-Codazzi and Ricci equations are

$$\begin{aligned} \gamma^\Sigma_A \gamma^K_B \gamma^P_\Gamma \gamma^T_\Delta \mathcal{R}_{\Sigma KPT} &= \sigma (K_{A\Delta} K_{\Gamma B} - K_{A\Gamma} K_{B\Delta}) + R_{AB\Gamma\Delta}, \\ \gamma^\Sigma_A \gamma^K_B \gamma^P_\Gamma \mathcal{R}_{\Sigma KPT} n^T &= D_A K_{B\Gamma} - D_B K_{A\Gamma}, \\ \gamma^\Lambda_M \gamma^K_N \mathcal{R}_{\Lambda PK\Sigma} n^P n^\Sigma &= K_M{}^\Sigma K_{\Sigma N} - \sigma N^{-1} D_M D_N N - N^{-1} \mathcal{L}_{N\underline{n}} K_{MN}, \end{aligned} \quad (2.22)$$

where D_M is the induced covariant derivative on each time slice (see definition in Section 2.2) and K_{MN} is the extrinsic curvature tensor of the Σ_τ hypersurface,

$$K_{MN} := \gamma^K_M \gamma^\Lambda_N \nabla_K n_\Lambda. \quad (2.23)$$

⁶ $\delta r^{MN}|_{\partial\mathcal{M}}$ implies only the induced metric on $\partial\mathcal{M}$ needs to be fixed. Also note that $\partial\mathcal{M} = \Sigma_{\tau_1} \cup \mathcal{B} \cup \Sigma_{\tau_2}$ as depicted in Figure 2.1.

⁷Raising the index we get $\underline{n} = N^{-1} \partial_\tau - N^{-1} N^\mu \partial_\mu$.

In Appendix A.1 we provide the proof of Eqs. (2.22).

Performing a $(D-1)+1$ decomposition of the fields $\varphi^{MNK\Lambda}$, $\mathcal{R}_{MNK\Lambda}$, $\varrho_{MNK\Lambda}$, using the projection tensor γ^M_N and Eqs. (2.22), the action (2.10) becomes

$$I_{\mathcal{L}} = \int_{\mathcal{M}} d^D x \mathcal{L} - \sigma \int d^D x \sqrt{|g|} \nabla_M (n^M K \cdot \Psi) - \mathcal{S}_{\text{GHY}}, \quad (2.24)$$

where⁸

$$\begin{aligned} \mathcal{L} = N\sqrt{h} & \left[\frac{1}{2} f(\varrho^{MN}{}_{K\Lambda}) + \frac{1}{2} \phi^{\mu\nu\kappa\lambda} (\sigma K_{\mu\lambda} K_{\kappa\nu} - \sigma K_{\mu\kappa} K_{\nu\lambda} + R_{\mu\nu\kappa\lambda} - \rho_{\mu\nu\kappa\lambda}) \right. \\ & + 2\sigma \phi^{\mu\nu\kappa} (D_\mu K_{\nu\kappa} - D_\nu K_{\mu\kappa} - \rho_{\mu\nu\kappa}) \\ & \left. + \sigma \Psi^{\mu\nu} ((K \cdot K)_{\mu\nu} - \sigma N^{-1} D_\mu D_\nu N + K K_{\mu\nu} - \Omega_{\mu\nu}) + \sigma N^{-1} K_{\mu\nu} \mathcal{L}_{N\underline{n}} \Psi^{\mu\nu} \right], \end{aligned} \quad (2.25)$$

and the “spatial” fields are

$$\begin{aligned} \rho_{\mu\nu\kappa\lambda} &:= \varrho_{\mu\nu\kappa\lambda}, \quad \rho_{\mu\nu\kappa} := \varrho_{\mu\nu\kappa P} n^P, \quad \Omega_{\mu\nu} := \varrho_{\mu K \nu \Lambda} n^K n^\Lambda, \\ \phi^{\mu\nu\kappa\lambda} &:= h^{\mu\mu'} h^{\nu\nu'} h^{\kappa\kappa'} h^{\lambda\lambda'} \varphi_{\mu'\nu'\kappa'\lambda'}, \quad \phi^{\mu\nu\kappa} := h^{\mu\mu'} h^{\nu\nu'} h^{\kappa\kappa'} \varphi_{\mu'\nu'\kappa' P} n^P, \\ \Psi^{\mu\nu} &:= 2\sigma h^{\mu\mu'} h^{\nu\nu'} n^K n^\Lambda \varphi_{\mu' K \nu' \Lambda}, \end{aligned} \quad (2.26)$$

with $A \cdot B := A_{\mu\nu} B^{\mu\nu}$, $(A \cdot B)_{\mu\nu} := A_{\mu\lambda} B^\lambda{}_\nu$. Note that the Riemannian symmetries of $\varphi^{MNK\Lambda}$, $\varrho_{MNK\Lambda}$ imply $\phi^{(\mu\nu)\lambda} = \Omega_{[\mu\nu]} = \Psi^{[\mu\nu]} = 0$ and $\phi^{\mu\nu\kappa\lambda}$ has the same symmetries as the Riemann tensor. In equation (2.25) we assume $f(\varrho_{MNK\Lambda})$ has been written in terms of the fields $\rho_{\mu\nu\kappa\lambda}$, $\rho_{\mu\nu\kappa}$, $\Omega_{\mu\nu}$, $h_{\mu\nu}$.

There are two boundary terms appearing in the Lagrangian (2.24). The first boundary term comes from the Gauss-Codazzi decomposition, and we will refer it as the Gauss-Codazzi boundary term, while the second is the generalized GHY term (2.12) introduced at the beginning of this section. Assuming the boundary is of the form $\partial\mathcal{M} = \Sigma_{\tau_1} \cup \mathcal{B} \cup \Sigma_{\tau_2}$ as shown in Figure 2.1, we can use the divergence theorem (2.2) on the Gauss-Codazzi boundary term and check that it exactly cancels the GHY term on Σ_{τ_1, τ_2} . Meanwhile it vanishes on the boundary $\mathcal{B}$, since we assume the *orthogonality relation* $g^{MN} n_M r_N = 0$. Thus the only boundary term surviving in the form (2.24) is the GHY term (2.12) on the boundary $\mathcal{B}$, labeled as $\mathcal{S}_{\text{GHY}}^{\mathcal{B}}$

$$I_{\mathcal{L}} = \int_{\mathcal{M}} d^D x \mathcal{L} - \mathcal{S}_{\text{GHY}}^{\mathcal{B}}, \quad (2.27)$$

where $\mathcal{L}$ is given in (2.25). The GHY term expressed in terms of the spatial fields is

$$\mathcal{S}_{\text{GHY}}^{\mathcal{B}} = \int_{\mathcal{B}} d^{D-1} x \sqrt{|r|} \left[-2k_{\mu\nu} \phi^{\mu\gamma\nu\delta} r_\gamma r_\delta - 4\sigma r^\beta K_{\beta\alpha} \phi^{\alpha\gamma\delta} r_\gamma r_\delta - (D_{\underline{r}} \ln N) (r_\alpha r_\beta \Psi^{\alpha\beta}) \right], \quad (2.28)$$

where $\sqrt{|r|} = N\sqrt{s}$, $D_{\underline{r}} := r^\mu D_\mu$ and r_μ is the unit normal one form at $\mathcal{B}$ which is also tangent to Σ_τ due to the orthogonality assumption.

⁸From (2.20) we have $\mathcal{L}_{N\underline{n}} \Psi^{\mu\nu} = \dot{\Psi}^{\mu\nu} - \mathcal{L}_{\underline{N}} \Psi^{\mu\nu}$.

The Lagrangian (2.25) is first order in time derivatives since the constant time extrinsic curvature tensor is⁹

$$K_{\mu\nu} = \frac{1}{2N}(\partial_\tau h_{\mu\nu}) - \frac{1}{2N}\mathcal{L}_{\underline{N}}h_{\mu\nu}, \quad (2.29)$$

The non-vanishing canonical momentum densities are

$$p^{\mu\nu} := \frac{\delta\mathcal{L}}{\delta(\partial_\tau h_{\mu\nu})}, \quad \Pi_{\mu\nu} := \frac{\delta\mathcal{L}}{\delta(\partial_\tau \Psi^{\mu\nu})}, \quad (2.30)$$

and can be calculated from the action (2.27). We find

$$\begin{aligned} p^{\mu\nu} &= \sigma\sqrt{h}\left[\phi^{\mu\alpha\beta\nu}K_{\alpha\beta} - 2N^{-1}D_\beta\left(N\phi^{\beta(\mu\nu)}\right) + (\Psi \cdot K)^{(\mu\nu)} + \frac{1}{2}(K \cdot \Psi)h^{\mu\nu} \right. \\ &\quad \left. + \frac{1}{2}K\Psi^{\mu\nu} + \frac{1}{2}N^{-1}\mathcal{L}_{N\underline{n}}\Psi^{\mu\nu}\right], \\ \Pi_{\mu\nu} &= \sqrt{h}\sigma K_{\mu\nu}. \end{aligned} \quad (2.31)$$

The Hamiltonian integral is

$$\int_{\tau_1}^{\tau_2} d\tau H = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x (p^{\mu\nu}\partial_\tau h_{\mu\nu} + \Pi_{\mu\nu}\partial_\tau \Psi^{\mu\nu}) - I_{\mathcal{L}}, \quad (2.32)$$

where $I_{\mathcal{L}}$ is the action given in (2.27). We find

$$\int_{\tau_1}^{\tau_2} d\tau H = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \left[N^\mu \mathcal{C}_\mu + N\mathcal{C} + \sqrt{h}D_\mu V^\mu \right] + \mathcal{S}_{\text{GHY}}^{\mathcal{B}}, \quad (2.33)$$

where $\mathcal{S}_{\text{GHY}}$ is given in (2.28) and

$$V^\mu = 2\frac{p^{\mu\nu}}{\sqrt{h}}N_\nu - 2\frac{\Pi_{\alpha\beta}}{\sqrt{h}}\Psi^{\beta\mu}N^\alpha + \Psi^{\mu\beta}D_\beta N - ND_\beta\Psi^{\beta\mu}, \quad (2.34)$$

$$\begin{aligned} \mathcal{C}_\mu &= 2\sqrt{h}D_\nu\left(\frac{\Pi_{\mu\lambda}}{\sqrt{h}}\Psi^{\lambda\nu}\right) - 2\sqrt{h}h_{\mu\sigma}D_\nu\left(\frac{p^{\sigma\nu}}{\sqrt{h}}\right) + \sqrt{h}\left(\frac{\Pi_{\alpha\beta}}{\sqrt{h}}D_\mu\Psi^{\alpha\beta}\right), \\ \mathcal{C} &= \sqrt{h}\left(D_\mu D_\nu\Psi^{\mu\nu} - \frac{1}{2}f(\varrho^{MN}{}_{K\Lambda}) + \sigma\Psi \cdot \Omega\right) + \frac{1}{\sqrt{h}}(2\sigma p \cdot \Pi - \sigma\Psi \cdot \Pi \cdot \Pi - \sigma\Pi(\Psi \cdot \Pi)) \\ &\quad - \frac{\sqrt{h}}{2}\phi^{\mu\nu\kappa\lambda}\left(\frac{2\sigma}{h}\Pi_{\mu\lambda}\Pi_{\nu\kappa} + R_{\mu\nu\kappa\lambda} - \rho_{\mu\nu\kappa\lambda}\right) + 2\sigma\sqrt{h}\phi^{\mu\nu\kappa}\left(\rho_{\mu\nu\kappa} - 2\sigma D_\mu\left(\frac{\Pi_{\nu\kappa}}{\sqrt{h}}\right)\right). \end{aligned} \quad (2.35)$$

From equations (2.27), (2.32) and (2.33) we can express the action (2.10) in *canonical form*

$$I_{\text{ADM}} = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x (p^{\mu\nu}\partial_\tau h_{\mu\nu} + \Pi_{\mu\nu}\partial_\tau \Psi^{\mu\nu} - N\mathcal{C} - N^\mu \mathcal{C}_\mu) + B_{\underline{N}} + B_N, \quad (2.36)$$

⁹For all ambient components K_{MN} we find from (2.23) that $K_{MN} = \frac{1}{2N}\mathcal{L}_{N\underline{n}}\gamma_{MN} = \frac{1}{2N}\dot{\gamma}_{MN} - \frac{1}{2N}\mathcal{L}_{\underline{N}}\gamma_{MN}$.

where $B_{\underline{N}}$, B_N are ¹⁰

$$B_{\underline{N}} = \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2} x \sqrt{s} N^\mu \left(2 \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} - 2 h_{\mu\sigma} \frac{p^{\sigma\nu}}{\sqrt{h}} \right) r_\nu, \quad (2.37)$$

$$B_N = \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2} x \sqrt{s} N (D_\lambda \Psi^{\lambda\nu} - \Psi^{\nu\lambda} D_\lambda \ln N) r_\nu - \mathcal{S}_{\text{GHY}}^{\mathcal{B}}, \quad (2.38)$$

and $\mathcal{C}, \mathcal{C}_\mu$ are given in (2.35) and $\mathcal{S}_{\text{GHY}}^{\mathcal{B}}$ in (2.28). The corresponding canonical path integral is

$$Z = \int \mathcal{D}N \mathcal{D}N^\mu \mathcal{D}h_{\mu\nu} \mathcal{D}p^{\mu\nu} \mathcal{D}\Psi^{\mu\nu} \mathcal{D}\Pi_{\mu\nu} \mathcal{D}[\Omega_{\mu\nu}, \rho_{\mu\nu\kappa\lambda}, \rho_{\mu\nu\kappa} \phi^{\mu\nu\kappa\lambda} \phi^{\mu\nu\kappa}] e^{-I_{\text{ADM}}}, \quad (2.39)$$

where I_{ADM} is given in (2.36). The path integral (2.39) corresponds to the canonical ensemble [54] of $f(\text{Riemann})$ gravity, since part of the boundary data for a well-defined variational problem requires fixing the induced metric on $\partial\mathcal{M}$, as can be verified from Eq. (2.17). We will see this explicitly in the next section.

In Appendix A.3 we reduce the path integral (2.39) to the familiar one of General Relativity when $f^{\text{GR}}(\varrho^{MN}{}_{K\Lambda}) = -\frac{\sigma}{8\pi G_N} (g^{MP} g^{N\Sigma} \varrho_{MNP\Sigma} - 2\Lambda)$.

2.4.1 Local Symmetries

Our analysis is suitable for the evaluation of path integrals, with fixed data on a factorization surface, in the semiclassical approximation. In going beyond the leading order, one must identify the gauge symmetries of the action and fix them. We will not do so here, but we make some brief remarks about this procedure. This is also important for the question whether the auxiliary formulation and original $f(\text{Riemann})$ theory are equivalent off-shell.

In the canonical formalism of GR, the constraints are first class and depend on the canonical variables only. They generate the symmetries of the canonical variables and their Poisson algebra also determines the transformation rules of the lapse and the shifts [70]. We have written the $f(\text{Riemann})$ path integral in the Hamiltonian formalism retaining auxiliary fields, which are extra variables that are not canonical. The status of local symmetries of the off-shell action is more obscure in this formulation of $f(\text{Riemann})$ than it is in ordinary GR.

By solving the equation of motion of the canonical momenta, one can fall back on the Lagrangian formulation (2.10) and write down the path integral with a manifestly diffeomorphism-invariant action. Another option can be the elimination of the auxiliary fields by solving some of their equations of motion. This way one gets an action that depends only on the lapse and the shifts in addition to the canonical variables. The gauge symmetries of this action can be identified as in canonical GR.

In either case, the auxiliary variables are of great use for reading off continuity properties in the path integral factorization, and the Hamiltonian formalism is useful – although not essential – for working out the microcanonical boundary terms. It would be interesting future work to evaluate the path integral in the auxiliary field formulation beyond the leading saddle point approximation.

¹⁰We use the notation B_N , $B_{\underline{N}}$ since they reduce to the General Relativity analogs [69] (see Appendix A.3).

2.5 $f(\text{Riemann})$ Path Integral Factorization

In this section we find the continuity conditions for the path integral factorization of $f(\text{Riemann})$ gravity theories in both canonical and microcanonical ensemble. For this purpose we need to isolate all radial derivatives appearing in the purely bulk form of the canonical action (2.36), which correspond to the canonical ensemble. (Again we stress that “radial” is used as a general term describing a spatial coordinate, constant values of which will label factorization surfaces. By a change of coordinatization we can work with any factorization surface orthogonal to the temporal foliation.) To achieve this we employ another ADM type decomposition of the induced metric and the rest of spatial fields on Σ_τ along a foliation of constant $\rho(x^\mu)$ surfaces. This allows us to implement a Gauss-Codazzi type decomposition of the canonical action and convert it to purely bulk form such that the ρ -derivatives appear explicitly. The purely bulk form is also important from another perspective; it will allow us to define the microcanonical ensemble by adding the necessary boundary terms to the canonical action.

2.5.1 Purely bulk form of I_{ADM}

In Section 2.4 we derived the canonical action I_{ADM} given in Eq. (2.36) for the $f(\text{Riemann})$ Lagrangian (2.10). The canonical action I_{ADM} is obviously not in purely bulk form due to the existence of the boundary terms B_N and $B_{\underline{N}}$ given in Eqs. (2.37)-(2.38). Both these boundary terms can be cancelled by surface terms produced from some of the bulk terms after integration by parts.

To isolate these terms we consider a foliation of the time slices Σ_τ by surfaces of constant radial coordinate $\rho(x^\mu)$. We then employ another ADM decomposition of the induced metric $h_{\mu\nu}$ as follows

$$ds^2|_{\Sigma_\tau} := h_{\mu\nu}dx^\mu dx^\nu = P^2 d\rho^2 + s_{ab}(dx^a + \beta^a d\rho)(dx^b + \beta^b d\rho), \quad (2.40)$$

where P is the radial lapse, $\underline{\beta}$ is the radial shift vector and s_{ab} is the induced metric on the codimension-2 surface $\mathcal{S}_\tau = \Sigma_\tau \cap \mathcal{T}$. The Latin indices label the coordinates x^a on $\mathcal{S}_\tau$. The unit one form r_μ and dual vector $\underline{r}$ normal to ρ -surfaces on Σ_τ are

$$r = P d\rho, \quad \underline{r} = P^{-1} \underline{\partial}_\rho - P^{-1} \underline{\beta}, \quad (2.41)$$

where $\underline{\beta} = \beta^a \underline{\partial}_a$ and $r_\mu r^\mu = 1$. The $\mathcal{S}_\tau$ -projector $r_{\mu\nu} = h_{\mu\nu} - r_\mu r_\nu$ allows us to define the *radial extrinsic curvature tensor*

$$k_{\mu\nu} = r^\sigma{}_\mu r^\lambda{}_\nu D_\sigma r_\lambda. \quad (2.42)$$

One can show that $k_{\mu\nu}$ contains radial derivatives of s_{ab} only. The Riemann tensor on Σ_τ has the familiar Gauss-Codazzi-Ricci decomposition (analogous to Eqs. (2.22))

$$\begin{aligned} r^\sigma{}_\alpha r^\kappa{}_\beta r^\rho{}_\gamma r^\tau{}_\delta R_{\sigma\kappa\rho\tau} &= (k_{\alpha\delta} k_{\gamma\beta} - k_{\alpha\gamma} k_{\beta\delta}) + {}^{(2)}R_{\alpha\beta\gamma\delta} \\ r^\sigma{}_\alpha r^\kappa{}_\beta r^\rho{}_\gamma R_{\sigma\kappa\rho\tau} r^\tau &= {}^{(2)}D_\alpha k_{\beta\gamma} - {}^{(2)}D_\beta k_{\alpha\gamma} \\ r^\lambda{}_\mu r^\kappa{}_\nu R_{\lambda\rho\kappa\sigma} r^\rho r^\sigma &= -k_\mu{}^\sigma k_{\sigma\nu} + {}^{(2)}D_\mu a_\nu - a_\mu a_\nu - D_{\underline{r}} k_{\mu\nu} - 2r_{(\mu} k_{\nu)\lambda} a^\lambda, \end{aligned} \quad (2.43)$$

where ${}^{(2)}D_\mu$ and ${}^{(2)}R_{\alpha\beta\gamma\delta}$ denote the induced covariant derivative and Riemann tensor on $\mathcal{S}_\tau$ and $a_\mu := r^\sigma D_\sigma r_\mu$

is the *radial acceleration*. One can check that a has the general expression

$$a_\mu = -^{(2)}D_\mu \ln P, \quad (2.44)$$

and thus it does not contain any radial derivatives. From the Gauss-Codazzi-Ricci equations in Eqs. (2.43) we can spot where the radial derivatives are hiding in the Riemann tensor components $R_{\mu\nu\kappa\lambda}$. Most of them are contained inside the radial extrinsic curvature tensor $k_{\mu\nu}$ which contain radial derivatives of s_{ab} . There is also a radial derivative of the extrinsic curvature tensor itself that appears as the fourth term in the third equation of (2.43). Additional details of the codimension-2 Gauss-Codazzi decomposition above can be found in Appendix A.2.

To make other radial derivatives in the canonical action visible we first use the radial projector $r_{\mu\nu}$ to decompose (some of) the canonical fields into tensors tangent to $\mathcal{S}_\tau$ similar to the ADM decomposition of the metric field $h_{\mu\nu}$ in (2.40). We find

$$\begin{aligned} N^\mu &= N|^\mu + r^\mu N_0 \\ \Pi_{\mu\nu} &= \Pi|_{\mu\nu} + 2r_{(\mu} \Pi_{\nu)}^\perp + r_\mu r_\nu \Pi_0 \\ \Psi^{\mu\nu} &= \Psi|_{\mu\nu} + 2r^{(\mu} \Psi_{\nu)}^\perp + r^\mu r^\nu \Psi_0 \\ \phi^{\mu\nu\kappa} &= \phi|_{\mu\nu\kappa} + 2r^{[\nu} \phi_{2,|}^{\mu]\kappa} + \left(\phi_{1,|}^{\mu\nu} + 2r^{[\nu} \phi_{\perp}^{\mu]} \right) r^\kappa \\ \phi^{\mu\nu\kappa\lambda} &= \theta|_{\mu\nu\kappa\lambda} + 2\theta|_{\mu\nu}^{[\kappa} r^{\lambda]} + 2\theta_{\perp}^{\kappa\lambda[\mu} r^{\nu]} + 2r^\nu \theta_{\perp}^{\mu[\kappa} r^{\lambda]} + 2r^\mu \theta_{\perp}^{\nu[\lambda} r^{\kappa]}, \end{aligned} \quad (2.45)$$

where we defined the fields

$$\begin{aligned} N|^\mu &:= r^\mu_{\mu'} N^{\mu'}, \quad N_0 := r_\lambda N^\lambda, \\ \Pi|_{\mu\nu} &:= r^\mu_{\mu'} r^\nu_{\nu'} \Pi^{\mu'\nu'}, \quad \Pi_\mu^\perp := r^\mu_{\mu'} \Pi^{\mu'\lambda} r_\lambda, \quad \Pi_0 := \Pi_{\alpha\beta} r^\alpha r^\beta, \\ \Psi|_{\mu\nu} &:= r^\mu_{\mu'} r^\nu_{\nu'} \Psi^{\mu'\nu'}, \quad \Psi_\mu^\perp := r^\mu_{\mu'} \Psi^{\mu'\lambda} r_\lambda, \quad \Psi_0 := \Psi^{\alpha\beta} r_\alpha r_\beta, \\ \theta|_{\mu\nu\kappa\lambda} &:= r^\mu_{\mu'} r^\nu_{\nu'} r^\kappa_{\kappa'} r^\lambda_{\lambda'} \phi^{\mu'\nu'\kappa'\lambda'}, \quad \theta_{\perp}^{\mu\nu\kappa} := r^\mu_{\mu'} r^\nu_{\nu'} r^\kappa_{\kappa'} \phi^{\mu'\nu'\kappa'\lambda} r_\lambda, \quad \theta_{\perp}^{\mu\nu} := r^\mu_{\mu'} r^\nu_{\nu'} \phi^{\mu'\alpha\nu'\beta} r_\alpha r_\beta, \\ \phi|_{\mu\nu\kappa} &:= r^\mu_{\mu'} r^\nu_{\nu'} r^\kappa_{\kappa'} \phi^{\mu'\nu'\kappa'}, \quad \phi_{1,|}^{\mu\nu} := r^\mu_{\mu'} r^\nu_{\nu'} \phi^{\mu'\nu'\lambda} r_\lambda, \quad \phi_{2,|}^{\mu\nu} := r^\mu_{\mu'} r^\nu_{\nu'} \phi^{\mu'\lambda\nu'} r_\lambda, \quad \phi_{\perp}^\mu := r^\mu_{\mu'} \phi^{\mu'\alpha\beta} r_\alpha r_\beta. \end{aligned} \quad (2.46)$$

The symbol $|$ above serves as a reminder that the field is tangent to $\mathcal{S}_\tau$, e.g. $r_\lambda \Psi|^\lambda{}_\nu = r_\lambda \Psi|^\mu{}_\nu = 0$ and similarly for the other fields. The symmetries of the spatial fields are inherited to the projected fields above, e.g. $\phi_{1,|}^{(\mu\nu)} = 0$ since $\phi^{(\mu\nu)\kappa} = 0$. All other spatial fields that do not appear above need not be decomposed since they do not participate with terms that have radial derivatives in the canonical action.

The Gauss-Codazzi equations in Eqs. (2.43) together with the projected fields in Eqs. (2.46) can be used to express the canonical action (2.36) in purely bulk form. For economic purposes we present a shortened

version of the purely bulk form up to a function $\mathcal{G}(\text{fields})$, namely

$$\begin{aligned}
I_{\text{ADM}} = & \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x (p^{\mu\nu} \partial_\tau h_{\mu\nu} + \Pi_{\mu\nu} \partial_\tau \Psi^{\mu\nu}) \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} (D_{\underline{r}} N) \left[2 {}^{(2)}D_\lambda \Psi|^\lambda - k_{\alpha\beta} \Psi|^\alpha{}^\beta + k \Psi_0 - 2a_\lambda \Psi|^\lambda + D_{\underline{r}} \Psi_0 + 2k_{\alpha\beta} \theta|^\alpha{}^\beta + 4\phi|^\lambda \frac{\Pi_\lambda}{\sqrt{h}} \right] \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} \left[(D_{\underline{r}} N|^\mu) \left(2 \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2 \frac{h_{\mu\sigma} p^{\sigma\nu} r_\nu}{\sqrt{h}} \right) + D_{\underline{r}} N_0 \left(\frac{2r^\mu \Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - \frac{2r_\mu p^{\mu\nu} r_\nu}{\sqrt{h}} \right) \right] \\
& - 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N \phi_{2,|}^{\nu\kappa} D_{\underline{r}} \left(\frac{\Pi_{\nu\kappa}}{\sqrt{h}} \right) + 2 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N k_{\mu\nu} D_{\underline{r}} \theta|^\mu{}^\nu \\
& + 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N (D_{\underline{r}} \phi|^\mu) \frac{\Pi_\mu}{\sqrt{h}} \\
& - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N_0 \left(\frac{\Pi_{\alpha\beta}}{\sqrt{h}} D_{\underline{r}} \Psi|^\alpha{}^\beta + 2 \frac{\Pi_\beta}{\sqrt{h}} D_{\underline{r}} \Psi|^\beta + \frac{\Pi_0}{\sqrt{h}} D_{\underline{r}} \Psi_0 \right) \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} \mathcal{G}(k_{\mu\nu}, a_\mu, \text{other field combinations with no radial derivatives}) .
\end{aligned} \tag{2.47}$$

The detailed derivation of (2.47) is presented in Appendix A.4 where the exact form of the function $\mathcal{G}(\text{fields})$ is also given (see Eq. (A.53)). The blue terms in (2.47) are all the terms that contain radial derivatives.

Let us make some important remarks. First, note that the blue terms in (2.47) contain radial (spatial) covariant derivatives and not partial derivatives of the form $A\partial_\rho B$. One could wonder whether there are more radial derivatives in the Christoffel symbols $\gamma^\lambda_{\mu\nu}$ of the spatial metric $h_{\mu\nu}$. From the radial ADM decomposition in Eq. (2.40) one can show that the Christoffel symbols needed contain only (partial) radial derivatives of the induced metric $\partial_\rho s_{ab}$. The reason is that the radial covariant derivatives appearing in (2.47) act only on tensors tangential to $\mathcal{S}_\tau$, which the subscript symbol $|$ signifies, as explained below Eqs. (2.46).¹¹

Second, from the purely bulk form of the canonical action we can easily read off the continuity conditions required for factorization as described in Section 2.3. Since our analysis assumes the factorization surface $\mathcal{T}$ intersects the time slices Σ_τ orthogonally, we have $N_0|_{\mathcal{T}} = 0$ (see Eq. (A.24)), and the purely bulk form in Eq. (2.47) implies that the following fields¹² need to be continuous in ρ across $\mathcal{T}$

$$\mathcal{B}_i^{\text{canonical}} = \{N, N^a, s_{ab}, \theta|^{ab}, \phi|^a, \Psi_0, \text{ and } \Pi_{ab}^|/\sqrt{h}\} \text{ at } \mathcal{T}. \tag{2.48}$$

For the variational problem of the canonical action I_{ADM} the set $\mathcal{B}_i^{\text{canonical}}$ is indeed the set of fields that need to be fixed at the boundary. As we have mentioned, there is a one-to-one relationship between the continuous fields $\mathcal{B}_i$ required for factorization and the boundary conditions in the variational problem.

In a classical boundary value problem, the fields in Eq. (2.48) represent a minimal set of data that must be fixed in order to have a well-defined functional derivative of the canonical action. However, the classical equations of motion will only admit a solution if the boundary data for $\Pi_{ab}^|$ is consistent with Hamilton's first equation (2.30). Relatedly, we have to impose Hamilton's first equation to achieve equivalence with the

¹¹For $D_{\underline{r}}$ acting on tangential tensors on $\mathcal{S}_\tau$ the Christoffels $\gamma^\rho_{\rho\rho}$, $\gamma^a_{\rho\rho}$ are not used. These are the only components that contain radial derivatives of the radial lapse P in (2.40), as we can see explicitly from the formulas given in Eq. (A.20).

¹²Without loss of generality in Eq. (2.48) we replaced lowercase greek $\mu, \nu, \dots$ indices with latin $a, b, \dots$ on $\mathcal{S}_\tau$.

Lagrangian form of the boundary conditions in Eq. (2.17) (using Eq. (2.46) to express (2.17) in terms of projected fields.) In terms of the projected fields the relevant component reads

$$\frac{\Pi_{ab}^|}{\sqrt{h}} = \frac{\sigma}{N} \left(\frac{1}{2} (\partial_\tau s_{ab}) - h_{c(a} {}^{(2)}D_{b)} N^c - k_{ab} N_0 \right). \quad (2.49)$$

The above equation together with the orthogonality condition $N_0|_{\mathcal{T}} = 0$ show that the momenta $\Pi_{ab}^|/\sqrt{h}$ depend only on the boundary metric (and tangential derivatives thereof) that is already fixed in the Lagrangian variational problem. The canonical and Lagrangian problems variational problems are consistent with each other when Hamilton's first equation holds.

Since the boundary metric is fixed in the variational problem, the canonical action in Eq. (2.36) together with the boundary terms in Eqs. (2.37)-(2.38) correspond to the canonical ensemble [54].¹³ In the next section we utilize the purely bulk form in Eq. (2.47) to define the microcanonical ensemble for $f(\text{Riemann})$ gravity theories and find the set $\mathcal{B}_i^{\text{micro}}$.

2.5.2 Microcanonical ensemble for $f(\text{Riemann})$

The canonical ensemble for $f(\text{Riemann})$ corresponds to the canonical action (2.36). The purely bulk form for the canonical ensemble was found in Eq. (2.47) and the continuity conditions across the factorization surface were found in Eq. (2.48). In this section we derive the microcanonical ensemble [64].

The prescription basically amounts to adding appropriate boundary terms to the action that change the boundary conditions on $\mathcal{B}$ from fixed N and shift N^a to fixed *energy-surface density* (Brown-York quasilocal energy) ϵ_{BY} and *momentum surface-density* J_a , while all the other boundary conditions remain the same. From the purely bulk form in Eq. (2.47) we find that the variation among classical solutions of I_{ADM} is

$$\begin{aligned} \delta(I_{\text{ADM}}|_{\text{on-shell}}) &= \int_{\Sigma_{\tau_1}, \Sigma_{\tau_2}} d^{D-1}x (p^{\mu\nu} \delta h_{\mu\nu} + \Pi_{\mu\nu} \delta \Psi^{\mu\nu}) \\ &\quad - \int_{\tau_1}^{\tau_2} d\tau \int_{\mathcal{S}_\tau} d^{D-2}x \sqrt{s} [\epsilon_{\text{BY}} \delta N - J_a \delta N^a - J^{ab} \delta s_{ab}] \\ &\quad + \int_{\tau_1}^{\tau_2} d\tau \int_{\mathcal{S}_\tau} d^{D-2}x \sqrt{s} \left[2N k_{ab} \delta \theta^{ab} + 4N \left(\frac{\Pi_a^|}{\sqrt{h}} \right) \delta \phi_a^| + (D_\tau N) \delta \Psi_0 \right], \end{aligned} \quad (2.50)$$

where

$$\begin{aligned} \epsilon_{\text{BY}} &= -2 {}^{(2)}D_\lambda \Psi^|_\lambda + k_{\alpha\beta} \Psi^{\alpha\beta}_| - k \Psi_0 + 2a_\lambda \Psi^|_\lambda - D_\tau \Psi_0 - 2k_{\alpha\beta} \theta^{|\alpha\beta} - 4\phi^|_\lambda \frac{\Pi^|_\lambda}{\sqrt{h}} - 4\phi^{\mu\nu}_{2,|} \left(\Pi^|_{\mu\nu} / \sqrt{h} \right), \\ J_a &= 2 \frac{\Pi_{a\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2h_{a\sigma} \frac{p^{\sigma\nu}}{\sqrt{h}} r_\nu - 4\sigma s_{af} {}^{(2)}D_b \left(\phi^{(bf)}_{2,|} \right). \end{aligned} \quad (2.51)$$

In the derivation of Eq. (2.50) we used Hamilton's equation (2.49) for the term $\phi_{2,|}^{ab} \delta(\Pi_{ab}^|/\sqrt{h})$, which is responsible for the final terms in the expressions for ϵ_{BY} , J_a in Eq. (2.51). The precise form of J^{ab} is quite complicated but is not needed for our purposes. Note that in the derivation of (2.50) we have assumed the orthogonality condition $N_0|_{\mathcal{T}} = 0$ and as usual dropped total derivatives of the form ${}^{(2)}D_a v^a$. It is now clear from Eq. (2.50) that in order to define the microcanonical action one needs to add the following boundary

¹³This justifies the notation $\mathcal{B}_i^{\text{canonical}}$ in Eq. (2.48).

terms in the canonical action (2.36)

$$I_{\text{micro}} = I_{\text{ADM}} + \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2}x \sqrt{s} [N \epsilon_{\text{BY}} - J_a N^a] . \quad (2.52)$$

From Eq (2.36) and (2.37), (2.38) the microcanonical action (2.52) becomes

$$I_{\text{micro}} = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x (p^{\mu\nu} \partial_\tau h_{\mu\nu} + \Pi_{\mu\nu} \partial_\tau \Psi^{\mu\nu} - NC - N^\mu \mathcal{C}_\mu) + B_{\text{micro}} , \quad (2.53)$$

where the boundary term B_{micro} is

$$B_{\text{micro}} = - \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2}x \sqrt{s} \left[4N \phi_{2,|}^{ab} \frac{\Pi_{ab}^|}{\sqrt{h}} - 4\sigma s_{af} N^a {}^{(2)}D_b \left(\phi_{2,|}^{(bf)} \right) \right] . \quad (2.54)$$

It is interesting to note the appearance of the boundary term above vanishes in GR [64] because in that case we have $\phi_{2,|}^{ab} = 0$.

The variation of $I_{\text{micro}}|_{\text{on-shell}}$ given in Eq. (2.53) is

$$\begin{aligned} \delta (I_{\text{micro}}|_{\text{on-shell}}) &= \int_{\Sigma_{\tau_1}, \Sigma_{\tau_2}} d^{D-1}x (p^{\mu\nu} \delta h_{\mu\nu} + \Pi_{\mu\nu} \delta \Psi^{\mu\nu}) \\ &+ \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2}x \sqrt{s} [N \delta \epsilon_{\text{BY}} - N^a \delta J_a - J^{ab} \delta s_{ab}] \\ &+ \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2}x \sqrt{s} \left[2N k_{ab} \delta \theta_{|}^{ab} + 4N \left(\frac{\Pi_a^|}{\sqrt{h}} \right) \delta \phi_{|}^a + (D_{\underline{r}} N) \delta \Psi_0 \right] , \end{aligned} \quad (2.55)$$

where we used Eq. (2.50).

The corresponding microcanonical path integral is

$$Z^{\text{micro}} = \int \mathcal{D}N \mathcal{D}N^\mu \mathcal{D}h_{\mu\nu} \mathcal{D}p^{\mu\nu} \mathcal{D}\Psi^{\mu\nu} \mathcal{D}\Pi_{\mu\nu} \mathcal{D}[\Omega_{\mu\nu}, \rho_{\mu\nu\kappa\lambda}, \rho_{\mu\nu\kappa} \phi^{\mu\nu\kappa\lambda} \phi^{\mu\nu\kappa}] e^{-I_{\text{micro}}} . \quad (2.56)$$

The purely bulk form of the microcanonical action in Eq. (2.53) is

$$\begin{aligned}
I_{\text{micro}} = & \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x (p^{\mu\nu} \partial_\tau h_{\mu\nu} + \Pi_{\mu\nu} \partial_\tau \Psi^{\mu\nu}) \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \sqrt{h} N \underline{D}_\tau \epsilon_{\text{BY}} - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N^a \underline{D}_\tau J_a + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N \underline{k} \epsilon_{\text{BY}} \\
& - 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} \phi_{2,|}^{\nu\kappa} \underline{D}_\tau \left(N(\Pi_{\nu\kappa}^{|}/\sqrt{h}) + \sigma^{(2)} D_{(\nu} N_{\kappa)}^{|} \right) \\
& + 2 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N k_{\mu\nu} \underline{D}_\tau \theta_{|}^{\mu\nu} + 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N (\underline{D}_\tau \phi_{|}^\mu) \frac{\Pi_\mu^{|}}{\sqrt{h}} \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} (\underline{D}_\tau N_0) \left(2r^\mu \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2r_\mu \frac{p^{\mu\nu}}{\sqrt{h}} r_\nu \right) \\
& - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N_0 \left(\frac{\Pi_{\alpha\beta}^{|}}{\sqrt{h}} \underline{D}_\tau \Psi_{|}^{\alpha\beta} + 2 \frac{\Pi_\beta^{|}}{\sqrt{h}} \underline{D}_\tau \Psi_{|}^\beta + \frac{\Pi_0}{\sqrt{h}} \underline{D}_\tau \Psi_0 \right) \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} \mathcal{G}'(k_{\mu\nu}, a_\mu, \text{other field combinations with no radial derivatives}) ,
\end{aligned} \tag{2.57}$$

where ϵ_{BY} , J_a were defined in (2.51). The precise form of $\mathcal{G}'$ is modified compared to the function $\mathcal{G}$ in Eq. (2.47) and its precise form is not needed, but is presented in full in Appendix A.4 (see Eq. (A.57)). Similarly to the canonical ensemble the blue terms in Eq. (2.57) are all the terms that contain radial derivatives. From the purely bulk form of the microcanonical action we find the continuous fields across $\mathcal{T}$ are

$$\mathcal{B}_i^{\text{micro}} = \{ \epsilon_{\text{BY}}, J_a, s_{ab}, \theta_{|}^{ab}, \phi_{|}^a, \Psi_0, \text{ and } N(\Pi_{ab}^{|}/\sqrt{h}) + \sigma^{(2)} D_{(a} N_{b)}^{|} \} \quad \text{at } \mathcal{T}, \tag{2.58}$$

while the lapse N and shift N^a can be discontinuous, just as in Einstein gravity. One way to impose continuity of $N(\Pi_{ab}^{|}/\sqrt{h}) + \sigma^{(2)} D_{(a} N_{b)}^{|}$ would be to impose Hamilton's first equation (2.49) as a surface condition for (2.17), upon which continuity of $N(\Pi_{ab}^{|}/\sqrt{h}) + \sigma^{(2)} D_{(a} N_{b)}^{|}$ becomes continuity of $\frac{1}{2} \sigma \partial_\tau s_{ab}$, which is already continuous according to (2.58). Since we are discussing factorization, however, which can take place on any surface, this would essentially mean imposing Hamilton's first equation for $\Pi_{ab}^{|}$ everywhere. The precise continuity condition is somewhat more general.¹⁴

The fields in Eq. (2.58) were defined in Eqs. (2.46), (2.26) and the currents $\epsilon_{\text{BY}}, J_a$ in Eq. (2.51). With these continuity properties on the fields the microcanonical action in (2.52) satisfies the additivity property (2.6) and the corresponding path integral (2.56) factorizes

$$Z^{\text{micro}} = \int_{\mathcal{T}} D\mathcal{B}_i^{\text{micro}} Z(\mathcal{M}_1 | \mathcal{B}_i^{\text{micro}}) Z(\mathcal{B}_i^{\text{micro}} | \mathcal{M}_2). \tag{2.59}$$

Let us emphasize the role of the discontinuous lapse allowed by the microcanonical path integral. Such a lapse discontinuity is actually realized semiclassically in a number of simple settings involving horizons out of equilibrium. For example, it allows us to make sense of Euclidean SdS_4 as a saddle point of a constrained path integral, exhibiting two smooth horizons but a finite lapse discontinuity. The constraint amounts to fixing the fields $\mathcal{B}_i^{\text{micro}}$ on the surface $\mathcal{T}$. We will see this in detail in the next section. Finally, as we mentioned above the microcanonical ensemble also allows for discontinuous shift N^a . Although this is not needed for

¹⁴A practical approach to imposing it would be to change path integral variables from $\Pi_{ab}^{|}$ to Z_{ab} that are related by $\Pi_{ab}^{|}/\sqrt{h} = N^{-1} Z_{ab} - \sigma^{(2)} D_{(a} N_{b)}^{|}$, and then enforce continuity of Z_{ab} .

the SdS geometry it is probably important for other geometries, like the (complex Euclidean continuation of) the Kerr-de Sitter spacetime [71].

2.6 Constrained states in EFT

Now we consider a constrained path integral, obtained by inserting a delta functional to Z^{micro} that fixes the microcanonical data (2.58) to the values $\underline{B}_i^{\text{micro}}$ on a codimension-1 surface $\mathcal{T}$. The constrained path integral then becomes

$$Z^{\text{constrained}} = Z(\mathcal{M}_1 | \underline{B}_i^{\text{micro}}) Z(\underline{B}_i^{\text{micro}} | \mathcal{M}_2), \quad (2.60)$$

i.e., Eq. (2.59) with the integral omitted. Similar expressions hold for other types of surface constraints.

The saddle point approximation may then be applied to each of the two factors separately. Since the boundary data $\mathcal{B}_i$ does not generally include all the field data, the semiclassical solution may be discontinuous on $\mathcal{M}$. However, the continuity requirements are inferred from a classical gravitational action, which is scale dependent when coupled to quantum fields. For this reason it is important to establish the continuity conditions implied by general Wilsonian effective actions.

The canonical and microcanonical factorization properties of the leading truncation of the effective action, Einstein gravity, were discussed in [18]. Here we have shown that some of the most physically interesting properties – namely the lack of continuity restrictions on N and N^a – are robust against ultraviolet quantum corrections, in the sense that the more general effective actions studied in the previous section have the same properties. In particular, a low-curvature, discontinuous- N or $-N^a$ saddle point solution to a constrained path integral in Einstein gravity, or to a constrained path integral with a more general Wilsonian gravitational effective action, should provide a good approximation to a saddle point solution with similar discontinuities arising from a new gravitational effective action obtained by a small change in renormalization scale.

(Since we have only worked with the $f(\text{Riemann})$ class of effective actions, which includes Einstein gravity but not all possible higher dimension operators, and moreover since the local effective action does not incorporate all infrared quantum effects, we cannot draw a completely general conclusion about quantum corrections to solutions of Einstein gravity. However, as discussed in the introduction, $f(\text{Riemann})$ is sufficiently general to discuss factorization provided one performs suitable field and coupling redefinitions, at least through the first few orders in the derivative expansion.)

Furthermore, we may use the Euclidean effective action to compute quantum corrections, parameterized by the effective couplings, to the thermodynamic potentials of constrained states. Examples include the free energies of out-of-equilibrium black holes in asymptotically flat or AdS space at finite temperature, and the Schwarzschild de Sitter black hole, which is always a constrained state and always out of equilibrium. Let us work out the latter in $D = 4$, including general curvature-squared terms in the effective action.

We have a path integral over metrics on a manifold of S^4 topology, and a microcanonical constraint applied on a surface corresponding to a fixed “radial” coordinate. The path integral then factorizes into the form (2.60).

We can now apply the saddle point approximation on each factor separately. What are values $\underline{B}_i^{\text{micro}}$ should we fix on $\mathcal{T}$? With malice aforethought we will fix them to be the SdS boundary data. To find their precise form, first note that including operators up to dimension 4, the function $f(\text{Riemann})$ in Eq. (2.8) takes the

form

$$f(\mathcal{R}^{MN}{}_{K\Lambda}) = -\frac{1}{8\pi G_N} (\mathcal{R} - 2\Lambda) + b_1 \mathcal{R}_{MNK\Lambda} \mathcal{R}^{MNK\Lambda} + b_2 \mathcal{R}_{MN} \mathcal{R}^{MN} + b_3 \mathcal{R}^2, \quad (2.61)$$

with $\Lambda > 0$. The Einstein-Hilbert case $b_1 = b_2 = b_3 = 0$ was analyzed in [18]. The auxiliary field Lagrangian $I_{\mathcal{L}}$ corresponding to the above $f(\text{Riemann})$ action is given in Eq. (2.10) with

$$f(\varrho^{MN}{}_{K\Lambda}) = -\frac{1}{8\pi G_N} (\varrho - 2\Lambda) + b_1 \varrho_{MNK\Lambda} \varrho^{MNK\Lambda} + b_2 \varrho_{MN} \varrho^{MN} + b_3 \varrho^2, \quad (2.62)$$

where $\varrho = g^{MP} g^{N\Sigma} \varrho_{MNP\Sigma}$.

The last two equations of motion in Eq. (2.16) become

$$\begin{aligned} \varphi^{MNK\Lambda} &= -\frac{1}{16\pi G_N} (g^{MK} g^{N\Lambda} - g^{NK} g^{M\Lambda}) + 2b_1 \mathcal{R}^{MNK\Lambda} + b_3 \mathcal{R} (g^{MK} g^{N\Lambda} - g^{NK} g^{M\Lambda}) \\ &\quad + \frac{1}{2} b_2 (\mathcal{R}^{N\Lambda} g^{MK} - \mathcal{R}^{M\Lambda} g^{NK} - \mathcal{R}^{NK} g^{M\Lambda} + \mathcal{R}^{MK} g^{N\Lambda}), \\ \varrho_{MNK\Lambda} &= \mathcal{R}_{MNK\Lambda}. \end{aligned} \quad (2.63)$$

Substituting the above into the first equation of motion in (2.16), $E^{MN} = 0$, we find the following metric equation of motion for the EFT Lagrangian in (2.61),

$$\begin{aligned} &-\frac{1}{8\pi G_N} (G^{MN} + \Lambda g^{MN}) - \frac{1}{2} g^{MN} \left((b_2 + 4b_1) \mathcal{R}_{AB} \mathcal{R}^{AB} + (b_3 - b_1) \mathcal{R}^2 - (b_2 + 4b_3) \nabla_K \nabla^K \mathcal{R} \right) \\ &- 2(b_2 + 4b_1) \mathcal{R}_{AB} \mathcal{R}^{MABN} + 2(b_3 - b_1) \mathcal{R} \mathcal{R}^{MN} - (b_2 + 2b_1 + 2b_3) \nabla^M \nabla^N \mathcal{R} \\ &+ (4b_1 + b_2) \nabla^K \nabla_K \mathcal{R}^{MN} = 0. \end{aligned} \quad (2.64)$$

It is now easy to check that general (D=4) Einstein manifolds satisfy the equation of motion if we substitute $R_{MN} = \Lambda g_{MN}$ into Eq. (2.64). Thus SdS is a solution of the gravitational EFT up to curvature-squared operators. To find the constrained values of $\mathbf{B}_i^{\text{micro}}$ we need to compute them for the SdS geometry. Before we present the result let us describe the SdS geometry in more detail.

The Euclidean SdS geometry is

$$ds^2 = \mathcal{N}^2 f(\rho) d\tau^2 + f(\rho)^{-1} d\rho^2 + \rho^2 d\Omega^2, \quad f(\rho) = 1 - 2M/\rho - \rho^2/L^2, \quad (2.65)$$

where $L = \sqrt{3/\Lambda}$ is the de Sitter radius, M the black hole mass parameter and $d\Omega^2$ the S_2 metric element. We impose Euclidean time periodicity $\tau \sim \tau + \beta$ with $\beta = 1/T_b$. We also define the lapse function $\mathcal{N} = 1 + (T_b/T_c - 1)\theta(\rho - r)$ that keeps track of the lapse discontinuity. $T_b = \frac{L^2 - 4r_b^2}{4\pi L^2 r_b}$ is the temperature of the black hole horizon located at $\rho = r_b$, and $T_c = \frac{3r_c^2 - L^2}{4\pi L^2 r_c}$ is the temperature of the cosmological horizon at $\rho = r_c$. The factorization surface $\mathcal{T}_r$ is located at radial coordinate $\rho = r$. The Heaviside step function that appears in lapse function $\mathcal{N}$ indicates that for the black hole side $\rho < r$, we have $\mathcal{N} = 1$, while for the cosmological side $\rho > r$, we have $\mathcal{N} = \frac{T_b}{T_c}$. Note that although both sides have the same Euclidean time periodicity $\beta = 1/T_b$, both horizons are smooth due to the different lapse functions on each side.

The continuation of the discontinuous Euclidean solution (2.65) to Lorentzian singnature can be done as follows [18]. First we rescale the Euclidean time periodicity of the cosmological side such that the lapse is continuous but the time periodicity is not. We then continue $\tau \rightarrow t$ on both sides which erases the

discontinuity in the periodicity by decompactifying t in both regions. The end result is the Lorentzian SdS solution with mass M .

Let us compute the factorization surface values $\underline{B}_i^{\text{micro}}$ (2.58) of the continuous fields for the SdS geometry given in Eq. (2.65). We find

$$\begin{aligned} \epsilon_{\text{BY}}|_{\text{SdS}} &= -2r^{-1}f(r)^{1/2} \left(-\frac{1}{8\pi G_N} + 2\Lambda(4b_3 + b_2) \right) + 2b_1f(r)^{1/2} (2r^{-1}f''(r) + f'''(r)) \\ J_a|_{\text{SdS}} &= 0 \quad , \quad s_{ab}|_{\text{SdS}} = r^2 t_{ab} \quad , \quad \Psi_0|_{\text{SdS}} = -\frac{1}{8\pi G_N} + 2\Lambda(4b_3 + b_2) - 2b_1f''(r) \\ \theta_{|}^{ab}|_{\text{SdS}} &= r^{-2} \left[\left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) - b_1r^{-1}f'(r) \right] t^{ab} \quad , \quad \phi_{|}^a|_{\text{SdS}} = 0 \\ N(\Pi_{ab}^{|}/\sqrt{h}) + {}^{(2)}D_{(a}N_{b)}|_{\text{SdS}} &= 0 \quad , \end{aligned} \tag{2.66}$$

where t_{ab} are the unit round metric components in (θ, φ) coordinates and $f(\rho)$ is given in Eq. (2.65). The details of the proof of Eqs. (2.66) are presented in Appendix A.5. The constrained path integral (2.60) with $\underline{B}_i^{\text{micro}}$ fixed to (2.66) on $\mathcal{T}$ will have the SdS geometry (2.65) on both sides of $\mathcal{T}$ as a saddle point solution. Then in the leading saddle approximation, if there are no saddles of lower action, $Z^{\text{constrained}} \approx Z_{\text{SdS}}^{\text{constrained}}$. The geometry is a genuine saddle because the geometry solves the equations of motion everywhere off of the constraint surface – in particular, both horizons are smooth.

Thus SdS can be interpreted as a constrained state of the gravitational path integral with an effective action up through arbitrary curvature-squared operators. Beyond this order in the derivative expansion of the effective action, the saddle point geometry will receive corrections from the effective couplings. Our results from the previous section show that if the corrections are of $f(\text{Riemann})$ type then there is no obstruction to maintaining a discontinuous lapse function, so we expect that the change in the saddle point geometry will be a small effect, well controlled in the derivative and semiclassical expansions.

Finally, let us compute the curvature squared corrections to the entropy of the constrained SdS state. We need to compute the on-shell value of the microcanonical action (2.53). Although the momenta $p^{\mu\nu}, \Pi_{\mu\nu}$, time derivatives $\partial_\tau h_{\mu\nu}, \partial_\tau \Psi^{\mu\nu}$ and the boundary terms in (2.53) vanish for SdS, one cannot conclude that the on-shell action vanishes. The reason is that the time foliation breaks down at the horizons $r_{b,c}$. To calculate the on-shell action we will use the method of [63] which we refer to as the *BTZ prescription*.

The BTZ prescription is as follows: first we draw infinitesimal boundaries $\mathcal{T}_b$ and $\mathcal{T}_c$ around the black hole and cosmological horizons. To compute the on-shell action we use the ADM form of the microcanonical action for the region between $\mathcal{T}_b$ and $\mathcal{T}_c$, while for the regions inside the boundaries $\mathcal{T}_b$ and $\mathcal{T}_c$ we use the Lagrangian form of the microcanonical action. This means that we use the microcanonical action (2.52) in these infinitesimal regions, but instead of I_{ADM} , we use the Lagrangian form (2.27). For the large bulk region in between $\mathcal{T}_b$ and $\mathcal{T}_c$, the microcanonical action vanishes on-shell for SdS. To see this note that on-shell the lapse and shift constraints should hold $\mathcal{C} = \mathcal{C}_\mu = 0$ and since $\partial_\tau h_{\mu\nu}, \partial_\tau \Psi^{\mu\nu}$ vanish for SdS, the bulk terms in Eq. (2.53) also vanish. The boundary terms B_{micro} in Eq. (2.54) also vanish because $\Pi_{ab}^{|}|_{\text{SdS}} = N^a|_{\text{SdS}} = 0$. For the infinitesimal horizon regions “inside” $\mathcal{T}_b$ and $\mathcal{T}_c$, one can show that the boundary terms in Eq. (2.52) vanish for the infinitesimal surfaces $\mathcal{T}_{b,c}$. The bulk terms in (2.27) also vanish because the curvature is finite but the integration volume goes to zero. The entirety of the classical action is thus due to generalized GHY

terms $\mathcal{S}_{\text{GHY}}^{\mathcal{B}}$ on $\mathcal{T}_{b,c}$. From Eq. (2.12) and the SdS metric (2.65), we find

$$\begin{aligned} I_{\text{micro}}^{\text{SdS}} &= -\frac{\pi r_b^2 + \pi r_c^2}{G_N} + 64\pi^2 b_1 + 48\pi^2 \frac{r_b^2 + r_c^2}{L^2} (4b_3 + b_2) \\ &= -\frac{A_b + A_c}{4G_N} + 64\pi^2 b_1 + 12\pi (4b_3 + b_2) \frac{A_b + A_c}{L^2}, \end{aligned} \quad (2.67)$$

where the horizons r_b, r_c satisfy $f(r_{b,c}) = 0$ with $f(\rho)$ given in (2.65) and $A_{b,c} = 4\pi r_{b,c}^2$. We see that the result is independent of the location r where the constraint was imposed. The entropy is minus the total on-shell action.

We have verified by explicit computation that the same result for the entropy is obtained using Wald's formula [61], [62]. Namely, we find

$$\begin{aligned} S_W^{\text{SdS}} &:= -2\pi \oint_{\text{horzn.}} d\theta d\varphi \sqrt{s} \hat{\epsilon}^{MN} \hat{\epsilon}_{K\Lambda} \frac{\partial \mathcal{L}}{\partial \mathcal{R}^{MN}_{K\Lambda}} \\ &= -I_{\text{micro}}^{\text{SdS}} \end{aligned} \quad (2.68)$$

where $\mathcal{L} = -\frac{1}{2}f(\mathcal{R}^{MN}_{K\Lambda})$ ¹⁵ and $f(\mathcal{R}^{MN}_{K\Lambda})$ was given in Eq. (2.61), $s_{ab} = r^2 t_{ab}$ is the cod-2 induced metric at the horizon, $\hat{\epsilon}_{MN} = \nabla_M \tilde{\xi}_N$ and $\tilde{\xi}^M$ is the Killing field that is null at the horizon with normalized surface gravity $\kappa := -\frac{1}{2}\nabla_A \tilde{\xi}_B \nabla^A \tilde{\xi}^B|_{\text{horzn.}} = 1$. Some details of the computation are presented at the end of Appendix A.5.

Let us make some remarks on the relation between BTZ and Wald's method of calculating the entropy. The relation with the ordinary Gibbons-Hawking Euclidean approach was already pointed by Wald [61], [72]. The equivalence of the former to the BTZ method then implies that we have two methods, one Lorentzian and one Euclidean, for computing horizon entropies just from data localized near the horizons. Thus there should be a geometric demonstration that the two computations are equivalent for geometries with Killing horizons. Let us assume spherical topology for the Euclidean horizon (or bifurcate horizon in Lorentzian signature). Then the BTZ surface $\mathcal{T}_r$ enclosing the Euclidean horizon r_h is $S_{\beta_r}^1 \times S^{D-2}$, where β_r denotes the size of $S_{\beta_r}^1$. In the horizon limit β_r shrinks to zero and the horizon has finite area. For $f(\text{Riemann})$ Lagrangians, we can compare Wald's formula in Eq. (2.68) with the generalized GHY term in Eqs. (2.12), (2.13), and using the on-shell value for $\varphi_{MNK\Lambda}$ given in Eq. (2.16) we find that for the two formulas to match we must have

$$\lim_{r \rightarrow r_h} \beta_r r_{[M} \mathcal{K}_{A][B} r_{N]} = \lim_{r \rightarrow r_h} \left(-\frac{\pi}{2} \nabla_M \tilde{\xi}_A \nabla_B \tilde{\xi}_N \right), \quad (2.69)$$

where the brackets on the left hand side mean anti-symmetrization and the right hand side is computed using the appropriate Euclidean continuation of the Killing vector $\tilde{\xi}$. We note that both sides of Eq. (2.69) should be computed in a coordinate system that is well-defined at the Euclidean horizon, e.g. Kruskal-Szekeres coordinates, where the Riemann components are finite. It is easy to check directly that Eq. (2.69) holds for the Schwarzschild horizon.

Returning to SdS, finally we can estimate that the nucleation probability of black hole of mass M in the de Sitter background is

$$P = Z^{\text{constrained}} / Z \sim e^{S_{\text{SdS}} - S_{\text{dS}}} = e^{-I_{\text{micro}}^{\text{SdS}} - \frac{\pi L^2}{G_N} + 16\pi^2 (4b_1 + 12b_3 + 3b_2)}, \quad (2.70)$$

¹⁵The minus sign in $\mathcal{L} = -\frac{1}{2}f(\mathcal{R}_{MNK\Lambda})$ is due to rotation from Euclidean to Lorentzian signature.

where $I_{\text{micro}}^{\text{SdS}}$ was given in (2.67).

2.7 Conclusions

Let us summarize our main results and sketch some possible directions for future progress.

The primary general result is the derivation of the continuity properties of the gravitational fields in factorizations of the path integral, corresponding to Dirichlet and microcanonical factorizations, in a large class of gravitational effective Lagrangians. These continuity properties can be used to implement the semiclassical approximation with fixed data on a factorization surface, including the insertion of constraints.

Our main application of the general result was to the Euclidean Schwarzschild-de Sitter geometry, where the detailed continuity properties of the factorized path integral are already necessary to make semiclassical sense of the geometry in Einstein gravity. By showing that the continuity properties of Einstein gravity extend in a simple way to a large class of more general effective actions, we conclude that the discontinuity present in Euclidean Schwarzschild-de Sitter is not an accident of working in the leading truncation of the effective Lagrangian, and something SdS-like can be expected to persist after taking into account the quantum fluctuations of massive matter fields. At the next-to-leading truncation, SdS itself remains a solution, and the BTZ method of computing the on-shell action – which reduces to a horizon-localized integral in the Euclidean signature – matches the Lorentzian signature result obtained by Wald’s formula.

Let us now discuss some possible extensions of this work.

As discussed above, we consider effective actions of the $f(\text{Riemann})$ type. This restricts the EFT operators to the parity-even sector with no explicit covariant derivatives. (In some cases, or at least through low orders, it is possible to perform field redefinitions and integrations-by-parts to rewrite the action in the $f(\text{Riemann})$ form, but we are not aware of a general proof that this can be extended to arbitrary order.) At a fixed order in the derivative expansion, we believe it should be possible to incorporate other classes of terms by introducing a finite number of additional auxiliary fields, but it would be interesting to exhibit this construction explicitly.

In addition, as briefly mentioned at the end of Section 2.5, one could investigate other types of factorizations apart from fixing canonical or microcanonical data. For example, one could fix the surface angular momentum density J_a along with ϵ_{BY} to non-zero values. It would be interesting to examine whether the Kerr-de Sitter metrics, which are complex metrics in Euclidean signature, provide a consistent semiclassical approximation of the contribution to the partition function.

The factorization methodology can be applied in other out-of-equilibrium situations for asymptotically AdS/flat spacetimes. Such situations can arise when black holes have different temperature from surrounding matter. In these cases, the Euclidean time periodicity at infinity plays analogous role to the Euclidean periodicity at the cosmological horizon for SdS. Similarly, geometries with lapse discontinuities will arise as saddle points of a constrained path integral.

Finally, we have confined our analysis to the leading order in the semiclassical expansion. Going beyond the leading order introduces many interesting technical and conceptual complications. Some have been discussed in Sec. 2.4.1. Issues of ellipticity that affect ordinary one-sided boundaries in gravity [73] are not a problem for factorization surfaces (essentially because they are two-sided, and “boundary moving diffs” should be treated as genuine diffs), but they do highlight the technical issue of gauge fixing in the presence of

a factorization surface. Another interesting related question concerns the renormalization of boundary terms, and thus of semiclassical boundary conditions: are all relevant boundary conditions radiatively stable?

We hope to address these matters in future work. Nearly a half century after the discovery of the Gibbons-Hawking entropy, the gravitational path integral continues to pose as many questions as it answers.

Chapter 3

Breakdown of Field Theory in Near-Horizon Regions

Work in this chapter is based on “Breakdown of Field Theory in Near-Horizon Regions” [74]. The paper was authored by Thomas Banks (NHETC and Department of Physics & Rutgers University), Patrick Draper (UIUC) and Manthos Karydas (UIUC).

3.1 Introduction

In this chapter we will consider asymptotically flat Schwarzschild black holes in $D > 2$ spacetime dimension and a minimally coupled real scalar field ϕ . We will always consider Quantum Field Theory (QFT) as an Effective Field Theory (EFT) with a UV and possibly IR cutoff. The background geometry will be fixed with metric given by the Schwarzschild-Tangherlini metric [75]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2 \quad , \quad f(r) \equiv 1 - \frac{r_s^{D-3}}{r^{D-3}} = 1 - \frac{\omega_D G_N M}{r^{D-3}} \quad , \quad (3.1)$$

where¹

$$\omega_D = \frac{16\pi}{(D-2)\text{Vol}(S^{D-2})} \quad . \quad (3.2)$$

We will set $\hbar = c = 1$ in this chapter and define the Planck length in D -dimensions to be

$$\ell_p \equiv G_N^{\frac{1}{D-2}} \quad . \quad (3.3)$$

The action for the minimally coupled scalar field is

$$S = -\frac{1}{2} \int d^D x \sqrt{-g} (g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2) \quad . \quad (3.4)$$

It is important to distinguish between two type of states, the Hartle-Hawking (HH) state [21] and Boulware state [77] (see also the discussion in [78], [79]). The Boulware state is the one annuled by the annihilation

¹For the form of the metric Eq. (3.1) see e.g. [76]

operators a_{Kill} associated with the time translation Killing vector ∂_t that becomes time translation very far from the black hole. Near the black hole horizon ∂_t acts as a Lorentz boost. The HH state is defined as the state annulled by a_{Krus} , the annihilation operators associated with “Kruskal modes”. These are positive-frequency modes in the Kruskal lightlike coordinate U [80].² The HH state is empty of particles for free falling observers crossing the event horizon. In this state the infalling observer sees “no drama” as is expected if one believes in a strong version of the equivalence principle. In terms of the Boulware basis the HH state can be written as a thermofield double of Boulware modes. When we restrict outside the black hole the density matrix is thermal with temperature given by the Hawking temperature of the black hole.

Physically the HH state describes the black hole in thermal equilibrium with the scalar field. The Hawking radiation from the black hole is balanced by the energy flux of the thermal gas outside the black hole. The HH state itself exhibits small back-reaction on the geometry. From a Euclidean path integral point of view this point is widely known. Thermal fields can be studied on curved backgrounds and the UV divergences can be cut off and absorbed into the renormalization of couplings in the gravitational effective action [83], [84]. The leading-power law divergences are absorbed into the cosmological constant, Newtons constant and the Gibbons-Hawking-York boundary term. In terms of the renormalized couplings, the total energy and entropy of the black hole receive very small residual corrections.

The small back-reaction is rather less obvious in the Hilbert space picture³, see e.g. [86]–[91] and references therein. The back-reaction issue has been discussed extensively in ’t Hooft’s seminal work on the brick wall model [92]. ’t Hooft proposed that the Bekenstein-Hawking entropy might be understood as a cutoff version of the thermal entropy of quantum fields outside the horizon. The thermal entropy is the entanglement entropy of the HH state when we restrict to the outside of the black hole [93], [94] and it is proportional to the area when a cutoff is present. ’t Hooft found that the thermal energy of the near horizon modes⁴ is of order one fraction of the black hole mass when the thermal entropy saturates the Bekenstein-Hawking entropy of the black hole. The large back-reaction created interpretational issues of the brick wall model [81], [95]–[97]. The confusion is related to the existence of two methods of computing the entropy of a black hole including one-loop effects, called “on-shell” or “off-shell” methods respectively (see e.g. [98]–[100]).

Consistent with path integral reasoning, Susskind and Uglum [101] argued that the same quantum fluctuations that produce an area law divergence in the entanglement entropy also renormalize G_N such that the generalized entropy is finite, namely $S = A/(4G_N^{\text{bare}}) + (\text{entropy of Quantum fields}) = A/(4G_N^{\text{ren}}) + (\text{finite terms})$. This was demonstrated explicitly by Demers, Lafrance and Myers [102] using a covariant regularization scheme which allowed them to get regularized expressions for the generalized entropy in which the brick wall can be removed. For technical reasons described in Section 3.3 the same calculation for the total energy has not been demonstrated, but nonetheless the expectation is that it must hold. In both the Euclidean path integral picture and Hilbert space picture, accounting for state-independent renormalizations of the gravitational couplings, the HH state has only small residual back-reaction on the background geometry.

Having established that HH is an acceptable state for perturbative QFT analysis, the question we want to address is about the “size” of the Hilbert space that EFT can be trusted. There are two kind of bounds that put restrictions on the EFT Hilbert space, entropic bounds and back-reaction bounds. In their seminal work

²Another way to define the HH state is through Euclidean path integral preparation, see e.g. [81], [82].

³By Hilbert space picture we here mean canonical quantization in curved background, see [85] for a nice review.

⁴Near horizon modes with large angular momentum are the dominant contribution to the density of states and hence to the entropy and energy. We will see this in detail in Section 3.3.

[20], Cohen-Kaplan-Nelson (CKN) noted that energetic back-reaction bounds impose stronger constraints on the EFT Hilbert space than entropic bounds. They considered hot boxes in flat space and found conditions before the box collapses to a black hole. Such strong gravitational back-reaction is beyond theoretic control. Our goal is to formulate a CKN type bound on near horizon QFT for the static observers. Instead of a hot box we will consider a state where HH thermally populated modes are cooled down to the Boulware vacuum. We will review the CKN arguments in Section 3.2 along with the connection to our construction.

The CKN bound provides us with the motivation to study large deviations of the HH state. Similarly, Page’s argument [103], [104] showed that if the black hole is a quantum system with Hilbert space dimension $\dim \mathcal{H}_{\text{black hole}} = \exp(A/(4G_N))$, Hawking radiation after the Page time must purify the early radiation. One possibility [105], [106] is that quantum gravity effects modify the HH state by disentangling pairs of Boulware modes near the horizon. The basic point of our construction is that one can construct a one-parameter family of deviations from the HH state using ’t Hooft’s brick wall model. Another motivation of such modification is related to the ER = EPR resolution [107] of the firewall paradox. A complicated operation in the radiation [108] should disturb the interior Boulware modes since their Hilbert spaces is contained the radiation Hilbert space, and measurements act as projectors. Our HH state modification could be thought roughly as a “proxy” for such an effect although it concerns the density matrix of the outside Boulware modes.

In contrast to [102], [109], we leave ’t Hooft’s brick wall finite, but on a shorter distance scale than the covariant regulator. In Section 3.3 we will see that this can be interpreted as a model for a class of deviations from the HH state in which a controllable fraction of the near horizon modes are removed from the thermal bath, modifying the thermal entropy. We will also explicitly see how perturbative renormalization fails to completely cancel the ultraviolet sensitivity of the thermodynamic quantities. We find that when the deviation from HH state is $O(1)$, in the sense of an $O(1)$ change of the ratio $\Delta S/S$ in the thermal entropy of near-horizon modes, energetic back-reaction implies that the QFT treatment breaks down below length scales of order $r_s^{2/D} \ell_p^{\frac{D-2}{2}}$. This provides complementary evidence for findings of [110], [111] and more recently of [112]. Our point of view is that excitations of near-horizon QFT modes with wavelengths shorter than $r_s^{2/D} \ell_p^{\frac{D-2}{2}}$ are not reliably described by QFT. As we will see this implies that the amount of black hole entropy that can be attributed to the entanglement entropy of quantum field is at most $(A/\ell_p^{D-2})^{\frac{D-2}{D}}$.

The organization of this chapter is as follows. In Section 3.2 we describe the Cohen-Kaplan-Nelson bound and its connection with our approach near horizons. In Section 3.3 we introduce the brick wall model together with Pauli-Villars covariant regularization method. The brick wall model allows us modify the density of states near the black hole horizon, see Eq. (3.40). Next, in Section 3.4 we discuss thermodynamics of the physical massless scalar field outside the black hole and explain well known results regarding renormalization of the HH state. In Section 3.5 we give our large back-reaction criterion and compute the distance scale that this criterion implies (see Eq. (3.56)) for large deviations of the HH state. Finally, in Section 3.6 we discuss our results and in particular we comment on implication of Casini’s bound [113] to our construction.

3.2 Cohen-Kaplan-Nelson bound

Cohen-Kaplan-Nelson [20] considered the subspace of the full effective field theory Hilbert space before back-reaction due to gravitational attraction becomes strong.

Consider an effective quantum field theory in a box of size L with UV cutoff Λ in D spacetime dimensions.

Conventional effective quantum field theory is expected to be capable of describing a system at a temperature T , provided the temperature is below the UV cutoff $T \leq \Lambda$. As long as $T \gg 1/L$, such a system has thermal energy and entropy

$$M \sim L^{D-1} T^D, \quad S \sim L^{D-1} T^{D-1}. \quad (3.5)$$

The Holographic Bekenstein bound [12], [13], [19] postulates that the QFT entropy can never exceed the Bekenstein-Hawking entropy S_{BH} with area replaced by the surface area of the box in Planck units

$$L^{D-1} T^{D-1} \lesssim S_{\text{BH}} \equiv (L/\ell_p)^{D-2}, \quad (3.6)$$

where ℓ_p was defined in Eq. (3.3). We ignore $O(1)$ prefactors in the right hand side of Eq. (3.6) which depend on the geometry of the box (e.g. rectangular versus spherical). Keeping the temperature T fixed the above the Bekenstein bound Eq. (3.6) can always be violated by making the box very large. The interpretation is that conventional QFT greatly overestimates the number of states of the system for such large volumes. Quantum gravity effects should step in and correct the counting of states, e.g. by introducing small overlaps into states that were otherwise considered exactly orthogonal.

CKN observed that one can impose a stronger constraint in the EFT compared to the Bekenstein bound Eq. (3.6) when one considered gravitational back-reaction. The back-reaction constraint comes by requiring that the length of the box L is larger the corresponding Schwarzschild radius of the system (which can be found from Eq. (3.1))

$$L_S \equiv \omega_D^{\frac{1}{D-3}} \ell_p^{\frac{D-2}{D-3}} M^{\frac{1}{D-3}}, \quad (3.7)$$

where ω_D is a dimensionless number defined in Eq. (3.2) and M is the mass of the thermal QFT box. L_S is the Schwarzschild radius of a black hole with mass equal to the energy of the box. We thus need to impose

$$L \gtrsim L_S \quad (3.8)$$

This is natural because otherwise, the system would have collapsed to a black hole, indicating strong gravitational back-reaction and conventional EFT breakdown. From Eq. (3.8), Eq. (3.7) and Eq. (3.5) we find ⁵

$$\ell_p^{\frac{D-2}{D-3}} T^{\frac{D}{D-3}} L^{\frac{2}{D-3}} \lesssim 1. \quad (3.9)$$

It is to see that the CKN bound Eq. (3.9) is far more restrictive than the Bekenstein bound Eq. (3.6). Saturating the bound Eq. (3.9) we have $T \sim \ell_p^{\frac{2-D}{D}} L^{-\frac{2}{D}}$ and the entropy of the box that we find from Eq. (3.5)

$$S_{\text{max}} \simeq S_{\text{BH}}^{\frac{D-1}{D}}. \quad (3.10)$$

Since $\frac{D-1}{D} = 1 - \frac{1}{D} < 1$ this is stronger than Eq. (3.6) for large entropies.

The important lesson taken from the CKN bound is that energy related back-reaction bounds are stronger than entropic bounds. We will see the same lesson is true for black hole horizons.

The CKN bound considers hot boxes on flat spacetime until they collapse to a black hole due to its gravitational attraction. In our setup we are interested in how deviations of the HH state around a D -dimensional black

⁵We ignore $O(1)$ constant pre-factor $\omega_D^{1/D-3}$ in Eq. (3.9).

hole change the background geometry by a significant amount that we will define precisely later. For now let us explain in few words main setup and its relation to the CKN bound described above. Consider a D -dimensional Schwarzschild black hole with metric given in Eq. (3.1). The “vacuum” state will be taken to be the HH state which as we mentioned in the introduction has small back-reaction to the geometry and is a well defined state around which perturbative effective quantum field theory can be trusted. We want to understand for what type of large deviations of this state back-reaction becomes large. In the case of CKN the vacuum was the usual flat space vacuum and the deviation was the thermal state with temperature T . For us the “vacuum” is the HH state which is already thermal outside the horizon of the black hole $\rho_{r>r_s} \sim e^{-\beta H}$ with H the Hamiltonian of the action in Eq. (3.4) for the metric in Eq. (3.1) and $\beta = 1/T_H$ with $T_H \sim 1/r_s$ the Hawking temperature of the black hole. Heating up the box in our set-up means the following: We introduce a brick wall at proper distance ρ_h from the horizon. The distance ρ_h will parametrize a modification of the density of states Δg as shown in Figure 3.1. The thermally populated modes in Δg will be projected to their Boulware vacuum modifying the HH state. Such a one parameter family of HH state modification has a cooling effect since it reduces the Killing (thermal) Boulware energy. Our large back-reaction criterion will be that the change of the energy density is of order the energy density of the HH state near the black hole horizon. We will see this in the next section (Section 3.3).

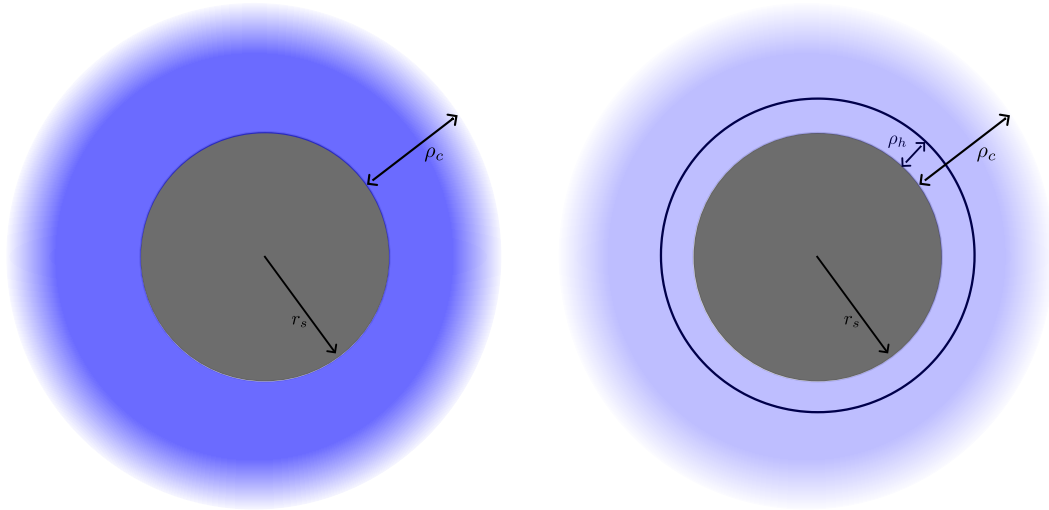


Figure 3.1: (Left): The black hole horizon at coordinate distance r_s . Modes of angular momenta $\ell \sim r_s/\rho_c$ are localized in a radial shell of proper radius ρ_c depicted as a blue halo. (Right): We introduce a brick wall at proper distance ρ_h from the horizon that parametrizes the modification of density of states Δg . The depletion of the modes introduced by the brick wall ρ is shown by reducing the brightness of the right halo compared to the left. Our criterion for large back-reaction is that the energy density reduction of the halo becomes of the same order $\sim 1/r_s^2$ as the HH state.

3.3 The Brick Wall Model

In this section we introduce the brick wall model and find the modified density of states due to the presence of the wall. We slightly generalize the Schwarzschild-Tangherlini metric given in (3.1) to

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, \quad (3.11)$$

where $f(r_s) = 0$ defines the location of the horizon. Besides the form of $f(r)$ given in (3.1) we keep the form of $f(r)$ more general such that our analysis is applicable to any event horizon, eg. in pure de Sitter space, Schwarzschild-de Sitter or anti-de Sitter black holes.

We consider just one physical massless minimally coupled scalar field. Since we are going to use massive fields in Pauli-Villars (PV) regularization [83], [114] we keep the mass non-zero in Eq. (3.4). The wave equation derived from that action is

$$\frac{1}{\sqrt{g}}\partial_M(\sqrt{-g}g^{MN}\partial_N\phi) - m^2\phi = 0. \quad (3.12)$$

Any solution can be written as a linear combination of the following modes

$$\phi = e^{-i\omega t} Y_{\ell,D} \phi(r), \quad (3.13)$$

where $Y_{\ell,D}$ (we have suppressed the dependence on the angles) are the $S^{(D-2)}$ hyper-spherical harmonics with angular momentum ℓ

$$\nabla_{S^{D-2}} Y_{\ell,D} = -\ell(\ell + D - 3) Y_{\ell,D}, \quad \nabla_{S^{D-2}} \equiv \text{Laplacian on } S^{D-2}. \quad (3.14)$$

The number of degenerate states for each ℓ eigenvalue is

$$N_{\ell,D} = \frac{(D + 2\ell - 3)(D + \ell - 4)!}{\ell!(D - 3)!}. \quad (3.15)$$

The radial mode $\phi(r)$ satisfies the second order differential equation

$$\frac{\omega^2}{f(r)}\phi(r) + \frac{1}{r^{D-2}}\partial_r(r^{D-2}f(r)\partial_r\phi(r)) - \frac{\ell(\ell + D - 3)}{r^2}\phi(r) - m^2\phi(r) = 0. \quad (3.16)$$

In the leading order WKB approximation we set $\phi(r) = e^{i\Phi(r)}$ and drop Φ' and Φ'' terms in Eq. (3.16), retaining Φ'^2 . WKB is a good approximation in the near horizon region [115]. We find

$$\Phi'(r)^2 = \frac{1}{f(r)} \left(\frac{\omega^2}{f(r)} - \frac{\ell(\ell + D - 3)}{r^2} - m^2 \right). \quad (3.17)$$

The brick wall model [92] amounts to setting the following boundary conditions on the field

$$\phi(r_s + h) = 0, \quad h/r_s \ll 1, \quad (3.18)$$

$$\phi(L) = 0, \quad L \gg r_s. \quad (3.19)$$

We are going to consider only near horizons modes in this analysis and as such the IR boundary condition Eq. (3.19) will not play any important role. The radial mode is oscillatory for $\Phi'(r)^2 > 0$ which determines some bounds on the angular momentum ℓ_{max} and radial distance and r_{max} .⁶ To find these bounds we first consider the near horizon region, namely we approximate

$$f(r) \simeq f'(r_s)(r - r_s) \quad (\text{near horizon region}), \quad (3.20)$$

⁶In the case of the de Sitter static patch, we have $\text{sgn}(f'(r_s)) = -1$, $r_s = L$, $h < 0$, and r_{max} actually corresponds to a minimum value of the radial coordinate.

and expand Eq. (3.17) using Eq. (3.20):

$$\Phi'(r)^2 \simeq \frac{r_s^2 \omega^2 - (r - r_s) f'(r_s) [\ell(\ell + D - 3) + m^2 r_s^2]}{r_s^2 (r - r_s)^2 f'(r_s)^2}. \quad (3.21)$$

We allowed for large mass and angular momenta ℓ of order $(r - r_s)^{-1}$. We can also replace factors of $|f'(r_s)|$ with the horizon temperature T_H

$$|f'(r_s)| = 4\pi T_H \equiv T. \quad (3.22)$$

The positivity condition on $\Phi'(r)$ implies the following bounds

$$r_{max} \simeq r_s + \frac{\omega^2 r_s^2}{T \text{sgn}(f'(r_s))(\ell^2 + m^2 r_s^2)}, \quad \ell_{max}^2 \simeq \frac{\omega^2 r_s^2}{T|h|} - m^2 r_s^2. \quad (3.23)$$

The near horizon approximation requires:

1) $\ell, m r_s \gg 1$ when $\omega^2 \sim T/r_s$ such that r_{max} is still close to the horizon. Therefore we must have $\ell_{max} \gg 1$ which implies

$$r_s^2 \left(\frac{1}{\rho_h^2} - \frac{1}{\rho_c^2} \right) \gg 1, \quad (3.24)$$

where $\rho_h \equiv \sqrt{r_s |h|}$ is the proper distance of the brick wall from the horizon and $\rho_c \equiv m^{-1}$ is the PV UV distance scale. Note that Eq. (3.26) is satisfied as long as

$$\begin{aligned} \frac{\rho_h}{\rho_c} < 1 \quad \text{and} \quad \rho_c \ll r_s \quad (\text{massive field}), \\ \rho_h \ll r_s \quad (\text{massless physical field}). \end{aligned} \quad (3.25)$$

2) $\ell_{max} > 0$ implies

$$\omega > \frac{\rho_h}{\rho_c} \sqrt{\frac{T}{r_s}}, \quad (3.26)$$

and below this frequency are approximations break down. The above constraints will be sufficient for our purposes.

The number of modes below a frequency ω is

$$\begin{aligned} N_{<\omega} &\approx \frac{\text{sgn}(f'(r_s))}{\pi} \sum_{\ell} N_{\ell,D} \int_{r_s+h}^{r_{max}(\ell,\omega,h)} dr \Phi'(r) \\ &= \frac{\text{sgn}(f'(r_s))}{\pi} \int^{\ell_{max}(\omega,h)} d\ell \frac{2\ell^{D-3}}{\Gamma(D-2)} \int_{r_s+h}^{r_{max}(\ell,\omega,h)} dr \Phi'(r) \\ &= \frac{2\omega}{\pi T} \int^{\ell_{max}(\omega,h)} d\ell \frac{2\ell^{D-3}}{\Gamma(D-2)} \left[\tanh^{-1} \left(\sqrt{\frac{\ell_{max}^2 - \ell^2}{\ell_{max}^2 + m^2 r_s^2}} \right) - \sqrt{\frac{\ell_{max}^2 - \ell^2}{\ell_{max}^2 + m^2 r_s^2}} \right], \end{aligned} \quad (3.27)$$

where in the second line we converted the sum over the modes ℓ into an integral using Eq. (3.15) assuming $\ell \gg 1$, and in the third line we performed the radial integral using Eq. (3.21) and Eq. (3.23).

For the physical massless field we find

$$N_{\text{physical}, < \omega} \simeq a_0 \left(\frac{\omega}{T} \right)^{D-1} (Tr_s)^{\frac{D-2}{2}} \left(\frac{r_s}{\rho_h} \right)^{D-2}, \quad (3.28)$$

where $a_0 = 1/[2^{D-3}(D-2)\Gamma(\frac{D-1}{2})\Gamma(\frac{D+1}{2})]$.

For the massive PV fields let us introduce the small dimensionless ratios

$$z_i \equiv m_i \sqrt{|h|r_s} \equiv m_i \rho_h, \quad i : (\text{labels each PV field}). \quad (3.29)$$

The mode number Eq. (3.27) for each massive PV field is

$$\begin{aligned} N_{i, < \omega} &= \left(\frac{\omega}{T} \right)^{D-1} (Tr_s)^{\frac{D-2}{2}} \left(\frac{r_s}{\rho_h} \right)^{D-2} f(b_i, D), \\ f(b_i, D) &= \frac{4b_i^{D-2}}{\pi\Gamma(D-2)} \int_0^1 dx (x^{D-3}) \left[\tanh^{-1}(b_i \sqrt{1-x^2}) - b_i \sqrt{1-x^2} \right], \\ b_i^2 &\equiv 1 - \frac{T}{\omega^2 r_s} z_i^2. \end{aligned} \quad (3.30)$$

The x integral is finite. For frequencies such that Eq. (3.26) is strongly satisfied, the N_i can be expanded around $b_i \approx 1$ in the qualitative form

$$\begin{aligned} N_i &= \left(\frac{\omega}{T} \right)^{D-1} (r_s T)^{\frac{D-2}{2}} \left(\frac{r_s}{\rho_h} \right)^{D-2} \sum_{k=0}^{\infty} a_k c_i^k \\ c_i^2 &\equiv (1 - b_i^2) = \frac{T}{\omega^2 r_s} z_i^2 \end{aligned} \quad (3.32)$$

where the a_k are dimension-dependent numerical coefficients. This has the same form as N_0 , modified by a power series in $c_i \sim z_i$. In even D , there are also $\log(c)$ terms. For example, in four dimensions one finds

$$\begin{aligned} \sum_k a_k c_i^k &= \frac{2(\sqrt{1-c^2}(1+2c^2) - 3c^2 \tanh^{-1}(\sqrt{1-c^2}))}{3\pi} \\ &= \frac{2}{3\pi} + \frac{c^2(2\log(\frac{c}{2})+1)}{\pi} - \frac{c^4}{4\pi} - \frac{c^6}{48\pi} + \dots \end{aligned} \quad (3.33)$$

For low frequencies near the boundary of the inequality (3.26), or equivalently $b_i \sim 0$, the expansion (3.32) breaks down. One can instead expand in small b_i and in this way find that $N_i \sim b_i^{D+1}$, indicating a strong suppression of the density of states in the infrared.

We consider modifications of the HH state of one physical massless scalar field and a specific (dimension dependent) collection of massive Pauli-Villars regulator fields. The PV fields suffice to cancel the formal divergences in the mode sums $N_{reg} = N_{\text{physical}, < \omega} + \sum_i N_{i, < \omega}$ even when $\rho_h \rightarrow 0$ to achieve a covariant regularization of the HH state [102]. We will keep ρ_h finite, however, to parametrize modifications of the HH state. The mass and number of each PV field is adjusted such that the leading term of order ρ_h^{2-D} in each individual mode number (see Eq. (3.28), Eq. (3.30)). In fact the role of the PV fields is to cancel all of the sub-leading divergent terms when $\rho_h \rightarrow 0$. What remains in N_{reg} from the series in Eq. (3.32) is $k = D - 2$, which are terms independent of ρ_h , and $k > D - 2$, which are terms proportional to positive powers of ρ_h .

The PV masses are all of order the UV regulator distance scale $m_i \sim 1/\rho_c$. For example in $D = 4$ we require two anticommuting regulator fields of mass $1/\rho_c$, two commuting fields of mass $\sqrt{3}/\rho_c$, and one anticommuting field of mass $2/\rho_c$. The regulated state count is

$$N_{reg}^{D=4} = \frac{r_s^2}{\pi\rho_c^2} \left[\frac{2\omega}{T} \log\left(\frac{27}{16}\right) + \frac{T}{4\omega^3 r_s^2} \left(\frac{\rho_h^4}{\rho_c^4}\right) + \dots \right] \quad (3.34)$$

$$= \log\left(\frac{27}{16}\right) \left(\frac{2}{\pi}\right) \left(\frac{r_s}{\rho_c}\right)^2 \left(\frac{\omega}{T}\right) + () \quad (3.35)$$

for ω not too small. For small ω of order $\sqrt{T/r_s}(\rho_h/\rho_c)$, the small- ρ_h expansion breaks down, and the actual N_{reg} (which can be obtained from (3.33) by summing over PV fields) is strongly suppressed.

For general D , the regulated state count has the small- ρ_h expansion

$$N_{reg} = \bar{a}_{D-2} \left(\frac{r_s}{\rho_c}\right)^{D-2} \frac{\omega}{T} + \Delta N, \quad (3.36)$$

$$\Delta N \equiv \left(\frac{\omega}{T}\right)^{D-1} (r_s T)^{\frac{D-2}{2}} \left(\frac{r_s}{\rho_h}\right)^{D-2} \sum_{k=D-1}^{\infty} \bar{a}_k \left(\frac{T}{r_s \omega^2}\right)^{\frac{k}{2}} \left(\frac{\rho_h}{\rho_c}\right)^k$$

for some modified coefficients $\bar{a}_k$. For $D = 4$ we can see from Eq. (3.35) that δN in Eq. (3.36) starts from $k = 6$.

The regularized density of states is

$$g(\omega) = dN_{reg}/d\omega \equiv g_0 + \Delta g. \quad (3.37)$$

Let us estimate g by retaining the leading terms in the ρ_h series. For $D = 4$ we find

$$g(\omega) = \frac{2r_s^2}{\pi\rho_c^2 T} \left(\log\left(\frac{27}{16}\right) - \frac{3}{8(r_s T)^2 (\omega/T)^4} \left(\frac{\rho_h}{\rho_c}\right)^4 + \dots \right). \quad (3.38)$$

For general D , it has the form

$$g(\omega) \sim \underbrace{r_s \left(\frac{r_s}{\rho_c}\right)^{D-2} X(\omega/T, r_s T)}_{\equiv g_0} + \underbrace{r_s \left(\frac{r_s}{\rho_c}\right)^{D-2} \left(\frac{\rho_h}{\rho_c}\right)^2 Y(\omega/T, r_s T) + \dots}_{\equiv \Delta g} \quad (3.39)$$

$$\equiv g_0 + \Delta g \quad (3.40)$$

for some functions $X, Y, \dots$ (The $k = D - 1$ term in ΔN , if it exists, is independent of ω .)

We will think of the first term, g_0 , which is independent of h , as the complete regularized density of states. It is dominated by angular ℓ modes of order (r_s/ρ_c) , rather than $\ell_{max} \sim (r_s/\rho_h)$, due to the Pauli-Villars cutoff. This will be important below. (It is evident, e.g., from the form of the individual N_i in Eq. 3.30, which scale as $\ell_{max}^{D-2} \sim (r_s/\rho_h)^{D-2}$. The leading term in N_{reg} scales instead as $(r_s/\rho_c)^{D-2}$, reflecting the cancellation of contributions from wavelengths shorter than the regulator scale $1/\rho_c$.)

The second term, Δg , which is of order h , represents the “nonthermal density of states.” In thermodynamic

sums we can interpret it as the density of states that are over- or under-populated with respect to the Bose-Einstein distribution corresponding to the HH state. We may think of Δg as encoding a smooth modification of a fraction $(\rho_h/\rho_c)^2$ of the states at every relevant energy scale, projecting them to the Boulware ground state. This is the sense in which the brick wall model can encode a one-parameter family of deviations from the HH state.

3.4 Thermodynamics and Renormalization

The density of states g_ω found in Eq. (3.40) allows us to compute the free energy, entropy making use of the fact that in the HH state the modes with frequency ω are thermally occupied with temperature $\beta = 1/T_H$

$$F = \frac{1}{\beta} \int_0^\infty d\omega g(\omega) \log(1 - e^{-\beta\omega}) \equiv F_0 + \Delta F, \quad (3.41)$$

$$U = \int_0^\infty d\omega g(\omega) \frac{\omega}{e^{\beta\omega} - 1} \equiv U_0 + \Delta U, \quad (3.42)$$

$$S = \beta^2 \partial_\beta F \equiv S_0 + \Delta S. \quad (3.43)$$

In the above expression F_0, U_0, S_0 are computed with the unmodified density of states $g_0(\omega)$ (see Eq. (3.40)) while $\Delta F, \Delta U, \Delta S$ are the modification due to the brick wall coming from Δg . For example, ΔU most directly corresponds to a modification of the regulated density of states, relative to the regulated $h = 0$ theory, but we could just as well think of it as a one-parameter family of excitations of the Hartle-Hawking state, parametrizing a deviation of some degrees of freedom out of the thermal state.

Note that F_0, U_0, S_0 are regulated expression that diverge when the UV scale ρ_c is taken to zero. However such divergences can be absorbed into the renormalization of the coupling constants of the gravitational action $\Lambda_0, G_{N,0}, \dots$. This was shown explicitly for the generalized entropy [102]

$$S_{tot} = \frac{A}{4G_{N,0}^{4D}} + S_0, \quad (S_0 \text{ given in Eq. (3.45)}), \quad (3.44)$$

based on a suggestion by Susskind and Uglum [101]. From the one-loop effective action the authors of [102] found that for the Schwarzschild black hole in $D = 4$ the renormalized Newton's constant $G_{N,r}^{4D}$ is related to the bare $G_{N,0}^{4D}$ by

$$\frac{1}{G_{N,r}^{4D}} = \frac{1}{G_{N,0}^{4D}} + \frac{\log(27/16)}{6\pi\rho_c^2}. \quad (3.45)$$

From (3.45) and S_0 in (3.43) one finds

$$\frac{A}{4G_{N,0}^{4D}} + S_0 = \frac{A}{4G_{N,r}^{4D}}, \quad (A : \text{Area of black hole horizon}), \quad (3.46)$$

and in this sense the UV divergent term in the entropy (of the HH state) is absorbed in the renormalization of Newton's constant.

The same is generally not quite true for the energy of the HH state. The total set of UV-sensitive contributions

to the energy, including bare gravitational contributions, has the form:

$$U_{tot} = [V\Lambda_0/G_{N,0}] + \left[\sum \frac{\omega}{2}\right] + [U_0 + \Delta U] + \left[\frac{r_s^{D-3}F(r_s T)}{G_{N,0}}\right]. \quad (3.47)$$

The first term contains the bare cosmological constant, Λ_0 , and the spatial volume V . The second term is the vacuum energy from the quantum fields. In leading-order WKB it has cutoff sensitivity $\sim V/\rho_c^D$ which exactly cancels the perturbative renormalization of the Λ_0 . The last term is the “bare gravitational mass,” with some function F that depends on the black hole in question.

For example, for the Schwarzschild black hole in $D = 4$ we easily obtain U_0 by inserting (3.38) into (3.42). We obtain

$$U_0 = \frac{r_s \log(27/16)}{48\pi\rho_c^2} \quad (3.48)$$

to be compared with the perturbative renormalization of the Newton constant Eq. (3.45) appearing in $M = r/2G$. We find that the bare mass renormalization is four times larger than the thermal energy in the field. This, however, we think is technical rather than profound. The Euclidean path integral on the smooth cigar geometry tells us that all UV sensitivity must be absorbed by counterterms in the HH state. It is difficult to establish a precise cancellation in the Hilbert space approach, but we have indications that NNLO WKB to the vacuum energy should be able to fill this gap. A careful sub-leading WKB calculation must be performed to establish this with certainty. This is the reason we included the vacuum energy and c.c. explicitly in the decomposition (3.47), instead of implicitly cancelling them.

In any case, it is a general property of the HH state that the ultraviolet contribution to U_0 , scaling as $(r_s/\rho_c)^{D-2}$, cancels against various other contributions, including the vacuum energy and $G_{N,0}$.

The important term for is ΔU coming from modifying the density of states due to the brick wall. The point is that ΔU is not absorbed by the counterterms, which are held fixed as we adjust ρ_h .⁷ ΔU will cause back-reaction to the geometry due to Einstein equations. In the next section we calculate ΔU and propose a criterion for large back-reaction.

3.5 Back-reaction Criterion

To calculate the energy modification ΔU we substitute Eq. (3.40) into Eq. (3.42). We find

$$\begin{aligned} \Delta U &\sim \left(\frac{1}{r_s}\right) \left(\frac{r_s}{\rho_c}\right)^{D-2} \left(\frac{\rho_h}{\rho_c}\right)^n Z(\omega/T, r_s T, \beta\omega) \\ Z &\equiv r_s \int_0^\infty d\omega Y(\omega/T, r_s T) \left(\frac{r_s \omega}{e^{\beta\omega} - 1}\right), \end{aligned} \quad (3.49)$$

where Z is a dimensionless function and for some n that depends on the dimension. We have omitted the lower limit of integration in Z as a reminder that the $z \sim \rho_h/\rho_c$ expansion breaks down for $\omega \rightarrow 0$, but the density of states is also very suppressed in this limit.

⁷There could also be a contribution to the ρ_h -dependence in the energy from the vacuum energy, if we took the brick wall seriously as removing modes entirely. Since we are treating the modes as still present, just not in their thermal state, the vacuum energy is not sensitive to ρ_h .

The dominant contribution to the integral comes from $\beta\omega \sim 1$. The power n might depend on the dimension. In $D = 4$ we find $n = 4$, and for $r_s \sim 1/T \sim \beta$,

$$\Delta U^{D=4} \sim \frac{r_s \rho_h^4}{\rho_c^6}. \quad (3.50)$$

Finally let us now describe our back-reaction criterion. Let us focus on the example of D -dimensional Schwarzschild. By definition back-reaction means that ΔU significantly changes the background geometry. Typical near-horizon curvatures are of order $1/r_s^2$ so the back-reaction is large and uncontrollable when the energy density $\rho = T_0^0$ coming from ΔU is of this order, namely

$$G_{N,r} T_0^0 \sim \frac{1}{r_s^2} \quad (\text{Back-reaction Criterion}) \quad (3.51)$$

In Section 3.3 we found that the regularized density of states for the near-horizon modes is dominated by angular modes

$$\ell_{\text{dom}} \sim r_s / \rho_c. \quad (3.52)$$

Inserting ℓ_{dom} into Eq. (3.23), we find that the dominant modes are localized in a radial shell of depth $(r_{\text{max}} - r_s) \sim r_s (\rho_c / r_s)^2$. The back-reaction criterion 3.51 becomes

$$\frac{G_{N,r} |\Delta U|}{(r_s^{D-2})(r_s (\rho_c / r_s)^2)} \sim \frac{1}{r_s^2}. \quad (3.53)$$

If we introduce the "renormalized" Planck length $\ell_p^{D-2} = G_{N,r}$ from Eq. (3.49) we have

$$G_{N,r} \Delta U \sim r_s^{D-3} \left(\frac{\ell_p}{\rho_c} \right)^{D-2} \left(\frac{\rho_h}{\rho_c} \right)^n. \quad (3.54)$$

Inserting Eq. (3.54) into Eq. (3.53) we find a breakdown of the QFT computation around a scale

$$\rho_c \sim \rho_h^{\frac{n}{D+n}} r_s^{\frac{2}{D+n}} \ell_p^{\frac{D-2}{D+n}}. \quad (3.55)$$

For a large excitation of the HH state, with $\rho_h \sim \rho_c$, there is an $O(1)$ deviation of the thermodynamic state of the static scalar modes: $\Delta g \sim g_0$, $\Delta U \sim U_0$, $\Delta S \sim S_0$, etc. We may simply estimate the back-reaction by replacing ΔU with U_0 in the back-reaction criterion. For the Schwarzschild geometry, we find a breakdown of QFT at

$$\rho_c \sim \rho_h \sim r_s^{\frac{2}{D}} \ell_p^{\frac{D-2}{D}}. \quad (3.56)$$

This coincides with the scale found in [110], [111] and more recently of [112]. The scale Eq. (3.56) is parametrically larger than the Planck scale. In $D = 4$ for example $\rho_c = \sqrt{r_s \ell_p} \gg \ell_p$.

At the distance scale Eq. (3.56) we can compute the entropy ΔS that can be attributed to the entanglement entropy of quantum field. We find

$$\Delta S \sim A^{\frac{D-2}{2}}, \quad A = \left(\frac{r_s}{\ell_p} \right)^{D-2}. \quad (3.57)$$

It is interesting to note the different exponent in the area compared to the CKN bound Eq. (3.10).

3.6 Discussion

In this chapter we have derived a CKN type bound near black hole horizons. We considered a brick wall inspired toy model that gives a one-parameter family of deviations of the Hartle-Hawking state. In Section 3.5 we proposed a criterion for strong back-reaction. For large excitations of the HH state we found that strong back-reaction happens at a scale that is parametrically larger than the Planck scale.

Our motivation was to find constraints on the QFT Hilbert space for which semiclassical QFT can be trusted. For CKN type bounds one should in principle consider all possible modifications of the HH states before back-reaction becomes large. In this regime semiclassical QFT can be trusted. Here we considered a one parameter family of modification of HH state with significant change of both the entropy and the energy, namely $\Delta S/S \sim \Delta U/U = O(1)$. We found that the EFT description breaks down at surprising long scales from the black hole horizon. At first sight it seems plausible that one with consider other states with large modification to the entropy without a similar large change in the energy. Then such states could be describable in semiclassical field theory even very close to the horizon. However, Casini's version [113] of the Bekenstein bound [116] shows that such state modifications does not exist if it comes from the near horizon modes. Casini's bound when applied for relative to the HH state (see Eq (28) of [113]) implies that entropy changes for states that come from modifying near horizon modes at the distance scale $\rho_c \sim r_s^{\frac{2}{D}} \ell_p^{\frac{D-2}{D}}$ cannot exceed $A^{(D-2)/2}$, where A is the area of the black hole horizon in Planck units.

Finally, it would be interesting to extend our results to other types of horizons. Our formalism is general enough to allow for such analysis, e.g. for (Anti)-de Sitter Schwarzschild black holes and cosmological horizons. Of special interest are Rindler and causal diamond horizons. We point out however that depending on the case a different back-reaction criterion than Eq. (3.51) needs to be implemented. For example in Rindler and causal diamond horizons the curvature is zero if we assume flat spacetime, and the right-hand side of Eq. (3.51) fails. One should instead use the following heuristic principle: Back-reaction is large when the horizon is shifted by the localization scale of the near horizon modes (see below Eq. (3.52)).

Chapter 4

Bubble nucleation in gravity: A semiclassical tunneling analysis

Work in this chapter is based on “Vacuum Decay in Time-Dependent Backgrounds” [117]. The paper was authored by Patrick Draper, Manthos Karydas and Hao Zhang, all affiliated with the Department of Physics at the University of Illinois at Urbana-Champaign.

In this chapter we will analyze the problem of bubble nucleation in gravity. We will consider the setting where a true vacuum bubble nucleates inside a false vacuum region. Most of the work presented in this chapter builds upon Section 5 of [117], but here we provide significantly more detailed calculations. While the main focus of that paper was an analysis of vacuum decay in time-dependent backgrounds, here we will focus on a gravitational setting where the gravitational field is static but gravitational back-reaction of the bubble is incorporated. We will comment on relaxing the static condition at the end of Section 4.4.

The organization of this chapter is as follows. In Section 4.1 (taken from Section 2 of [117]) we describe a new perspective on semiclassical tunneling that was presented in [117]. The main differences with the usual Euclidean approach are; the introduction of a generalized time contour, and a reformulation of the variational problem, from fixed initial position and fixed energy to fixed wavepacket-like initial and final states at fixed times. The same perspective will be applied on bubble nucleation with gravity. To do this, in Section 4.2 we find the equation of motion of the bubble from the second Israel junction condition [118]. In Section 4.3 we derive the effective action for the bubble. Our construction slightly generalizes the effective action constructed by Visser [119]. As a consistency check we derive the same equations of motions that we derived from the junction condition. Finally in Section 4.4, making use of the techniques described in Section 4.1 we analyze bubble nucleation using the bubble effective action derived in Section 4.3. We close the section with some comments on how to incorporate time dependance. In Section 4.5 we conclude with a summary of the main points presented in this chapter.

4.1 Semiclassical Formulation

In the usual semiclassical treatment of vacuum decay [23]–[25], one searches for saddle points of the Euclidean path integral that interpolate between the false vacuum and a bubble of true vacuum, or in quantum mechanics, between the false vacuum and a point outside the well.

To get a different perspective on the problem, let us consider real-time quantum mechanical transition amplitudes of the form

$$\begin{aligned} \langle \psi_f(R, T_f) | \psi_i(R, T_i) \rangle &= \mathcal{N} \int dR_i dR_f \int_{R(T_i)=R_i}^{R(T_f)=R_f} DR e^{iS/\hbar}, \\ S &= S_i(R_i) - S_f(R_f) + \int_{T_i}^{T_f} dt L(\partial_t R, R, t). \end{aligned} \quad (4.1)$$

The initial and final states are given by wavefunctions parametrized as $\psi(R) = e^{iS_{i,f}(R)/\hbar}$, where $S_{i,f}$ are complex functions. In tunneling problems these wavefunctions should be localized inside and outside of a false vacuum well. We have written explicit factors of $\hbar$ to indicate that we are interested in states where both the real and imaginary parts of $S_{i,f}$ contain contributions of order one in the $\hbar$ expansion, like coherent states, but subsequently we will drop the $\hbar$ s. A similar starting point was used to study complexified path integral trajectories in [120].

Next, we perform a general rotation of the time contour, $t \rightarrow e^{i\gamma}\tau$. The Euclidean continuation corresponds to $\gamma = -\pi/2$, but it turns out to be useful to keep γ arbitrary. This technique was applied to the symmetric double well tunneling problem in [121].

After rotation of the time contour, the path integral appearing in the amplitude is

$$\begin{aligned} &\int dR_i dR_f \int_{R(T_i)=R_i}^{R(T_f)=R_f} DR e^{-S_c}, \\ S_c &= -iS_i(R_i) + iS_f(R_f) - i \int_{T_i}^{T_f} d\tau L_c(\partial_\tau R, R, \tau) \\ L_c(\partial_\tau R, R, \tau) &= e^{i\gamma} L(e^{-i\gamma} \partial_\tau R, R, e^{i\gamma} \tau). \end{aligned} \quad (4.2)$$

In the leading semiclassical approximation we are interested in solutions to the bulk Euler-Lagrange equation, $\delta L_c / \delta R - \partial_\tau (\delta L_c / \delta \partial_\tau R) = 0$. The equation of motion can be replaced by the conservation of energy condition if L does not depend explicitly on t .

Finally, we relax the Dirichlet boundary conditions used in ordinary semiclassical treatments of the path integral. Since we have introduced general initial and final states, we can consider unrestricted boundary variations. Stationarity of the action then gives rise to the following boundary relations:

$$\begin{aligned} p_i &\equiv \left. \frac{\delta L_c}{\delta \partial_\tau R} \right|_{\tau=T_i} = S'_i(R_i) \\ p_f &\equiv \left. \frac{\delta L_c}{\delta \partial_\tau R} \right|_{\tau=T_f} = S'_f(R_f). \end{aligned} \quad (4.3)$$

Eq. (4.3) will be important for understanding the various semiclassical tunneling solutions we obtain below.

These conditions relate the initial and final momenta of the semiclassical trajectories to the gradient of the wavefunction at the initial and final positions. A totally generic initial and final wavefunction, and a given solution $R(\tau)$, will not in general satisfy Eq. (4.3), and so will not provide a single saddle point approximation to the full path integral, including the initial and final states as described above. Our approach will be to find solutions to the bulk equations of motion first, and then use Eq. (4.3) to infer wavefunctions with consistent properties, such that those solutions provide genuine saddle points of the full path integral. These are generally weak constraints on the initial and final wavefunctions, only constraining their gradients at points, but we must determine them after the fact.

These self-consistency conditions have very natural physical interpretations. For example, the real part of the momentum at the semiclassical endpoints has to match the momentum of the initial and final state wavefunctions in the eikonal approximation.

At leading order in the semiclassical expansion, then, the task is to solve the equations of motion subject to the boundary conditions (4.3). The canonical variables in the solution are generally complex-valued. In addition to the $e^{i\gamma}$, $S_{i,f}$ are complex if, say, the states defined on real $R_{i,f}$ are modeled as Gaussian wavepackets or some similar perturbative ground state. To estimate a total decay probability, one must also extremize over semiclassically inequivalent final states, and integrate over any zero modes.

Although this is an acceptable formulation of a tunneling problem, generally it is not what is done in standard, time-independent tunneling problems. One instead solves a variational problem with fixed energy:

$$E_E = E_{FV} \tag{4.4}$$

where E_E is the Euclidean energy and $E_{FV} = V(R_{FV})$ is the value of the potential in the classical false vacuum R_{FV} . Typically there is a solution to Eq. (4.4) that starts at R_{FV} and reaches the classical turning point $R = R_{TP}$ with zero momentum at some later time, which is determined by the solution.

From such a solution, we may work backward and identify problems of the type (4.1)-(4.3) to which it is also a solution. In a corresponding problem of the type (4.1)-(4.3), one needs a low-energy initial state, like the perturbative ground state wavefunction in the false vacuum, for which $S'_i(R_{FV}) = 0$, and similarly the final state can be some zero-momentum wavepacket peaked at R_{TP} . These requirements are typical of time-independent problems analytically continued to Euclidean time.

We will see that other final states arise in time-independent problems with arbitrary time continuation parameters γ . Such a case will be vacuum decay with gravity. In the next section we set up the problem and find the bubble equations of motion. Then we find the bubble effective action that will be used later in the semiclassical tunneling analysis as described above. Before we couple to gravity let us describe semiclassical tunneling for a toy Lagrangian. This toy model is instructive because it turns out the bubble effective action in the weak gravity limit will take similar form.

4.1.1 Toy model

We consider toy model Lagrangians of the form¹

$$L = -R_0 R^{p-1} \sqrt{1 - (\partial_t R)^2} + R^p. \tag{4.5}$$

¹In section (4.4) we show that the bubble effective action takes such form in the $G_N \rightarrow 0$ limit.

Since Eq. (4.5) does not depend explicitly on time there is a conserved energy. After continuation $t \rightarrow e^{i\gamma}\tau$ with $\tau \in \mathbb{R}$, the conserved energy is

$$E = e^{i\gamma} \left(\frac{R_0 R^{p-1}}{\sqrt{1 - e^{-2i\gamma}(\partial_\tau R)^2}} - R^p \right). \quad (4.6)$$

There is a false vacuum at $R_{FV} = 0$ with $E_{FV} = 0$, and the classical turning point is $R_{TP} = R_0$. It is also informative to write the energy in terms of the momentum $P = e^{-i\gamma} R_0 R^{p-1} \left(\partial_\tau R / \sqrt{1 - e^{-2i\gamma}(\partial_\tau R)^2} \right)$,

$$E = e^{i\gamma} \left(\sqrt{P^2 + R_0^2 R^{2(p-1)}} - R^p \right). \quad (4.7)$$

R is generally not real, so we seek a zero energy solution for which R is real at $\tau = 0$. We find

$$R = \Theta(\tau - R_0 \csc \gamma) \sqrt{R_0^2 + (e^{i\gamma}\tau - R_0 \cot \gamma)^2}. \quad (4.8)$$

The particle sits at the $R = 0$ false vacuum until $\tau = \tau_0 \equiv R_0 \csc \gamma$, then moves in the complex R plane to the nucleation point at $\tau = 0$,

$$R(0) = R_0 |\csc \gamma|. \quad (4.9)$$

Curiously, the nucleation point only coincides with the classical turning point in the Euclidean case $\gamma = -\pi/2$.

We have assumed that the initial time is early enough, $T_i < R_0 \csc \gamma$, so that the instanton can “fit.” Apart from this constraint, the action of the semiclassical solution is independent of T_i . The particle sits as long as it needs to in the false vacuum, and then at $\tau = R_0 \csc \gamma$ it starts the jump to the nucleation point.

To better understand the solutions for $\gamma \neq -\pi/2$, we can ask what wavepacket problems of the form (4.1)-(4.3) they also solve. Let us examine the boundary conditions (4.3). The final momentum of the solutions (4.8) is

$$p_f = -R_0^p |\csc \gamma|^{p-1} \cot \gamma. \quad (4.10)$$

In particular, it is real, and so is related to the gradient of the phase of the final wavefunction by Eq. (4.3). For wavepackets the gradient of the phase is just the classical momentum P .

Furthermore is easily seen that the *real-time* energy, Eq. (4.7) with $\gamma = 0$, vanishes for all final states labeled by the real positions $R \rightarrow R(0)$ given in Eq. (4.9) and real momenta $P = p_f$ given in Eq. (4.10). This is because Eq. (4.7) only depends explicitly on γ in an overall multiplicative factor, and Eq. (4.3) instructs us to map the semiclassical trajectory momentum p_f and the semiclassical final state momentum P to each other. The final states at $\tau = 0$ correspond to different points in phase space, related by real-time classical evolution.

We conclude that different values of γ correspond to equivalent tunneling processes in an interesting way. For $-\pi < \gamma < -\pi/2$, the classical state is a particle somewhere to the right of the potential barrier, moving up the hill toward the classical turning point. For $-\pi/2 < \gamma < 0$, the particle is moving away from the turning point. Classical evolution, either forward or backward in time, takes the particle to the state at rest at the turning point, which is the nucleation point for $\gamma = -\pi/2$. We may interpret this as meaning that different γ (in the range $-\pi/2 < \gamma < 0$) correspond to different times at which a bubble nucleates at rest. The semiclassical path integral computes the amplitude to evolve from “no bubble” at T_i to “some bubble”

at T_f , and different γ appear to naturally account for the fact that we would also find a nonzero semiclassical amplitude, to nucleate a bubble with a smaller p_f , if we chose earlier T_i .

Consistent with this interpretation, all of the trajectories have the same on-shell value of the real part of the action. For example, for $p = 3$, one finds

$$\text{Re} \left[-i \int_{T_i}^{T_f} d\tau L_c(\partial_\tau R, R, \tau) \right] = \frac{\pi R_0^4}{16} \quad (4.11)$$

independent of γ .

4.2 Junction Conditions

We now switch gears and consider bubble nucleation when we include gravitational effects. Since this is a tunneling problem the semiclassical analysis described in Section 4.1 will be applied here also. In this section we will set up the problem and find the equation of motion of the bubble.

The 4D Einstein-Hilbert action coupled to a brane² is

$$S_{1,2}^{\text{EHGH}} = \frac{1}{16\pi G_N} \int_{\mathcal{M}_{1,2}} d^4x \sqrt{|g|} (\mathcal{R}_{1,2} - 2\Lambda_{1,2}) + \epsilon \frac{1}{8\pi G_N} \int_{\mathcal{B}_{1,2}} d^3y \sqrt{|h|} K_{1,2} - \mu \int_{\mathcal{T}} d^3y \sqrt{|h|}, \quad (4.12)$$

where $\mathcal{M}_{1,2}$ are the portions of spacetime corresponding to true and false vacuum respectively (cf. Figure 4.1), $\mathcal{B}_{1,2} = \partial\mathcal{M}_{1,2}$ and $\mathcal{T}$ is the world-volume of the brane, K is the trace of the extrinsic curvature tensor $K_{MN} = \gamma^A_M \gamma^B_N \nabla_A n_B$ with the normal covector n_M with $n_M n^M = \epsilon$ defined as outward pointing. For timelike boundaries we have $\epsilon = +1$ and the projector is defined $\gamma_{MN} = g_{MN} - \epsilon n_M n_N$.

The junction conditions [118] are

$$ds_{\mathcal{M}_1}^2|_{\mathcal{T}} = ds_{\mathcal{M}_2}^2|_{\mathcal{T}}, \quad (4.13)$$

$$T_{ab}^1 + T_{ab}^2 - \mu h_{ab} = 0, \quad (4.14)$$

where y^a label coordinates on $\mathcal{T}$ and

$$T_{ab}^i = -\frac{1}{8\pi G_N} (K_{ab}^i - K^i h_{ab}) \quad i = 1, 2, \quad (4.15)$$

with the convention that the normal vector is outward pointing in each region $\mathcal{M}_{1,2}$. Taking the trace of Eq. (4.13) we find

$$K^1 + K^2 = 12\pi G_N \mu, \quad (4.16)$$

where $K^i \equiv h^{ab} K_{ab}^i$ with $i = 1, 2$. Substituting Eq. (4.16) into Eq. (4.13) and Eq. (4.14) we find

$$ds_{\mathcal{M}_1}^2|_{\mathcal{T}} = ds_{\mathcal{M}_2}^2|_{\mathcal{T}}, \quad (4.17)$$

$$K_{ab}^1 + K_{ab}^2 - 4\pi G_N \mu h_{ab} = 0. \quad (4.18)$$

²We use the word brane or bubble interchangeably in this chapter.

Our next goal is to calculate Eq. (4.18). We will use the setup of [122] (see also [123]) because it is useful for calculating the extrinsic curvature tensor. Choose a neighborhood $N \in \mathcal{M}$ of $\mathcal{T}$ so that any point $p \in N$ lies on one and only one geodesic orthogonal to $\mathcal{T}$. We parametrize the timelike surface $\mathcal{T}$ with coordinates

$$y^a = (s, \theta, \varphi), \quad (4.19)$$

where s is the proper time for worldlines of fixed angular coordinates on the brane. A point $p \in N$ is described by coordinates $x^M = (\eta, y^a)$ where η is the proper distance from $\mathcal{T}$ to p in the positive direction along a geodesic passing through p . The positive η direction is taken to be from dS_T to dS_F in the direction of $\underline{n}$ as shown in Figure 4.1.³ In these Gaussian normal coordinates the ambient metric in the neighborhood N takes the following form

$$ds^2 = d\eta^2 + h_{ab}(s, \eta) dy^a dy^b. \quad (4.20)$$

The bubble hypersurface $\mathcal{T}$ is specified by setting $\eta = 0$ and has induced metric

$$ds^2|_{\mathcal{T}} = h_{ab} dy^a dy^b = -ds^2 + \gamma_{mk} dx^m dx^k. \quad (4.21)$$

Rotational invariance further implies that the metric (4.20) takes the form

$$ds^2 = d\eta^2 - h_{ss}^{\pm}(\eta, s) ds^2 + r_{\pm}^2(\eta, s) d\Omega^2 \quad (4.22)$$

where $h_{ss}^{\pm}(\eta = 0, s) = -1$ and $\pm$ correspond to positive/negative η and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and $r_{\pm}^2(\eta = 0, s) \equiv R^2(s)$.

In the Gaussian normal coordinates the unit vector $\underline{n}$ normal to $\mathcal{T}$ hypersurface is given by

$$n^M = (n^\eta, n^s, n^\theta, n^\varphi) = (1, 0, 0, 0), \quad (4.23)$$

and the extrinsic curvature of the $\mathcal{T}$ hypersurface is

$$K_{ab} = \frac{1}{2} \partial_\eta h_{ab} = \frac{1}{2} n^M \partial_M h_{ab}, \quad (4.24)$$

when evaluated at $\eta = 0$. We will use Eq. (4.24) to compute the extrinsic curvature components and substitute back to the Junction condition Eq. (4.18). Before we show this let us describe the static coordinates systems $(t_{dS_T}, r, \theta, \varphi)$ and $(t_{dS_F}, r, \theta, \varphi)$ that cover the region $\mathcal{M}_1$ and $\mathcal{M}_2$ as shown in Figure 4.1. We assume We assume that the relevant true and false vacuum spacetimes are approximately de Sitter in the static slicing with the following metrics

$$ds_i^2 = -f_i(r_i) dt_i^2 + \frac{1}{f(r_i)} dr_i^2 + r_i^2 d\Omega_i^2, \quad f(r_i) = 1 - \frac{r_i^2}{L_i^2}, \quad (4.25)$$

where the subscripts $i = \{dS_T, dS_F\}$ refer to the true and false vacuum spacetimes (cf. Figure 4.1).

The first junction condition Eq. (4.17) implies continuity of the induces metric on $\mathcal{T}$ which we set to

$$r_1 = r_2 = R, \quad \text{location of } \mathcal{T}. \quad (4.26)$$

³The positive direction is different from the normal vector definition used for the extrinsic curvatures in Eq. (4.15).

Matching the static metrics (4.25) with (4.31) on the brane implies

$$f_i(R)\dot{t}_i(s) = \sigma_i \sqrt{f_i(R) + \dot{R}^2(s)}, \quad i = \{dS_T, dS_F\}, \quad (\text{1st junction condition}). \quad (4.27)$$

In general the value of each $\sigma_{dS_{T,F}}$ could be ± 1 . The unit normal vector $\underline{n}$ components in static coordinates in the order $(\eta, s, \theta, \varphi)$ are ⁴

$$n^M = (f_i^{-1}(R)\dot{R}(s), f_i(R)\dot{t}_i(s), 0, 0) = (f_i^{-1}(R)\dot{R}(s), \sigma_i \sqrt{f_i(R) + \dot{R}^2(s)}, 0, 0). \quad (4.28)$$

To get from the first to the second equality above we used the first junction condition Eq. (4.27). The extrinsic curvature components $K_{\theta\theta}^i, K_{\varphi\varphi}^i$ are then computed from (4.24) using (4.22) and (4.28)

$$\begin{aligned} K_{\theta\theta}^i &= R(s)f_i(R(s))\dot{t}_i(s) = \sigma_i R(s) \sqrt{f_i(R(s)) + \dot{R}^2(s)}, \\ K_{\varphi\varphi}^i &= R(s)\sin^2\theta \dot{t}_i = \sin^2\theta K_{\theta\theta}^i. \end{aligned} \quad (4.29)$$

The $\theta\theta$ component of the second junction condition (4.18) is

$$\sigma_{dS_T} \sqrt{f_{dS_T}(R(s)) + \dot{R}^2(s)} - \sigma_{dS_F} \sqrt{f_{dS_F}(R(s)) + \dot{R}^2(s)} - 4\pi G_N \mu R(s) = 0, \quad (\text{2nd junction cond.}) \quad (4.30)$$

where $\sigma_{dS_T/dS_F} = \pm 1$ and $f_{dS_{T,F}}(R)$ were defined in (4.25). Eq. (4.30) is the main result of this section. It turns out the $s-s$ component of the second junction condition (4.18) is just the proper time derivative of Eq. (4.30), thus Eq. (4.30) is the only independent differential equation whose solution gives us the motion of the brane $R(s)$.

In the next section we will derive an effective action for the brane and analyze the tunneling problem for this action. We will see that the equation of motion of the effective action is the junction condition (4.18).

4.3 Bubble effective action

In this section we will show the derivation of an effective action for the motion of the bubble. We will generalize slightly the effective action constructed by Visser [119]. The same action was used in [124] to compute decay rates using euclidean methods. The effective action for the bubble is found by substituting the on-shell metrics Eq. (4.25) in the true and false regions back into action while keeping the degrees of freedom for the bubble off-shell.

The first step is to introduce the lapse $N(\lambda)$ on the bubble as a degree of freedom that we can vary. The timelike parameter λ is not necessarily the proper time (cf. Figure 4.1). The induced metric on the bubble is thus taken to be

$$ds^2|_{\mathcal{T}} = -N^2(\lambda)d\lambda^2 + R(\lambda)^2 d\Omega^2, \quad (4.31)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. In the case where one gauge fixes $\lambda = s$ we get $N(\lambda) = 1$ and the metric Eq. (4.31) takes the form of Eq. (4.20) of the previous section. Consider now the static coordinate systems

⁴For details of the derivation of Eq. (4.28) see [122].

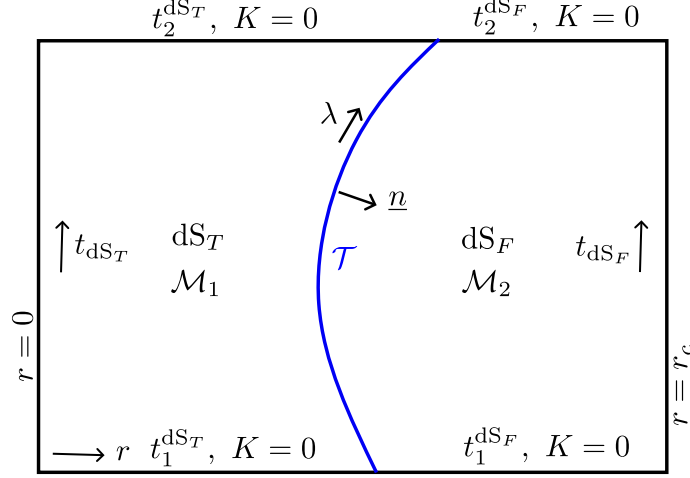


Figure 4.1: Inside true vacuum $\mathcal{M}_1$ dS_T region and outside false vacuum $\mathcal{M}_2$ dS_F and the membrane $\mathcal{T}$ shown in blue. The boundary $\partial\mathcal{M}$ consists of the spacelike surfaces at $t_{1,2}^{dS_{T,F}}$ and a timelike surface at the cosmological horizon r_c of dS_F . We denote a timelike parameter λ for an observer on the brane, which we take to be the proper time in Section 4.2 but keep arbitrary in Section 4.3. The cutoff time like surface is at the cosmological horizon $r = r_c$. The extrinsic curvature traces at $t_1^{dS_T/dS_F}$, $t_2^{dS_T/dS_F}$ turn out to be zero and do not contribute to the Gibbons-Hawking York boundary terms. We ignore corner terms.

for the true and false vacuum spacetime regions Eq. (4.25). The normalized 4-velocity in static coordinates is

$$U^M = (U^t, U^r, U^\theta, U^\varphi) = N(\lambda)^{-1}(t'(\lambda), R'(\lambda), 0, 0) \quad (4.32)$$

where prime denotes differentiation with λ . Similarly to what we did in Section 4.2 the first junction condition Eq. (4.13) implies⁵

$$t'_i(\lambda) = \sigma_i \sqrt{f_i(R(\lambda))^{-1}N(\lambda)^2 + f_i(R(\lambda))^{-2}R'(\lambda)^2}, \quad i = \{dS_T, dS_F\}, \quad (\text{1st junction condition}) \quad (4.33)$$

where $\sigma_i = \pm 1$, the f_i functions were defined in Eq. (4.25). Note that Eq. (4.33) reduces to Eq. (4.27) when we gauge fix $N = 1$ which sets $\lambda = s$. The unit vector $\underline{n}$ components normal to $\mathcal{T}$, in static coordinates in the order $(\eta, \lambda, \theta, \varphi)$ are

$$n^M = N^{-1}(\lambda)(f_i^{-1}(R(\lambda))R'(\lambda), f_i(R(\lambda))t'_i(\lambda), 0, 0) = N(\lambda)^{-1}(f_i^{-1}(R(\lambda))R'(\lambda), \sigma_i \sqrt{f_i(R)N(\lambda)^2 + R'(\lambda)^2}, 0, 0). \quad (4.34)$$

To get from the first to the second equality above we used the first junction condition Eq. (4.33).

The $K_{\theta\theta}^i, K_{\varphi\varphi}^i$ components of the extrinsic curvature on $\mathcal{T}$ can be computed from Eq. (4.24) using Eq. (4.34) and Eq. (4.22) where instead of η we have λ together with the lapse $N(\lambda)$. We find

$$\begin{aligned} K_{\theta\theta}^i &= N(\lambda)^{-1}R(\lambda)f(R(\lambda))t'_i(\lambda) = \sigma_i R(s) \sqrt{f_i(R(\lambda)) + N(\lambda)^{-2}R'(\lambda)^2}, \\ K_{\varphi\varphi}^i &= \sin^2\theta K_{\theta\theta}^i. \end{aligned} \quad (4.35)$$

⁵We assume $f_i(r) > 0$.

To find $K_{\lambda\lambda}^i$ we can use the following formula derived in [122]

$$K_{\lambda\lambda}^i = -N^2 n_N a_i^N, \quad (4.36)$$

where $a_i^N := U_i^M \nabla_M U_i^N$ and U_i^M is the proper velocity vector for an observer on the brane $\mathcal{T}$. If we use the matching condition Eq. (4.33) we find

$$\begin{aligned} a_i^t &= -N(\lambda)^{-3} N'(\lambda) t'_i(\lambda) + N(\lambda)^{-2} t''_i(\lambda) + \frac{\partial_R f_i(R)}{f(R(\lambda))} N(\lambda)^{-2} t'_i(\lambda) R'(\lambda), \\ a_i^r &= -N(\lambda)^{-3} N'(\lambda) R'(\lambda) + N^{-2} R''(\lambda) + \frac{1}{2} \partial_R f(R), \\ a^\theta &= a^\varphi = 0. \end{aligned} \quad (4.37)$$

From Eq. (4.36) and Eq. (4.37), Eq. (4.34) we find

$$K_{\lambda\lambda}^i = -\frac{1}{f_i(R(\lambda)) t'_i(\lambda)} \left[N(\lambda) R''(\lambda) + N(\lambda)^3 \frac{1}{2} \partial_R f_i(R) - N'(\lambda) R'(\lambda) \right], \quad (4.38)$$

or equivalently⁶

$$K_{\lambda\lambda}^i = -N \left[\frac{d}{d\lambda} \left[\sinh^{-1} \left(\frac{R'(\lambda)}{\sigma_i \sqrt{N(\lambda)^2 f_i(R(\lambda))}} \right) \right] + \frac{1}{2} \partial_R f_i(R) t'_i(\lambda) \right], \quad (4.39)$$

where we used the first junction condition (4.33). From Eq. (4.35), Eq. (4.39) and Eq. (4.31) the extrinsic curvature trace $K = h^{ab} K_{ab}^i$ is

$$K = N(\lambda)^{-1} \left(\frac{d}{d\lambda} \left[\sinh^{-1} \left(\frac{R'(\lambda)}{\sigma_i \sqrt{N(\lambda)^2 f_i(R)}} \right) \right] + \frac{1}{2} \partial_R f(R) t'(\lambda) \right) + 2R(\lambda)^{-1} N(\lambda)^{-1} f_i(R(\lambda)) t'(\lambda). \quad (4.40)$$

We have calculated all the necessary ingredients to find the bubble effective action from the action Eq. (4.12). The spacetime region consists of $\mathcal{M}_1 \cup \mathcal{M}_2$ whose boundary $\partial\mathcal{M}$ consists of the surfaces at $t_{1,2}^{\text{dS},F}$ and a timelike surface which we set at the cosmological horizon $r = r_c$ as shown in Eq. (4.1). After some calculation we find from Eq. (4.12) the bubble effective action

$$\begin{aligned} S_{\text{bubble}}^{\text{eff}} &= \frac{4\pi}{8\pi G_N} \int d\lambda \left[2R(\lambda) R'(\lambda) \sinh^{-1} \left(\frac{R'(\lambda)}{\sigma_{\text{dS}F} \sqrt{N^2 f_{\text{dS}F}}} \right) - 2RR' \sinh^{-1} \left(\frac{R'(\lambda)}{\sigma_{\text{dS}T} \sqrt{N^2 f_{\text{dS}T}}} \right) \right. \\ &\quad \left. + 2\sigma_{\text{dS}T} R \sqrt{f_{\text{dS}T} N^2 + R'^2} - 2\sigma_{\text{dS}F} R \sqrt{f_{\text{dS}F} N^2 + R'^2} - 8\pi G_N \mu N R^2 \right], \end{aligned} \quad (4.41)$$

where $R'(\lambda) = \frac{dR}{d\lambda}$. This action slightly generalizes that of Visser [119] by the inclusion of a metric degree of freedom on the brane, N which allows manifest reparametrization invariance. Varying this degree of freedom, $\frac{\delta S_{\text{bubble}}}{\delta N} = 0$, we find ⁷

$$\sqrt{f_{\text{dS}T} + N^{-2} R'(\lambda)^2} - \sqrt{f_{\text{dS}F} + N^{-2} R'(\lambda)^2} - 4\pi \mu G_N R(\lambda) = 0. \quad (4.42)$$

⁶This trick was taken from [119].

⁷We have chosen $\sigma_{\text{dS},F} = +1$ which is the physically relevant case when $0 < R < L_F < L_T$, with same time orientation inside and outside the bubble, and for small enough positive μ . For an extensive analysis of different cases see [125] and references therein.

Note that from our choice of $\sigma_{dS_{T,F}} = +1$ Eq. (4.42) matches the second junction condition Eq. (4.30) when $\lambda = s$. Eq. (4.42) is the only independent differential equation, since it turns out that the Euler-Lagrange equation obtained by varying R is simply the λ derivative of (4.42). Keeping the lapse in the action is important to obtain the first-order equation (4.42) from first principles. Having derived it as an Euler-Lagrange equation, we may now fix the gauge on the brane by setting $N = 1$, which identifies λ with the proper time s along worldlines of fixed angular coordinates on the brane.

4.4 Tunneling analysis

To analyse the problem of bubble nucleation as we described in Section 4.1 as a tunneling problem we need to find the energy of the bubble, namely the Hamiltonian for the action Eq. (4.41). First, the canonical momentum is

$$P(R) = \frac{1}{G_N} R \left(\sinh^{-1} \left(\frac{R'(\lambda)}{\sigma_{dS_F} \sqrt{N^2 f_{dS_F}}} \right) - \sinh^{-1} \left(\frac{R'(\lambda)}{\sigma_{dS_T} \sqrt{N^2 f_{dS_T}}} \right) \right). \quad (4.43)$$

We can use $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$ to rewrite the momenta (4.43) as

$$P(R) = \frac{1}{G_N} R \ln \left(\sqrt{\frac{f_{dS_T}}{f_{dS_F}}} \left(\frac{R'(\lambda) + \sqrt{N^2 f_{dS_F} + R'(\lambda)^2}}{R'(\lambda) + \sqrt{N^2 f_{dS_T} + R'(\lambda)^2}} \right) \right). \quad (4.44)$$

The Hamiltonian $H = P(R)R'(\lambda) - \mathcal{L}$ is found from Eq. (4.43) and Eq. (4.41)

$$H = \frac{\pi}{G_N} \left[R \sigma_{dS_F} \sqrt{f_{dS_F} N^2 + R'(\lambda)^2} - R \sigma_{dS_T} \sqrt{f_{dS_T} N^2 + R'(\lambda)^2} + 4\pi G_N \mu N R^2 \right]. \quad (4.45)$$

We now set $N = 1$, $\sigma_{dS_{T,F}} = +1$ and continue $(\lambda \equiv s) \rightarrow e^{i\gamma'} s$. (We use γ' instead of γ because the proper time is not exactly the same as the coordinate time which was continued in Section 4.1; we will derive the relationship below.) As we described in Section 4.1 the continuation prescription is [117]

$$L \rightarrow L_c = e^{i\gamma'} L(e^{-i\gamma'} \partial_s R, R, e^{i\gamma'} s), \quad E \rightarrow E_c = e^{i\gamma'} E(e^{-i\gamma'} \partial_s R, R, e^{i\gamma'} s) \quad (4.46)$$

From Eq. (4.45) and Eq. (4.46) we find the energy

$$E_c = e^{i\gamma'} \frac{\pi}{G_N} \left[R \sqrt{f_{dS_F} + e^{-2i\gamma'} \dot{R}^2} - R \sqrt{f_{dS_T} + e^{-2i\gamma'} \dot{R}^2} + 4\pi G_N \mu R^2 \right]. \quad (4.47)$$

Setting $N = 1$ and $(\lambda \equiv s) \rightarrow e^{i\gamma'} s$ into Eq. (4.42) we note that the energy vanishes on-shell, indicative of time reparametrization invariance. To find the classical solutions we first note that $R(s) = 0$ is a trivial solution with vanishing energy. To find the other solution branch we rearrange (4.42) into the form

$$e^{-2i\gamma'} \left(\frac{dR}{ds} \right)^2 + V(R) = 0, \quad (4.48)$$

where $V(R) = 1 - \alpha^2 R^2$, and $\alpha^2 = L_F^{-2} + \frac{(L_F^{-2} - L_T^{-2} - 16\pi^2 G_N^2 \mu^2)^2}{64\pi^2 G_N^2 \mu^2}$. The solution with $R(s_0) = 0$ is

$$R(s) = \alpha^{-1} \cosh \left[\alpha e^{i\gamma'} (s - s_0) + i(\pi/2) \right]. \quad (4.49)$$

The initial proper time $s_0 = \frac{\pi}{2} \alpha^{-1} \csc(\gamma')$ is the largest negative time for which $\text{Im } R(0) = 0$. The final position at $s = 0$ is $R(0) = \alpha^{-1} \cosh((\pi/2) \cot \gamma')$ which falls within the classical allowed region $R > \alpha^{-1}$ for any γ' . The real time energy vanishes for the final position and momenta, and final states with different γ' are connected to each other at the classical level, by real time evolution.

The on-shell action is given by the following integral $-i \int_{T_i}^{T_f} ds L_c = -ie^{i\gamma} \int_0^{R(0)} P_c(R) dR$, where $P_c(R)$ is the on-shell continued momenta

$$P_c(R)|_{\text{on shell}} = \frac{1}{G_N} R \ln \left[\frac{\sqrt{1 - L_T^{-2} R^2}}{\sqrt{1 - L_F^{-2} R^2}} \left(\frac{\sqrt{\alpha^2 R^2 - 1} + R \sqrt{\alpha^2 - L_F^{-2}}}{\sqrt{\alpha^2 R^2 - 1} + R \sqrt{\alpha^2 - L_T^{-2}}} \right) \right], \quad (4.50)$$

which was computed from Eq. (4.44) using Eq. (4.49).

If we set $N = 1$ and expand (4.41) to leading order in $G_N/R^2 \rightarrow 0$ we find the following action [124]

$$S_{\text{bubble}}^{\text{eff}} = \int ds \left(\frac{4}{3} \pi R^3 (\epsilon_F - \epsilon_T) \sqrt{1 + \dot{R}^2} - 4\pi \mu R^2 \right). \quad (4.51)$$

We can also write Eq. (4.51) into a more familiar form if we express it in terms of Lorentzian time t instead of proper time s . The exact relation is $dt = \sqrt{f(R)^{-1} + f(R)^{-2} \left(\frac{dR}{ds} \right)^2} ds$ which to leading order in $G_N \rightarrow 0$ becomes the familiar time dilation relation

$$dt \approx ds \sqrt{1 + \left(\frac{dR}{ds} \right)^2}. \quad (4.52)$$

In terms of Lorentzian time t the Lagrangian of (4.51) is proportional to Eq. (4.5) with $p = 3$ and $R_0 = \frac{3\mu}{\epsilon_F - \epsilon_T} \approx \alpha^{-1}$.

Similarly we can recover the solution (4.8) from (4.49) in the $R^2/G_N \rightarrow 0$ limit. If we continue the time dilation relation Eq. (4.52) we find

$$e^{i\gamma} d\tau \approx e^{i\gamma'} ds \sqrt{1 + e^{-2i\gamma'} \left(\frac{dR}{ds} \right)^2}. \quad (4.53)$$

If we now integrate the time dilation relation, use $\alpha^{-1} \approx R_0$ to leading order and the initial conditions s_0, τ_0 we find

$$e^{i\gamma} \tau - R_0 \cot \gamma = \alpha^{-1} \sinh \left(\alpha e^{i\gamma'} s - \frac{\pi}{2} \cot \gamma' \right). \quad (4.54)$$

Making use of (4.54) it is easy to check that the solution (4.49) reduces to (4.8). The relation between the coordinate time continuation parameter γ and the proper time parameter γ' can also be found, if we set $\tau = s = 0$ into (4.54). We find $\cot \gamma = \sinh \left(\frac{\pi}{2} \cot \gamma' \right)$, which together with (4.54) implies $R(0) = R_0 |\csc \gamma|$ as expected.

Now let us include some time dependence. A field theory model that can be mapped more or less straightfor-

wardly to a time-dependent generalization of (4.41) is given by

$$V = \frac{1}{2}[\lambda_0 + (g/a)\chi](\phi^2 - a^2)^2 - (\epsilon/2a)\phi + V_0 + V(\chi). \quad (4.55)$$

As in the nongravitational example, and for similar reasons, we consider the hierarchy $\lambda_0 a^4 \gg V_0 \gg (V(\chi), \epsilon) > 0$, so that the bubble profile is dominated by the ϕ^4 potential, and the state of the gravitational fields in the true and false vacua can be approximated by de Sitter space. We assume χ and ϕ are weakly coupled, which means both that g is small and that the metric in the bubble background does not substantially alter the background solution for χ , which is consistent with the small ϵ approximation.

The relevant rolling solution for χ at $\mathcal{O}(g^0)$ is homogeneous in the flat slicing of dS. The static slicing is more convenient for the present problem, so χ is a function of the form $\chi(t + L \log \sqrt{1 - r^2/L^2})$, with $L^2 \approx 3M_p^2/V_0$ and r and t the static radial and time coordinates, respectively.

This model is obviously not the most general case, since the background spacetimes are approximated as time-independent and the underlying field theory model is certainly fine-tuned. Its main virtue is that the effective action is almost entirely recycled from the time-independent case. In the thin-wall limit the bubble dynamics of the model maps to a membrane effective action of the form (4.41) with $\mu \rightarrow \mu(t_{dS_F}, R)$. The spacetime dependence of the tension appears at $\mathcal{O}(g)$ and is inherited from the background solution for χ in static coordinates.

If we set lapse variation of (4.41) to zero with μ time dependent, rearrange and apply a λ derivative on both sides we find

$$\frac{d}{d\lambda} \left[\frac{R}{\left(\frac{\partial q}{\partial N}\right)} \left(2\sqrt{f_{dS_F} + N^{-2}R'(\lambda)^2} - 2\sqrt{f_{dS_F} + N^{-2}R'(\lambda)^2} - \tilde{\mu}R \right) \right] = -NR^2 \frac{\partial \tilde{\mu}}{\partial t_{dS_F}}, \quad (4.56)$$

where we defined $q := \sqrt{f_{dS_F}^{-1}N^2 + f_{dS_F}^{-2}R'(\lambda)^2}$ and $\tilde{\mu} := 8\pi G_N \mu$. Eq. (4.42) is a first order equation analogous to a conservation of energy law. Eq. (4.56) is a second order equation describing the non-conservation of energy by the varying μ .

From this point one may proceed to rotate the time contour and seek suitable tunneling solutions connecting the vacuum to nucleated bubble states. Similarly to the nongravitational cases, at leading order (in the derivative of μ with respect to the flat time slicing) corrections to the tunneling probability may be found from the unperturbed solutions (4.49) and extremization over γ .

4.5 Summary

In this chapter we have analyzed vacuum decay with bubble nucleation taking into account gravitational back-reaction. We approximated the spacetimes in the true (inside the bubble) and false (outside the bubble) vacua as de Sitter spaces in static slicing with different cosmological constants, cf. Figure 4.1.

In Section 4.1 we described our methodology behind tunneling effects in the semiclassical approximation. In Section 4.2 we found the Israel junction conditions across the brane $\mathcal{T}$. The second junction Eq. (4.18) provides us with a first order differential equation for the motion of the brane.

In Section 4.3 we found the bubble effective action Eq. (4.41). A novel point in our construction (compared to [119]) is that we kept the lapse degree of freedom on $\mathcal{T}$ unfixed. Keeping the lapse degree of freedom in the effective action is important because the equation of motion derived from its variation Eq. (4.42) gives us the second junction condition derived in Section 4.2.

In Section 4.4 we analyzed semiclassical tunneling along the lines of Section 4.1 for the bubble effective action. Finally we ended the section with some comments on including time dependance in this model and found the equation of motion for the bubble Eq. (4.56) in this case. We leave a more complete analysis of the time-dependent case for future work.

Chapter 5

Weyl connection and holographic Weyl anomaly

Work in this chapter is based on “Obstruction tensors in Weyl geometry and holographic Weyl anomaly” [126], authored by Weizhen Jia (UIUC) and Manthos Karydas (UIUC).

In this chapter, we present several results originally published in [126], along with new material not included in that publication. Our main goal is to investigate the holographic Weyl anomaly using the Weyl-Fefferman-Graham gauge introduced in [43], which served as the primary basis for our work. We begin in Section 5.1 by introducing the necessary geometric background of Conformal/Weyl geometry required for the subsequent sections. In Section 5.2, we provide a general introduction to the Weyl anomaly, both with and without the Weyl connection. The bulk of our findings are presented in Section 5.3. Additionally, Appendix B contains useful formulas and supplementary material for this chapter.

5.1 Weyl geometry

In this section we introduce the basic ingredients of Weyl geometry [39]–[42], [127], [128]. We will first introduce Riemannian geometry, then define conformal geometry and finally we extend conformal geometry to Weyl geometry.

The fundamental structures of **Riemannian geometry** is a manifold $\mathcal{M}$ together with a metric g_{ij} , $(\mathcal{M}, g)$. Given the existence of the metric another fundamental geometric quantity is the Levi-Civita (LC) connection ∇ . The LC connection is constructed out of the metric (and partial derivatives) and allows one to parallel transport vectors. The fundamental symmetry group for any manifold are diffeomorphisms, namely coordinate transformations $x'^i = x'^i(x^j)$. The building blocks of Riemannian geometry are Riemannian tensors $T^{ij\dots}_{kl\dots}$ that transform covariant under coordinate transformations

$$T'^{ij\dots}_{kl\dots} = \left(\frac{\partial x'^i}{\partial x^r} \frac{\partial x'^j}{\partial x^s} \dots \right) \left(\frac{\partial x^p}{\partial x'^k} \frac{\partial x^q}{\partial x'^l} \dots \right) T^{rs\dots}_{pq\dots} \quad (5.1)$$

One of the most important tensors is the Riemann curvature tensor $\hat{R}^i_{jkl} = \partial_k \hat{\Gamma}^i_{jl} - \partial_l \hat{\Gamma}^i_{jk} + \hat{\Gamma}^i_{km} \hat{\Gamma}^m_{lj} -$

$\overset{\circ}{\Gamma}_{lm}^i \overset{\circ}{\Gamma}_{kj}^m$ which captures the curvature of space (or spacetime in General Relativity). There is a simple procedure to create new Riemannian tensors. One can act with any number of covariant derivatives $\overset{\circ}{\nabla}$ on a Riemannian tensor to create another Riemannian tensor. For example in the case of the Riemann curvature tensor a plethora of new Riemannian tensors can be constructed, e.g.

$$\overset{\circ}{\nabla}_{m_1} \dots \overset{\circ}{\nabla}_{m_n} \overset{\circ}{R}_{jkl}^i \overset{\circ}{\nabla}_{k_1} \dots \overset{\circ}{\nabla}_{k_l} \overset{\circ}{R}_{fgh}^e. \quad (5.2)$$

Moreover, one can make use of the metric to contract any two indices to create another Riemannian tensor.¹ In the case of scalar tensors Weyl's classical invariant theory for the orthogonal group showed that all scalar tensors arise by the operations mentioned above.

In **conformal geometry** the symmetry group is extended to include both diffeomorphisms defined above and Weyl transformations of the metric

$$g_{ij} \rightarrow \mathcal{B}(x)^{-2} g_{ij}. \quad (5.3)$$

The symmetry group is now $\text{Diff}(M) \ltimes \text{Weyl}$. The Weyl symmetry defines a conformal class of metrics denoted by $[g]$. Two metrics belong in the same conformal class if one metric is a smooth positive multiple of the other. The fundamental structure of conformal geometry is a manifold $\mathcal{M}$ endowed with a conformal class of metrics $[g]$, $(\mathcal{M}, [g])$.

The building blocks of conformal geometry are conformal tensors, which are Riemannian tensors that transform covariantly under the Weyl transformation $g_{ij} \rightarrow \mathcal{B}(x)^{-2} g_{ij}$, namely

$$T^{ij\dots}_{ab\dots} \rightarrow \mathcal{B}(x)^{w_T} T^{ij\dots}_{ab\dots}, \quad w_T \in \mathbb{R}. \quad (5.4)$$

The real number w_T is called the *Weyl weight* of the conformal tensor $T^{ij\dots}_{ab\dots}$. By definition we take the Weyl weight of the metric tensor to be $w_{g_{ij}} = -2$. The Riemann curvature tensor is not a conformal tensor since it does not transform covariantly under a Weyl transformation. One can construct a conformal tensor from the Riemann curvature tensor, called the Weyl tensor², defined as

$$\overset{\circ}{W}_{ikjl} = \overset{\circ}{R}_{ikjl} + g_{il} \overset{\circ}{P}_{kj} + g_{kj} \overset{\circ}{P}_{il} - g_{ij} \overset{\circ}{P}_{kl} - g_{kl} \overset{\circ}{P}_{ij}, \quad (5.5)$$

where

$$\overset{\circ}{P}_{ij} = \frac{1}{d-2} \left(\overset{\circ}{R}_{ij} - \frac{1}{2(d-1)} \overset{\circ}{R} g_{ij} \right) \quad (5.6)$$

is the Schouten tensor. The Weyl weight of the Weyl tensor according to (5.4) turns out to be $w_{W_{ikjl}} = -2$.

One might have expected that it would be easy to construct new conformal tensors from the Weyl tensor using similar process in Riemannian tensors. Unfortunately, this is not the case, and the main difficulty is that acting with the LC connection $\overset{\circ}{\nabla}$ on a conformal tensor does not create another conformal tensor. Conformal tensors are thus quite scarce. The only other known conformal tensors besides the Weyl tensor

¹In the case of Riemann curvature tensor it is possible that due to Bianchi/Ricci identities not all of the tensors constructed are linearly independent.

²The Weyl tensor is a conformal tensor in any spacetime dimension $d \geq 4$ and vanishes identically in $d = 2, 3$.

are the Cotton tensor [129] (conformal only in 3d) and the Bach tensor [130] (conformal only in 4d)

$$\mathring{C}_{ijk} = \mathring{\nabla}_k \mathring{P}_{ij} - \mathring{\nabla}_j \mathring{P}_{ik} \quad , \quad w_{\mathring{C}_{ijk}} = 0 \quad \text{in } 3d, \quad (5.7)$$

$$\mathring{B}_{ij} = \mathring{\nabla}^k \mathring{C}_{ijk} + \mathring{P}^{kl} \mathring{W}_{ikjl} \quad , \quad w_{\mathring{B}_{ij}} = 2 \quad \text{in } 4d. \quad (5.8)$$

Through the work of Fefferman and Graham [44], [131], [132] a systematic method to find conformally covariant tensors in even $d = 2k$, ($k = 2, 3, \dots$) spacetime dimension, called *obstruction tensors* $\mathring{\mathcal{O}}_{ij}^{(2k)}$, was proposed. The obstruction tensors are generalizations of the Bach tensor $\mathring{\mathcal{O}}_{ij}^{(4)} \equiv \mathring{B}_{ij}$ in higher dimensions. For example for $k = 3$ one has the following obstruction tensor

$$\begin{aligned} \mathring{\mathcal{O}}_{ij}^{(6)} = & \mathring{\nabla}^k \mathring{\nabla}_k \mathring{B}_{ij} - 2 \mathring{W}_{kijl} \mathring{B}^{kl} - 4 \mathring{P} \mathring{B}_{ij} \\ & + 2(d-4) (2 \mathring{P}^{kl} \mathring{\nabla}_i \mathring{C}_{(ij)k} + (\mathring{\nabla}_l \mathring{P}) \mathring{C}_{(ij)}^{\quad l} - \mathring{C}^{kl} \mathring{C}_{ljk} + \frac{1}{2} \mathring{C}_i^{kl} \mathring{C}_{jkl} - \mathring{W}_{kijl} \mathring{P}_m^k \mathring{P}^{ml}) \end{aligned} \quad (5.9)$$

where $\mathring{P} = g^{ij} \mathring{P}_{ij}$. We note that the obstruction tensors $\mathring{\mathcal{O}}_{ij}^{(2k)}$ are symmetric traceless Riemannian tensors for any dimension d . They are divergence-free conformal tensors only in $d = 2k$ dimension with Weyl weight $w_{\mathring{\mathcal{O}}_{ij}^{(2k)}} = 2k - 2$. We will define $\mathring{\mathcal{O}}_{ij}^{(2k)}$ through the Fefferman-Graham expansion in Subsection 5.3.1.

In **Weyl geometry** the fundamental structure is a manifold $\mathcal{M}$ together with a *Weyl class* $[g, a]$, $(\mathcal{M}, [g, a])$. The Weyl class $[g, a]$ is an extension of the conformal class $[g]$ used in conformal geometry. It is defined as the equivalence class of pairs

$$(g, a) \sim (\mathcal{B}(x)^{-2} g, a - d \ln \mathcal{B}(x)), \quad (5.10)$$

with g the metric and a an one-form called the *Weyl connection* [39]–[41]. The fundamental symmetry (besides diffeomorphism) in Weyl geometry is the following Weyl transformation

$$g_{ij} \rightarrow \mathcal{B}(x)^{-2} g_{ij} \quad , \quad a_i \rightarrow a_i - \partial_i \ln \mathcal{B}(x). \quad (5.11)$$

The building blocks of Weyl geometry are *Weyl-covariant tensors*, namely tensors that transform as

$$T^{ij\dots}_{ab\dots} \rightarrow \mathcal{B}(x)^{w_T} T^{ij\dots}_{ab\dots} \quad , \quad w_T \in \mathbb{R} \quad , \quad \text{under } (g_{ij} \rightarrow \mathcal{B}(x)^{-2} g_{ij} \quad , \quad a_i \rightarrow a_i - \partial_i \ln \mathcal{B}(x)), \quad (5.12)$$

where $w_T \in \mathbb{R}$ is the Weyl-weight. Note that Weyl-covariant tensors generically depend on some combination of the metric g_{ij} , the Weyl connection a_i and derivatives thereof.

The introduction of the Weyl connection a allows one the Weyl-Levi-Civita (Weyl-LC) connection $\hat{\nabla}$, which we describe next. The canonical way to define the LC connection $\hat{\nabla}$ is as the unique torsionless and metric compatible connection. Due to the existence of the Weyl connection form a_i we can define a new connection ∇ that is also torsion-free but now satisfies the Weyl-metricity condition [41], [43] condition. In a coordinate system these two conditions become³

$$\Gamma^i_{[jk]} = 0 \quad , \quad (\text{Torsion-free condition}) \quad (5.13)$$

$$\nabla_k g_{ij} = 2a_k g_{ij} \quad , \quad (\text{Weyl-metricity condition}), \quad (5.14)$$

³We assume a coordinate basis $\underline{e}_i = \underline{\partial}_i$ on the manifold.

which has as unique solution the following connection coefficients

$$\Gamma^i_{jk} = \mathring{\Gamma}^i_{jk} - (a_j \delta^i_k + a_k \delta^i_j - g^{im} a_m g_{jk}). \quad (5.15)$$

An important property of Γ^i_{jk} is that it is invariant under the Weyl transformation (5.11). One can check this using the Weyl transformation of the LC connection, given in Appendix B.1, and is consequence of Weyl covariance of the torsion-free condition (5.13) and Weyl-metricity condition (5.14) under (5.11).

Suppose we are given a Weyl-covariant tensor $T^{ij\cdots}_{ab\cdots}$ with Weyl weight $w_{T^{ij\cdots}_{ab\cdots}}$. We now define the Weyl-LC connection $\hat{\nabla}$ as

$$\hat{\nabla}_k T^{ij\cdots}_{ab\cdots} \equiv \nabla_k T^{ij\cdots}_{ab\cdots} + a_k w_{T^{ij\cdots}_{ab\cdots}} T^{ij\cdots}_{ab\cdots}, \quad (5.16)$$

and for ∇ we use the connection coefficients (5.15). From the Weyl invariant property of Γ^i_{jk} mentioned below (5.15) it is easy to verify that $\hat{\nabla}_k T^{ij\cdots}_{ab\cdots}$ is again a Weyl-covariant tensor with the same Weyl weight as $T^{ij\cdots}_{ab\cdots}$. This is the most important property of the Weyl-LC connection $\hat{\nabla}$. One can create new Weyl-covariant tensors acting with $\hat{\nabla}$ and contracting with the metric, very similar in spirit to what one does in Riemannian geometry.

To find the analog of the Riemann curvature in Weyl geometry we can use ∇ and define it as

$$R^a_{bcd} = \partial_c (\Gamma^a_{db}) - \partial_d (\Gamma^a_{cb}) + \Gamma^f_{db} \Gamma^a_{cf} - \Gamma^f_{cb} \Gamma^a_{df}. \quad (5.17)$$

Using the definition above that R^a_{bcd} is a Weyl-covariant tensor Weyl weight $w_{R^a_{bcd}} = 0$. Contracting the indices we can define the Ricci tensor $R_{ab} = R^f_{afb}$ with Weyl weight $w_{R_{ab}} = 0$ and Ricci scalar $R = g^{ab} R_{ab}$ with Weyl weight $w_R = 2$. From (5.15) we can write R^a_{bcd} , R_{ab} in terms of the LC analogs $\mathring{R}^a_{bcd}$ and Weyl connection a_i . One finds [43]

$$R^a_{bcd} = \mathring{R}^a_{bcd} + \mathring{\nabla}_d a_b \delta^a_c - \mathring{\nabla}_c a_b \delta^a_d + \left(\mathring{\nabla}_d a_c - \mathring{\nabla}_c a_d \right) \delta^a_b + \mathring{\nabla}_c a^a g_{db} - \mathring{\nabla}_d a^a g_{bc} \quad (5.18)$$

$$+ a_b (a_d \delta^a_c - a_c \delta^a_d) + a^a (a_c g_{db} - a_d g_{cb}) + a^2 (g_{cb} \delta^a_d - g_{db} \delta^a_c),$$

$$R_{ab} = \mathring{R}_{ab} - \frac{d}{2} f_{ab} + (d-2) \left(\mathring{\nabla}_{(a} a_{b)} + a_a a_b \right) + \left(\mathring{\nabla} \cdot a - (d-2) a^2 \right) g_{ab}, \quad (5.19)$$

$$R = \mathring{R} + 2(d-1) \mathring{\nabla} \cdot a - (d-1)(d-2) a^2, \quad (5.20)$$

where $\mathring{\nabla} \cdot a = g^{ij} \mathring{\nabla}_i a_j$, $a^2 = g^{ij} a_i a_j$ and $f_{ab} = \mathring{\nabla}_a a_b - \mathring{\nabla}_b a_a = \partial_a a_b - \partial_b a_a$. Note that f_{ab} is also a Weyl covariant tensor of Weyl weight $w_{f_{ab}} = 0$.

Finally we define the *Weyl-Riemann curvature* $\hat{R}^a_{bcd}$

$$\hat{R}^a_{bcd} = R^a_{bcd} + \delta^a_b f_{cd}, \quad (5.21)$$

which is a Weyl-covariant tensor with Weyl weight $w_{\hat{R}^a_{bcd}} = 0$. It turns out that $\hat{R}^a_{bcd}$ is the curvature tensor defined from the Weyl-LC connection $\hat{\nabla}$. Suppose we have a basis of vectors e_a that is not necessarily a coordinate basis. If we assume they have Weyl weight $w_{e_a} = +1$ under (5.11), we can find the connection coefficients $\hat{\Gamma}^i_{jk}$ from the definition (5.16). One can show⁴ that the corresponding curvature tensor for $\hat{\Gamma}^i_{jk}$ is

⁴We refer to [126] for more details.

the Weyl-Riemann curvature tensor $\hat{R}^a_{bcd}$ given in Eq. (5.21).

We can also define the Weyl-Schouten tensor $\hat{P}_{ij}$ with $w_{\hat{P}_{ij}} = 0$, Weyl-Cotton tensor $\hat{C}_{ijk}$ with $w_{\hat{C}_{ijk}} = 0$ and Weyl-Bach tensor $\hat{B}_{ij}$ with $w_{\hat{B}_{ij}} = 2$

$$\hat{P}_{ij} = \frac{1}{d-2} \left(\hat{R}_{ij} - \frac{1}{2(d-1)} \hat{R} g_{ij} \right) \quad (5.22)$$

$$\hat{C}_{ijk} = \hat{\nabla}_k \hat{P}_{ij} - \hat{\nabla}_j \hat{P}_{ik} \quad (5.23)$$

$$\hat{B}_{ij} = \hat{\nabla}^k \hat{\nabla}_k \hat{P}_{ij} - \hat{\nabla}^k \hat{\nabla}_j \hat{P}_{ik} - \hat{W}_{kjim} \hat{P}^{mk}, \quad (5.24)$$

where $\hat{W}^i_{jkm}$ is the Weyl-Weyl tensor, namely the traceless part of $\hat{R}^i_{jkm}$. It is noteworthy that the Weyl-Cotton, Weyl-Bach tensor are Weyl-covariant tensors in any dimension in contrast to the Cotton (5.7) and Bach tensors (5.7) that are conformal only in $3d$ and $4d$ respectively. Similarly, obstruction tensors can be generalized to Weyl-obstruction tensors $\hat{\mathcal{O}}^{(2k)}_{ij}$, which are Weyl-covariant tensors with Weyl weight $w_{\hat{\mathcal{O}}^{(2k)}_{ij}} = 2k - 2$. We will see how to construct Weyl-obstruction tensors using the Weyl-ferfferman-Graham expansion in Section 5.3.3 and more rigorously using the Weyl-ambient metric in Chapter 6.

5.2 Weyl anomaly and cohomology

In the previous section we explained the basic ingredients of conformal and Weyl geometry. In this section we will define the Weyl anomaly [133], [134] for a Quantum field theory on a conformal and Weyl manifold. We also discuss the cohomological properties of the Weyl anomaly [135]–[142] and derive the Weyl-Ward identity.

Similar to Section 5.1 we will first consider the Weyl anomaly for a QFT in the case of conformal geometry and then extend it to Weyl geometry. We note that the constructions in this subsection are valid regardless of holography.

Consider a d -dimensional field theory⁵ on a background endowed with conformal geometry $(\mathcal{M}, [g])$. The path integral $Z[g_{ij}]$ is a functional of the background metric g_{ij}

$$Z[g_{ij}] = \int [\mathcal{D}\phi]_{g_{ij}} e^{-S[g_{ij}, \phi]}, \quad (5.25)$$

where we consider conformally coupled “matter fields” with Weyl weight w_ϕ , namely we assume the classical action to be Weyl invariant⁶

$$S[\mathcal{B}(x)^{-2} g_{ij}, \mathcal{B}(x)^{w_\phi} \phi] = S[g_{ij}, \phi]. \quad (5.26)$$

The Weyl anomaly comes from [135]–[142] an additional exponential factor arising in the path integral after applying a Weyl transformation (5.3)

$$Z[g_{ij}] = e^{-\mathcal{A}[\mathcal{B}(x); g_{ij}]} Z[g_{ij}/\mathcal{B}(x)^2]. \quad (5.27)$$

The origin of the anomalous exponent is the non-invariance of the path integral measure $[\mathcal{D}\phi]_{g_{ij}}$ in Eq. (5.25)

$$[\mathcal{D}(\mathcal{B}(x)^{w_\phi} \phi)]_{\mathcal{B}(x)^{-2} g_{ij}} = e^{-\mathcal{A}[\mathcal{B}(x); g_{ij}]} [\mathcal{D}\phi]_{g_{ij}}. \quad (5.28)$$

⁵In this section we will work in the Euclidean signature. We also adopt natural units where $c = \hbar = 1$.

⁶In holography [11] $S[g_{ij}, \phi]$ would be the action for $\mathcal{N} = 4$ super-Yang Mills in curved background g_{ij} in $d = 4$ dimensions.

If we now apply two successive Weyl transformations of the measure and using Eq. (5.28) we obtain⁷

$$[D((\mathcal{B}''\mathcal{B}')^{w_\phi}\phi)]_{(\mathcal{B}''\mathcal{B}')^{-2}g_{ij}} = [D(\mathcal{B}'^{w_\phi}\phi)]_{\mathcal{B}'^{-2}g_{ij}} e^{\mathcal{A}[\mathcal{B}'';\mathcal{B}'^{-2}g_{ij}]} = [D\phi]_{g_{ij}} e^{\mathcal{A}[\mathcal{B}';g_{ij}]} e^{\mathcal{A}[\mathcal{B}'';\mathcal{B}'^{-2}g_{ij}]} \quad (5.29)$$

$$\equiv e^{\mathcal{A}[\mathcal{B}''\mathcal{B}';g]} [D\phi]_{g_{ij}}. \quad (5.30)$$

Therefore the anomalous exponent $\mathcal{A}[\mathcal{B}, g_{ij}]$ satisfies the *1-cocycle condition* [139]–[142]

$$\mathcal{A}[\mathcal{B}''\mathcal{B}'; g_{ij}] = \mathcal{A}[\mathcal{B}'; g_{ij}] + \mathcal{A}[\mathcal{B}''; g_{ij}/(\mathcal{B}')^2]. \quad (5.31)$$

For infinitesimal $\mathcal{B}, \mathcal{B}'$ near the identity and the fact that the Weyl group is abelian, namely $\mathcal{A}[\mathcal{B}'\mathcal{B}''; g_{ij}]$ the 1-cocycle condition in Eq. (5.31) gives the celebrated Wess-Zumino (WZ) consistency conditions [143], [144] for the Weyl anomaly.

The cohomological origin of the anomaly can be made precise by defining a coboundary operator $\Delta_{\mathcal{B}}$ [136] (see also [142]) acting on functionals $\mathcal{F}[g_{ij}]$ as

$$\Delta_{\mathcal{B}} \circ \mathcal{F}[g_{ij}] = \mathcal{F}[\mathcal{B}^{-2}g_{ij}] - \mathcal{F}[g_{ij}]. \quad (5.32)$$

$\Gamma[\mathcal{B}; g_{ij}] \equiv \Delta_{\mathcal{B}} \circ \mathcal{F}[g_{ij}]$ can be regarded as a one-form with respect to the finite shift Weyl group, and is a natural generalization of the concept of a one-form in differential geometry to the case of the infinite dimensional functional space of metrics. Acting on one-forms $\Gamma^{(1)}[\mathcal{B}; g_{ij}]$ the coboundary is defined as

$$\Delta_{\mathcal{B}''} \circ \Gamma^{(1)}[\mathcal{B}'; g_{ij}] = \Gamma^{(1)}[\mathcal{B}''; \mathcal{B}'^{-2}g_{ij}] - \Gamma^{(1)}[\mathcal{B}''\mathcal{B}'; g_{ij}] + \Gamma^{(1)}[\mathcal{B}'; g_{ij}]. \quad (5.33)$$

The definition of the operator $\Delta_{\mathcal{B}}$ can also be generalized to $\Delta_{\mathcal{B}_{k+1}}$ acting on k -forms $\Gamma^{(k)}[\mathcal{B}_1, \dots, \mathcal{B}_k; g_{ij}]$ ⁸. It produces a $(k+1)$ -form from a k -form, and it can be shown that it is nilpotent, namely it satisfies

$$(\Delta^2) \equiv \Delta_{\mathcal{B}_1} \circ \Delta_{\mathcal{B}_2} = 0. \quad (5.34)$$

Given the nilpotent operator $\Delta_{\mathcal{B}}$ any one-form $\Gamma^{(1)}[\mathcal{B}_1; g_{ij}]$ is *closed* when it satisfies

$$\Delta_{\mathcal{B}_2} \circ \Gamma^{(1)}[\mathcal{B}_1; g_{ij}] = 0 \quad (\text{closed}). \quad (5.35)$$

Moreover, any one-form is *exact* when

$$\Gamma^{(1)}[\mathcal{B}_1; g_{ij}] = \Delta_{\mathcal{B}_1} \circ \mathcal{F}_{\text{local}}[g_{ij}] \quad (\text{exact}), \quad (5.36)$$

and $\mathcal{F}_{\text{local}}[g_{ij}]$ is some *local, single-valued* scalar functional of the metric. Closed forms that are not exact define the *non-trivial cohomology* of the Weyl group.

The important point is that the closed property (5.35), using the definition (5.33), is exactly the 1-cocycle condition (5.31) when $\Gamma^{(1)}[\mathcal{B}_1; g_{ij}] = \mathcal{A}[\mathcal{B}; g_{ij}]$. Thus the anomalous exponent $\mathcal{A}[\mathcal{B}; g_{ij}]$ is a closed form and represents a genuine anomaly when it is not exact according to the definition (5.36). If it were exact there

⁷From now on we suppress the x dependence in $\mathcal{B}(x)$ for convenience.

⁸See [136] for the original definition for any group and more recently [142] where the definition makes use of the abelian nature of Weyl transformations.

would exist local-single valued counterterms that could be added to the action that shift the anomalous exponent to zero. Thus the anomalous one-forms $\mathcal{A}[\mathcal{B}; g_{ij}]$ belong to the non-trivial cohomology of the Weyl group.

Expanding Eq. (5.27) to first order under an infinitesimal Weyl transformation near the identity $\mathcal{B}(x) \approx 1 + \epsilon(x)$ we find the anomalous Weyl-Ward identity

$$\langle g_{ij} T^{ij}(x) \rangle = \frac{1}{\sqrt{\det g^{(0)}}} \frac{\delta \mathcal{A}}{\delta \ln \mathcal{B}(x)} \Big|_{\mathcal{B}(x)=1, g_{ij}(x)}, \quad (5.37)$$

where we defined the vacuum expectation value (vev) of the energy-momentum tensor operator $T^{ij}(x)$ as

$$\langle T^{ij}(x) \rangle = \frac{2}{\sqrt{\det g^{(0)}}} \frac{\delta S_{\text{eff}}}{\delta g_{ij}(x)}, \quad S_{\text{eff}} \equiv -\ln Z[g_{ij}]. \quad (5.38)$$

The right hand side of (5.37) was classified in any dimension into Type A and Type B anomalies [145]. Type A contains the topological Euler density in d dimensions, while Type B contains conformal invariant scalar quantities. The general solution of the (finite) 1-cocycle condition (5.31) that implies the Ward identity (5.37) was found in 2d by Polyakov [146], [147] and in 4d by Riegert [137] (see also [141]). In [142] a method was proposed to find the general solution in any even dimension. The results for 2d and 4d are ⁹

$$\mathcal{A}_{2d} = c_2 \int d^2 x \sqrt{\det g} \left[\ln \mathcal{B} \mathring{\square} \ln \mathcal{B} + \mathring{R} \ln \mathcal{B} \right] \quad (5.39)$$

$$\begin{aligned} \mathcal{A}_{4d} = a_4 \int d^4 x \sqrt{\det g} & \left[\ln \mathcal{B} (\mathring{R}^{ij} \mathring{R}_{ij} - \frac{1}{3} \mathring{R}^2) - 2 \mathring{G}^{ij} \mathring{\nabla}_i \ln \mathcal{B} \mathring{\nabla}_j \ln \mathcal{B} + 2 (\mathring{\nabla}_k \ln \mathcal{B} \mathring{\nabla}^k \ln \mathcal{B}) \mathring{\square} \ln \mathcal{B} \right. \\ & \left. - (\mathring{\nabla}_k \ln \mathcal{B} \mathring{\nabla}^k \ln \mathcal{B})^2 \right] + c_4 \int d^4 x \sqrt{\det g} \ln \mathcal{B} \mathring{W}_{ijkl} \mathring{W}^{ijkl}, \end{aligned} \quad (5.40)$$

where $\mathring{\square} = g^{ij} \mathring{\nabla}_i \mathring{\nabla}_j$ and $\mathring{G}_{ij} = \mathring{R}_{ij} - \frac{1}{2} \mathring{R} g_{ij}$. We see that in 2d there is only one central charge (c_2) and in 4d there are two central charges (a_4, c_4). These depend on the specific action we are considering.

In the rest of this section we extend all of the above results for a QFT on a conformal manifold to a QFT on a Weyl manifold. The steps are analogous to the conformal manifold case with slight modifications. Consider a d -dimensional field theory on a background endowed with Weyl geometry $(\mathcal{M}, [g, a])$. The path integral $Z[g_{ij}, a_i]$ is now a functional of both the background metric g_{ij} and Weyl connection a_i

$$Z[g_{ij}, a_i] = \int [\mathcal{D}\phi]_{g_{ij}, a_i} e^{-S[g_{ij}, a_i, \phi]}, \quad (5.41)$$

where we consider Weyl coupled “matter fields” with Weyl weight w_ϕ , namely we assume the classical action to be Weyl invariant under the Weyl transformation

$$S[\mathcal{B}(x)^{-2} g_{ij}, a_i - \partial_i \ln \mathcal{B}(x), \mathcal{B}(x)^{w_\phi} \phi] = S[g_{ij}, a_i, \phi]. \quad (5.42)$$

A prime example of such theory is the minimally coupled Weyl-scalar field with Weyl weight $w_\phi = \frac{1}{2}(d-2)$ and action [43]

$$S[g_{ij}, a_i, \phi] = -\frac{1}{2} \int d^d x \sqrt{\det g} g^{ij} \hat{\nabla}_i \phi \hat{\nabla}_j \phi, \quad (5.43)$$

⁹We assume we can drop total derivative terms in this chapter.

where $\hat{\nabla}_i \phi = \partial_\mu \phi + w_\phi a_i \phi$ is the Weyl-LC connection introduced in Eq. (5.16).

The Weyl anomaly comes from the anomalous transformation of the the path integral

$$Z[g_{ij}, a_i] = e^{-\mathcal{A}[\mathcal{B}(x); g_{ij}, a_i]} Z[g_{ij}/\mathcal{B}(x)^2, a_i - \partial_i \ln \mathcal{B}(x)]. \quad (5.44)$$

The origin of the anomalous exponent is again due to non-invariance of the path integral measure

$$[\mathcal{D}(\mathcal{B}(x)^{w_\phi} \phi)]_{\mathcal{B}(x)^{-2} g_{ij}, a_i - \partial_i \ln \mathcal{B}} = e^{-\mathcal{A}[\mathcal{B}(x); g_{ij}, a_i]} [\mathcal{D}\phi]_{g_{ij}, a_i}. \quad (5.45)$$

Applying two successive Weyl transformations the 1-cocycle condition takes the following form

$$\mathcal{A}[\mathcal{B}'' \mathcal{B}'; g_{ij}, a_i] = \mathcal{A}[\mathcal{B}'; g_{ij}, a_i] + \mathcal{A}[\mathcal{B}''; g_{ij}/(\mathcal{B}')^2, a_i - \partial_i \ln \mathcal{B}']. \quad (5.46)$$

Similar to the conformal case from the abelian property of the Weyl transformations (5.11) the infinitesimal form of the 1-cocycle condition implies the WZ consistency conditions for the Weyl anomaly. Once again we can define the nilpotent coboundary operator $\Delta_{\mathcal{B}}$ acting on functionals $\mathcal{F}[g_{ij}, a_i]$ and one-forms $\Gamma^{(1)}[\mathcal{B}; g_{ij}, a_i]$ as

$$\Delta_{\mathcal{B}} \circ \mathcal{F}[g_{ij}, a_i] = \mathcal{F}[\mathcal{B}^{-2} g_{ij}, a_i - \partial_i \ln \mathcal{B}] - \mathcal{F}[g_{ij}, a_i], \quad (5.47)$$

$$\Delta_{\mathcal{B}''} \circ \Gamma^{(1)}[\mathcal{B}'; g_{ij}] = \Gamma^{(1)}[\mathcal{B}''; \mathcal{B}'^{-2} g_{ij}, a_i - \partial_i \ln \mathcal{B}'] - \Gamma^{(1)}[\mathcal{B}'' \mathcal{B}'; g_{ij}, a_i] + \Gamma^{(1)}[\mathcal{B}'; g_{ij}, a_i]. \quad (5.48)$$

It is not hard to extend the action of the coboundary operator on k -forms $\Gamma^{(k)}[\mathcal{B}_1, \dots, \mathcal{B}_k; g_{ij}, a_i]$, but this is not needed for our purposes. Any one-form $\Gamma^{(1)}[\mathcal{B}_1; g_{ij}, a_i]$ is *closed* when it satisfies

$$\Delta_{\mathcal{B}_2} \circ \Gamma^{(1)}[\mathcal{B}_1; g_{ij}, a_i] = 0 \quad (\text{closed}). \quad (5.49)$$

Moreover, any one-form is *exact* when

$$\Gamma^{(1)}[\mathcal{B}_1; g_{ij}, a_i] = \Delta_{\mathcal{B}_1} \circ \mathcal{F}_{\text{local}}[g_{ij}, a_i] \quad (\text{exact}), \quad (5.50)$$

and $\mathcal{F}_{\text{local}}[g_{ij}, a_i]$ is some *local, single-valued* scalar functional of the metric.

The 1-cocycle condition is equivalent to the statement that $\mathcal{A}[\mathcal{B}; g_{ij}, a_i]$ is closed, and is a genuine anomaly if it is not exact. Thus the anomalous one-forms $\mathcal{A}[\mathcal{B}; g_{ij}, a_i]$ belong to the non-trivial cohomology of the Weyl group. It would be interesting to find a complete basis for the non-trivial cohomology of the Weyl group above, namely the number of independent central charges. Note that the solutions of the non-trivial cohomology group for conformal geometry (e.g. Eq. (5.39), Eq. (5.40)) carry over in Weyl geometry, even though they do not depend on the Weyl connection. Thus, we expect that when the Weyl connection is included the space of non-trivial anomalies to be larger compared to the case where the Weyl connection is turned off. However in the context of holography, the Weyl structure in the bulk only modifies the cohomological space trivially, namely it does not enlarge the dimension of the non-trivial cohomological space. We will discuss this in detail in Subsection 5.3.4.

Expanding Eq. (5.44) to first order under an infinitesimal Weyl transformation near the identity $\mathcal{B}(x) \approx 1 + \epsilon(x)$

we find the anomalous Weyl-Ward identity

$$\left. \frac{1}{\sqrt{\det g}} \frac{\delta \mathcal{A}}{\delta \ln \mathcal{B}(x)} \right|_{\mathcal{B}(x)=1, g_{ij}, a_i} = \langle T^{ij}(x) g_{ij}(x) + \hat{\nabla}_k J^k(x) \rangle, \quad (5.51)$$

where we defined the vev of the energy-momentum tensor operator $T^{ij}(x)$ and the Weyl current operator $J^i(x)$

$$\langle T^{ij}(x) \rangle = \frac{2}{\sqrt{\det g}} \frac{\delta S_{\text{eff}}}{\delta g_{ij}(x)}, \quad \langle J^i(x) \rangle = -\frac{1}{\sqrt{\det g}} \frac{\delta S_{\text{eff}}}{\delta a_i(x)}, \quad S_{\text{eff}} \equiv -\ln Z[g_{ij}, a_i]. \quad (5.52)$$

Note that according to the definitions above the energy-momentum tensor and Weyl current are Weyl-covariant tensors with Weyl weights $w_{T^{ij}} = d + 2$ and $w_{J^i} = d$.

5.3 Holographic Weyl anomaly

We start this section by describing the holographic Weyl anomaly [148]–[150] using the well-known Fefferman-Graham (FG) expansion of the bulk metric. In Subsection 5.3.2 we use briefly describe the *dilatation operator method* [151]–[153] for a holographic theory with Einstein dual. We then compute the Weyl-Ward identity in 8d extending the results found in [153]. In Subsection 5.3.3 we give a brief review of the Weyl-Fefferman-Graham (WFG) formalism introduced in [43] and find the corresponding Weyl-anomaly in this context. In Appendix B.3 we prove prescription (5.100) for the 3d holographic energy-momentum tensor in the WFG gauge, using the method of [149]. In Subsection 5.3.4 we see that the holographic Weyl anomaly in the WFG gauge only trivially modifies the corresponding cocycle in the FG gauge, and in Appendix B.4 we prove this statement (see Eq. (5.108)) for 2d, 4d boundary dimension.

5.3.1 Fefferman-Graham Gauge and Obstruction tensors

Consider a d -dimensional holographic field theory dual to vacuum Einstein theory in the $(d + 1)$ -dimensional bulk. According to the Fefferman-Graham (FG) theorem [131] the metric of a $(d + 1)$ -dimensional asymptotically locally AdS (AlAdS) spacetime can always be expressed in the following form

$$ds^2 = L^2 \frac{dz^2}{z^2} + g_{ij}(z; x) dx^i dx^j, \quad i, j = 0, \dots, d - 1, \quad (5.53)$$

where the coordinate z can be considered as a “radial” coordinate, and $z = 0$ is the “location” of the conformal boundary. When $g_{ij} = L^2 \eta_{ij} / z^2$ with η_{ij} the flat metric, this represents the Poincaré metric for AdS_{d+1} spacetime. Near the conformal boundary, h_{ij} can be expanded with respect to z as follows [43], [154]:

$$g_{ij}(z; x) = \frac{L^2}{z^2} \left[g_{ij}^{(0)}(x) + \frac{z^2}{L^2} g_{ij}^{(2)}(x) + \dots \right] + \frac{z^{d-2}}{L^{d-2}} \left[\pi_{ij}^{(0)}(x) + \frac{z^2}{L^2} \pi_{ij}^{(2)}(x) + \dots \right]. \quad (5.54)$$

In the FG expansion $g_{ij}^{(0)}$ serves as the “canonical” representative of the conformal class $[g_{ij}^{(0)}]$ of the boundary manifold sourcing the energy-momentum tensor of the dual field theory on the boundary, while $\pi_{\mu\nu}^{(0)}$ corresponds to the expectation value of the energy-momentum tensor. The boundary manifold is a conformal manifold and the Weyl transformation of the boundary geometry is induced by a bulk diffeomorphism called the

Penrose-Brown-Henneaux (PBH) transformation transformation [37], [38]¹⁰.

The two series in the metric expansion (5.54) reflect the fact that Einstein's equations are second order differential equations. Once $g_{ij}^{(0)}$ is given, each term in the first series can be determined by solving the bulk vacuum Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ with $(\mu, \nu) = (z, i)$, and negative cosmological constant $\Lambda = -d(d-1)/(2L^2)$. Similarly, once $\pi_{ij}^{(0)}$ is given, the second series will be determined. However, $\pi_{ij}^{(0)}$ is not completely arbitrary but is actually constrained by the Einstein equations. To be more specific, the zz -component of the Einstein equations tells us that $\pi_{ij}^{(0)}$ is traceless while the zi -components indicate that it is also divergence-free $\bar{\nabla}^i \pi_{ij}^{(0)}$.

In the metric expansion (5.54) we treat d formally as a variable $d \in \mathbb{C}$ [43], [154]. When Einstein equations are satisfied, $g_{ij}^{(2k)}$ is given in terms of tensors constructed from $g_{ij}^{(0)}$ and has a first order pole at $d = 2k$. For example, for $k = 1, 2$ one finds [44]

$$g_{ij}^{(2)} = -L^2 \dot{P}_{ij}^{(0)}, \quad g_{ij}^{(4)} = -\frac{L^4}{4(d-4)} \dot{B}_{ij}^{(0)} + \frac{1}{4} \dot{P}_{ki}^{(0)} \dot{P}_{(0)j}^k, \quad g_{ij}^{(6)} = -\frac{L^6}{24(d-6)(d-4)} \dot{\mathcal{O}}_{ij}^{(6)} + \frac{L^6}{6(d-4)} \dot{B}_{ik}^{(0)} \dot{P}_{(0)j}^k, \quad (5.55)$$

where $\dot{P}_{ij}^{(0)}$ is the Schouten tensor (5.6) and $\dot{B}_{ij}^{(0)}$ is the Bach tensor (5.8) and $\dot{\mathcal{O}}_{ij}^{(6)}$ was defined in (5.9). The superscript (0) in a quantity is there to denote that it is computed from the boundary metric $g_{ij}^{(0)}$ in the expansion (5.54). From Eq. (5.55) and Eq. (5.6) we see that $g_{ij}^{(2)}$ has a first order pole at $d = 2$, $g_{ij}^{(4)}$ has a first order pole at $d = 4$ and $g_{ij}^{(6)}$ at $d = 6$.

For any integer $k \geq 2$, the coefficient of the pole term in $g_{ij}^{(2k)}$ is proportional to the ‘‘obstruction tensor’’ [44] (see also Section 5.1) denoted by $\dot{\mathcal{O}}_{ij}^{(2k)}$ ¹¹

$$g_{ij}^{(2k)} = \frac{c_{(2k)}}{d-2k} \dot{\mathcal{O}}_{ij}^{(2k)} + \tilde{g}_{ij}^{(2k)}, \quad c_{(2k)} = -\frac{L^{2k}}{2^{2k-3}k!} \frac{\Gamma(d/2 - k + 1)}{\Gamma(d/2 - 1)}. \quad (5.56)$$

Note that the tensor $\dot{\mathcal{O}}_{ij}^{(2k)}$ as defined from Eq. (5.56) is well-defined in any dimension (see e.g. Eq. (5.9) for $\mathcal{O}_{ij}^{(6)}$), but is equal to the obstruction tensor as defined in [44], [132] when $d = 2k$.

For convenience, we can also absorb the d -dependent factors in $g_{ij}^{(2k)}$ by introducing Graham's extended obstruction tensor $\dot{\Omega}_{ij}^{(k)}$ ($k \geq 1$):

$$\dot{\Omega}_{ij}^{(1)} = -\frac{1}{d-4} \dot{B}_{ij}, \quad \dot{\Omega}_{ij}^{(2)} = \frac{1}{(d-6)(d-4)} \dot{\mathcal{O}}_{ij}^{(6)}, \quad \dots \quad (5.57)$$

The extended obstruction tensor $\Omega_{ij}^{(k)}$ was precisely defined in [155] in the context of the ambient metric. The general relation between the obstruction tensor and extended obstruction tensor is

$$\dot{\Omega}_{ij}^{(k)} = \frac{(-1)^k}{2^k} \frac{\Gamma(d/2 - k - 1)}{\Gamma(d/2 - 1)} \dot{\mathcal{O}}_{\mu\nu}^{(2k+2)} \quad (k \geq 1). \quad (5.58)$$

When d approaches even dimension $d \rightarrow 2k^-$ the two series in Eq. (5.54) merge¹² and a logarithmic term

¹⁰See Subsection 5.3.3 for more comments on the PBH transformation.

¹¹The normalization factor $c_{(2k)}$ has been chosen so that the obstruction tensor agrees with the convention of [44].

¹²See [126] for more details on how to merge the two series in even d .

appears in the metric expansion [148]–[150]

$$g_{ij}(z; x) = \sum_{n=0}^{k-1} g_{ij}^{(2n)} \left(\frac{z}{L} \right)^{2n-2} + \gamma_{ij}^{(2k)} \left(\frac{z}{L} \right)^{2k-2} - c_{(2k)} \left(\frac{z}{L} \right)^{2k-2} \ln \left(\frac{z}{L} \right) \mathring{O}_{ij}^{(2k)} + O((z/L)^d). \quad (5.59)$$

This is also the origin of the name obstruction tensor, as it obstructs the existence of a formal power series expansion. The vacuum Einstein equations do not completely determine $\gamma_{ij}^{(2k)}$ but constraint its trace and divergence (see [149] for more details).

The FG expansion (5.54) allows one to compute the anomaly cocycle $\mathcal{A}^{\text{FG}}[\mathcal{B}, g_{ij}^{(0)}]$ of the boundary theory. One can either start by first computing the anomalous Weyl-Ward identity (5.37) for holographic theories [148] and then “integrate it” to find $\mathcal{A}^{\text{FG}}[\mathcal{B}, g_{ij}^{(0)}]$ as was done in [137]. A recent more general prescription is to use the bulk on-shell action in dimensional regularization as described in [142]. For holographic theories with bulk Einstein dual, the anomaly cocycle takes the form given in (5.39) with $c_2 = -\frac{L}{16\pi G_N}$ in $2d$ and (5.40) with $a_4 = -\frac{L^3}{64\pi G_N}$, $c_4 = 0$ in $4d$, namely

$$\mathcal{A}_{2d}^{\text{FG}}[\mathcal{B}; g_{ij}^{(0)}] = -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} \left[\ln \mathcal{B} \mathring{\square} \ln \mathcal{B} + \mathring{R}_{(0)} \ln \mathcal{B} \right], \quad (5.60)$$

$$\mathcal{A}_{4d}^{\text{FG}}[\mathcal{B}; g_{ij}^{(0)}] = -\frac{L^3}{64\pi G_N} \int d^4x \sqrt{\det g_{(0)}} \left[\ln \mathcal{B} (\mathring{R}_{(0)}^{ij} \mathring{R}_{ij}^{(0)} - \frac{1}{3} \mathring{R}_{(0)}^2) - 2 \mathring{G}_{(0)}^{ij} \mathring{\nabla}_i \ln \mathcal{B} \mathring{\nabla}_j \ln \mathcal{B} \right. \quad (5.61)$$

$$\left. + 2 (\mathring{\nabla}_k \ln \mathcal{B} \mathring{\nabla}^k \ln \mathcal{B}) \mathring{\square} \ln \mathcal{B} - (\mathring{\nabla}_k \ln \mathcal{B} \mathring{\nabla}^k \ln \mathcal{B})^2 \right]. \quad (5.62)$$

Note that $\mathring{R}_{ij}^{(0)}$ above is the LC Ricci tensor of the boundary metric $g_{ij}^{(0)}$.

5.3.2 Dilatation Operator Method

In this section we will derive the holographic anomalous Weyl-Ward identity (5.37) for even d , and up to $8d$ using the recursive algorithm of [153] which we will refer to as the *dilatation operator method*. For completeness we start with a brief review of the algorithm of the dilatation operator method. We then apply the algorithm one step further than [153] and compute the holographic anomalous Weyl-Ward identity in $8d$.

Review of the Algorithm

We start by using the metric in the FG gauge (5.53)

$$ds^2 = dr^2 + g_{ij}(r; x) dx^i dx^j, \quad i, j = 0, 1, \dots, d-1, \quad (5.63)$$

where we changed the radial bulk coordinate by setting $r = -L \ln(z/L)$. The conformal boundary is now “located” at $r = +\infty$. The Einstein-Hilbert action in the $d+1$ bulk with Gibbons-Hawking-York boundary term is

$$S_{\text{EH-GH}} = \frac{1}{2\kappa^2} \int_{\mathcal{M}} dr d^d x \sqrt{\det g_{\mu\nu}} (R(g_{\mu\nu}) - 2\Lambda) + \frac{1}{\kappa^2} \int_{\partial M_{r_c}} d^d x \sqrt{\det g_{ij}} K, \quad (5.64)$$

where $\kappa^2 = 8\pi G_N$, $\Lambda = -\frac{d(d-1)}{2L^2}$, and ∂M_{r_c} is a cutoff surface at some large value of r_c .¹³ Taking a metric

¹³We abuse notation and call the cutoff surface ∂M_{r_c} even though it is not necessarily the boundary of M .

variation and evaluating the result on-shell we obtain

$$\delta S_{\text{EH-GHY}}^{\text{o.s}} = \int_{\partial M_{r_c}} d^d x \pi^{ij} \delta g_{ij}, \quad \pi^{ij} = \frac{1}{2\kappa^2} \sqrt{\det g} (Kg^{ij} - K^{ij}), \quad (5.65)$$

where $K_{ij} = \frac{1}{2} \partial_r g_{ij}(r; x)$ is the extrinsic curvature tensor in the FG gauge, and $g_{ij}(r; x)$ is the induced metric (see Eq. (5.63)) on ∂M_{r_c} . The boundary tensor density π^{ij} appears in many different contexts; it was defined in [54] and was later used in [156] to define a boundary stress tensor in an asymptotically AdS spacetime (with the inclusion of necessary counterterms to cancel divergences). It also appears as the conjugate momenta in the ADM formalism [157].¹⁴

The rr - and ri -components of the Einstein equations, i.e. $G^{rr} + \Lambda g^{rr} = 0$ and $G^{ri} = 0$, can be written in terms of the conjugate momenta as follows:

$$\frac{2\kappa^2}{\sqrt{\det g}} \left(\pi^i_j \pi^j_i - \frac{1}{d-1} \pi^2 \right) + \frac{1}{2\kappa^2} \sqrt{\det h} (\dot{R} - 2\Lambda) = 0, \quad (5.66)$$

$$\dot{\nabla}_k \pi^{ki} = 0, \quad (5.67)$$

where $\pi \equiv g_{ij} \pi^{ij}$, $\dot{\nabla}$ is the LC connection of the induced metric g_{ij} on ∂M_{r_c} and $\dot{R}$ is the LC Ricci scalar of g_{ij} . Note that the indices are raised and lowered using the induced metric g_{ij} . Eq. (5.66) and (5.67) are the well-known Hamiltonian and momentum constraints in the ADM language [157].

The dilatation operator method of solving the constraint equations uses an asymptotic expansion of the conjugate momenta in terms of the induced metric. One assumes a Hamilton-Jacobi functional $S[g]$ such that

$$\pi^{ij} = \frac{\delta S[h]}{\delta g_{ij}}, \quad S[g] \equiv \int_{\partial M_{r_c}} d^d x \mathcal{L}[g], \quad (5.68)$$

where $S[g]$ is a local diffeomorphic invariant functional of the induced metric.¹⁵

The momentum constraint (5.67) is now trivially satisfied. The Hamiltonian constraint (5.66) can be solved asymptotically by writing

$$\mathcal{L}[g] = \sum_{k=0} \mathcal{L}_{(2k)}[g], \quad \delta_D \mathcal{L}_{(2k)}[g] = (d-2k) \mathcal{L}_{(2k)}[g], \quad (5.69)$$

where δ_D is the *dilatation operator* [151] (acting on metric functionals), defined as

$$\delta_D \equiv \int d^d x \left[2g_{ij} \frac{\delta}{\delta g_{ij}} \right]. \quad (5.70)$$

It is useful to keep in mind that for an AlAdS spacetime the radial derivative ∂_r asymptotes the dilatation operator, i.e. $\delta_D \sim \frac{\partial}{\partial r}$ (see [153]). We can view the dilatation operator expansion as another asymptotic expansion near the conformal boundary since $g_{ij} \sim e^{2r/L} g_{ij}^{(0)} + \dots$.

The expansion (5.69) together with (5.68) implies an expansion of π^{ij} in terms of the dilatation weight,

¹⁴The sign difference in the definition of $\pi^{\mu\nu}$ in [157] arises because here we consider the radial evolution which is in a spacelike direction.

¹⁵In [153] the functional $S[g]$ was used to derive the boundary terms for a well-defined bulk variational problem. Since these are tangential remarks for the calculation of the Weyl anomaly, we simply neglect them and refer the reader there for more details.

namely

$$\pi_{(2k)}^{ij} = \frac{\delta}{\delta g_{ij}} \int_{\partial M_{r_c}} d^d x \mathcal{L}_{(2k)}, \quad \pi^{ij} = \sum_{k=0} \pi_{(2k)}^{ij}. \quad (5.71)$$

The $\mathcal{L}_{(2k)}[g]$ are defined only up to total derivative terms in ∂M_{r_c} . To set up the recursive algorithm we need the following relation:

$$\pi_{(2k)} = \frac{L}{2} \mathcal{Q}_{(2k)}, \quad \pi_{(2k)} = g_{ij} \pi_{(2k)}^{ij}, \quad (5.72)$$

where we defined the *Q-curvature* [158]–[161] (see [162] for a short review)

$$\mathcal{Q}_{(2k)} = \frac{d-2k}{L} \mathcal{L}_{(2k)}. \quad (5.73)$$

The Q-curvature is the main quantity we are interested in, since it is proportional to the Weyl anomaly in a specific even dimension [see (5.76)]. Plugging (5.71) into (5.66) and making use of (5.72) we find

$$\mathcal{Q}_{(2k)} = \frac{2\kappa^2}{\sqrt{\det g}} \sum_{m=1}^{k-1} \left(\pi_{(2m)}^i{}_j \pi_{(2k-2m)}^j{}_i - \frac{1}{d-1} \pi_{(2m)} \pi_{(2k-2m)} \right), \quad (5.74)$$

where we used the initial values

$$\mathcal{Q}_{(2)} = \frac{\sqrt{\det g}}{2\kappa^2} \mathring{R}, \quad \pi_{(0)}^{ij} = \frac{(d-1)}{2\kappa^2 L} \sqrt{\det g} g^{ij}. \quad (5.75)$$

Equations (5.74), (5.71) and the initial values (5.75), are enough to fix the iterative algorithm of the dilatation operator method. Expanding on this a little more, given the value of $\mathcal{Q}_{(2)}$ we can use (5.71) to find $\pi_{(2)}^{ij}$. Since we now have $\pi_{(2)}^{ij}$ and $\pi_{(0)}^{ij}$, we can find $\mathcal{Q}_{(4)}$ from (5.74). The process can then be iterated to compute $\pi_{(2k)}^{ij}$ and $\mathcal{Q}_{(2k)}$ to any order. The recursive algorithm has been solved up to $\pi_{(4)}^{ij}$ and $\mathcal{Q}_{(6)}$ in [153]. In the next section we will push the calculation one step further for finding $\pi_{(6)}^{ij}$ and $\mathcal{Q}_{(8)}$. This will enable us to find the Weyl anomaly in $8d$.

Results and Anomaly up to $8d$

We have explained the algorithm for solving the Hamiltonian and momentum constraints. We now focus on the Q-curvature, which is expressed in terms of the conjugate momenta in (5.74). The Weyl anomaly in d -dimension corresponds to the Q-curvature for $d = 2k$ [153]

$$\left. \frac{\delta \mathcal{A}_{2kd}^{\text{FG}}}{\delta \ln \mathcal{B}} \right|_{\mathcal{B}(x)=1, g_{ij}(x)} = -L \mathcal{Q}_{(2k)}^{d=2k}, \quad (5.76)$$

where the subscript $2kd$ for $k = 1, 2, \dots$ means $2d, 4d, \dots$

We now present the results of the algorithm presented above. First, we review the results for $\pi_{(2k)}^{ij}$ and $\mathcal{Q}_{(2k)}$

up to $k = 3$:¹⁶

$$\begin{aligned}\pi_{(0)}^{ij} &= \frac{(d-1)}{2\kappa^2 L} \sqrt{\det g} g^{ij}, \\ \pi_{(2)}^{ij} &= -\frac{\sqrt{\det g} L}{2\kappa^2} \left(\dot{P}^{ij} - \dot{P} g^{ij} \right), \\ \pi_{(4)}^{ij} &= -\frac{\sqrt{\det g} L^3}{2\kappa^2 (d-4)(d-2)} \left[\dot{B}^{ij} + (d-4) \left(\dot{P}^i_k \dot{P}^{kj} - \dot{P} \dot{P}^{ij} - \frac{1}{2} g^{ij} (\text{tr}(\dot{P}^2) - \dot{P}^2) \right) \right],\end{aligned}\tag{5.77}$$

and

$$\begin{aligned}\mathcal{Q}_{(2)} &= \frac{\sqrt{\det g}}{2\kappa^2} \dot{R}, \\ \mathcal{Q}_{(4)} &= \frac{\sqrt{\det g} L^2}{2\kappa^2} \left[\text{tr}(\dot{P}^2) - \dot{P}^2 \right], \\ \mathcal{Q}_{(6)} &= \frac{\sqrt{\det g} L^4}{\kappa^2 (d-4)(d-2)} \left[\text{tr}(\dot{P} \dot{B}) + (d-4) \left(\text{tr}(\dot{P}^3) - \frac{3}{2} \dot{P} \text{tr}(\dot{P}^2) + \frac{1}{2} \dot{P}^3 \right) \right].\end{aligned}\tag{5.78}$$

$\mathcal{Q}_{(2)}$, $\mathcal{Q}_{(4)}$ and $\mathcal{Q}_{(6)}$ correspond to the Weyl anomaly in $2d$, $4d$ and $6d$ respectively according to Eq. (5.76). Using $\mathcal{Q}_{(6)}$ in (5.78) we can calculate $\pi_{(6)}^{ij}$ from (5.71) by taking a metric variation of $\mathcal{Q}_{(6)}$. After a lengthy calculation we obtain

$$\begin{aligned}\pi_{(6)}^{ij} &= -\frac{L^5}{\kappa^2 (d-6)(d-4)(d-2)^2} \left[\dot{\mathcal{O}}_{(6)}^{ij} + (d-6) \left(\dot{P}^{(i}_k \dot{B}^{j)k} - \dot{P} \dot{B}^{ij} - 2 \dot{P}_{mk} \dot{\nabla}^k \dot{C}^{(ij)m} \right. \right. \\ &\quad - (\dot{\nabla}_k P) \dot{C}^{(ij)k} + \dot{C}^{mik} \dot{C}^j_m - \frac{1}{2} \dot{C}^{imk} \dot{C}^j_{mk} + \dot{P} \dot{W}^i_{ab} \dot{P}^{ab} + \frac{1}{2} \dot{\nabla}^k \dot{\nabla}_k (\dot{P}^{ib} \dot{P}^j_b - \dot{P} \dot{P}^{ij}) \\ &\quad - \frac{(d-2)}{4(d-1)} \dot{\nabla}^i \dot{\nabla}^j (\text{tr}(\dot{P}^2) - \dot{P}^2) - \frac{1}{4(d-1)} g^{ij} \dot{\nabla}^k \dot{\nabla}_k (\text{tr}(\dot{P}^2) - \dot{P}^2) + (d-4) \dot{P}^i_a \dot{P}^{ab} \dot{P}^j_b \\ &\quad - \frac{(3d^2 - 12d + 8)}{4(d-1)} \dot{P}^{ij} (\text{tr}(\dot{P}^2) - \dot{P}^2) - (d-4) \dot{P} \dot{P}^i_k \dot{P}^{kj} - \frac{1}{2} g^{ij} (\text{tr}(\dot{P} \dot{B}) + (d-4) \text{tr}(\dot{P}^3) \\ &\quad \left. \left. - \frac{(3d^2 - 14d + 10)}{2(d-1)} \dot{P} \text{tr}(\dot{P}^2) + \frac{(d^2 - 4d + 2)}{2(d-1)} \dot{P}^3 \right) \right],\end{aligned}\tag{5.79}$$

where $\dot{\mathcal{O}}_{(6)}^{ij}$ is the LC obstruction tensor defined in (5.9). We have also checked that $\pi_{(6)}^{ij}$ is divergence-free in any dimension, as is required by (5.67).¹⁷ In Appendix B.2 we provide some metric variation formulas that were used in the above calculation.

By plugging (5.79) and (5.77) into (5.74) we find after some reorganization the expression of $\mathcal{Q}_{(8)}$ as follows:

$$\begin{aligned}\mathcal{Q}_{(8)} &= \frac{2L^6 \sqrt{\det g}}{\kappa^2 (d-6)(d-4)(d-2)^2} \left[\frac{(d-2)}{2(d-4)} \text{tr}(\dot{P} \dot{\mathcal{O}}_{(6)}) + (d-6) \left(\frac{3}{4(d-4)} \text{tr}(\dot{B}^2) + \frac{5d-16}{2(d-4)} \text{tr}(\dot{P}^2 \dot{B}) \right. \right. \\ &\quad - \frac{5d-16}{2(d-4)} \dot{P} \text{tr}(\dot{P} \dot{B}) - \frac{5d^2 - 20d + 8}{16(d-1)} \dot{P}^4 + \frac{16-5d}{2} \dot{P} \text{tr}(\dot{P}^3) + \frac{15d^2 - 62d + 40}{8(d-1)} \dot{P}^2 \text{tr}(\dot{P}^2) \\ &\quad \left. \left. - \frac{13d^2 - 44d + 24}{16(d-1)} (\text{tr}(\dot{P}^2))^2 + \frac{7d-20}{4} \text{tr}(\dot{P}^4) \right) + \dot{\nabla}_m K^m \right],\end{aligned}\tag{5.80}$$

¹⁶We use the notation $\text{tr}(AB) = A^i_j B^j_i$ and $\dot{P} = g^{ij} \dot{P}_{ij}$.

¹⁷This calculation was facilitated by the Mathematica package diffgeo.m by Matthew Headrick.

where $K^m = \frac{(d-6)}{2(d-4)}K_0^m + \frac{(d-6)}{2}K_1^m + \frac{(d-6)}{2}K_2^m + \frac{(d-6)(d-2)}{4(d-1)}K_3^m$, with

$$\begin{aligned} K_0^m &:= \dot{P}^{ab}\dot{\nabla}^m\dot{B}_{ab} - \dot{B}_{ab}\dot{\nabla}^m\dot{P}^{ab} + \dot{B}^{mb}\dot{\nabla}_b\dot{P} - \dot{P}\dot{\nabla}_a\dot{B}^{am}, \\ K_1^m &:= (\dot{P}^{mk}\dot{P}_k^b - \dot{P}\dot{P}^{mb})\dot{\nabla}_b\dot{P} - \dot{P}\dot{\nabla}_b(\dot{P}^{ba}\dot{P}_a^m - \dot{P}\dot{P}^{bm}), \\ K_2^m &:= \dot{P}_{ab}\dot{\nabla}^m(\dot{P}^{ak}\dot{P}_k^b - \dot{P}\dot{P}^{ab}) - (\dot{\nabla}^m\dot{P}_{ab})(\dot{P}^{ak}\dot{P}_k^b - \dot{P}\dot{P}^{ab}), \\ K_3^m &:= (\text{tr}(\dot{P}^2) - \dot{P}^2)\dot{\nabla}^m\dot{P} - \dot{P}^{mk}\dot{\nabla}_k(\text{tr}(\dot{P}^2) - \dot{P}^2). \end{aligned} \quad (5.81)$$

If we plug $d = 8$ into (5.80) and use Eq. (5.76), we find that the holographic Weyl anomaly $\mathcal{A}_{8d}^{\text{FG}}$ in 8d is

$$\begin{aligned} \left. \frac{\delta \mathcal{A}_{8d}^{\text{FG}}}{\delta \ln \mathcal{B}} \right|_{\mathcal{B}(x)=1, g_{ij}(x)} &= -L\mathcal{Q}_{(8)}^{\text{d}=8} = -\frac{L^7\sqrt{\det g}}{48\kappa^2} \left[\frac{1}{4}\text{tr}(\dot{P}\dot{\mathcal{O}}_{(6)}) + \frac{1}{8}\text{tr}(\dot{B}^2) + 2\text{tr}(\dot{P}^2\dot{B}) - 2\dot{P}\text{tr}(\dot{P}\dot{B}) \right. \\ &\quad \left. + 6\text{tr}(\dot{P}^4) - 3\text{tr}(\dot{P}^2)^2 + 6\dot{P}^2\text{tr}(\dot{P}^2) - 8\dot{P}\text{tr}(\dot{P}^3) - \dot{P}^4 + \dot{\nabla}_m K^m \right]. \end{aligned} \quad (5.82)$$

5.3.3 Weyl-Fefferman-Graham Gauge

In this section we give a brief review of the Weyl-Fefferman-Graham (WFG) formalism introduced in [43].

The Fefferman-Graham ansatz (5.53) is quite convenient for calculations, especially in the context of holographic renormalization [150]. In this setup, one can induce a Weyl transformation of the boundary metric (5.3) by a bulk diffeomorphism, coined the PBH transformation [37], [38],

$$z \rightarrow z' = z/\mathcal{B}(x), \quad x^i \rightarrow x'^i = x^i + \xi^i(z; x), \quad (5.83)$$

where $\xi^i(z; x)$ vanish at the boundary $z = 0$. The functions $\xi^i(z; x)$ can be found (infinitesimally) in terms of $\mathcal{B}(x)$ by the constraint that the form of the FG ansatz is preserved under the transformation. However, under the PBH transformation, the subleading terms in the FG expansion (5.54) do not transform in a Weyl-covariant way. The source of this complication is the compensating diffeomorphisms $\xi^i(z; x)$ introduced for preserving the FG ansatz.

This above-mentioned issue motivated the authors of [43] to replace the FG ansatz with

$$ds^2 = L^2 \left(\frac{dz}{z} - a_i(z; x)dx^i \right)^2 + g_{ij}(z; x)dx^i dx^j, \quad i, j = 0, \dots, d-1, \quad (5.84)$$

which was named the Weyl-Fefferman-Graham ansatz. With the additional Weyl structure $a_i(z; x)$ added, the form of the WFG ansatz is now preserved under the Weyl diffeomorphism

$$z \rightarrow z' = z/\mathcal{B}(x), \quad x^i \rightarrow x'^i = x^i, \quad (5.85)$$

provided the fields $a_i(z; x)$ and $g_{ij}(z; x)$ also transform as

$$a_i(z; x) \rightarrow a'_i(z'; x) = a_i(\mathcal{B}(x)z'; x) - \partial_i \ln \mathcal{B}(x), \quad g_{ij} \rightarrow g'_{ij}(z'; x) = g_{ij}(\mathcal{B}(x)z'; x). \quad (5.86)$$

Thus, we can now induce a Weyl transformation on the boundary and preserve the form of the metric without introducing $\xi^i(z; x)$. Note that according to the FG theorem, any AlAdS spacetime can always be expressed in the FG form, and so (5.84) can be transformed into (5.53) under a suitable diffeomorphism. This indicates

that $a_i(z; x)$ is actually pure gauge in the bulk, which has significant implications for the boundary Weyl current (see discussion below Eq. (5.101)) and the Weyl anomaly as we will explain in Subsection 5.3.4. Another way of going back to the FG gauge is to simply set $a_i(z; x)$ to zero; in this perspective, the FG gauge is nothing but a special case of the WFG gauge with a fixed gauge.

The main utility of the WFG gauge is that all the terms (except one) in the z -expansions of $g_{ij}(z; x)$ and $a_i(z; x)$ transform as Weyl tensors under Weyl diffeomorphisms. To see this, let us expand g_{ij} and a_i near $z = 0$:

$$g_{ij}(z; x) = \frac{L^2}{z^2} \left[g_{ij}^{(0)}(x) + \frac{z^2}{L^2} g_{ij}^{(2)}(x) + \cdots \right] + \frac{z^{d-2}}{L^{d-2}} \left[\pi_{ij}^{(0)}(x) + \frac{z^2}{L^2} \pi_{ij}^{(2)}(x) + \cdots \right], \quad (5.87)$$

$$a_i(z; x) = \left[a_i^{(0)}(x) + \frac{z^2}{L^2} a_i^{(2)}(x) + \cdots \right] + \frac{z^{d-2}}{L^{d-2}} \left[p_i^{(0)}(x) + \frac{z^2}{L^2} p_i^{(2)}(x) + \cdots \right]. \quad (5.88)$$

In the FG gauge where a_i is turned off, the FG expansion only includes (5.87), and the subleading terms $\gamma_{ij}^{(2k)}$ in the first series are determined solely by the boundary induced metric $\gamma_{ij}^{(0)}$ and its derivatives. Now with the extra series (5.88), $\gamma_{ij}^{(2k)}$ will also depend on $a_i^{(0)}$, $a_i^{(2)}$, $a_i^{(4)}$, etc. From the transformations (5.86) under a Weyl diffeomorphism, one finds the transformation of each term in the expansions (5.87) and (5.88) as follows [43]:

$$g_{ij}^{(2k)}(x) \rightarrow g_{ij}^{(2k)}(x) \mathcal{B}(x)^{2k-2}, \quad \pi_{ij}^{(k)}(x) \rightarrow \pi_{ij}^{(2k)}(x) \mathcal{B}(x)^{d-2+2k}, \quad (5.89)$$

$$a_i^{(2k)}(x) \rightarrow a_i^{(2k)}(x) \mathcal{B}(x)^{2k} - \delta_{k,0} \partial_i \ln \mathcal{B}(x), \quad p_i^{(2k)}(x) \rightarrow p_i^{(2k)}(x) \mathcal{B}(x)^{d-2+2k}. \quad (5.90)$$

Indeed, we see that almost all the terms in the expansions transform Weyl-covariantly. The only exception is $a_i^{(0)}$, which transforms inhomogeneously under Weyl transformation, and thus does not have a definite Weyl weight. All the other terms in the expansions (5.87) and (5.88) can be viewed as tensor fields on the boundary and we can easily read off their Weyl weights from the power of $\mathcal{B}(x)$ appearing in (5.89) and (5.90).

The role of $a_i^{(0)}$ as a Weyl connection on the boundary was shown in [43]. To show this they computed the LC connection coefficients $\mathring{\nabla}$ associated to the following vector basis $\{\underline{D}_z, \underline{D}_i\}$ in the bulk

$$e_z = \frac{z}{L} \partial_z \equiv \underline{D}_z, \quad \underline{D}_i = \partial_i + z a_i(z; x) \partial_z. \quad (5.91)$$

The basis vectors $\{\underline{D}_i\}$ form a d -dimensional distribution in the bulk. The Lie brackets of these basis vectors are

$$[\underline{D}_i, \underline{D}_j] = L f_{ij} \underline{D}_z, \quad [\underline{D}_z, \underline{D}_i] = L \varphi_i \underline{D}_z, \quad (5.92)$$

where $\varphi_i \equiv D_z a_i$ and $f_{ij} \equiv D_i a_j - D_j a_i$ (D_z and D_i represent taking the derivatives along e_z and e_i). According to the Frobenius theorem, the condition for the distribution spanned by $\{\underline{D}_i\}$ to be integrable is that $[\underline{D}_i, \underline{D}_j] = 0$, i.e. $f_{ij} = 0$. In this case, this distribution defines a hypersurface. For instance, in the FG gauge where a_i is turned off, the distribution $\{\underline{D}_i\}$ becomes $\{\partial_i\}$, and generates a foliation of constant- z surfaces. The LC connection coefficients $\mathring{\nabla}$ induced on the distribution $\{\underline{D}_i\}$ (see [163]) can be read off from

$$\mathring{\nabla}_{\underline{D}_i} \underline{D}_j = \mathring{\Gamma}_{ij}^k \underline{D}_k + \mathring{\Gamma}_{ij}^z \underline{D}_z. \quad (5.93)$$

Expanding $\overset{\circ}{\Gamma}_{ij}^k$ with respect to z , at the leading order one finds that

$$\overset{\circ}{\Gamma}_{(0)}^k{}_{ij} = \frac{1}{2} g_{(0)}^{km} (\partial_i g_{jm}^{(0)} + \partial_j g_{im}^{(0)} - \partial_m g_{ij}^{(0)}) - (a_i^{(0)} \delta_j^k + a_j^{(0)} \delta_i^k + a_m^{(0)} g_{(0)}^{km} g_{ij}^{(0)}). \quad (5.94)$$

Note that the above are exactly the Weyl connection coefficients defined in Eq. (5.15) where $g_{ij}^{(0)}$ and $a_i^{(0)}$ replace g_{ij} , a_i . This was one of the key points made in [43].

The Weyl transformations (5.89), (5.90) induced by the bulk diffeomorphism (5.85) in the WFG gauge (5.84) and the induced Weyl connection (5.94) make it apparent that the WFG gauge induced a Weyl geometry in the boundary. Similar to Section 5.3.1 we can define Weyl-obstruction tensors mentioned in Section 5.1 using the WFG metric expansion (5.87). By solving the bulk Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ with $\Lambda = -d(d-1)/(2L^2)$ order by order in the WFG gauge, we find that $g_{ij}^{(2k)}$ still has the same form as Eq. (5.56), except that the obstruction tensor $\overset{\circ}{\mathcal{O}}_{ij}^{(2k)}$ is now promoted to the *Weyl-obstruction tensor* $\hat{\mathcal{O}}_{ij}^{(2k)}$. Unlike $\overset{\circ}{\mathcal{O}}_{ij}^{(2k)}$, which is only Weyl-covariant in $2k$ -dimension, the Weyl-obstruction tensors $\hat{\mathcal{O}}_{ij}^{(2k)}$ are Weyl-covariant with a weight $2k-2$ in any dimension; that is, under a Weyl transformation $(g_{ij}^{(0)}, a_i^{(0)}) \rightarrow \mathcal{B}^{-2} g_{ij}^{(0)}, a_i^{(0)} - \partial_i \ln \mathcal{B}$ it transforms in any d as $\hat{\mathcal{O}}_{ij}^{(2k)} \rightarrow \mathcal{B}(x)^{2k-2} \hat{\mathcal{O}}_{ij}^{(2k)}$.

For example, from leading and subleading Einsteins equations we find

$$g_{ij}^{(2)} = -\frac{L^2}{d-2} \left(\hat{R}_{(ij)}^{(0)} - \frac{1}{2(d-1)} \hat{R}^{(0)} g_{ij}^{(0)} \right), \quad (5.95)$$

$$g_{ij}^{(4)} = -\frac{L^4}{4(d-4)} \hat{\mathcal{O}}_{ij}^{(4)} + \frac{L^4}{4} \hat{P}^k{}_i \hat{P}_{kj} - \frac{L^2}{2} \hat{\nabla}_{(i}^{(0)} a_{j)}^{(2)} \quad (5.96)$$

where $\hat{\mathcal{O}}_{ij}^{(4)}$ is the Weyl-obstruction tensor for $d=4$, namely the Weyl-Bach tensor $\hat{B}_{ij}$, given by

$$\hat{\mathcal{O}}_{ij}^{(4)} = \hat{B}_{ij} = \hat{\nabla}_k^{(0)} \hat{\nabla}_{(0)}^k \hat{P}_{ij} - \hat{\nabla}_k^{(0)} \hat{\nabla}_j^{(0)} \hat{P}_i{}^k - \hat{W}_{kjim}^{(0)} \hat{P}^{mk}. \quad (5.97)$$

The hatted quantities and Weyl-connection $\hat{\nabla}$ were defined in Section 5.1. If we compare (5.97) with the corresponding result (5.55) in the FG case, we see that the form of the expression stays almost the same, with all the LC quantities now being promoted to the corresponding Weyl quantities. Besides, in the WFG gauge $g_{ij}^{(4)}$ also has an additional term involving $a_i^{(2)}$, which does not contribute to the pole at $d=4$.

It is also convenient to define the *extended Weyl-obstruction tensor* $\hat{\Omega}_{\mu\nu}^{(k)}$ as the Weyl-covariant version of the extended obstruction tensor defined in (5.57). For example, for $k=1$ and $k=2$ we have

$$\hat{\Omega}_{ij}^{(1)} = -\frac{1}{d-4} \hat{B}_{ij}, \quad \hat{\Omega}_{ij}^{(2)} = \frac{1}{(d-6)(d-4)} \hat{\mathcal{O}}_{ij}^{(6)}. \quad (5.98)$$

Similar to the FG case, the Weyl-obstruction tensor $\hat{\mathcal{O}}_{ij}^{(2k+2)}$ is also proportional to the residue of the extended Weyl-obstruction tensor $\hat{\Omega}_{ij}^{(k)}$. Both the Weyl-obstruction tensors and the extended Weyl-obstruction tensors can be defined following [132], [155] by promoting the ambient metric to the “Weyl-ambient metric”. We will discuss this in detail in Chapter 6.

Let us now focus on a holographic field theory dual to the vacuum Einstein theory in the $(d+1)$ -dimensional bulk. The holographic dictionary provides the relation between the on-shell classical bulk action S_{bulk} and

quantum effective action S_{bdr} of the field theory on the boundary [164]:

$$\exp(-S_{bulk}[g; g_{(0)}, a_{(0)}]) = \exp(-S_{bdr}^{\text{eff}}[g_{(0)}, a_{(0)}]) , \quad (5.99)$$

where $g_{(0)}$ and $a_{(0)}$ are the boundary values of $g_{ij}(z; x)$ and $a_i(z; x)$ as shown in (5.87) and (5.88). Since a_i is pure gauge in the bulk, $a_\mu^{(0)}$ could be gauged away and hence it is not expected to source any current on the boundary. The role of the $a_i^{(0)}$, however, is important since it makes the energy-momentum tensor along with all the geometric quantities on the boundary Weyl-covariant.

For any d we conjecture the boundary energy-momentum tensor vec is given by

$$\langle \kappa^2 T_{ij} \rangle = \frac{d}{2L^2} \pi_{ij}^{(0)} + \hat{\nabla}_{(i}^{(0)} p_{j)}^{(0)} , \quad (5.100)$$

where $\kappa^2 = 8\pi G_N$ and $\pi_{ij}^{(0)}$ $p_i^{(0)}$ appear in the metric expansion (5.87). In Appendix B.3 we derive the vev of the boundary energy-momentum tensor in the WFG gauge (5.84) for $d = 3$ boundary dimension¹⁸, following the prescription of [149].

In holographic theories with even d that have bulk Einstein duals the anomalous Weyl-Ward identity (5.51) becomes

$$\frac{1}{\sqrt{g^{(0)}}} \frac{\delta \mathcal{A}}{\delta \ln \mathcal{B}(x)} = \langle T^{ij}(x) g_{ij}^{(0)} \rangle . \quad (5.101)$$

Note that compared to Eq. (5.51) the Weyl current disappears. In the boundary field theory, the Weyl current J_i vanishes identically since the the mode $a_i(z; x)$ is pure gauge in the bulk WFG expansion (5.84). This is a special feature of holographic Weyl field theories with bulk Einstein duals. It would be interesting to find a holographic example where a non-trivial Weyl current is sourced at the boundary.¹⁹

The method of extracting the Weyl anomaly in dimensional regularization for the WFG was shown in [43] following analogous prescription for the FG gauge given in [142]. Briefly, one expands the bulk on-shell Einstein-Hilbert action as follows

$$\begin{aligned} S_{EH}^{\text{o.s.}} &= \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{\det g_{\mu\nu}} (R(g_{\mu\nu}) - 2\Lambda) \Big|_{\text{o.s.}} \\ &= \frac{2\Lambda}{\kappa^2(d-1)} \int_{\mathcal{M}} d^{d+1}x \sqrt{\det g_{\mu\nu}} \Big|_{\text{o.s.}} \\ &= -\frac{dL^{-2}}{\kappa^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{\det g_{\mu\nu}} \Big|_{\text{o.s.}} \\ &= -\frac{dL^{-1}}{\kappa^2} \int_{\mathcal{M}} \frac{dz}{z} d^d x \sqrt{\det g_{ij}} \Big|_{\text{o.s.}} , \end{aligned} \quad (5.102)$$

where $\kappa^2 = 8\pi G_N$, $\Lambda = -\frac{d(d-1)}{2L^2}$, from the first to the second line above we used the vacuum Einstein equations $R(g_{\mu\nu}) = 2\Lambda \frac{d+1}{d-1}$ and from the third to the fourth line we used $\sqrt{\det g_{\mu\nu}} = (L/z) \sqrt{\det g_{ij}}$ from (5.84).

¹⁸The case of even d is more subtle since the two series mix and a logarithmic term appears.

¹⁹An interesting question is whether a non-trivial Weyl current is sourced if we include higher curvature corrections in the gravitational action.

From the WFG metric expansion (5.87) we have that

$$\sqrt{\det g_{ij}} = \left(\frac{L}{z}\right)^d \sqrt{-\det \gamma^{(0)}} \left(1 + \frac{1}{2} \left(\frac{z}{L}\right)^2 X^{(1)} + \frac{1}{2} \left(\frac{z}{L}\right)^4 X^{(2)} + \dots + \frac{1}{2} \left(\frac{z}{L}\right)^d Y^{(1)} + \dots\right), \quad (5.103)$$

we expand the bulk on-shell action (5.102) as [43]

$$S_{EH}^{\text{o.s.}} = -\frac{L^{-2}}{\kappa^2} \int_{\mathcal{M}} \frac{dz}{z} \left(\frac{L}{z}\right)^{d+1} \left(1 + \frac{1}{2} \left(\frac{z}{L}\right)^2 X^{(1)} + \frac{1}{2} \left(\frac{z}{L}\right)^4 X^{(2)} + \dots + \frac{1}{2} \left(\frac{z}{L}\right)^d Y^{(1)} + \dots\right) \Big|_{\text{o.s.}} vol_{\Sigma}, \quad (5.104)$$

where $vol_{\Sigma} \equiv \sqrt{\det g^{(0)}} dx^1 \wedge \dots \wedge dx^d$.

It can be shown that ²⁰ the anomalous part $\mathcal{A}_{2kd}^{\text{WFG}}$ of the of the $2k$ -dimensional boundary theory is

$$\mathcal{A}_{2kd}^{\text{WFG}} = \frac{k}{\kappa^2 L} \int \ln \mathcal{B} X_{d=2k}^{(k)} vol_{\Sigma}. \quad (5.105)$$

Therefore, to find the Weyl anomaly in $2k$ -dimension, we only have to compute $X^{(k)}$ coming from the expansion of $\sqrt{\det g_{ij}}$. In the next section we will give explicit results for $\mathcal{A}$ in 2d, 4d and discuss cohomological properties of the holographic Weyl anomaly.

5.3.4 WFG cohomology is FG cohomology

The prescription for $\mathcal{A}_{2kd}^{\text{WFG}}$ given in Eq. (5.105) in the previous section allows us to compute the anomaly for 2d, 4d using the expansions (5.87), (5.88) and solving Einsteins equations order by order. One finds [43]

$$\mathcal{A}_{2d}^{\text{WFG}}[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}] = -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} \ln \mathcal{B} \hat{R}_{(0)}, \quad (5.106)$$

$$\mathcal{A}_{4d}^{\text{WFG}}[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}] = -\frac{L}{8\pi G_N} \int d^4x \sqrt{\det g_{(0)}} \ln \mathcal{B} \left[\frac{L^2}{8} \left(\hat{R}_{ij}^{(0)} \hat{R}_{(0)}^{ij} - \frac{1}{3} \hat{R}_{(0)}^2 \right) + \hat{\nabla} \cdot a^{(2)} \right]. \quad (5.107)$$

For the results $\mathcal{A}_{6d,8d}^{\text{WFG}}$ in 6d, 8d see [126].

Note that the integrands appearing in both $\mathcal{A}_{2d,4d}^{\text{WFG}}$ above have the form $\sqrt{\det g_{(0)}} \ln \mathcal{B} \mathcal{X}$ where $\mathcal{X}$ is Weyl-covariant scalar tensor of zero Weyl weight under the Weyl transformation (5.11) in 2d, 4d respectively. It is easy to see this if we consider the following Weyl weights $w_{g_{ij}^{(0)}} = -2$, $w_{\hat{R}_{ij}^{(0)}} = 0$ and $w_{a_i^{(2)}} = 2$. We expect this to be a generic feature of the holographic Weyl anomaly in WFG gauge in any higher even d . It is also straightforward to check that that $\mathcal{A}_{2d,4d}^{\text{WFG}}$ satisfy the 1-cocycle condition (5.46), as expected.

Let us now investigate the WFG holographic cocycles (5.106), (5.107) in relation to the FG holographic cocycles given in (5.60), (5.61). We mentioned in Section 5.2 (see the discussion below Eq. (5.50)) that the cocycles in conformal geometry carry over to Weyl geometry. Thus, the FG cocycles are also solutions also solutions of the 1-cocycle equation (5.46). A natural question is whether the FG and WFG anomaly cocycles belong to the same non-trivial cocycle or not.

We will show for 2d, 4d that the WFG cocycles (5.106), (5.106) represent the same cocycle as in FG (5.60), (5.61),

²⁰We refer the reader to [43], [142] for more details.

namely we will find local single-valued functionals $S_{2d,4d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}]$ such that²¹

$$\mathcal{A}_{2d,4d}^{\text{WFG}}[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}] = \mathcal{A}_{2d,4d}^{\text{FG}}[\mathcal{B}; g_{ij}^{(0)}] + \Delta_{\mathcal{B}} \circ S_{2d,4d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}], \quad (5.108)$$

where the operator $\Delta_{\mathcal{B}}$ was defined in Eq. (5.47). We also have the following formulas

$$\begin{aligned} S_{2d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}] &= -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} g_{(0)}^{ij} a_i^{(0)} a_j^{(0)}, \\ S_{4d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}] &= -\frac{L^3}{16\pi G_N} \left[\frac{1}{2} \int d^4x \sqrt{\det g_{(0)}} \mathring{G}_{(0)}^{ij} a_i^{(0)} a_j^{(0)} + \int d^4x \sqrt{\det g_{(0)}} a_{(0)}^i a_{(0)}^j \mathring{\nabla}_i^{(0)} a_j^{(0)} \right. \\ &\quad - \frac{1}{2} \int d^4x \sqrt{\det g_{(0)}} (g_{(0)}^{ij} a_i^{(0)} a_j^{(0)}) (g_{(0)}^{mn} \mathring{\nabla}_m^{(0)} a_n^{(0)}) + \frac{3}{4} \int d^4x \sqrt{\det g_{(0)}} (g_{(0)}^{ij} a_i^{(0)} a_j^{(0)})^2 \\ &\quad \left. - 2L^{-2} \int d^4x \sqrt{\det g_{(0)}} g_{(0)}^{ij} a_i^{(0)} a_j^{(2)} \right]. \end{aligned} \quad (5.110)$$

In Eq. (5.110) above $\mathring{G}_{(0)}^{ij}$ is the Einstein tensor of $g_{ij}^{(0)}$.

As we mentioned below Eq. (5.50) anomaly cocycles belong to the non-trivial cohomology of the nilpotent operator $\Delta_{\mathcal{B}}$. These are closed forms $\Delta_{\mathcal{B}} \circ \mathcal{A} = 0$ that are identified only up to an exact form $\mathcal{A} \sim \mathcal{A} + \Delta_{\mathcal{B}} \circ \mathcal{F}$ defined in Eq. (5.36). Eq. (5.108) shows that the contribution from the Weyl structure in the bulk appears as *cohomologically trivial* modifications. In Appendix B.4 we provide the proof of Eq. (5.108).

We conjecture the reason behind this fact is that the mode $a_i(z; x)$ in (5.84) is pure gauge due to the FG theorem and thus we expect (5.108) to hold in all higher even d . This is also expected, as the bulk theory is identical in both FG and WFG gauges, implying a single central charge. Furthermore, by gauge fixing the Weyl data $a_i(z; x)$ to zero, the WFG should reduce to the FG gauge. It would be interesting however if Eq. (5.108) could be proven in any d by a general argument that does not rely on brute force computations.

5.4 Summary

In this chapter we started in Section 5.1 by introducing Conformal geometry $(\mathcal{M}, [g])$ and Weyl geometry $(\mathcal{M}, [g, a])$ as the basic geometric constructions. In Weyl geometry, besides the metric, the Weyl connection one form a_i is included as part of the background geometric quantities. The fundamental symmetry besides diffeomorphisms is Weyl rescalings of the metric $g_{ij} \rightarrow \mathcal{B}(x)^{-2} g_{ij}$ for conformal manifolds and $(g_{ij}, a_i) \rightarrow (\mathcal{B}(x)^{-2} g_{ij}, a_i - \partial_i \ln \mathcal{B}(x))$ for Weyl manifolds. We saw that the basic building blocks for conformal geometry are conformal tensors and Weyl-covariant tensors for Weyl geometry. Special attention was paid to a special class of such tensors called Obstruction tensors.

In Section 5.2 we considered conformal field theories on a conformal manifolds and extended them to Weyl-field theories on a manifold endowed with Weyl geometry. We defined the anomalous Weyl-Ward identity and demonstrated that the anomaly functional satisfies the 1-cocycle condition (5.31), (5.46) which can be interpreted as part of the cohomology associated with Weyl transformations. The general solution of the 1-cocycle condition for Weyl-field theories (5.46) in different dimensions is an interesting open research problem.

²¹We assume total derivative terms can be dropped.

In Section 5.3 we shifted our focus to holography. We demonstrated how the Fefferman-Graham expansion of the bulk metric can define the Obstruction tensor and calculate the holographic Weyl anomaly using the holographic dictionary. We then considered the Weyl-Fefferman-Graham expansion [43] of the bulk metric and found that it induces a Weyl geometry in the conformal boundary. Weyl transformations were shown to be induced by a specific bulk diffeomorphism. We proposed a formula for the holographic energy-momentum tensor of the dual field theory and found that the induced Weyl current vanishes identically. This vanishing result is attributed to the Weyl data being a pure gauge degree of freedom, following Fefferman-Graham's seminal theorem for AlAdS spacetimes that satisfy Einstein equations. In Appendix B.3 we proved the holography energy-momentum [149] proposal in the WFG gauge for 3d. Similar to the FG expansion, one can compute Weyl-obstruction tensors and the holographic Weyl anomaly from the WFG expansion. Finally in Section 5.3.4, we found that the WFG cocycle can be shifted to the FG cocycle by adding suitable exact forms, which we proved in Appendix B.4 for 2d, 4d boundary dimension. An interesting question remains whether this can be proven for general dimension d .

Chapter 6

The Weyl-Ambient Metric

Work in this chapter is based on “Weyl-ambient geometries” [165], authored by Weizhen Jia (UIUC), Manthos Karydas (UIUC) and Robert G. Leigh (UIUC).

In Chapter 5 we introduced the basic elements of Weyl geometry. We found that Weyl geometry is a natural extension of conformal geometry with Weyl covariance mediated by a Weyl connection. In holographic context we also saw that the Weyl-Fefferman-Graham gauge induces a Weyl geometry in the boundary. From the WFG metric expansion [43] we defined special Weyl-covariant tensors called Weyl-obstruction tensors. In this chapter, we generalize the Fefferman-Graham ambient construction [44] for conformal manifolds to a corresponding construction for Weyl manifolds. The *Weyl-ambient construction* places the results of Chapter 5 into a solid geometric framework. We introduce the Weyl-ambient metric motivated by the Weyl-Fefferman-Graham gauge. We then promote a conformal manifold into a Weyl manifold by assigning a Weyl connection to the principal $\mathbb{R}_+$ -bundle realizing a Weyl structure. We show that the Weyl structure admits a well-defined initial value problem, which determines the Weyl-ambient metric. Through the Weyl-ambient construction, we also investigate Weyl-covariant tensors on the Weyl manifold and define extended Weyl-obstruction tensors explicitly.

This chapter is organized as follows. In Section 6.1 we provide an introductory overview, highlighting the relevant literature concerning this topic. In Section 6.2 we first briefly review the ambient metric of Fefferman-Graham before we introduce the Weyl-ambient metric $\tilde{g}$ at the end. To get some intuition on the construction, we start from the example of the flat ambient metric and then generalize to Ricci-flat ambient metrics. Various coordinate systems presented in Section 6.2 are described in Appendix C.1. In Section 6.3 we consider a bottom-up construction of the Weyl-ambient metric.¹ We show rigorously that the Weyl-ambient metric has a well-defined perturbative initial value problem, where the Ricci-flatness condition plays the role of the equation of motion. To show this we follow [44] closely and generalize their definitions properly for the Weyl-ambient space. We also extend some major theorems of [44] with suitable modifications for the Weyl-ambient space. The details of some proofs are presented in Appendix C.2. We then discuss how the Weyl-covariant tensors on $(\mathcal{M}, [g, a])$ can be derived from the Riemann tensor of $(\tilde{\mathcal{M}}, \tilde{g})$, and define the extended Weyl-obstruction tensors as a special example. Finally, our results are summarized in Section 6.4. [39]–[42], [127], [128]

¹The top-down construction will not be presented in this thesis. It can be found in Section 3 of [165].

6.1 Introduction

Conformal geometry is a very rich area of mathematics with its history deeply intertwined with that of physics. Historically, the subject was initiated at the beginning of the twentieth century with the work of Weyl [39], Cartan [127] and Thomas [128]. In physics, there have been numerous applications of conformal geometry, from conformal compactification [166] and conformal gravity [167] to the AdS/CFT correspondence [11], [164].

The fundamental structure appearing in conformal geometry is a manifold M endowed with a *conformal class* of metrics $[g]$. Two metrics belong in the same conformal class $[g]$ if one metric is a smooth positive multiple of the other. Local rescalings of the metric tensor by an arbitrary smooth positive function are called *Weyl transformations*. Compared to pseudo-Riemannian manifolds (M, g) , conformal manifolds are endowed with an enlarged symmetry group with both diffeomorphisms and Weyl transformations, denoted by $\text{Diff}(M) \ltimes \text{Weyl}$. A tensor T on a conformal manifold $(M, [g])$ is said to be conformally covariant if it transforms covariantly under a Weyl transformation:

$$T \rightarrow \mathcal{B}(x)^{w_T} T, \quad \text{when} \quad g \rightarrow \mathcal{B}(x)^{-2} g, \quad (6.1)$$

where w_T is the Weyl weight of the tensor T . On the physics side, conformal-covariant tensors appear as expectation values of operators in conformal field theories coupled to a background metric. As an important example, the expectation value of the trace of the energy-momentum tensor acquires an anomalous term after quantization, namely the celebrated Weyl anomaly [133]. By investigating the effective action in dimensional regularization, Deser and Schwimmer [145] made a conjecture regarding the possible candidates for the Weyl anomaly, which are global conformal invariants. This conjecture was later proven by Alexakis [168].

Just as diffeomorphism-covariant quantities, i.e., tensors, on pseudo-Riemannian manifolds can easily be constructed out of the metric, Riemann tensor and covariant derivatives, one might expect to find conformal-covariant tensors on conformal manifolds. However, unlike the abundance of diffeomorphism-covariant quantities on (M, g) , it is significantly harder to construct conformal-covariant tensors on $(M, [g])$. Before the work of Fefferman and Graham, known examples of conformal tensors were the Weyl tensor in any dimension, the Cotton tensor [129] in $3d$ and the Bach tensor [130] in $4d$. In their seminal work [44], [131] Fefferman and Graham introduced the ambient metric construction based on previous work by Fefferman [169], which provided a systematic method of finding conformal-covariant tensors. The basic idea of the construction was to associate a $(d+2)$ -dimensional ambient pseudo-Riemannian manifold to a d -dimensional conformal manifold. One can then find a specific class of ambient diffeomorphisms that induces Weyl transformations on the conformal manifold. In the context of AdS/CFT, diffeomorphisms that induce a Weyl transformation of the boundary metric are the Penrose-Brown-Henneaux (PBH) transformations [37]. Thus, conformal-covariant tensors can descend from ambient Riemannian tensors, and their Weyl transformations can be derived from certain ambient diffeomorphisms.

An important outcome of the ambient construction was to define extended obstruction tensors from covariant derivatives of the ambient Riemann tensor [155]. Obstruction tensors are the generalization to higher (even) dimension of the Bach tensor. For each even dimension, the corresponding obstruction tensor is the only irreducible conformal-covariant tensor in that dimension [132]. Defined through the ambient space, the k^{th} extended obstruction tensor $\Omega_{ij}^{(k)}$ has a simple pole at $d = 2k + 2$, whose residue is the obstruction tensor in

that dimension. For example, the first obstruction tensor reads

$$\Omega_{ij}^{(1)} = -\frac{1}{d-4}B_{ij}, \quad (6.2)$$

where B_{ij} is the Bach tensor, namely the obstruction tensor in $4d$.

A different perspective on conformal geometry was introduced by Weyl [39], whose idea was to make the physical scale a local quantity. The Weyl connection was introduced so that one can transport the physical scale between two points of the manifold. Although Weyl's initial attempt to identify the Weyl connection with the electromagnetic gauge field failed, the consistent mathematical structure he introduced was developed further in [40], [41]. In this approach, a Weyl connection a is introduced on the conformal manifold which transforms together with the metric g under a Weyl transformation. One can modify the conformal class $[g]$ to a Weyl class $[g, a]$, which is the equivalence class formed by the pairs $(g, a) \sim (\mathcal{B}(x)^{-2}g, a - d \ln \mathcal{B}(x))$. This defines a Weyl manifold $(M, [g, a])$, and the conformal geometry is promoted to Weyl geometry [40]–[42]. Equivalently, a Weyl connection can be thought of as a connection on the Weyl structure, which is a principal bundle with the Weyl symmetry group as the structure group [40].

Similarly to a conformal-covariant tensor, one can define a Weyl-covariant tensor T on a Weyl manifold $(M, [g, a])$ to be a tensor that transforms covariantly under a Weyl transformation:

$$T \rightarrow \mathcal{B}^{w_T}(x)T, \quad \text{when} \quad g \rightarrow \mathcal{B}(x)^{-2}g, \quad a \rightarrow a - d \ln \mathcal{B}(x). \quad (6.3)$$

Although conformal-covariant tensors on a conformal manifold $(M, [g])$ are hard to find, Weyl-covariant tensors on a Weyl manifold $(M, [g, a])$ can be constructed quite easily. Recall that on a pseudo-Riemannian manifold (M, g) , one can define a Levi-Civita (LC) connection ∇ , and it is well-known that diffeomorphism-covariant quantities can be constructed from the metric, Riemann curvature, and covariant derivatives of the Riemann curvature. On a Weyl manifold $(M, [g, a])$, one can define a Weyl-Levi-Civita connection $\hat{\nabla}$, and a plethora of Weyl-covariant quantities can similarly be constructed from the metric, Weyl-Riemann curvature, and Weyl-covariant derivatives $\hat{\nabla}$ of the Weyl-Riemann curvature. This indicates that the $\text{Diff}(M) \ltimes \text{Weyl}$ symmetry is manifested more naturally on a Weyl manifold, and the representation has a similar structure as that of $\text{Diff}(M)$ on pseudo-Riemannian manifolds. There are corresponding notions of Weyl metricity, Weyl torsion and a uniqueness theorem giving a Weyl-LC connection [40].

A significant step towards incorporating the Weyl connection in the formalism of AdS/CFT was taken in [43], which was further investigated in [126]. By modifying the Fefferman-Graham (FG) ansatz of an asymptotically locally AdS (AlAdS) spacetime to the Weyl-Fefferman-Graham (WFG) form, it was shown [43] that the bulk LC connection induces a Weyl connection on the conformal boundary. Thus, the AlAdS bulk geometry in the WFG gauge induces a Weyl geometry instead of only a conformal geometry on the conformal boundary. Applying the holographic dictionary [164], they also calculated the holographic Weyl anomaly in the WFG gauge, and found it can be organized in a Weyl-covariant fashion. In the FG ambient construction, the conformal boundary $(M, [g])$ of a $(d+1)$ -dimensional AlAdS bulk is associated with a $(d+2)$ -dimensional ambient space, and the AlAdS bulk in the FG gauge can be considered as a hypersurface in the ambient space.

A natural question to ask is whether such a construction exists for the conformal boundary as a Weyl manifold. In this chapter we will provide such a construction. We introduce the Weyl-ambient space $(\tilde{M}, \tilde{g})$

as a modification of the FG ambient space, in which the AlAdS bulk in the WFG gauge is a hypersurface and its boundary is associated with a codimension-2 Weyl manifold $(M, [g, a])$.

Notation

We will label the indices in a d -dimensional manifold $\mathcal{M}$ by lowercase Latin letters $i, j, \dots$, in a $(d+1)$ -dimensional AlAdS bulk by lowercase Greek letters $\mu, \nu, \dots$, and in a $(d+2)$ -dimensional ambient space $\tilde{\mathcal{M}}$ by uppercase Latin letters $I, J, \dots$. The vectors on $\mathcal{M}$ are denoted by $\underline{U}, \underline{V}$, on the Weyl structure $\mathcal{P}_W$ over $\mathcal{M}$ are denoted by $\underline{u}, \underline{v}$, and on the ambient manifold $\tilde{\mathcal{M}}$ are denoted by $\underline{\mathcal{U}}, \underline{\mathcal{V}}$.

Unless otherwise indicated, in Section 6.3 we mainly use the ambient coordinate system $\{t, x^i, \rho\}$, and the indices are $I = 0, 1, \dots, d, \infty$, where 0 labels the t -component and ∞ labels the ρ -component. The notation $(0, x^i, \infty)$ is also used for the components in a trivialization $\mathcal{P}_W \times \mathbb{R} \simeq \mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}$, even without specifying a choice of coordinates on $\mathcal{M}$. The above-mentioned notation is summarized in Table 6.1.

Dimension	Manifold	Vectors	Indices
d	$\mathcal{M}$	$\underline{U}, \underline{V}$	$i, j, \dots \quad \{x^i\} \quad i = 1, \dots, d$
$d+1$	AlAdS_{d+1}		$\mu, \nu, \dots \quad \{x^\mu\} = \{z, x^i\} \quad i = 1, \dots, d$
$d+1$	$\mathcal{P}_W$	$\underline{u}, \underline{v}$	
$d+2$	$\tilde{\mathcal{M}}$	$\underline{\mathcal{U}}, \underline{\mathcal{V}}$	$I, J, \dots$ In the frame $\{\mathbf{e}^I\} = \{\mathbf{e}^+, \mathbf{e}^i, \mathbf{e}^-\}$, $I = +, 1, \dots, d, -$. In the coordinates $\{x^I\} = \{t, x^i, \rho\}$, $I = 0, 1, \dots, d, \infty$.

Table 6.1: Notation for this chapter. Table taken from [165].

6.2 Ambient Metrics

In this section we will review the FG ambient metric and introduce the Weyl-ambient metric. To build up some intuition, we begin with the flat ambient metric and then generalize to Ricci-flat ambient metrics.

6.2.1 Flat Ambient Metrics

The simplest example of an ambient space is the flat ambient space. Consider the $(d+2)$ -dimensional Minkowski spacetime $\mathbb{R}^{1, d+1}$ with the metric

$$\eta = -(dX^0)^2 + \sum_{i=1}^{d+1} (dX^i)^2. \quad (6.4)$$

One can describe $(d+1)$ -dimensional Euclidean AdS spaces as the following codimension-1 hyperboloids:²

$$(X^0)^2 - R^2 = L^2, \quad R^2 = \sum_{i=1}^{d+1} (X^i)^2, \quad (6.5)$$

²One can also take the signature in (6.4) to be $(2, d)$. Then, g^+ will be the Lorentzian signature AdS spacetime and the δ_{ij} in (6.10) becomes η_{ij} . More generally, if one takes the signature in (6.4) to be $(p, d+2-p)$, then the signature of g^+ will be $(p-1, d+2-p)$.

where L represents the AdS radius. The hyperboloids with different L form a one-parameter family of hypersurfaces foliating the interior of the future light cone, denoted by $\mathcal{N}^+$, emanating from the origin of the Lorentzian coordinate system $\{X^0, X^i\}$. Then, one can also write the Minkowski metric (6.4) in the following “cone” form:

$$\eta = -d\ell^2 + \frac{\ell^2}{L^2}g^+, \quad \ell > 0, \quad (6.6)$$

where the coordinate $\ell = \sqrt{(X^0)^2 - R^2}$, and g^+ is the $(d+1)$ -dimensional Euclidean AdS metric. Now the Euclidean AdS space with AdS length L is represented by the hyperbola $\ell = L$. The metric g^+ can be expressed in the Fefferman-Graham (FG) form in the following different ways (see Appendix C.1 for details):

$$g_S^+ = \frac{L^2}{z^2} \left(dz^2 + L^2 \left(1 - \frac{1}{4} (z/L)^2 \right)^2 d\Omega_d^2 \right), \quad 0 < z < 2L, \quad (6.7)$$

$$g_F^+ = \frac{L^2}{z^2} (dz^2 + \delta_{ij} dx^i dx^j), \quad i = 1, \dots, d, \quad z > 0. \quad (6.8)$$

The metric (6.6) with $g^+ = g_S^+$ or g_F^+ is defined in the whole interior of the light cone $\mathcal{N}^+$,³ while their AdS boundaries have different topologies. It is easy to see that the AdS boundary at $z \rightarrow 0^+$ of g_S^+ in (6.7) is (conformally) a d -sphere while that of g_F^+ in (6.8) is (conformally) flat.

While the metric (6.6) is singular in the limit $z \rightarrow 0^+$ with ℓ fixed, it is well-defined when taking both z and ℓ to zero with z/ℓ fixed. To make this evident we introduce a new coordinate system $\{t, x^i, \rho\}$, called the *ambient coordinate system*, with $t = \ell/z$ and $\rho = -z^2/2$. First we look at the metric (6.6) with g_S^+ in (6.7), which in the ambient coordinate system becomes

$$\eta = 2\rho dt^2 + 2t dt d\rho + t^2 \left(1 + \frac{\rho}{2L^2} \right)^2 L^2 d\Omega_d^2. \quad (6.9)$$

The coordinate patch of $\{\ell, x^i, z\}$ which covers the interior of the light cone surface $\mathcal{N}^+$, corresponds to $t \in (0, \infty)$, $\rho \in (-2L^2, 0)$ (see Figure 6.1). However, it is apparent now that the limit $\rho \rightarrow 0^-$ of the above metric is well-defined, and thus we can extend the coordinate patch of $\{t, x^i, \rho\}$ to include an open neighborhood of the surface $\mathcal{N}^+$ at $\rho = 0$. Hence, $\mathcal{N}^+$ is parametrized by $\{t, x^i\}$, where $t \in \mathbb{R}_+$ and x^i are the coordinates of the d -sphere S_d . In other words, $\mathcal{N}^+$ can be regarded as a line bundle over S^d whose fibres are parametrized by t .

Suppose ϕ is a function on $\mathbb{R}^{1,d+1}$, which defines a hypersurface Σ by the locus of points $p \in \mathbb{R}^{1,d+1}$ such that $\phi(t, x^i, \rho)|_p = 0$. In order to find the intersection $\Sigma \cap \mathcal{N}^+$, one can set $\rho = 0$ and solve for t as a function $t(x^i)$ of the d -sphere coordinates from $\phi(t, x, \rho = 0) = 0$. The pullback metric on the intersection submanifold is $\eta|_{\Sigma \cap \mathcal{N}^+} = t(x)^2 L^2 d\Omega_d^2$. The function $t(x)$ depends on the choice of function ϕ (which is arbitrary) that defines Σ , and thus we see that the pullback metric is conformally equivalent to the metric of S_d . An example is to take $\phi = \ln t$, and to consider the pull back of the metric at $\rho = 0$, $t = 1$, namely $\eta|_{\rho=0, t=1} = L^2 d\Omega_d^2$. If we perform a diffeomorphism $t = \mathcal{B}(x)^{-1} t'$ and pull back the metric at $\rho = 0$, $t' = 1$, then we find $\eta|_{\rho=0, t'=1} = \mathcal{B}(x)^{-2} L^2 d\Omega_d^2$. Therefore, at $\rho = 0$ we have a *conformal class* $[g]$ of d -dimensional metrics, and the $(d+2)$ -dimensional Minkowski metric expressed in (6.9) is said to be the ambient metric of $[g]$. This implies that the null surface $\mathcal{N}^+$ at $\rho = 0$ is associated with a metric bundle, which will be important for the formal construction later in Section 6.3.

³Note that for Lorentzian signature AdS spacetime, the metric (6.6) with g^+ given in Poincare coordinates g_F^+ (6.8) only covers half of the interior of the future light cone.

Similarly, the metric (6.6) with g_F^+ in (6.8) can also be expressed in the ambient coordinates as

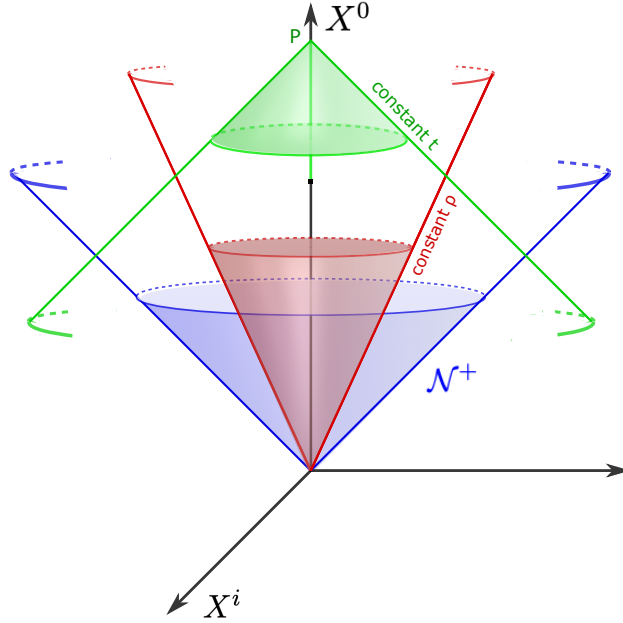


Figure 6.1: Sketch of a constant- ρ surface (red) and a constant- t surface (green) of the flat ambient metric (6.9) in the Lorentzian coordinate system $\{X^0, X^i\}$. Constant- t surfaces are past directed light cones. Changing t moves the apex P of the cone along the X^0 -axis. Constant- ρ surfaces are future directed timelike cones. When $\rho \rightarrow 0^-$ the constant ρ surface becomes the light cone $\mathcal{N}^+$ (blue). Figure taken from [165].

$$\eta = 2\rho dt^2 + 2t dt d\rho + t^2 \delta_{ij} dx^i dx^j, \quad i = 1, \dots, d. \quad (6.10)$$

In this case, the original coordinate patch of $\{\ell, x^i, z\}$ corresponds to $t \in (0, \infty)$, $\rho \in (-\infty, 0)$, and the null surface $\mathcal{N}^+$ is again covered by the $\{t, x^i, \rho\}$ system at $\rho = 0$. Intersecting the null surface with a hypersurface and taking the pullback metric on the intersection, we now obtain a d -dimensional metric $ds^2 = t(x)^2 \delta_{ij} dx^i dx^j$ that is conformally flat. This metric is also in the conformal class $[g]$ but the topology is different from the d -dimensional metric obtained from (6.9). Note that the flat ambient metric in either (6.9) or (6.10) is homogeneous of degree 2 with respect to the t -coordinate; that is, under a constant scaling $t \rightarrow st$ the metric transforms as $\eta \rightarrow s^2 \eta$, or in the infinitesimal form,

$$\mathcal{L}_{\underline{T}} \eta = 2\eta, \quad \underline{T} = t \underline{\partial}_t. \quad (6.11)$$

We will retain this property also for Ricci-flat ambient metrics and the Weyl-ambient metric.

6.2.2 Ricci-Flat Ambient Metrics

The flat ambient metric combines hyperbolic metrics and their conformal boundaries in a unified framework. Before we describe its utility, we will review the generalization of flat ambient metrics to Ricci-flat ambient metrics. This will allow us to consider $(d+1)$ -dimensional asymptotically locally Anti-de Sitter (AlAdS) spaces which are especially relevant in holographic theories.

The main observation that allows an extension to Ricci-flat ambient metrics is that (6.6) can be generalized

in the following form:

$$\tilde{g} = -d\ell^2 + \frac{\ell^2}{L^2} g_{\mu\nu}^+(x) dx^\mu dx^\nu, \quad \mu, \nu = 1, \dots, d+1, \quad \ell > 0, \quad (6.12)$$

where now $g^+(x)$ is an arbitrary $(d+1)$ -dimensional metric independent of ℓ . We will refer to this $(d+1)$ -dimensional geometry as the “bulk”. The ambient Ricci tensor $\tilde{Ric}(\tilde{g})$ can be decomposed in terms of the Ricci tensor of g^+ as [44], [170]

$$\tilde{Ric}(\tilde{g}) = Ric(g^+) + \frac{d}{L^2} g^+. \quad (6.13)$$

The right-hand side of the above equation can also be written as $G_{\mu\nu}(g^+) + \Lambda g_{\mu\nu}^+$ with $\Lambda = -\frac{d(d-1)}{2L^2}$. Therefore, when the ambient metric $\tilde{g}$ is Ricci-flat, g^+ is an Einstein metric and thus satisfies the vacuum Einstein equations.

According to the Fefferman-Graham theorem [131], [170], any AlAdS Einstein metric can be expressed in the Fefferman-Graham form⁴

$$g^+ = L^2 \frac{dz^2}{z^2} + \frac{L^2}{z^2} \gamma_{ij}(x, z) dx^i dx^j, \quad i, j = 1, \dots, d, \quad z > 0, \quad (6.14)$$

which is a generalization of (6.8). Then, by a coordinate transformation $t = \ell/z$ and $\rho = -z^2/2$, the metric (6.12) takes the form

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 \gamma_{ij}(x, \rho) dx^i dx^j, \quad t > 0. \quad (6.15)$$

We can see that the flat ambient metrics (6.9) and (6.10) are nothing but special cases of (6.15) when $\tilde{g} = \eta$. The codimension-2 metric is now generalized to an arbitrary $\gamma_{ij}(x, z)$ whose corresponding g^+ in (6.14) is an Einstein metric.

Note that the advantages of the ambient coordinate system $\{t, x^i, \rho\}$ mentioned before for the flat ambient space are now carried over to the Ricci-flat case. One can see that the surface at $\rho = 0$ is still a null hypersurface, denoted by $\mathcal{N}$, which is a coordinate singularity in the original $\{\ell, x^i, z\}$ coordinate system. Hence, the ambient coordinate system permits one to extend the spacetime region to include an open neighborhood of the null surface $\mathcal{N}$. Denoting the extended spacetime manifold as $\tilde{\mathcal{M}}$, then $\mathcal{N}$ is a hypersurface in $\tilde{\mathcal{M}}$ parametrized by $\{t, x^i\}$, which furnishes a conformal class $[\gamma]$ of codimension-2 metrics. Suppose $\mathcal{M}$ is a d -dimensional manifold equipped with the conformal class $[\gamma]$, then $(\tilde{\mathcal{M}}, \tilde{g})$ is called the $(d+2)$ -dimensional ambient space of $(\mathcal{M}, [\gamma])$.

Being part of the Ricci-flat ambient space, $\mathcal{N}$ can be regarded as an initial value surface. Then given the initial data $\gamma_{ij}(x, \rho)|_{\rho=0}$, the Ricci-flatness condition can be used to “propagate” the metric beyond the initial surface to a neighborhood around $\rho = 0$. That is, the Ricci-flatness condition $\tilde{Ric}(\tilde{g}) = 0$ is a set of differential equations for $\tilde{g}_{ij}(x, \rho)$, which can be solved iteratively in a series around $\rho = 0$ given the initial value $\tilde{g}(x, \rho)|_{\rho=0}$. The initial value problem for the Ricci-flat ambient space has been defined and evaluated rigorously in [44], the results of which will be carried over to the Weyl-ambient space in Section 6.3.

⁴Note the change in notation compared to Chapter 5, namely $\gamma_{ij}(z; x)$ defined here corresponds to $g_{ij}(z; x)$ defined in (5.53).

6.2.3 Weyl-Ambient Metric

Now we are ready to introduce the Weyl-ambient metric. We start from the $(d+2)$ -dimensional ambient metric in the form of (6.12). The expression of g^+ in (6.14) is the FG ansatz for an AlAdS spacetime, which is not preserved under a Weyl diffeomorphism $z \rightarrow z/\mathcal{B}(x)$, $x^i \rightarrow x^i$. This motivated the authors of [43] to introduce the Weyl-Fefferman-Graham (WFG) gauge (see Subsection 5.3.3) by adding an additional mode a_i to (6.14) as follows⁵

$$g_{\text{WFG}}^+ = L^2 \left(\frac{dz}{z} - a_i(x, z) dx^i \right)^2 + \frac{L^2}{z^2} \gamma_{ij}(x, z) dx^i dx^j, \quad z > 0. \quad (6.16)$$

Now we substitute the g^+ in (6.12) with the WFG ansatz (6.16), then transforming back to the ambient coordinates $\{t, x^i, \rho\}$, we obtain the *Weyl-ambient metric*⁶

$$\tilde{g} = 2\rho dt^2 + 2t^2 d\rho \left(\frac{dt}{t} + a_i(x, \rho) dx^i \right) + t^2 g_{ij}(x, \rho) dx^i dx^j, \quad t > 0, \quad (6.17)$$

where $g_{ij}(x, \rho) := \gamma_{ij}(x, \rho) - 2\rho a_i(x, \rho) a_j(x, \rho)$. We call the pseudo-Riemannian space $(\tilde{\mathcal{M}}, \tilde{g})$ a *Weyl-ambient space*. Having the form of the Weyl-ambient metric, the *ambient Weyl diffeomorphism*⁷

$$t' = \mathcal{B}(x)t, \quad x'^i = x^i, \quad \rho' = \mathcal{B}(x)^{-2}\rho \quad (6.18)$$

induces a change in the constituents a_i and γ_{ij} of the form

$$a'_i(x', \rho') = a_i(x, \rho) - \partial_i \ln \mathcal{B}(x), \quad \gamma'_{ij}(x', \rho') = \mathcal{B}(x)^{-2} \gamma_{ij}(x, \rho). \quad (6.19)$$

If we regard the AlAdS bulk as a hypersurface of the Weyl-ambient space, the above transformation gives rise to the Weyl diffeomorphism which preserves the WFG ansatz. In addition, we want to point out that just as the ambient metric (6.15) is homogeneous with respect to t , the homogeneity property (6.11) also pertains for the Weyl-ambient metric (6.17) since both $a_i(x, \rho)$ and $\gamma_{ij}(x, \rho)$ are independent of t . This homogeneity property will be repeatedly used throughout this chapter. In the following we use this property in order to show how an induced Weyl class arises from the Weyl-ambient metric; it is also crucial for the bottom-up construction and for proving Proposition 6.1.

The Ricci-flatness condition $\tilde{Ric}(\tilde{g}) = 0$ for the Weyl-ambient metric (6.17), similar to that for the ambient metric (6.15), is a set of differential equations for $\tilde{g}_{ij}(x, \rho)$ which can be solved order by order in a neighborhood of $\rho = 0$ given the initial value $\tilde{g}_{ij}(x, \rho)|_{\rho=0}$. To be precise, in a neighborhood of $\rho = 0$ we can expand γ_{ij}

⁵Note again the change in notation compared to Chapter 5, namely $\gamma_{ij}(z; x)$ defined here corresponds to $g_{ij}(z; x)$ defined in (5.84).

⁶The form of the metric (6.17) with a_i independent of ρ has been introduced in [171] in the context of Ricci gauging. We thank Omar Zanusso for pointing this out to us.

⁷In terms of the coordinates ℓ, z , the ambient Weyl diffeomorphism acts as $(\ell', x'^i, z') = (\ell, x^i, \mathcal{B}(x)^{-1}z)$.

and a_i as⁸

$$\gamma_{ij}(x, \rho) = \gamma_{ij}^{(0)}(x) + \gamma_{ij}^{(1)}(x)\rho + \gamma_{ij}^{(2)}(x)\rho^2 + \cdots, \quad (6.20)$$

$$a_i(x, \rho) = a_i^{(0)}(x) + a_i^{(1)}(x)\rho + a_i^{(2)}(x)\rho^2 + \cdots. \quad (6.21)$$

From the equation $\tilde{Ric}(\tilde{g}) = 0$, one can solve for $\gamma_{ij}^{(n)}(x)$ in terms of $\gamma_{ij}^{(k)}(x)$ and $a_i^{(k)}(x)$ with k up to $n - 1$. However, the modes $a_i^{(n)}(x)$, $\forall n$ are not determined by the Ricci flatness condition and hence we regard $a_i(x, \rho)$ as input data. This initial value problem will be examined in detail in Section 6.3 after the Weyl-ambient space is defined in terms of the Weyl structure, and the ansatz in (6.17) will be shown to be the uniquely determined Weyl-ambient metric for any given $\gamma_{ij}^{(0)}(x)$ and $a_i(x, \rho)$.

Let us now present explicit results. Starting from the metric (6.17), and solving $\tilde{Ric}(\tilde{g}) = 0$ order by order using the expansions (6.20), (6.21) (which is equivalent to solving the Einstein equations in the AlAdS bulk) to find⁹

$$\gamma_{ij}^{(1)} = 2\hat{P}_{(ij)} = 2\hat{P}_{ij} - f_{ij}^{(0)}. \quad (6.22)$$

$$\gamma_{ij}^{(2)} = \hat{\Omega}_{ij}^{(1)} + \hat{P}^k{}_i \hat{P}_{kj} + \hat{\nabla}_{(i}^{(0)} a_{j)}^{(1)}, \quad (6.23)$$

where $f_{ij}^{(0)} = \partial_i a_j^{(0)} - \partial_j a_i^{(0)}$, and $\hat{P}_{ij}$ is the Weyl-Schouten tensor (5.22) on $(M, [\gamma^{(0)}, a^{(0)}])$. Treating d as an continuous complex variable, the solution for each $\gamma_{ij}^{(k \geq 2)}$ has a pole at $d = 2k$ represented by $\hat{\Omega}_{ij}^{(k-1)}$. For now one should simply regard $\hat{\Omega}_{ij}^{(k-1)}$ in the above equations as denoting the pole terms of $\gamma_{ij}^{(k)}$ at $d = 2k$ ($\hat{P}_{ij}$ also represents the ‘‘pole’’ of $\gamma_{ij}^{(1)}$ at $d = 2$, which identically vanishes in $2d$). In Subsection 6.3.2 we will recognize them as extended Weyl-obstruction tensors through a precise definition. In terms of $\gamma_{ij}^{(0)}$, these quantities can be written as

$$\hat{P}_{ij} = \frac{1}{d-2} \left(\hat{R}_{ij}^{(0)} - \frac{\hat{R}^{(0)}}{2(d-1)} \gamma_{ij}^{(0)} \right), \quad (6.24)$$

$$\hat{\Omega}_{ij}^{(1)} = \frac{1}{d-4} \left(-\hat{\nabla}_k^{(0)} \hat{\nabla}_{(0)}^k \hat{P}_{ij} + \hat{\nabla}_k^{(0)} \hat{\nabla}_j^{(0)} \hat{P}_i{}^k + \hat{W}_{kjl}^{(0)} \hat{P}^{lk} \right), \quad (6.25)$$

where $\hat{W}_{(0)jkl}^i$ is the Weyl curvature tensor and $\hat{C}_{ijk} \equiv \hat{\nabla}_k^{(0)} \hat{P}_{ij} - \hat{\nabla}_j^{(0)} \hat{P}_{ik}$ is the Weyl-Cotton tensor (5.23). Note that indices are lowered with $\gamma_{ij}^{(0)}$ as necessary.

From the transformation (6.19) and the expansions (6.20) and (6.21), we can see that $\gamma_{ij}^{(k \geq 0)}$ and $a_i^{(k \geq 1)}(x)$ transform covariantly under the ambient Weyl diffeomorphism (6.18), with Weyl weights $2k - 2$ and $2k$, respectively:

$$\gamma_{ij}^{(k \geq 0)}(x) \rightarrow \mathcal{B}(x)^{2k-2} \gamma_{ij}^{(k \geq 0)}(x), \quad a_i^{(k \geq 1)}(x) \rightarrow \mathcal{B}(x)^{2k} a_i^{(k \geq 1)}(x). \quad (6.26)$$

On the other hand, $a_i^{(0)}$ transforms as $a_i^{(0)} \rightarrow a_i^{(0)} - \partial_i \ln \mathcal{B}$. Therefore, we should anticipate that $a_i^{(0)}$ can be interpreted as a Weyl connection on the codimension-2 geometry. An important result of [43] was to show that the bulk metric of an AlAdS spacetime in the WFG gauge provides a Weyl geometry on the conformal

⁸There will be a second series starting from the $\rho^{d/2}$ order in the expansion (6.20):

$$\gamma_{ij}(x, \rho) = (\gamma_{ij}^{(0)}(x) + \gamma_{ij}^{(1)}(x)\rho + \cdots) + \rho^{d/2}(\pi_{ij}^{(0)}(x) + \pi_{ij}^{(1)}(x)\rho + \cdots).$$

However, to solve for the second series in γ_{ij} order by order one needs the data $\pi_{ij}^{(0)}$ of the ambient space. This is related to holographic energy-momentum of the boundary theory and also to obstruction tensors as explained in Subsection 5.3.1.

⁹Note that the $\gamma_{ij}^{(k)}$ and $a_i^{(k)}$ defined here correspond to $(-2)^k g_{ij}^{(2k)}/L^{2k}$ and $(-2)^k a_i^{(2k)}/L^{2k}$ in Subsection 5.3.3, respectively.

boundary as we explained in Subsection 5.3.3.

Closing this section, we remark that the codimension-1 surface $\mathcal{N}$ at $\rho = 0$ is again a null surface parametrized by (t, x) with $t \in \mathbb{R}_+$, just like the case of the ambient metric (6.15). This surface in fact has the structure of a line bundle with each fibre parametrized by t , which turns out to be a principal bundle with the structure group $\mathbb{R}_+$. The new ingredient a_i in the Weyl-ambient metric (6.17) induces naturally a connection on this principal bundle, represented by $a_i^{(0)} = a_i|_{\rho=0}$.

6.3 Weyl-Ambient Space

6.3.1 Formal Theory

In this section we will present a geometric interpretation of the Weyl-ambient metric (6.17) as well as the Weyl connection therein in terms of a bottom-up construction. By “bottom-up” we mean to construct a $(d+2)$ -dimensional Weyl-ambient space from a d -dimensional manifold $\mathcal{M}$. The majority of this section is based on Section 2 and Section 3 of [44] where a more detailed exposition of the ambient construction can be found. We will generalize the main definitions and theorems there with the inclusion of a Weyl connection on the principal $\mathbb{R}_+$ -bundle. The resulting Weyl structure together with the metric bundle, viewed as an associated bundle, will be then used to define the Weyl-ambient metric. For this section to be self-contained we repeat some of the definitions and proofs of [44] when necessary, while generalizing them appropriately.

We start with a d -dimensional manifold $\mathcal{M}$ and introduce a principal $\mathbb{R}_+$ -bundle $\mathcal{P}_W$ over $\mathcal{M}$ that we call a *Weyl structure*.¹⁰

Definition 6.1. Given a d -dimensional manifold $\mathcal{M}$, a *Weyl structure* is a $(d+1)$ -dimensional manifold $\mathcal{P}_W$ together with the structure group $\mathbb{R}_+$, which is equipped with

(2.1) a free right action $\delta : \mathcal{P}_W \times \mathbb{R}_+ \rightarrow \mathcal{P}_W$, such that $\delta_s(p) = p \cdot s$, $\forall p \in \mathcal{P}_W, s \in \mathbb{R}_+$;

(2.2) a projection map $\pi : \mathcal{P}_W \rightarrow \mathcal{M}$, such that $\pi(p) = \pi(p \cdot s)$, $\forall p \in \mathcal{P}_W, s \in \mathbb{R}_+$;

(2.3) a local trivialization $T_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}_+$ for each open set $U_i \subset \mathcal{M}$ with $T_i(p) = (\pi(p), t_i(p))$, where $t_i : \pi^{-1}(U_i) \rightarrow \mathbb{R}_+$ satisfies $t_i(p \cdot s) = t_i(p) \cdot s$ for all $s \in \mathbb{R}_+$.

For brevity, suppose $U_i \subset \mathcal{M}$ has local coordinates $\{x^i\}$, we can express a point $p \in \mathcal{P}_W$ as (x, t) with $t \in \mathbb{R}_+$.

A connection on the Weyl structure can be described as follows. First we note that the push forward $\pi_* : T\mathcal{P}_W \rightarrow T\mathcal{M}$ defines the vertical sub-bundle $V \subset T\mathcal{P}_W$ given at any point $p \in \mathcal{P}_W$ by

$$V_p = \ker(\pi_*) \equiv \{\underline{v} \in T_p\mathcal{P}_W \mid \pi_*(\underline{v}) = \underline{0}\}. \quad (6.27)$$

In the present case V_p is a one-dimensional vector space spanned by the fundamental vector field which generates the group action along the fibres; in the local trivialization, it is expressed as $\underline{T} = t\partial_t$. From the perspective of $\mathcal{P}_W$, we can then think of the action of $\mathbb{R}_+$ as corresponding to a dilatation of the fibres. To assign a connection on $\mathcal{P}_W$ is to specify a horizontal sub-space $H_p \subset T_p\mathcal{P}_W$ such that $T_p\mathcal{P}_W = H_p \oplus V_p$ at any p . In the local trivialization given above, the horizontal bundle can be described as the span of vectors of the form

¹⁰We use this name since $\mathcal{P}_W$ can be regarded as a G -structure of the frame bundle, in which the structure group is reduced from $GL(d, \mathbb{R})$ to $\mathbb{R}_+$.

$\underline{D}_i = \partial_i - a_i(x)t\partial_t$.¹¹ Equivalently, it can be described as the kernel of a form $n := t^{-1}dt + a_i(x)dx^i \in T^*\mathcal{P}_W$, i.e.

$$H_p := \{\underline{u} \in T_p\mathcal{P}_W \mid i_{\underline{u}}n = 0\} \quad \forall p \in \mathcal{P}_W. \quad (6.28)$$

We note that under the Abelian group action $(x, t(x)) \mapsto (x, t'(x)) = (x, t(x)s(x))$, we have

$$n' = n + (a'_i(x) - a_i(x) + \partial_i \ln s(x))dx^i, \quad (6.29)$$

and so we see that the coefficients $a_i(x)$ transform as connection coefficients. Note also that it is natural to introduce the projector $\mathbf{a} : T\mathcal{P}_W \rightarrow V$ as

$$\mathbf{a} = t\partial_t \otimes (t^{-1}dt + a_i(x)dx^i), \quad (6.30)$$

which is an alternative way to express the connection on $\mathcal{P}_W$. We will refer to both $\mathbf{a}$ and $a_i(x)$ as the *Weyl connection*.

This line bundle has an important representation given by a conformal class of metrics. Indeed, all the non-trivial representations are one-dimensional, and thus a representation of $\mathbb{R}_+$ is given by specifying a Weyl weight w . We call the corresponding associated bundle $\mathcal{E}_w$ and its sections respond to the group action as

$$T_x \mapsto s(x)^w T_x. \quad (6.31)$$

Equivalently, this determines the transition functions on the associated bundle.

Suppose a conformal class $[g]$ of smooth metrics of signature (p, q) is given on $\mathcal{M}$, in which any two representatives g and g' are related by a smooth function $\mathcal{B}(x)$ as $g'_x = \mathcal{B}(x)^{-2}g_x$, where g_x is the value of g at a point $x \in \mathcal{M}$. Then, $(\mathcal{M}, [g])$ is a conformal manifold. One can define a *metric bundle* $\mathcal{G}$ as follows [44]:

Definition 6.2. A *metric bundle* $\mathcal{G}$ is the collection of pairs (x, h) where $h = s^2g_x$, $\forall s \in \mathbb{R}_+$ and $\forall x \in \mathcal{M}$, which is equipped with

(3.1) a dilatation map $\tilde{\delta}_s : \mathcal{G} \rightarrow \mathcal{G}$ such that $\tilde{\delta}_s(x, h) = (x, s^2h)$, $\forall s \in \mathbb{R}_+$.

(3.2) a projection map $\tilde{\pi} : \mathcal{G} \rightarrow \mathcal{M}$ such that $(x, h) \mapsto x$;

This definition simply identifies a conformal class of metrics with a bundle associated to the Weyl structure given by the weight $w = -2$ representation of $\mathbb{R}_+$. We note that it is isomorphic to the Weyl structure $\mathcal{P}_W$, as is any non-trivial associated bundle of $\mathcal{P}_W$.¹² Under a trivialization, assigning an isomorphism between $\mathcal{P}_W$ and the metric bundle $\mathcal{G}$ can be thought of as a choice of representative g of the conformal class $[g]$ if we identify

$$(x, t) \in U_i \times \mathbb{R}_+ \quad \text{with} \quad (x, t^2g_x) \in \mathcal{G}. \quad (6.32)$$

Given $g \in [g]$, for any $p \in \mathcal{P}_W$, by means of the corresponding $(x, h) \in \mathcal{G}$ one can define a symmetric tensor

¹¹Here we have required that $a(x)$ be independent of t in order to make the Weyl-ambient metric homogeneous of degree 2 with respect to t .

¹²Note that in [44], the metric bundle $\mathcal{G}$ itself is treated as the principal $\mathbb{R}_+$ -bundle through an isomorphism. Here we introduced the Weyl structure $\mathcal{P}_W$ and distinguish it from $\mathcal{G}$ in order to emphasize that a conformal class of metrics furnishes a representation of the group $\mathbb{R}_+$ with $w = -2$.

$\mathbf{g}_0$ of type $(0, 2)$ called the *tautological tensor* that acts on vector fields $\underline{w}_1, \underline{w}_2 \in T_p \mathcal{P}_W$ as follows:

$$\mathbf{g}_0(\underline{w}_1, \underline{w}_2) \equiv h(\pi_* \underline{w}_1, \pi_* \underline{w}_2), \quad (6.33)$$

which can be expressed as $\mathbf{g}_0 = t^2 \pi^* g$ under the identification in (6.32).

If we pick another representative $g'_x = \mathcal{B}(x)^{-2} g_x$ of the conformal class $[g]$, following the identification in (6.32), we obtain another isomorphism between $\mathcal{P}_W$ and $\mathcal{G}$ by identifying

$$(x, t') \in U_i \times \mathbb{R}_+ \quad \text{with} \quad (x, t'^2 g'_x) \in \mathcal{G}. \quad (6.34)$$

It is easy to see that the two isomorphisms are related by setting $t' = \mathcal{B}(x)t$. To preserve the horizontal subspace on $\mathcal{P}_W$, from (6.29) we can see that $a'_i(x)$ satisfies

$$a'_i(x) = a_i(x) - \partial_i \ln \mathcal{B}(x). \quad (6.35)$$

In the present circumstances, it is natural to replace the notion of conformal class $[g]$ by the *Weyl class* $[g, a]$, with the property

$$\forall (g, a), (g', a') \in [g, a], \quad \exists \mathcal{B}(x) \text{ such that } (g'_x, a'_x) = (\mathcal{B}(x)^{-2} g_x, a_x - d \ln \mathcal{B}(x)), \quad (6.36)$$

where d is the exterior derivative on $\mathcal{M}$.

Before we proceed to define the Weyl-ambient space based on the Weyl structure $\mathcal{P}_W$, we would like to make a few remarks. Recall that for the Weyl-ambient metric (6.17), the coordinates t and x^i parametrize a codimension-1 null hypersurface $\mathcal{N}$ located at $\rho = 0$. One can see that this surface is exactly a Weyl structure. The degenerate “induced metric” of $\tilde{g}$ on $\mathcal{N}$ is the tautological tensor, the induced metric $\gamma^{(0)}$ on $\mathcal{M}$ in the expansion (6.20) is a representative g in the conformal class, and the Weyl connection $a_i^{(0)}(x)$ on $\mathcal{M}$ in the expansion (6.21) is the $a_i(x)$ in (6.30). Thus, the Weyl class $[\gamma^{(0)}, a^{(0)}]$ corresponds to $[g, a]$ in this section, and $(\mathcal{M}, [g, a])$ defines a Weyl manifold. We will discuss more details of the role of the Weyl connection and the horizontal subspace it defines in Theorem 6.1 below. It is noteworthy that the projector in (6.30), which defines the Weyl connection on $\mathcal{P}_W$, is a special case of the construction presented in [172] with restricted diffeomorphisms.

Now we will define a Weyl-ambient space for a Weyl manifold generalizing the definition of a Fefferman-Graham ambient space for a conformal manifold introduced in [44]. Consider a $(d+2)$ -dimensional space $\tilde{\mathcal{M}}$ which looks at least locally like $\mathcal{P}_W \times \mathbb{R}$ where each point can be labeled by (p, ρ) with $\rho \in \mathbb{R}$. The inclusion map $\iota : \mathcal{P}_W \rightarrow \tilde{\mathcal{M}}$ is defined such that $p \mapsto (p, 0)$. By letting the map δ_s act only on $p \in \mathcal{P}_W$, we can extend δ_s to a map on $\tilde{\mathcal{M}}$, which commutes with ι . The vector field $\underline{T}$ which generates the Weyl group action is extended to a vector field $\underline{\mathcal{T}} = \iota_* \underline{T} = t \partial_t$ on $\tilde{\mathcal{M}}$.

Definition 6.3. Suppose $\mathcal{M}$ is a d -dimensional manifold equipped with a Weyl class $[g, a]$, and $\mathcal{P}_W$ is a Weyl structure over $\mathcal{M}$. A pseudo-Riemannian space $(\tilde{\mathcal{M}}, \tilde{g})$ is called the *Weyl-ambient space* for $(\mathcal{M}, [g, a])$ if

(4.1) $\tilde{\mathcal{M}}$ is a dilatation-invariant open neighborhood of $\mathcal{P}_W \times \{0\}$ in $\mathcal{P}_W \times \mathbb{R}$, and the pullback $\iota^* \tilde{g}$ is the tautological tensor $\mathbf{g}_0$ defined above;

(4.2) $\tilde{g}$ is a smooth metric on $\tilde{\mathcal{M}}$ of signature $(p+1, q+1)$, which is homogeneous of degree 2 on $\tilde{\mathcal{M}}$, i.e.,

$$\delta_s^* \tilde{g} = s^2 \tilde{g}, \forall s \in \mathbb{R}_+;$$

(4.3) $Ric(\tilde{g})$ vanishes to infinite order at every point of $\mathcal{P}_W \times \{0\}$.

Without condition (4.3), $(\tilde{\mathcal{M}}, \tilde{g})$ is called a *Weyl pre-ambient space* for $(M, [g, a])$. Note that the condition (3) in [44] is presented differently when d is even and odd, and $Ric(\tilde{g})$ has an obstruction in the order $O(\rho^{d/2-1})$ for even d . Here we take the dimension to be a continuous complex variable [43] as we explained in Chapter 5, and so the Ricci-flatness condition always holds to infinite order. The obstruction at even dimension will be manifested by the pole of the expansion of $\tilde{g}$ at even d , which is identified as the extended Weyl-obstruction tensor. For more details see Section 5.3.

Now we introduce the final ingredient in our Weyl-ambient construction—the *Weyl-normal form*, which is a generalization of the *normal form* defined in [44].

Definition 6.4. A Weyl pre-ambient space $(\tilde{M}, \tilde{g})$ for $(M, [g, a])$ is said to be in *Weyl-normal form* with acceleration $\underline{\mathcal{A}}$ if

(5.1) For each fixed $p \in \mathcal{P}_W$, the set of $\rho \in \mathbb{R}$ such that $(p, \rho) \in \tilde{M}$ is an open interval $I_p \in \mathbb{R}$ containing 0.

(5.2) For each $p \in \mathcal{P}_W$, the parametrized curve $C_p : I_p \rightarrow \tilde{\mathcal{M}}, \rho \mapsto (p, \rho)$ has a tangent vector $\underline{\mathcal{U}}$, whose acceleration $\underline{\mathcal{A}} \equiv \tilde{\nabla}_{\underline{\mathcal{U}}} \underline{\mathcal{U}}$ satisfies $\tilde{g}(\underline{\mathcal{T}}, \underline{\mathcal{A}}) = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of $(\tilde{\mathcal{M}}, \tilde{g})$.

(5.3) Let (t, x, ρ) represent a point in $\mathbb{R}_+ \times M \times \mathbb{R} \simeq \mathcal{P}_W \times \mathbb{R}$ under the local trivialization induced by g . Then, at each point $(t, x, 0) \in \mathcal{P}_W \times \{0\}$, the metric $\tilde{g}$ takes the form

$$\tilde{g}|_{\rho=0} = \mathbf{g}_0 + 2t^2(t^{-1}dt + a_i(x)dx^i)d\rho, \quad (6.37)$$

where $\mathbf{g}_0$ is the tautological symmetric tensor defined in (6.33).

Definition 6.4 is engineered for the purpose of generating the Weyl-ambient metric from the “initial surface” at $\rho = 0$. At $\rho = 0$, the Weyl-ambient metric we have seen in (6.17) has the form (6.37), which motivates condition (5.3). Since $\underline{\mathcal{T}} = t\partial_t$ everywhere in $\tilde{\mathcal{M}}$, condition (5.2) implies that the covector $\underline{\mathcal{A}}$ of the acceleration does not have a t -component. Furthermore, one can also parametrize the accelerated curve C_p such that $\tilde{g}(\underline{\mathcal{A}}, \underline{\mathcal{U}}) = 0$, and let $\underline{\mathcal{A}}$ have no ρ -component either.¹³ We will assume that ρ is such a parametrization. Note that in the special case where $\underline{\mathcal{A}} = 0$, the ρ -coordinate lines are geodesics, and condition (5.2) goes back to that of normal form in [44], while condition (5.3) will still be different as long as $a_i(x)$ are nonvanishing. The acceleration $\underline{\mathcal{A}}$ encodes all the higher modes $a_i^{(k \geq 1)}(x)$ in the expansion (6.21) of $a_i(x, \rho)$, as we will see in Lemma 6.1. In fact, if both $a_i(x)$ and $\underline{\mathcal{A}}$ are zero, the mode $a_i(x, \rho)$ in (6.17) vanishes.

The following Theorem is a generalization of Proposition 2.8 in [44].

Theorem 6.1. Let $(\mathcal{M}, [g, a])$ be a Weyl manifold, with (g, a) a representative of the Weyl class. Let $\mathcal{P}_W$ be the Weyl structure over $\mathcal{M}$, and $(\tilde{\mathcal{M}}, \tilde{g})$ be a Weyl pre-ambient space for $(\mathcal{M}, [g, a])$. Then, there exists a dilatation-invariant open set $\tilde{\mathcal{M}}' \subset \mathcal{P}_W \times \mathbb{R}$ containing $\mathcal{P}_W \times \{0\}$ on which there is a unique diffeomorphism $\phi : \tilde{\mathcal{M}}' \rightarrow \tilde{\mathcal{M}}$ commuting with dilatations with $\phi|_{\mathcal{P}_W \times \{0\}}$ being the identity map, such that the Weyl pre-ambient space $(\tilde{\mathcal{M}}', \phi^* \tilde{g})$ is in Weyl-normal form with acceleration $\underline{\mathcal{A}}'$.

¹³Suppose C_p has a parameter λ , then under a reparametrization $\lambda \rightarrow f(\lambda)$ we have $\underline{\mathcal{U}} \rightarrow f' \underline{\mathcal{U}}$, and the acceleration vector transforms $\underline{\mathcal{A}} \rightarrow f'^2 \underline{\mathcal{A}} + f' \underline{\mathcal{U}}(f) \underline{\mathcal{U}}$, and thus $\tilde{g}(\underline{\mathcal{A}}, \underline{\mathcal{U}})$ can always be set to zero for non-null $\underline{\mathcal{U}}$ by choosing an appropriate function f . For null $\underline{\mathcal{U}}$ the condition holds automatically.

This theorem indicates that given a representative pair (g, a) , any Weyl pre-ambient space can be put into Weyl-normal form by a diffeomorphism ϕ . $(\tilde{\mathcal{M}}, \tilde{g})$ and $(\tilde{\mathcal{M}}', \phi^* \tilde{g})$ are also said to be ambient-equivalent (see Definition 2.2 in [44] for the precise definition of ambient equivalence). For a proof of the theorem, see Appendix C.2.1.

Before we move on to the main result of this section, namely Theorem 6.2, let us introduce some useful notation. Given a local coordinate system $\{x^i\}$ ($i = 1, \dots, d$) on $\mathcal{M}$, the fibre coordinate t of $\mathcal{P}_W$ and the parameter ρ naturally defines an *ambient coordinate system* $\{t, x^i, \rho\}$ on $\tilde{\mathcal{M}}$. Later on, we will follow [44] and use $I, J, \dots = (0, i, \infty)$ to label the ambient coordinate indices, where 0 labels the t -component and ∞ labels the ρ -component. It is also convenient to interpret the notations $(0, i, \infty)$ as representing the components in a trivialization $\mathcal{P}_W \times \mathbb{R} \simeq \mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}$, even without specifying a choice of coordinates on $\mathcal{M}$.

We will now present Theorem 6.2, which is a natural generalization of Theorem 2.9 of [44], based on our definition of Weyl-normal form. As a corollary of this theorem, we will show that for a Weyl-ambient space in Weyl-normal form, the Weyl-ambient metric (6.17) emerges from the initial surface uniquely under the Ricci-flatness condition. We emphasize again that we consider the dimension d of the manifold $\mathcal{M}$ formally as a complex parameter, and do not need to distinguish between even and odd dimensions.

Theorem 6.2. *Let $(\mathcal{M}, [g, a])$ be a Weyl manifold, and let (g, a) be a representative in the Weyl class.*

- (A) *There exists a Weyl-ambient space $(\tilde{\mathcal{M}}, \tilde{g})$ for $(\mathcal{M}, [g, a])$ which is in Weyl-normal form with acceleration $\underline{\mathcal{A}}$.*
- (B) *Suppose that $(\tilde{\mathcal{M}}_1, \tilde{g}_1)$ and $(\tilde{\mathcal{M}}_2, \tilde{g}_2)$ are two Weyl-ambient spaces for $(\mathcal{M}, [g, a])$, both of which are in Weyl-normal form with acceleration $\underline{\mathcal{A}}$. Then $\tilde{g}_1 - \tilde{g}_2$ vanishes to infinite order at every point of $\mathcal{P}_W \times \{0\}$.*

The proof of Theorem 6.2 employs the following lemma.

Lemma 6.1. *Let $(\tilde{\mathcal{M}}, \tilde{g})$ be a Weyl pre-ambient space for $(\mathcal{M}, [g, a])$. Suppose for each $p \in \mathcal{P}_W$, the set of all $\rho \in \mathbb{R}$ such that $(p, \rho) \in \tilde{\mathcal{M}}$ is an open interval I_p containing 0. Let g be a metric in the representative (g, a) of the Weyl class, which provides a local trivialization $\mathcal{P}_W \times \mathbb{R} \simeq \mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}$. Then $(\tilde{\mathcal{M}}, \tilde{g})$ is in Weyl-normal form with acceleration $\underline{\mathcal{A}}$ if and only if one has on $\tilde{\mathcal{M}}$:*

$$\tilde{g}_{0\infty} = t, \quad \tilde{g}_{i\infty} = t^2 a_i(x, \rho), \quad \tilde{g}_{\infty\infty} = 0, \quad (6.38)$$

where $a_i(x, \rho) \equiv a_i(x) + t^{-2} \int_0^\rho \mathcal{A}_i(t, x, \rho') d\rho'$.

Proof. Suppose $\tilde{g}$ satisfies (6.38), then it follows from the condition $\iota^* \tilde{g} = \mathbf{g}_0$ for the pre-ambient space that $\tilde{g}|_{\rho=0}$ must have the form (6.37). Thus, all we have to prove is that for $\tilde{g}$ satisfying (6.37) at $\rho = 0$, the condition that the ρ -coordinate lines have acceleration $\underline{\mathcal{A}}$ with $\tilde{g}(\underline{\mathcal{T}}, \underline{\mathcal{A}}) = 0$ is equivalent to (6.38). The fact that the ρ -coordinate lines have an acceleration $\underline{\mathcal{A}}$ implies

$$\tilde{\Gamma}_{\infty\infty I} = \mathcal{A}_I, \quad (6.39)$$

where $\tilde{\Gamma}_{IJK} \equiv \tilde{g}_{KL} \tilde{\Gamma}_{IJ}^L$. The condition $\tilde{g}(\underline{\mathcal{T}}, \underline{\mathcal{A}}) = 0$ leads to $\mathcal{A}_0 = 0$. As we have mentioned, one can also parametrize the curve $C_p : I_p \rightarrow \tilde{\mathcal{M}}$ such that $\tilde{g}(\underline{\mathcal{U}}, \underline{\mathcal{A}}) = 0$, then we also have $\mathcal{A}_\infty = 0$, and thus

$\mathcal{A}_I = (\mathcal{A}_0, \mathcal{A}_i, \mathcal{A}_\infty) = (0, t^2 \varphi_i(x, \rho), 0)$. The functions $\varphi_i(x, \rho)$ are considered as external input and cannot be determined from the initial conditions. The factor t^2 is derived from the homogeneity property of $\tilde{g}$ and (6.39). If we set $I = \infty$ in (6.39) we get

$$\tilde{\Gamma}_{\infty\infty\infty} = \mathcal{A}_\infty = 0 \implies \partial_\rho g_{\infty\infty} = 0 \implies g_{\infty\infty} = 0, \quad (6.40)$$

where in the last step we used the initial condition $g_{\infty\infty}|_{\rho=0} = 0$. Similarly, setting $I = 0$ in (6.39) we find

$$\partial_\infty g_{\infty 0} = 0 \implies g_{\infty 0} = t, \quad (6.41)$$

where we used the initial condition $g_{0\infty}|_{\rho=0} = t$. Finally, setting $I = i$ yields

$$\partial_\rho g_{\infty i} = \mathcal{A}_i(t, \rho; x) \implies g_{\infty i} = t^2 a_i(x) + t^2 \int_0^\rho \varphi_i(\rho; x) d\rho \equiv t^2 a_i(\rho; x), \quad (6.42)$$

where we used the initial condition $\tilde{g}_{\infty i}|_{\rho=0} = t^2 a_i(x)$. $\square$

The main logic of the proof of Theorem 6.2 will follow part of Section 3 in [44]. To show part (A) of Theorem 6.2, namely the existence of the Weyl-ambient space $\tilde{\mathcal{M}}$ in Weyl-normal form, we need to show the following: for a Weyl manifold $(\mathcal{M}, [g, a])$, given a representative (g, a) of the Weyl class and $a_i(x, \rho)$ determined by $\underline{\mathcal{A}}$, there exists a metric $\tilde{g}$ on an open neighborhood $\tilde{\mathcal{M}}$ of $\mathcal{P}_W \times \{0\}$ with the following properties:

- (1) $\delta_s^* \tilde{g} = s^2 \tilde{g}$, $\forall s > 0$ (homogeneity property);
- (2) $\tilde{g} = t^2 g(x) + 2t^2(t^{-1}dt + a_i(x)dx^i)d\rho$ when $\rho = 0$;
- (3) $\tilde{g}_{0\infty} = t$, $\tilde{g}_{i\infty} = t^2 a_i(x, \rho)$, $\tilde{g}_{\infty\infty} = 0$;
- (4) $\tilde{Ric}(\tilde{g}) = 0$ to infinite order at $\rho = 0$.

The first property above is the homogeneity property which is still taken to be true for the Weyl-ambient metric. Property (3) is equivalent to condition (5.2) of Definition 6.4 due to Lemma 6.1, which indicates that $\tilde{g}_{I\infty}$ components are known, while the rest are now regarded as unknown functions. Property (2) can be considered as the initial data of these components at the initial surface at $\rho = 0$, while the Ricci-flatness property (4) is a system of partial differential equations that one can solve to find the metric components beyond the initial surface. We will show that this is a well defined initial value problem so that the unknown components of the Weyl-ambient metric can be uniquely determined in a series expansion in ρ , which will prove part (B) of Theorem 6.2. The complete proof can be found in Appendix C.2.2.

As an important corollary, we now show in Theorem 6.3 that the metric $\tilde{g}$ determined from Theorem 6.2 has exactly the form of the Weyl-ambient metric (6.17). First we need the following lemma.

Lemma 6.2. *Suppose a metric $\tilde{g}$ has the following form:*

$$\tilde{g}_{IJ} = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_{ij}(x, \rho) & t^2 a_j(x, \rho) \\ t & t^2 a_i(x, \rho) & 0 \end{pmatrix}. \quad (6.43)$$

Then the Ricci curvature of $\tilde{g}$ satisfies $\tilde{R}_{0I} = 0$.

Proof. For $\tilde{g}$ of the form (6.43), we can write the inverse metric as

$$\tilde{g}^{IJ} = \frac{1}{1+2\rho a^2} \begin{pmatrix} a^2 & -t^{-1}a^j & t^{-1} \\ -t^{-1}a^i & t^{-2}(1+2\rho a^2)g^{ij} - 2t^{-2}\rho a^i a^j & 2t^{-2}\rho a^i \\ t^{-1} & 2t^{-2}\rho a^j & -t^{-2}2\rho \end{pmatrix}, \quad (6.44)$$

and the Christoffel symbols $\tilde{\Gamma}_{IJK} = \tilde{g}_{KL}\tilde{\Gamma}^L_{IJ}$ are given by

$$\begin{aligned} \tilde{\Gamma}_{IJO} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -tg_{ij} & -ta_i \\ 1 & -ta_j & 0 \end{pmatrix}, \quad \tilde{\Gamma}_{IJ\infty} = \begin{pmatrix} 0 & ta_j & 0 \\ ta_i & -t^2(\frac{1}{2}\partial_\rho g_{ij} - \partial_{(i}a_{j)}) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \tilde{\Gamma}_{IJK} &= \begin{pmatrix} 0 & tg_{jk} & ta_k \\ tg_{ik} & t^2\Gamma_{ijk} & \frac{t^2}{2}(\partial_\rho g_{ik} + F_{ik}) \\ ta_k & \frac{t^2}{2}(\partial_\rho g_{jk} + F_{jk}) & t^2\partial_\rho a_k \end{pmatrix}, \end{aligned} \quad (6.45)$$

where $\Gamma_{ijk} = g_{kl}\Gamma^l_{ij}$ are the Christoffel symbols of $g_{ij}(x, \rho)$, and $F_{jk} = \partial_j a_k - \partial_k a_j$. Substituting (6.44) and (6.45) into (C.20) we can compute $\tilde{R}_{0I}$ explicitly and find that $\tilde{R}_{0I} = 0$. $\square$

Theorem 6.3. *Suppose $(\mathcal{M}, [g, a])$ is a Weyl manifold. Let $(\tilde{\mathcal{M}}, \tilde{g})$ be the unique ambient space for $(\mathcal{M}, [g, a])$ which is in Weyl-normal form with acceleration $\underline{A}$. Then, for any representative (g, a) , the uniquely determined metric $\tilde{g}$ has the following form*

$$\tilde{g} = 2\rho dt^2 + 2td\rho \left(\frac{dt}{t} + a_i(x, \rho) dx^i \right) + t^2 g_{ij}(x, \rho) dx^i dx^j, \quad (6.46)$$

where $a_i(x, \rho) \equiv a_i(x) + t^{-2} \int_0^\rho \mathcal{A}_i(t, x, \rho') dt$. This metric is exactly the Weyl-ambient metric introduced in (6.17).

Proof. Based on Theorem 6.2, all we have to prove is that $\tilde{g}_{00} = 2\rho$ and $\tilde{g}_{0i} = 0$ to all orders. Let $\tilde{g}^{(m)}$ be the m^{th} order of $\tilde{g}$, and let $\tilde{g}^{[k]}$ represent $\tilde{g}$ with all the orders higher than $O(\rho^k)$ in the ρ -expansion excluded, i.e. $\tilde{g} = \tilde{g}^{[k]} + O(\rho^{k+1})$. From (C.21) we find to the first order that $\tilde{g}_{00}^{[1]} = 2\rho$ and $\tilde{g}_{0i}^{[1]} = 0$. Assuming that $\tilde{g}_{00}^{[m-1]} = 2\rho$ and $\tilde{g}_{0i}^{[m-1]} = 0$, it follows from Lemma 6.2 that $\tilde{R}_{00}^{[m-1]} = \tilde{R}_{0i}^{[m-1]} = 0$. Then, from (C.26) we obtain that $\phi_{00} = \phi_{0i} = 0$, and hence $\tilde{g}_{00}^{(m)} = \tilde{g}_{0i}^{(m)} = 0$ [see (C.23)], $\forall m > 1$. Therefore, by induction we can deduce to infinite order that $\tilde{g}_{00} = 2\rho$ and $\tilde{g}_{0i} = 0$, which completes the proof. $\square$

6.3.2 Weyl-Ambient metric and Weyl-Obstruction tensors

A very useful property of the ambient metric introduced in [160] in the context of conformal geometry is the ability to construct conformal-covariant tensors from the ambient Riemann tensor, including the (extended) obstruction tensors, which were generalized to (extended) Weyl-obstruction tensors in Section 5.3. The Weyl-obstruction tensors on a d -dimensional Weyl manifold $(M, [\gamma^{(0)}, a^{(0)}])$ were defined in Subsection 5.3.3 as the poles of the metric expansion of γ_{ij} in the AIAdS bulk, or equivalently through the Ricci flat Weyl-ambient metric in Subsection 6.2.3. However, as we will see in this subsection, they can also be defined in a more explicit way from the Weyl-ambient space $(\tilde{M}, \tilde{g})$.

In this section we will show that appropriate ambient tensors constructed from the Weyl-ambient Riemann

tensor on $\tilde{\mathcal{M}}$ behave as Weyl-covariant tensors on $\mathcal{M}$, through which Weyl-obstruction tensors can again be defined as a special case. Then we will show that the Weyl-obstruction tensors defined in this way agree with the Weyl-obstruction tensors we defined previously in Subsection 5.3.3 and Subsection 6.2.3.

We have proven in the last subsection that for any pair of (g, a) on $\mathcal{M}$, there exists a unique Weyl-ambient space $(\tilde{\mathcal{M}}, \tilde{g})$ for the Weyl manifold $(\mathcal{M}, [g, a])$ where $\tilde{g}$ has the form of (6.17). In Subsection 6.2.3 we saw that the ambient Weyl diffeomorphism

$$(t', x'^i, \rho') = (\mathcal{B}(x)t, x^i, \mathcal{B}^{-2}(x)\rho) \quad (6.47)$$

induces a Weyl transformation on $\mathcal{M}$. Therefore, it is reasonable to expect that to find a Weyl-covariant tensor on $(\mathcal{M}, [g, a])$, we can find an ambient tensor which is covariant under an ambient Weyl diffeomorphism, and its pullback on M will be Weyl covariant. The first main result of this section is Proposition 6.1. This provides the Weyl transformations of tensors constructed from covariant derivatives of the Riemann tensor of a Weyl-ambient metric, from which we can see which tensors are Weyl covariant when pulled back to $\mathcal{M}$.

Proposition 6.1. *Suppose $(\tilde{\mathcal{M}}, \tilde{g})$ is the Weyl-ambient space for $(\mathcal{M}, [g, a])$, and let (g, a) and (g', a') be two representatives of $[g, a]$, with $g'_{ij} = \mathcal{B}^{-2}g_{ij}$ and $a'_i = a_i - \partial_i \ln \mathcal{B}$. Let $IJKLM_1 \dots M_r$ be a list of indices, s_0 of which are 0, s_M of which are x^i on M , and s_∞ of which are ∞ . Then, the following components of the covariant derivatives of the Riemann tensor $\tilde{R}_{ABCD}$ of $\tilde{g}$ in the trivialization defined by g satisfy the transformation law*

$$\tilde{R}'_{IJKL;M_1 \dots M_r}|_{\rho'=0, t'=1} = \mathcal{B}(x)^{2(s_\infty-1)} \tilde{R}_{ABCD;F_1 \dots F_r}|_{\rho=0, t=1} p^A_I \dots p^{F_r}_{M_r} \quad (6.48)$$

under an ambient Weyl diffeomorphism (6.47), where p^A_I is the matrix

$$p^I_J = \begin{matrix} & 0 & j & \infty \\ \begin{matrix} 0 \\ i \\ \infty \end{matrix} & \begin{pmatrix} 1 & \Upsilon_j & 0 \\ 0 & \delta^i_j & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}, \quad (6.49)$$

and $\Upsilon(x) \equiv -\ln \mathcal{B}(x)$, $\Upsilon_j \equiv \partial_j \Upsilon(x)$. $\tilde{R}'_{IJKL;M_1 \dots M_r}$ denotes covariant derivatives of the Riemann tensor of $\tilde{g}$ in the coordinates $X'^I = (t', x'^i, \rho')$ given by the trivialization provided by g' .

Proof. The logic for the proof of this Proposition follows the proof of Proposition 6.5 in [44] closely. We start by observing that the ambient Weyl diffeomorphism $\psi : (t', x'^i, \rho') \mapsto (t, x^i, \rho)$ has the following properties:

$$\psi(t', x'^i, 0) = (t' e^{\Upsilon(x)}, x'^i, 0), \quad \psi^* \tilde{g}|_{\rho'=0} = 2t' d\rho' dt' + t'^2 g'_{ij} dx'^i dx'^j + 2t'^2 a'_i dx'^i d\rho', \quad (6.50)$$

where the Weyl-ambient metric $\tilde{g}$ has the form of (6.17), and $g'_{ij} = \mathcal{B}(x)^{-2}g_{ij}$, $a'_i = a_i + \Upsilon_i$. The Jacobian $(\psi)^A_I = \left(\frac{\partial X'^I}{\partial X^A} \right)$ of this diffeomorphism is

$$(\psi)^I_J \equiv \begin{pmatrix} \psi^t_{t'} & \psi^t_{j'} & \psi^t_{\rho'} \\ \psi^i_{t'} & \psi^i_{j'} & \psi^i_{\rho'} \\ \psi^\rho_{t'} & \psi^\rho_{j'} & \psi^\rho_{\rho'} \end{pmatrix} = \begin{pmatrix} e^{\Upsilon(x)} & t' e^{\Upsilon(x)} \Upsilon_j & 0 \\ 0 & \delta^i_j & 0 \\ 0 & -2\rho' e^{-2\Upsilon(x)} \Upsilon_j & e^{-2\Upsilon(x)} \end{pmatrix}, \quad (6.51)$$

where $\Upsilon(x) \equiv -\ln \mathcal{B}(x)$ and $\Upsilon_i \equiv \partial_i \Upsilon(x)$. At $\rho' = 0$ the Jacobian matrix (6.51) reads

$$(\psi)^A_I|_{\rho'=0} = \begin{pmatrix} e^{\Upsilon(x)} & t' e^{\Upsilon(x)} \Upsilon_j & 0 \\ 0 & \delta^i_j & 0 \\ 0 & 0 & e^{-2\Upsilon(x)} \end{pmatrix}. \quad (6.52)$$

The above matrix can be written as the following matrix product:

$$(\psi)^A_I|_{\rho'=0} = d_1 p d_2, \quad (6.53)$$

with

$$p^I_J = \begin{pmatrix} 1 & \Upsilon_j & 0 \\ 0 & \delta^i_j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d_1 = \begin{pmatrix} t' e^{\Upsilon(x)} & 0 & 0 \\ 0 & \delta^i_j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} t'^{-1} & 0 & 0 \\ 0 & \delta^i_j & 0 \\ 0 & 0 & e^{-2\Upsilon(x)} \end{pmatrix}. \quad (6.54)$$

Since the Weyl-ambient metric is homogeneous of degree 2 under dilatations $\delta_s^* \tilde{g} = s^2 \tilde{g}$, it follows that the left-hand side of (6.48) satisfies

$$\tilde{R}'_{IJKL;M_1 \dots M_r}|_{\rho'=0, t'=1} = \mathcal{B}(x)^{s_0-2} \tilde{R}'_{IJKL;M_1 \dots M_r}|_{\rho'=0, t'=\mathcal{B}(x)}. \quad (6.55)$$

Under the ambient Weyl diffeomorphism (6.47) the covariant derivatives of the ambient Riemann curvature components transform tensorially as

$$\tilde{R}'_{IJKL;M_1 \dots M_r}|_{\rho', t'} = \tilde{R}_{ABCD;F_1 \dots F_r}|_{\rho, t} (\psi)^A_I \dots (\psi)^{F_r}_{M_r}. \quad (6.56)$$

Evaluating both sides of (6.56) at $\rho' = 0$, $t' = e^{-\Upsilon(x)}$ and using (6.54) we have

$$\tilde{R}'_{IJKL;M_1 \dots M_r}|_{\rho'=0, t'=e^{-\Upsilon(x)}} = \mathcal{B}(x)^{2s_0-s_0} \tilde{R}_{ABCD;F_1 \dots F_r}|_{\rho=0, t=1} p^A_I \dots p^{F_r}_{M_r}. \quad (6.57)$$

Plugging (6.55) into (6.57), we obtain (6.48). $\square$

Theorem 6.1 helps us to find Weyl-covariant tensors on $(\mathcal{M}, [g, a])$. First let us look at the case without derivatives. In the coordinate basis, the nonvanishing components of the Weyl-ambient Riemann tensor $\tilde{R}_{IJKL}$ are $\tilde{R}_{\infty jk\infty}$, $\tilde{R}_{\infty jkl}$ and $\tilde{R}_{ijkl}$. Evaluating at $\rho = 0$ and $t = 1$, they are

$$\tilde{R}_{\infty jk\infty}|_{\rho=0, t=1} = \hat{\Omega}_{jk}^{(1)}, \quad \tilde{R}_{\infty jkl}|_{\rho=0, t=1} = \hat{C}_{jkl}, \quad \tilde{R}_{ijkl}|_{\rho=0, t=1} = \hat{W}_{ijkl}. \quad (6.58)$$

Here $\hat{C}_{jkl}$ and $\hat{W}_{ijkl}$ are the Weyl-Cotton tensor and the Weyl curvature tensor on $\mathcal{M}$, respectively, and $\hat{\Omega}_{jk}^{(1)}$ for now simply denotes the tensor defined in (6.25). Then, applying (6.48) we get $\hat{C}'_{jkl} = \hat{C}_{jkl}$ and $\hat{W}'_{ijkl} = \mathcal{B}^{-2}(x) \hat{W}_{ijkl}$ under Weyl transformation as expected, we can also read off from (6.48) that the Weyl weight of $\hat{\Omega}_{jk}^{(1)}$ is +2.

Now we will define Weyl-obstruction tensors as the derivatives of $\tilde{R}_{\infty jk\infty}$.

Definition 6.5. Suppose k is a positive integer. The k^{th} extended Weyl-obstruction tensor $\hat{\Omega}_{ij}^{(k)}$ is defined as

$$\hat{\Omega}_{ij}^{(k)} = \tilde{R}_{\infty ij \infty; \underbrace{\infty \dots \infty}_{k-1}}|_{\rho=0, t=1}. \quad (6.59)$$

For $k = 1$ we can see from (6.58) that $\tilde{R}_{\infty jk \infty}|_{\rho=0, t=1} = \hat{\Omega}_{jk}^{(1)}$ is indeed the first extended Weyl-obstruction tensor.

From the symmetry of the Weyl-ambient Riemann tensor we can immediately see that $\hat{\Omega}_{ij}^{(k)}$ given by Definition 6.5 is symmetric. From the Ricci-flatness condition $\tilde{Ric}(\tilde{g}) = 0$ and the fact that $\tilde{R}_{0IJK} = 0$, we can see that $\hat{\Omega}_{ij}^{(k)}$ is traceless. Now we will show another important property of the extended Weyl-obstruction tensors defined in this way, namely that they are Weyl covariant.

Lemma 6.3. *The components of the Riemann tensor of the Weyl-ambient metric $\tilde{g}$ satisfy*

$$\tilde{R}_{IJK0; M_1 \dots M_r} = -\frac{1}{t} \sum_{s=1}^r \tilde{R}_{IJKM_s; M_1 \dots \hat{M}_s \dots M_r}, \quad (6.60)$$

where $\hat{M}_s$ means to remove M_s from the indices.

Proof. Computing the Christoffel symbols of the Weyl-ambient metric $\tilde{g}$ in (6.17), one finds $\tilde{\Gamma}_{j0}^i = \frac{1}{t} \delta_j^i$ and $\tilde{\Gamma}_{\infty 0}^\infty = \frac{1}{t}$. Differentiating $\mathcal{T} = t \partial_t$ we have $\mathcal{T}^I{}_{;J} = \delta^I_J$ and $\mathcal{T}^I{}_{;JK} = 0$, then

$$\begin{aligned} (\mathcal{T}^L \tilde{R}_{IJKL})_{; M_1 \dots M_r} &= \tilde{R}_{IJKM_1; M_2 \dots M_r} + (\mathcal{T}^L \tilde{R}_{IJKL, M_1})_{; M_2 \dots M_r} \\ &= \tilde{R}_{IJKM_1; M_2 \dots M_r} + \tilde{R}_{IJKM_2; M_2 \dots M_r} + (\mathcal{T}^L \tilde{R}_{IJKL, M_1 M_2})_{; M_3 \dots M_r} \\ &= \dots \\ &= \tilde{R}_{IJKM_1; M_2 \dots M_r} + \dots + \tilde{R}_{IJKM_r; M_1 \dots M_{r-1}} + \mathcal{T}^L \tilde{R}_{IJKL; M_1 \dots M_r}. \end{aligned}$$

The left-hand side of this equation vanishes since $R_{IJK0} = 0$, and thus the above equation leads to (6.60). $\square$

Proposition 6.2. *The extended Weyl-obstruction tensor $\hat{\Omega}_{ij}^{(k)}$ defined in (6.59) is a Weyl-covariant tensor with Weyl weight $2k$.*

Proof. According to Proposition 6.1, if we choose $(IJKL; M_1 \dots M_r) = (\infty, i, j, \infty; \underbrace{\infty \dots \infty}_{(k-1)})$, then $s_\infty = k+1$ and under a Weyl transformation we have

$$\tilde{R}'_{\infty ij \infty; \underbrace{\infty \dots \infty}_{k-1}}|_{\rho'=0, t'=1} = \mathcal{B}(x)^{2k} (\tilde{R}_{\infty ij \infty; \underbrace{\infty \dots \infty}_{k-1}} + \Upsilon_i \tilde{R}_{\infty 0j \infty; \underbrace{\infty \dots \infty}_{k-1}} + \Upsilon_j \tilde{R}_{\infty i0 \infty; \underbrace{\infty \dots \infty}_{k-1}})|_{\rho'=0, t'=1}. \quad (6.61)$$

It follows from Lemma 6.3 that

$$R_{\infty i0 \infty; \underbrace{\infty \dots \infty}_{k-1}} = \frac{k-1}{t} R_{\infty i \infty \infty; \underbrace{\infty \dots \infty}_{k-2}} = 0. \quad (6.62)$$

Therefore, we obtain from (6.61) that $\hat{\Omega}_{ij}^{(k)} = \mathcal{B}(x)^{2k} \hat{\Omega}_{ij}^{(k)}$ under a Weyl transformation, i.e. $\hat{\Omega}_{ij}^{(k)}$ is a Weyl-covariant tensor with Weyl weight $2k$. $\square$

6.4 Discussion

In this chapter we have generalized the ambient construction for conformal manifolds to that for Weyl manifolds. Inspired by the WFG gauge for AIAdS [43], we introduced the Weyl-ambient metric $\tilde{g}$ in (6.17). The metric $\tilde{g}$ and the LC connection on $\tilde{\mathcal{M}}$ give rise to a Weyl class $[\gamma^{(0)}, a^{(0)}]$ on M , in which a representative includes an induced metric $\gamma_{ij}^{(0)}$ together with a Weyl connection $a_i^{(0)}$. The ambient Weyl diffeomorphisms on $\tilde{\mathcal{M}}$ act as Weyl transformations on the $\mathcal{M}$. This enhances the codimension-2 conformal geometry in the usual ambient construction to a Weyl geometry $(\mathcal{M}, [\gamma^{(0)}, a^{(0)}])$.

In Section 6.3 we constructed the $(d+2)$ -dimensional Weyl-ambient space from a d -dimensional Weyl manifold $(\mathcal{M}, [g, a])$. We first introduced a Weyl structure $\mathcal{P}_W$ on $\mathcal{M}$ together with a Weyl connection. We then generalized the definition of ambient spaces to Weyl-ambient spaces, and proved that any Weyl-ambient space can be put in Weyl-normal form by a diffeomorphism. Besides assigning the Weyl connection a_i on $\mathcal{P}_W$, the ρ -coordinate lines of a Weyl-ambient space in Weyl-normal form are not required to be geodesics but can acquire an acceleration $\underline{\mathcal{A}}$. By taking the Weyl structure as an initial surface, we have shown that there exists a unique Weyl-ambient space in Weyl-normal form for any given Weyl manifold provided the data $(g_{ij}, a_i, \underline{\mathcal{A}})$ is given. The metric generated order by order from the initial value problem is exactly the $\tilde{g}$ we introduced in (6.17) from the top-down approach, where g_{ij} corresponds to $\gamma_{ij}^{(0)}$, and $(a_i, \underline{\mathcal{A}})$ corresponds to $a_i(x, \rho)$.

Based on the Weyl-ambient construction, we investigated Weyl-covariant quantities induced by the ambient tensors. As an important example, the extended Weyl-obstruction tensor $\hat{\Omega}_{ij}^{(k)}$ is defined through covariant derivatives of the ambient Riemann tensor. Compared with the extended obstruction tensor $\Omega_{ij}^{(k-1)}$, whose residue is only conformally covariant in $d = 2k$, the extended Weyl-obstruction tensor $\hat{\Omega}_{ij}^{(k-1)}$ is Weyl covariant in any dimension.

There are many possible extensions of this work. The Weyl-ambient space induces the $\text{Diff}(M) \ltimes \text{Weyl}$ symmetry on the codimension-2 manifold M , which can be regarded as an asymptotic corner symmetry [173], [174]. The algebra of corner symmetries and their Noether charges have been studied in [173], [175] (see also [176]), it is possible to apply the results therein to the Weyl-ambient space and study the asymptotic corner symmetries of the Weyl-ambient space. Moreover, since the surface $\mathcal{N}$ at $\rho = 0$ of the Weyl-ambient space is null, there is an induced Carroll structure [172], [177]. This is evident from the fact that the ambient Weyl diffeomorphism acts on the null surface as (a special case of) a Carrollian diffeomorphism. It would be interesting to see if the results of [178] for a Ricci-flat space with a Killing vector can be emulated for the Weyl-ambient space, which is Ricci-flat but possesses a conformal Killing vector $\underline{T} = t\partial_t$.

One also expects intriguing holographic applications of the Weyl-ambient construction, for example in the context of celestial holography [179]–[181] and codimension-2 holography [182], [183]. In particular, the $\text{Diff}(M) \ltimes \text{Weyl}$ symmetry on M corresponds to the Weyl-BMS symmetry on $\tilde{\mathcal{M}}$ [184] (with supertranslations turned off). Therefore, we expect that the Weyl-ambient construction will provide a new arena for realizing the holographic principle. The Weyl-ambient metric construction is part of a bigger program of introducing the Weyl connection back into physics. Viewed as an ordinary gauge symmetry, the Weyl symmetry can provide an organizing principle for constructing effective field theories (e.g., for conformal hydrodynamics). Weyl manifolds would be the proper geometric setup for such future investigations. More recently, the ambient construction was used to study correlators of CFTs on general curved backgrounds [185], [186]. We hope the Weyl-ambient geometries can be utilized in similar contexts.

Chapter 7

Conformal graphs as twisted partition functions

Work in this chapter is based on “Conformal graphs as twisted partition functions” [187]. The paper was authored by Manthos Karydas (UIUC), Songyuan Li, Anastasios C. Petkou affiliated with Aristotle U., Thessaloniki and Matthieu Villate (Aristotle U., Thessaloniki and Ecole Polytechnique, CPHT).

In this chapter we present some of the results originally published in [187] related to the recently discovered connection [45] between thermal partition functions and conformal ladder diagrams. One of the key results of [45] was that thermal expectation values of a $U(1)$ charge operator for a free complex massive scalar field with complex chemical potential are equal to the celebrated conformal diagrams calculated in [49], [50]. A key new result of [187] is the connection between twisted thermal partition functions of the same theory to L -loop conformal ladder graphs in two spacetime dimensions introduced in [51]. These results were explained to us in private communication by Enrico Olivucci. The origin of the correspondence is still unknown. However we hope this new perspective on graphs can provide novel new ways of computation. As an instructive application we will see how to reproduce the resummation of infinite number of loop diagrams [52] using the language of thermal partition functions.

7.1 Free Boson Gas in $d = 2L + 1$ dimensions

The thermal partition function of a complex scalar field with imaginary chemical potential¹

$$Z_d(\beta; m, \mu) = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} e^{-S_E} \quad , \quad S_E = \int_0^\beta d\tau \int d^{d-1}\vec{x} \left[|(\partial_\tau - i\mu)\phi|^2 + |\vec{\partial}\phi|^2 + m^2\phi^2 \right] . \quad (7.1)$$

Since the theory is gaussian the partition function can be calculated exactly [45], [188]. We will derive these results from a slightly different perspective that is more appropriate for our purposes. First consider the case

¹We set $c = \hbar = 1$ in this chapter.

$d = 1$ in Eq. (7.1). The real time action of Eq. (7.1) but with real chemical potential² is

$$S = \int dt \left(|\dot{\phi}|^2 + i\mu \left((\partial_t \phi) \bar{\phi} - \phi (\partial_t \bar{\phi}) \right) \right) + \mu^2 |\phi|^2 - m^2 |\phi|^2. \quad (7.2)$$

The canonical momenta are

$$\pi_\phi \equiv \frac{\partial \mathcal{L}}{\partial (\partial_t \phi)} = \partial_t \bar{\phi} + i\mu \bar{\phi} \quad , \quad \pi_{\bar{\phi}} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_t \bar{\phi})} = \partial_t \phi - i\mu \phi, \quad (7.3)$$

and for the Hamiltonian one finds

$$H = \pi_\phi \pi_{\bar{\phi}} + i\mu (\pi_\phi \phi - \pi_{\bar{\phi}} \bar{\phi}) + m^2 |\phi|^2. \quad (7.4)$$

We go quantum by promoting the Hamiltonian Eq. (7.4) to an operator

$$H = \hat{\pi}^\dagger \hat{\phi} + m^2 \hat{\phi}^\dagger \hat{\phi} + i\mu \left(\hat{\pi} \hat{\phi} - \hat{\phi}^\dagger \hat{\pi}^\dagger \right), \quad (7.5)$$

with canonical commutation relations $[\hat{\phi}, \hat{\pi}] = i$, $[\hat{\phi}^\dagger, \hat{\pi}^\dagger] = i$. If we transform to

$$\begin{aligned} \hat{\phi} &= \frac{1}{\sqrt{2m}} (\hat{\mathbb{b}} + \hat{\mathfrak{a}}^\dagger) \quad , \quad \hat{\pi} = i\sqrt{\frac{m}{2}} (\hat{\mathbb{b}}^\dagger - \hat{\mathfrak{a}}) \\ \hat{\phi}^\dagger &= \frac{1}{\sqrt{2m}} (\hat{\mathbb{b}}^\dagger + \hat{\mathfrak{a}}) \quad , \quad \hat{\pi}^\dagger = (-i)\sqrt{\frac{m}{2}} (\hat{\mathbb{b}} - \hat{\mathfrak{a}}^\dagger), \end{aligned} \quad (7.6)$$

with $[\hat{\mathfrak{a}}, \hat{\mathfrak{a}}^\dagger] = 1$, $[\hat{\mathbb{b}}, \hat{\mathbb{b}}^\dagger] = 1$ the Hamiltonian is $\hat{H} = m(\hat{\mathfrak{a}}^\dagger \hat{\mathfrak{a}} + \hat{\mathbb{b}}^\dagger \hat{\mathbb{b}} + 1) + \mu (\hat{\mathfrak{a}}^\dagger \hat{\mathfrak{a}} - \hat{\mathbb{b}}^\dagger \hat{\mathbb{b}} + 1)$. The path integral in Eq. (7.1) however computes the Weyl ordered Hamiltonian.³ The Weyl ordering affects only the following operator that appears in the Hamiltonian

$$\mathcal{Q} = i \left(\hat{\pi} \hat{\phi} - \hat{\phi}^\dagger \hat{\pi}^\dagger \right) \quad \rightarrow \quad \mathcal{Q}_W := \frac{i}{2} (\hat{\pi} \hat{\phi} + \hat{\phi} \hat{\pi}) + \text{hermitian conjugate} \quad (7.7)$$

$$= \left(\hat{\mathfrak{a}}^\dagger \hat{\mathfrak{a}} - \hat{\mathbb{b}}^\dagger \hat{\mathbb{b}} + 1 \right) \quad = \left(\hat{\mathfrak{a}}^\dagger \hat{\mathfrak{a}} - \hat{\mathbb{b}}^\dagger \hat{\mathbb{b}} \right). \quad (7.8)$$

From this point on $\hat{H}$ will refer to the Weyl ordered Hamiltonian, namely

$$\hat{H} = m(\hat{\mathfrak{a}}^\dagger \hat{\mathfrak{a}} + \hat{\mathbb{b}}^\dagger \hat{\mathbb{b}} + 1) + \mu \left(\hat{\mathfrak{a}}^\dagger \hat{\mathfrak{a}} - \hat{\mathbb{b}}^\dagger \hat{\mathbb{b}} \right). \quad (7.9)$$

The Euclidean path integral Eq. (7.1) computes the thermal trace of the $\hat{H}$ with imaginary chemical potential $\mu \rightarrow -i\mu$

$$Z_0 = \text{Tr}_{\mathcal{H}_{1,2}} \left[e^{-\beta(\hat{H}_0 + m^2 \hat{\mathcal{O}})} e^{-i\beta\mu \hat{\mathcal{Q}}} \right], \quad \mu, m \in \mathbb{R}, \quad (7.10)$$

where the operators H_0 , $\mathcal{O}$, $\mathcal{Q}$ in terms of position and momentum operators are⁴

$$\hat{H}_0 = \frac{1}{2} \hat{p}_1^2 + \hat{p}_2^2, \quad \hat{\mathcal{O}} = \frac{1}{2} (\hat{x}_1^2 + \hat{x}_2^2), \quad \hat{\mathcal{Q}} = \hat{p}_2 \hat{x}_1 - \hat{p}_1 \hat{x}_2. \quad (7.11)$$

²We will rotate to imaginary μ later.

³See e.g. [189] page 281.

⁴We used $\mathfrak{a} = \frac{1}{2\sqrt{m}} (m(x_1 - ix_2) + (p_2 + ip_1))$, $\mathbb{b} = \frac{1}{2\sqrt{m}} (m(x_1 + ix_2) - (p_2 - ip_1))$.

The total Hilbert space is the tensor product of the (Fock) Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The states can be labeled as $|n_1\rangle \otimes |n_2\rangle$ where $\hat{N}_{1,2} |n_{1,2}\rangle = n_{1,2} |n_{1,2}\rangle$ with $\hat{N}_1 = \hat{\alpha}^\dagger \hat{\alpha}$ and $\hat{N}_2 = \hat{\beta}^\dagger \hat{\beta}$.

It is convenient to represent combine m, μ into one complex parameter

$$z = e^{-\beta m - i\beta \mu}. \quad (7.12)$$

The partition function Z_0 in Eq. (7.10) can be easily computed and one finds

$$\ln Z_0 = \int_0^z \frac{dz'}{1-z'} + \int_0^{\bar{z}} \frac{dz'}{1-z'} - \int_{|z|}^1 \frac{dz'}{z'}. \quad (7.13)$$

In d dimensions the partition function is

$$Z_d = \text{Tr} \left(e^{-\beta \hat{H} - i\beta \mu \hat{Q}} \right), \quad (7.14)$$

where $\hat{H}$ and $\hat{Q}$ operators can be written in Fourier space as

$$\hat{H} = V_{d-1} \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}} \omega_{\vec{k}} \left(\hat{\alpha}_{\vec{k}}^\dagger \hat{\alpha}_{\vec{k}} + \hat{\beta}_{\vec{k}}^\dagger \hat{\beta}_{\vec{k}} + 1 \right), \quad (7.15)$$

$$\hat{Q} = V_{d-1} \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}} \left(\hat{\alpha}_{\vec{k}}^\dagger \hat{\alpha}_{\vec{k}} - \hat{\beta}_{\vec{k}}^\dagger \hat{\beta}_{\vec{k}} \right), \quad (7.16)$$

$$\omega_{\vec{k}} = \sqrt{m^2 + \vec{k}^2}, \quad (7.17)$$

where used the sum over the modes as an integral in momentum space $\sum_{\vec{k}} = V_{d-1} \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}}$ (see e.g. [190]) and $V_{d-1} = \ell^{d-1}$ is the volume of the box. The modes $\vec{k}$ behave a system of decoupled harmonic oscillators with frequency $\omega_{\vec{k}}$ and chemical potential μ . The logarithm of the partition function (7.1) in $d = 2L + 1$ can be written as

$$\ln Z_L = V_{d-1} \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1}} \ln Z_0(\omega_{\vec{k}}, \mu). \quad (7.18)$$

If we express the measure in spherical coordinates $\int d^{d-1} \vec{k} = S_{d-2} \int_0^\infty k^{d-2} dk$ we can write $\ln Z_L$ as

$$\ln Z_L = \frac{2\ell^{2L}}{(4\pi)^L (L-1)!} \int_m^\infty \omega d\omega (\omega^2 - m^2)^{L-1} \ln Z_0(\omega, \mu). \quad (7.19)$$

The above form is suggestive of the following picture. The system corresponds to decoupled harmonic oscillators (7.10) of same frequency ω and chemical potential μ but with a frequency dependent density of oscillators given by the $\rho_L(\omega; m) = \omega(\omega^2 - m^2)^{L-1}$. Implementing the well known Cauchy's repeated integral formula $\int_a^x \int_a^{\sigma_1} \dots \int_a^{\sigma_{n-1}} f(\sigma_n) d\sigma_n \dots d\sigma_2 d\sigma_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt$ we can also express $\ln Z_L$ as an iterated integral

$$\begin{aligned} \ln Z_L &= (-2\alpha^2)^L \int_0^{w_L=|z|} \frac{dw_{L-1}}{w_{L-1}} \ln w_{L-1} \int_0^{w_{L-1}} \frac{dw_{L-2}}{w_{L-2}} \ln w_{L-2} \dots \int_0^{w_1} \frac{dw_0}{w_0} \ln w_0 \ln Z_0 \\ &= (-2\alpha^2)^L \prod_{i=0}^{L-1} \left[\int_0^{w_{i+1}} \frac{dw_i}{w_i} \ln w_i \right] \ln Z_0, \end{aligned} \quad (7.20)$$

where $\ln Z_0$ is a function of $z, \bar{z}$ with $z = w_0 e^{-i\beta\mu}$, $w_L = |z|$ and $\alpha^2 = \ell^2/4\pi\beta^2$. The iterated integrals can be computed [45]

$$\ln Z_L = \alpha^{2L} \frac{(-1)^L L!}{2(2L+1)!} (2 \ln |z|)^{2L+1} + \alpha^{2L} \sum_{n=0}^L \frac{(2L-n)! (-2 \ln |z|)^n}{(L-n)! n!} 2\Re[\text{Li}_{2L+1-n}(z)], \quad (7.21)$$

where $\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ is the polylogarithm function. The formula Eq. (7.21) arises by subtracting the vacuum energy contribution $\ln Z_L|_{m=\mu=0, \beta=\infty}$ $m = \mu = 0$ from Eq. (7.20). The vacuum subtraction is enough to make the expressions finite.

We will find it very useful to introduce the following operators

$$\hat{\mathbf{D}} = \frac{1}{\beta^2} \frac{\partial}{\partial m^2} = \frac{1}{2 \ln |z|} (z \partial_z + \bar{z} \partial_{\bar{z}}), \quad (7.22)$$

$$\hat{\mathbf{L}} = \frac{i}{\beta} \frac{\partial}{\partial \mu} = (z \partial_z - \bar{z} \partial_{\bar{z}}). \quad (7.23)$$

Acting with $\hat{\mathbf{D}}, \hat{\mathbf{L}}$ on $\ln Z_L$ gives thermal expectation values of $\mathcal{O} = |\phi|^2$, $\mathcal{Q} = \phi^\dagger \overleftrightarrow{\mathcal{D}}_\tau \phi$. This can be easily checked from the euclidean path integral. Eq. (7.1). We find

$$\langle \mathcal{O} \rangle_L = -\beta \hat{\mathbf{D}} \ln Z_L = \beta \alpha^2 \ln Z_{L-1}, \quad (7.24)$$

$$\langle \mathcal{Q} \rangle_L = \hat{\mathbf{L}} \ln Z_L = -\hat{\mathbf{D}} \cdot \langle \mathcal{Q} \rangle_{L+1} / \alpha^2. \quad (7.25)$$

The operator $\hat{\mathbf{D}}$ is especially interesting since it lowers the dimension $d \rightarrow d-2$, or equivalently as we will see later it reduced the graph by one loop $L \rightarrow L-1$. Being first order, the differential equation (7.24) can be integrated. Later we will utilize this fact to find a compact expression for the summation of infinite number of loops [52]. A second order differential operator that will be of interest to us is the laplacian in m, μ parameters, namely

$$\hat{\Delta} = 4\beta^2 z \bar{z} \partial_z \partial_{\bar{z}} = \frac{\partial^2}{\partial m^2} + \frac{\partial^2}{\partial \mu^2}. \quad (7.26)$$

By explicit computation if one acts $\hat{\Delta}$ on Eq. (7.19) one finds

$$\hat{\Delta} \ln Z_L = -4\beta^2 L \alpha^2 \ln Z_{L-1}. \quad (7.27)$$

Using Eq. (7.24) on the right hand side of Eq. (7.27) we find the second order differential equation

$$\left[m^2 \hat{\Delta} - 4L\beta^2 m^2 \hat{\mathbf{D}} \right] \ln Z_L = 0. \quad (7.28)$$

An interesting observation is that the second order differential equation Eq. (7.28) is the Laplace-Beltrami operator of AdS_{2L+2} with metric

$$ds^2 = \frac{1}{m^2} \left(dm^2 + d\mu^2 + \sum_{i=1}^{2L} dx^i dx^i \right), \quad (7.29)$$

acting on function of just m and μ .

It is also important to note that the differential equations Eq. (7.24), Eq. (7.27) and Eq. (7.28) hold also

for $\langle \mathcal{Q} \rangle$ due to the fact that $[\hat{\mathbf{D}}, \hat{\mathbf{L}}] = 0$. As we will see shortly $\langle \mathcal{Q} \rangle$ also correspond to certain conformal diagrams. $\langle \mathcal{Q} \rangle_L$ can be computed from Eq. (7.25) and formula Eq. (7.21)

$$\langle \mathcal{Q} \rangle_L = \hat{\mathbf{L}} \ln Z_L = \alpha^{2L} \sum_{n=0}^L \frac{(2L-n)!(-2 \ln |z|)^n}{(L-n)!n!} 2i\Im[\text{Li}_{2L-n}(z)]. \quad (7.30)$$

Before moving to the next section let us summarize. The main results presented here are the formulas Eq. (7.21) and Eq. (7.30). These formulas are not new, but we have presented them from a novel perspective. We saw that the formula for the partition function can be generated from the $d = 1$ case by an iterative integral Eq. (7.20), which facilitated in deriving the first order differential equations we found in Eq. (7.24), Eq. (7.25). In the next section we will see exactly the same formulas appear in a very different context.

7.2 Conformal graphs as thermal partition functions

Consider the bi-scalar theory in D spacetime dimensions introduced in [51]

$$\mathcal{L} = N_c \text{Tr} \left[\phi_1^\dagger (-\partial^2)^\omega \phi_1 + \phi_2^\dagger (-\partial^2)^{\frac{D-2\omega}{2}} \phi_2 + a_{D,\omega}^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 \right], \quad (7.31)$$

where $\phi_{1,2}$ belong to the adjoint of $SU(N_c)$ and $\omega \in (0, D/2)$. The coupling $a_{D,\omega}^2$ is classically dimensionless. It was shown [51] that the theory is conformal and integrable at any D and $N_c = \infty$.⁵ The case $D = 4$, $\omega = 1$ corresponds to the original four-dimensional conformal fishnet CFT introduced in [191].

The non-local kinetic operators are defined as

$$(\partial_\mu \partial^\mu)^\beta f(x) = \frac{(-4)^\beta \Gamma(\frac{D}{2} + \beta)}{\pi^{D/2} \Gamma(-\beta)} \int \frac{d^D y f(y)}{|x - y|^{D+2\beta}}, \quad (7.32)$$

and the propagator is

$$(-\partial_\mu \partial^\mu)^\beta D(x) = \delta^{(D)}(x), \quad D(x - y) = \frac{\Gamma(\frac{D}{2} - \beta)}{4^\beta \pi^{D/2} \Gamma(\beta) |x - y|^{D-2\beta}}, \quad (7.33)$$

and from the Lagrangian Eq. (7.31) we find that the scaling dimensions are $\Delta_{\phi_1} = \frac{D}{2} - \omega$ and $\Delta_{\phi_2} = \omega$.

The objects of interest are the four-point functions

$$G_{D,\omega}^{(L)}(\{x_i\}) = \langle \text{Tr} \left[\phi_2^L(x_1) \phi_1(x_3) \phi_2^{\dagger L}(x_2) \phi_1^\dagger(x_4) \right] \rangle. \quad (7.34)$$

The infinite $N_c \rightarrow \infty$ contribution to Eq. (7.34) comes only from the diagram [192] shown in Figure 7.1 whose integral form is

$$G_{D,\omega}^{(L)}(\{x_i\}) = \int \prod_{i=1}^L d^D u_i \prod_{n=1}^{L+1} \frac{1}{|u_{n-1} - u_n|^{D-2\omega}} \prod_{n=1}^L \left(\frac{1}{|x_1 - u_n|^{2\omega}} \frac{1}{|u_n - x_2|^{2\omega}} \right). \quad (7.35)$$

Because of conformal covariance we can write $G_{D,\omega}^{(L)}(\{x_i\}) = (x_{12}^2)^{-L\omega} (x_{34}^2)^{-\frac{D-2\omega}{2}} \mathcal{F}(u, v)$ and $u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}$,

⁵To be precise in [51] the beta function was only computed for the isotropic case $\omega = D/4$. We will assume this holds for arbitrary ω, D .

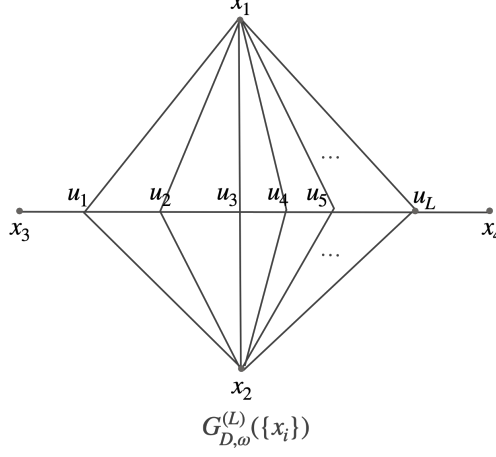


Figure 7.1: Conformal diagram corresponding to the four-point function $G_{D,\omega}^{(L)}(\{x_i\}) = \langle \text{Tr} [\phi_2^L(x_1) \phi_1(x_3) \phi_2^{\dagger L}(x_2) \phi_1^{\dagger}(x_4)] \rangle$.

$v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2}$ denote the cross-ratios. With foresight, we combine the two real variables u, v into one complex z

$$u = \frac{z\bar{z}}{|1-z|^2}, \quad \frac{u}{v} = z\bar{z}. \quad (7.36)$$

Similarly we can exploit conformal covariance further to write the 4-pt function [193]

$$G_{D,\omega}^{(L)}(\{x_i\}) = \frac{1}{(x_{12}^2)^{L\omega} (x_{34}^2)^{\frac{D-2\omega}{2}}} (|1-z|^2)^{\frac{D}{2}-\omega} I_{D,\omega}^{(L)}(z, \bar{z}), \quad (7.37)$$

where the function $I_{D,\omega}^{(L)}(z, \bar{z})$ can be computed from a simplified diagram that is formed from Figure 7.1 by sending the point $x_1 \rightarrow \infty$, ignoring propagators connected to x_1 , and then sending $x_4 \rightarrow 1, x_2 \rightarrow 0, x_3 \rightarrow z$. The graph for $I_{D,\omega}^{(L)}(z, \bar{z})$ is shown in Figure 7.2. An important point that we see explicitly shortly is that

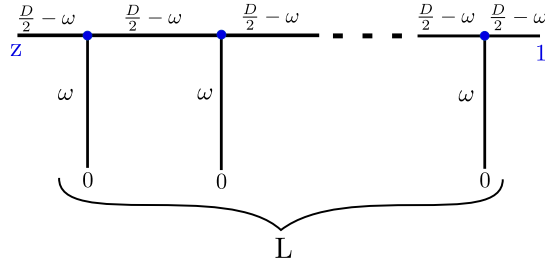


Figure 7.2: The simplified diagram for $I_L(z, \bar{z})$. Figure taken from [187].

the complex parameter z that for the diagrams depends on the cross-ratios will be exactly z introduced in Eq. (7.12) that depends on the mass and chemical potential (multiplied by β so that they are dimensionless) in the context of thermal partition functions. Moreover, the number L that counts the number of loops in the conformal graph Figure 7.2 is related to the dimension $d = 2L + 1$ for thermal partition functions.

The graph $I_{D,\omega}^{(L)}(z, \bar{z})$ (see e.g. [192], [194], [195]) in the limit $D \rightarrow 2, \omega \rightarrow 1$ can be written in the following

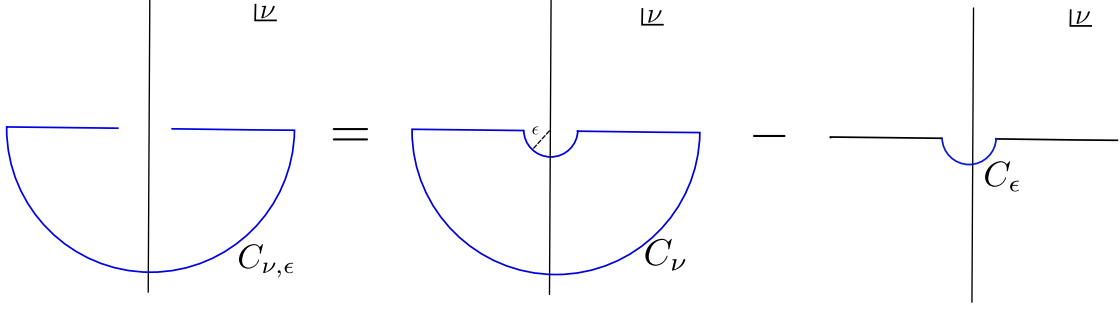


Figure 7.3: The contours $C_{\nu, \epsilon}$, C_{ν} and C_{ϵ} shown in blue. Since there are no poles inside C_{ν} the integral vanishes and the regularized integral of Eq. (7.42) can be computed from C_{ϵ} which is a semicircle of radius ϵ in the counterclockwise direction.

form

$$I_{2,1}^{(L)}(z, \bar{z}) = \tilde{a}_{2,1}^{2L} \sum_{m \in \mathbf{Z}} \int_{-\infty}^{\infty} d\nu \frac{(z\bar{z})^{i\nu} (z/\bar{z})^{m/2}}{(\frac{m^2}{4} + \nu^2)^{L+1}}, \quad (7.38)$$

where the limit is taken keeping fixed the ‘renormalized’ coupling $\tilde{a}_{D,\omega}$ which in terms of the original coupling $a_{D,\omega}$ in Eq. (7.31) is given by

$$\tilde{a}_{D,\omega} = a_{D,\omega} \frac{1}{\Gamma(D/2 - \omega)}. \quad (7.39)$$

Note for $\tilde{a}_{D,\omega}$ fixed $\alpha_{D,\omega}$ diverges in the $D \rightarrow 2$, $\omega \rightarrow 1$ limit. For $|z| < 1$ and $m \neq 0$ we can compute the integral in Eq. (7.38) by closing the contour from below in the complex ν plane

$$\sum_{m \neq 0} \int d\nu \frac{(z\bar{z})^{i\nu} (z/\bar{z})^{m/2}}{(\frac{m^2}{4} + \nu^2)^{L+1}} = \frac{2\pi}{L!} \sum_{n=0}^L \frac{(2L-n)! (-2 \log |z|)^n}{(L-n)! n!} 2\Re[L i_{2L+1-n}(z)]. \quad (7.40)$$

For $m = 0$ there is a pole in the origin and the integral below is formally divergent

$$\int d\nu \frac{(z\bar{z})^{i\nu}}{(\nu^2)^{L+1}}, \quad (7.41)$$

and a procedure to extract the finite piece is needed. We regularize the integral by cutting of the contour on the real axis to $C_{\nu, \epsilon} := (-\infty, \epsilon) \cup (\epsilon, +\infty)$

$$\int d\nu \frac{(z\bar{z})^{i\nu}}{(\nu^2)^{L+1}} \rightarrow \int_{C_{\nu, \epsilon}} d\nu \frac{(z\bar{z})^{i\nu}}{(\nu^2)^{L+1}}. \quad (7.42)$$

Closing the contour from below we can write $\int_{C_{\nu, \epsilon}} = \int_{C_{\nu}} - \int_{C_{\epsilon}}$ where the contours C_{ν} and C_{ϵ} are shown in Figure 7.3. Since $\int_{C_{\nu}}$ includes no poles the integral vanishes due to Cauchy’s theorem and we are left with

$$\begin{aligned} \int_{C_{\nu, \epsilon}} d\nu \frac{(z\bar{z})^{i\nu}}{(\nu^2)^{L+1}} &= - \int_{C_{\epsilon}} d\nu \frac{(z\bar{z})^{i\nu}}{(\nu^2)^{L+1}} \\ &= -i \int_{\pi}^{2\pi} d\theta \frac{\exp(2i(\ln |z|)\epsilon e^{i\theta})}{\epsilon^{2L+1} e^{i(2L+1)\theta}}. \end{aligned} \quad (7.43)$$

We can Taylor expand the exponential in the numerator for small ϵ , and we find $2L + 1$ divergent terms in

the $\epsilon \rightarrow 0$ limit which we discard and we are left with the finite term

$$-i \int_{\pi}^{2\pi} d\theta \frac{(2i \ln |z|)^{2L+1}}{(2L+1)!} = (-1)^L \pi \frac{(2 \ln |z|)^{2L+1}}{(2L+1)!}. \quad (7.44)$$

The divergences found above are analogous to the the divergences found in $\ln Z_L$ that are cancelled off by subtracting the vacuum energy contribution (see below Eq. (7.21)). Combining Eq. (7.40) and Eq. (7.44) into Eq. (7.38) and comparing with Eq. (7.21) we find

$$I_{2,1}^{(L)}(z, \bar{z}) = \frac{2\pi}{L!} \ln Z_L(z, \bar{z}), \quad (7.45)$$

if we identify $\tilde{a}_{2,1}^2 = \alpha^2$. Eq. (7.45) is one of the main results that connects conformal graphs with thermal partition functions. The number of loops L in the conformal graph corresponds to the dimension of the free boson Gas. Moreover the renormalized coupling $\tilde{a}$ is mapped to the dimensionless parameter α that is related to the volume of the box (see below Eq. (7.20) for the exact expression).

For $D = 4, \omega = 1$ a similar map for the thermal one-point functions $\langle \mathcal{Q} \rangle_L$ has also been found [45]

$$I_{4,1}^{(L)}(z, \bar{z}) = \frac{1}{L!} \frac{1}{z - \bar{z}} \langle \mathcal{Q} \rangle_L(z, \bar{z}), \quad (7.46)$$

when we set $a_{4,1}^2 = \alpha^2$. The conformal graphs $G_{4,1}^{(L)}$ are proportional to the celebrated Davydychev-Usuykina L -loop conformal ladder graphs [49], [50].

The physical origin of the correspondence summarized in Table 7.1 is currently not known. In the next chapter we will present an application of this novel perspective, namely the resummation of an infinite series of Davydychev-Usuykina ladder diagrams.

Thermal free Gas		Conformal graph
Dimension d	$\longleftrightarrow$ $d = 2L + 1$	Loop $2L+1$
mass - chemical potential $z = e^{-\beta m - i\beta \mu}$	$\longleftrightarrow$	cross ratios $u = \frac{z\bar{z}}{ 1-z ^2}, \frac{u}{v} = z\bar{z}$
Box size $\alpha^2 = \frac{\ell^2}{(4\pi\beta)^2}, V = \ell^d$	$\longleftrightarrow$ $\alpha^2 = \tilde{a}_{2,1}^2, a_{4,1}^2$	coupling $\tilde{a}_{2,1}^2, a_{4,1}^2$
Twisted Partition function $\ln Z_L(z, \bar{z})$	$\longleftrightarrow$ $I_{2,1}^{(L)}(z, \bar{z}) = \frac{2\pi}{L!} \ln Z_L(z, \bar{z})$	Conformal graph $I_{2,1}^{(L)}(z, \bar{z})$
Charge thermal 1-pt function $\langle \mathcal{Q} \rangle_L(z, \bar{z})$	$\longleftrightarrow$ $I_{4,1}^{(L)}(z, \bar{z}) = \frac{1}{L!} \frac{1}{z - \bar{z}} \langle \mathcal{Q} \rangle_L(z, \bar{z})$	Conformal Graph $I_{4,1}^{(L)}(z, \bar{z})$

Table 7.1: The correspondence between thermal partition functions and conformal graphs summarized.

7.2.1 Some interesting remarks

We will note two more interesting geometric connections in similar fashion to the Laplace-Beltrami interpretation of the differential equation Eq. (7.28). Our iterated formula Eq. (7.20) when applied to $\langle Q \rangle_L$ for $L = 1$ gives

$$\langle Q \rangle_1 = (-2\alpha^2) \int_0^{|z|} \frac{d|z'|}{|z'|} \ln |z'| \langle Q \rangle_0 = 4i\alpha^2 D(z), \quad (7.47)$$

where $D(z) = \text{Im} [\text{Li}_2(z) + \ln |z| \ln(1-z)]$ is the Bloch-Wigner function [196] that gives the volume of an ideal tetrahedron in three-dimensional hyperbolic space $\mathbb{H}^3$ with vertices in $\partial\mathbb{H}^3$ shown in Figure 7.4. Similarly $\langle Q \rangle_0$ has a geometric interpretation

$$\langle Q \rangle_0 / 4i = \frac{z - \bar{z}}{(1 - \bar{z})(1 - z)} = \frac{1}{2} ab \sin \phi, \quad z = \frac{b}{a} e^{i\phi}, \quad \cos \phi = \left(\frac{a^2 + b^2 - 1}{2ab} \right), \quad (7.48)$$

namely the area of triangle whose side lengths are a, b and 1, and ϕ the angle between a and b . It would be interesting to see if $\langle Q \rangle_L$ for $L > 1$ can also be given a geometric interpretation and similarly for $\langle Q \rangle_L$.

Another interesting connection is that $\langle Q \rangle_0 = -8\pi \rho_{m_0 \rightarrow m_1 + m_2}^{D=4}$ where $\rho_{m_0 \rightarrow m_1 + m_2}^D$ is the D -dimensional $1 \rightarrow 2$ decay phase space of relativistic massive particles. Since

$$\rho_{m_0 \rightarrow m_1 + m_2}^{D=4} = \frac{1}{8\pi} \sqrt{\lambda(1, m_1^2/m_0^2, m_1^2/m_0^2)} \quad (7.49)$$

with $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$ the Källén triangle function, we see that if we set $a = m_1/m_0$ and $b = m_2/m_0$, then $\langle Q \rangle_0$ represents the phase space for a virtual process with $\lambda < 0$ [197]. Then, our (7.20) is reminiscent to eq. (7) of [198] that gives a recurrent relationship for higher dimensional $1 \mapsto 2$ relativistic phase spaces.

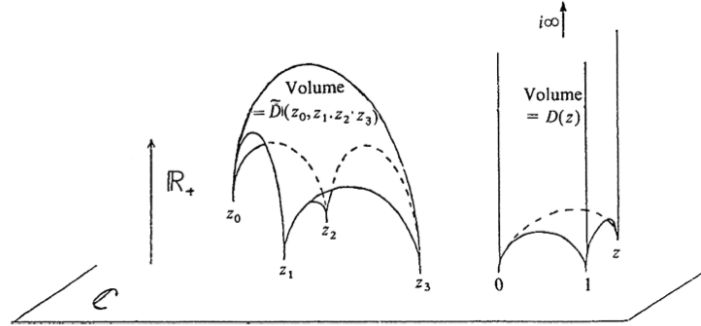


Figure 7.4: Left: Ideal tetrahedron in $\mathbb{H}^3$ with vertices in $\mathbb{H}^3$. Right: Special case where three of the vertices are $\infty, 0, 1$ and the volume is still finite $V(z) = D(z)$. Figure taken from [196].

7.3 Infinite loop resummation of ladder diagrams

The correspondence between conformal graphs and thermal partition functions can have interesting applications. In this section we will describe such an application, namely we will find an integral expression for the resummation of all loop Davydychev-Usuykina conformal graphs. This result was obtained in [52] but here we will derive it using the thermal free gas perspective of these graphs.

Figure 7.5: 4-point, L -loop diagrams in ϕ^3 theory. Figure taken from [52].

The goal is to reproduce the integral expression [52] of the infinite summation

$$\begin{aligned} \mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) &= \frac{1}{t} \left\{ 1 + \sum_{L=1}^{\infty} \left(-\frac{\kappa^2}{4} \right)^L \Phi^{(L)}(X, Y) \right\} \\ &= \frac{1}{t} \left\{ \frac{1}{2\sqrt{XY}} \int_{\ell/2}^{\infty} d\eta \frac{\sinh \eta J_0(\kappa \sqrt{\eta^2 - \frac{1}{4}\ell^2})}{(\cosh \eta - \cos \phi)^2} \right\}, \end{aligned} \quad (7.50)$$

where $\ell \equiv \ln \frac{X}{Y}$, $\phi = \arccos \left(\frac{X+Y-1}{\sqrt{4XY}} \right)$ and the dimensionless ratios

$$X \equiv \frac{k_1^2 k_3^2}{st}, \quad Y \equiv \frac{k_2^2 k_4^2}{st}, \quad \kappa^2 \equiv \frac{g^2}{4\pi^2 s}. \quad (7.51)$$

and $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$. Each $\Phi^{(L)}(X, Y)$ corresponds to the L -loop diagram shown in Figure 7.5. Such diagrams also arise in massless ϕ^3 theory with coupling g^2 . They can also be related to the graphs shown in Figure 7.1 using the dual diagram map [199].

From [52] if we identify

$$z = \sqrt{\frac{Y}{X}} e^{i\phi} \quad \text{with} \quad z = e^{-\beta m - i\beta \mu}, \quad (7.53)$$

we find

$$X = \frac{1}{|1-z|^2}, \quad Y = \frac{|z|^2}{|1-z|^2}, \quad \tilde{\mu} = \frac{|z|^2}{i} \left(\frac{1}{z(1-\bar{z})} - \frac{1}{\bar{z}(1-z)} \right), \quad (7.54)$$

where $\tilde{\mu} \equiv \sqrt{4XY - (X+Y-1)^2}$. In terms of m, μ we have

$$X = \frac{1}{2} e^{\beta m} \frac{1}{\cosh(\beta m) - \cos(\beta \mu)}, \quad Y = \frac{1}{2} e^{-\beta m} \frac{1}{\cosh(\beta m) - \cos(\beta \mu)}, \quad (7.55)$$

and

$$\tilde{\mu} = \frac{|\sin(\beta \mu)|}{\cosh(\beta m) - \cos(\beta \mu)}. \quad (7.56)$$

Form [45] and [52] we also find

$$i\tilde{\mu} \left(-\frac{\kappa^2}{4} \right)^L \Phi^{(L)}(X, Y) = \frac{1}{L!} \langle Q \rangle_L, \quad \tilde{\mu} = -i \langle Q \rangle_0 \quad \text{for } \pi < \mu < 2\pi, \quad (7.57)$$

if we set

$$\alpha^2 = -\frac{\kappa^2}{4}. \quad (7.58)$$

Let us now consider the following sum

$$\mathcal{B}[Q](\alpha^2) := \langle Q \rangle_0 + \sum_{L=1}^{\infty} \frac{1}{L!} \langle Q \rangle_L. \quad (7.59)$$

If we use Eq. (7.24) we find

$$\left(\hat{\mathbf{D}} + \alpha^2 \right) \ln Z = -\frac{1}{\beta} \langle \mathcal{O} \rangle_0, \quad (7.60)$$

where $\ln Z \equiv \sum_{L=0}^{\infty} \ln Z_L$. Applying the differential operator $\hat{\mathbf{L}}$ on both sides we find the first order differential equation

$$(\hat{\mathbf{D}} + \alpha^2)Q = \hat{\mathbf{D}}\langle Q \rangle_0, \quad (7.61)$$

where we defined

$$Q \equiv \sum_{L=0}^{\infty} \langle Q \rangle_L. \quad (7.62)$$

The differential equation Eq. (7.61) can be integrated to

$$Q = \beta^2 e^{-\beta^2 \alpha^2 m^2} \int^{m^2} e^{\beta^2 \alpha^2 \tilde{m}^2} \hat{\mathbf{D}}\langle Q \rangle_0 d\tilde{m}^2, \quad (7.63)$$

and from Eq. (7.57), Eq. (7.56) and $\hat{\mathbf{D}} = \frac{1}{\beta^2} \frac{\partial}{\partial m^2}$ we also find

$$\hat{\mathbf{D}}\langle Q \rangle_0 = \frac{i}{2\beta m} \frac{\sinh(\beta m) \sin(\beta \mu)}{(\cosh(\beta m) - \cos(\beta \mu))^2}. \quad (7.64)$$

What we have achieved so far is to find an integral expression for Q defined in Eq. (7.62). What we really want is an expression for Eq. (7.59). Note the existence of the L factorial in the denominators of each term in the sum. $\mathcal{B}[Q]$ is the Borel transform of the series Q . Thus our goal is to invert the Borel transform and derive $\mathcal{B}[Q]$ from Q .⁶ The Borel transform $\mathcal{B}A(t)$ of a series $A(z) = \sum_{k=0}^{\infty} a_k z^k$ is defined as

$$\mathcal{B}[A](t) := \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k. \quad (7.65)$$

Given the series $A(z)$ one can find $\mathcal{B}[A](t)$ and vice versa

$$\mathcal{B}[A](t) = \frac{1}{2\pi i} \int_C \frac{dz}{z} e^z A(t/z), \quad A(z) = \int_0^{\infty} dt e^{-t} \mathcal{B}[A](t/z), \quad (7.66)$$

where C is the Hankel contour shown in Figure 7.6. If we apply the left hand side of Eq. (7.66) to Eq. (7.63) after a short calculation we obtain⁷

$$\begin{aligned} \mathcal{B}[Q](t) &= \beta^2 \frac{1}{2\pi i} \int^{m^2} d\tilde{m}^2 (\hat{\mathbf{D}}\langle Q \rangle_0) \int_C \frac{du}{u} e^{u - \frac{t\beta^2}{u}(m^2 - \tilde{m}^2)} \\ &= \beta^2 \int^{m^2} d\tilde{m}^2 J_0(2\beta\sqrt{t(m^2 - \tilde{m}^2)}) (\hat{\mathbf{D}}\langle Q \rangle_0), \end{aligned} \quad (7.67)$$

⁶We thank Costas Bachas for pointing this out to us.

⁷We used the following integral representation of the Bessel function $J_{\nu}(z) = \frac{(\frac{1}{2}z)^{\nu}}{2\pi i} \int_C dt \frac{1}{t^{\nu+1}} e^{t - \frac{z^2}{4t}}$.

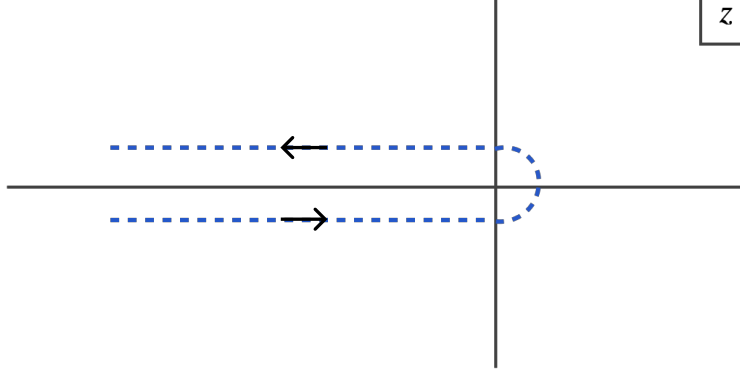


Figure 7.6: The Hankel contour C (blue) used in Eq. (7.66).

where $(\hat{\mathbf{D}}\langle Q \rangle_0)$ was defined in Eq. (7.64). Eq. (7.67) is the main result of this section. If we combine Eq. (7.67) with Eq. (7.65) and the identifications of Eq. (7.57) we find exactly Broadhurst's integral expression Eq. (7.50) after we set $t = -\frac{\kappa^2}{4}$, $\beta\tilde{m} = \eta$, $\ell = 2\beta m$ and set the lower bound of the integral Eq. (7.67) to be $+\infty$. The key equation in the above derivation is the first order differential equation Eq. (7.60) found for the thermal partition functions. Then our correspondence (cf. 7.1 for a summary) connects the thermal one-point functions to graphs and allow us to sum an infinite number of such graphs with increasing loop order.

7.4 Discussion

In this chapter we have described a connection between partition functions for a free thermal gas in one side and conformal ladder graphs on the other side [45], [187]. Some novel features of this connection is the that the dimension in the thermal side corresponds to loops in the diagrammatic side while the mass and chemical potential (divided by temperature to make them dimensionless) correspond to cross ratios. For a summary of the correspondence see Table 7.1. A simple explanation of the second order differential equation Eq. (7.28) is missing on the thermal side. More speculatively the interpretation of this second order differential equation as the Laplace-Beltrami differential operator on AdS_{2L+2} with metric Eq. (7.29) also hints on connections with holography.

In Section 7.3 we showed that the correspondence can have practical implications. We derived an integral expression (see Eq. (7.50)) for the resummation of infinite number of ladder diagrams (see Figure 7.5) that was originally found by Broadhurst et al. [52]. The key equation behind the derivation was the first order differential equation Eq. (7.60). From a technical standpoint we made use of the inverse Borel transform (see Eq. (7.65) and Eq. (7.66)). The same method can be applied to sum an infinite number of $D = 2$, $\omega = 1$ loop diagrams as shown in Figure 7.2 using the correspondence with partition functions Eq. (7.45). We leave this for future work.

An important future direction is to generalize this connection to include Fishnet type of diagrams. Such diagrams correspond to 4-pt functions $G_{D,\omega}^{(L,M)}(\{x_i\}) = \langle \text{Tr} [\phi_2^L(x_2)\phi_1^M(x_3)\phi_2^{\dagger M}(x_2)\phi_1^{\dagger M}(x_4)] \rangle$ whose formula in terms of a determinant has been found in [199]. Our previous diagrams correspond to the case $M = 1$. It would be interesting to investigate the thermal free gas analogs of such diagrams. Moreover, expressions of ladder diagrams for arbitrary spacetime dimension d have been recently found in [200]. It would very

interesting to map these expressions to thermal free gas thermodynamic quantities for any dimension.

The underlying physical mechanism behind the correspondence is currently not known. Due to the growing connections between both sides we believe such underlying mechanism exists and hope to understand better in the future.

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Appendix A

Appendices for Chapter 2

A.1 Proof of Gauss-Codazzi-Ricci Eqs. (2.22)

We follow [201] closely. If we use the induced covariant derivative on Σ_τ , D_M , defined in Eq. (2.1), we define the following ambient tensor $R^A_{B\Gamma\Delta}$ such that $\forall V^\Sigma$ we have

$$R^\Gamma_{MAB}V^M := [D_A, D_B]V^\Gamma, \quad (\text{A.1})$$

where $[D_A, D_B] := D_AD_B - D_BD_A$. Note that the tensor R^Γ_{MAB} is a tangential tensor to Σ_τ , namely if we contract the projector $\gamma^M_N = \delta^M_N - \sigma n^M n_N$ with any index, we get back the original tensor e.g. $\gamma^K_N R^M_{KAB} = R^M_{KAB}$. By construction, for vectors tangent to Σ_τ the tensor R^Γ_{MAB} is exactly the intrinsic Riemann tensor of the induced metric on Σ_τ .

From the definition (2.1) and using (2.23) we obtain

$$D_AD_BV^\Gamma = -\sigma K_{AB}\gamma^\Gamma_E n^\Delta \nabla_\Delta \tilde{V}^E + \sigma K_A{}^\Gamma V^E K_{EB} + \gamma^{A'}_A \gamma^\Delta_B \gamma^\Gamma_E \nabla_{A'} \nabla_\Delta \tilde{V}^E, \quad (\text{A.2})$$

where $\tilde{V}^M := \gamma^M_K V^K$.

From (A.2) and the fact that $K_{[AB]} = 0$ Eq. (A.1) becomes

$$R^\Gamma_{MAB}V^M = \sigma K_A{}^\Gamma K_{BE}V^E - \sigma K_B{}^\Gamma K_{AE}V^E + \gamma^{A'}_A \gamma^\Delta_B \gamma^\Gamma_E \gamma^\Sigma_{\Sigma'} \mathcal{R}^E_{\Sigma A' \Delta} V^{\Sigma'}, \quad (\text{A.3})$$

where we used $[\nabla_{A'}, \nabla_\Delta] \tilde{V}^\Sigma = \mathcal{R}^E_{\Sigma A' \Delta} \tilde{V}^\Sigma$ for the ambient Riemann tensor $\mathcal{R}^M_{NAB}$. Since Eq. (A.3) holds for every vector V^M we get the first equation in (2.22)

$$\gamma^\Sigma_A \gamma^K_B \gamma^P_\Gamma \gamma^T_\Delta \mathcal{R}_{\Sigma K P T} = \sigma (K_{A\Delta} K_{\Gamma B} - K_{A\Gamma} K_{B\Delta}) + R_{AB\Gamma\Delta}. \quad (\text{A.4})$$

To prove the second equation in (2.22) we use the definition of the ambient Riemann tensor $\mathcal{R}^M_{NAB}$

$$\nabla_M \nabla_N n^P - \nabla_N \nabla_M n^P = \mathcal{R}^P_{\Sigma M N} n^\Sigma. \quad (\text{A.5})$$

If we apply the projector γ^A_B on all three free indices in Eq. (A.5) and use $\nabla_M n^N = K_M^N + \sigma n_M n^K \nabla_K n^N$ we find the second equation in (2.22)

$$\gamma^M_A \gamma^N_B \gamma^P_\Gamma \mathcal{R}_{MNP\Gamma} n^T = D_A K_{B\Gamma} - D_B K_{A\Gamma}. \quad (\text{A.6})$$

To prove the third equation in (2.22) we will use the following useful relation

$$n^K \nabla_K n_M = -\sigma D_M \ln N, \quad (\text{A.7})$$

and

$$\nabla_M n_N = K_{MN} - n_M D_N \ln N. \quad (\text{A.8})$$

Proof of (A.7).

$$\begin{aligned} a_M &:= n^K \nabla_K n_M, \quad \text{use (2.19)}, \\ a_M &= n^K \nabla_K (\sigma N \nabla_M \tau), \\ a_M &= \sigma n^K \nabla_K N \nabla_M \tau + \sigma N n^K \nabla_K \nabla_M \tau, \quad \text{no torsion thus } \nabla_{[K,M]} \tau = 0, \\ a_M &= N^{-1} n^K \nabla_K N n_M + \sigma N n^K \nabla_M \left(\frac{\sigma}{N} n_K \right), \\ a_M &= n^K (\nabla_K \ln N) n_M - \sigma (\nabla_M \ln N), \\ a_M &= -\sigma (\delta^P_M - \sigma n^P n_M) \nabla_P \ln N, \\ a_M &= -\sigma \gamma^P_M \nabla_P \ln N, \\ a_M &= -\sigma D_M \ln N. \end{aligned} \quad (\text{A.9})$$

□

Starting from the definition of the ambient Riemann $(\nabla_A \nabla_B - \nabla_B \nabla_A) n^\Gamma = \mathcal{R}^\Gamma_{PAB} n^P$ and making use of (A.7) and (A.8) we find

$$-K_{B\Gamma} K_A^B - \gamma^\Gamma_\Delta \gamma^A_M n^B \nabla_B K_{A\Gamma} - \sigma N^{-1} D_A D_\Gamma N = \gamma^\Gamma_\Delta \gamma^A_M \mathcal{R}_{\Gamma PAB} n^P n^B. \quad (\text{A.10})$$

We can relate the second term above to the Lie derivative $\mathcal{L}_{N\underline{n}}$ using the following formula

$$\gamma^A_M \gamma^B_N n^P \nabla_P K_{AB} = N^{-1} \gamma^A_M \gamma^B_N \mathcal{L}_{N\underline{n}} K_{AB} - 2K_{AP} K^P_B. \quad (\text{A.11})$$

Substituting (A.11) into (A.10) we find

$$\gamma^\Gamma_\Delta \gamma^A_M \mathcal{R}_{\Gamma PAB} n^P n^B = K_{\Delta P} K^P_M - \sigma N^{-1} D_\Delta D_M N - N^{-1} \gamma^A_M \gamma^B_\Delta \mathcal{L}_{N\underline{n}} K_{AB}. \quad (\text{A.12})$$

Finally, for the last term above we need to show that $\mathcal{L}_{N\underline{n}} K_{AB}$ is a tangent tensor on Σ_τ , namely

$$\gamma^A_M \gamma^B_N \mathcal{L}_{N\underline{n}} K_{AB} = \mathcal{L}_{N\underline{n}} K_{MN}, \quad (\text{A.13})$$

which holds due to

$$\mathcal{L}_{N\underline{n}} \gamma^A_B = 0. \quad (\text{A.14})$$

From (A.14) and (A.12) we finally get the last equation in (2.22). This completes the proof of all three equations in (2.22).

A.2 Details of cod-2 Gauss-Codazzi

In this appendix we provide details of the codimension-2 Gauss-Codazzi decomposition presented in Subsection 2.5.1.

The ambient metric (2.18) is

$$ds^2 = \sigma N^2 d\tau^2 + h_{\mu\nu} (dx^\mu + N^\mu d\tau) (dx^\nu + N^\nu d\tau) , \quad (\text{A.15})$$

with inverse

$$g^{MN} \partial_M \otimes \partial_N = \frac{\sigma}{N^2} \underline{\partial}_\tau \otimes \underline{\partial}_\tau + (h^{\mu\nu} + \frac{\sigma}{N^2} N^\mu N^\nu) \underline{\partial}_\mu \otimes \underline{\partial}_\nu - \frac{2\sigma}{N^2} N^\mu \underline{\partial}_\mu \otimes \underline{\partial}_\tau . \quad (\text{A.16})$$

We now employ another ADM type decomposition along the “radial” coordinate ρ as follows

$$\begin{aligned} ds^2|_{\Sigma_\tau} &= h_{\mu\nu} dx^\mu dx^\nu \\ &= P^2 d\rho^2 + s_{ab} (dx^a + \beta^a d\rho) (dx^b + \beta^b d\rho) , \end{aligned} \quad (\text{A.17})$$

where $a = 1, \dots, D-2$ label the coordinate $\mathcal{S}_\tau$ (see Section 2.2). The inverse of (A.17) is

$$h^{\mu\nu} \underline{\partial}_\mu \otimes \underline{\partial}_\nu = \frac{1}{P^2} \underline{\partial}_\rho \otimes \underline{\partial}_\rho + (s^{ab} + \frac{1}{P^2} \beta^a \beta^b) \underline{\partial}_a \otimes \underline{\partial}_b - \frac{2}{P^2} \beta^a \underline{\partial}_a \otimes \underline{\partial}_\rho . \quad (\text{A.18})$$

In matrix form (the columns/rows are arranged in the order of $\{\rho, a\}$) the metric (A.17) and inverse metric (A.18) are

$$h_{\mu\nu} = \begin{pmatrix} P^2 + s_{ab} \beta^a \beta^b & s_{bc} \beta^c \\ s_{ac} \beta^c & s_{ab} \end{pmatrix} , \quad h^{\mu\nu} = \begin{pmatrix} \frac{1}{P^2} & -\frac{1}{P^2} \beta^b \\ -\frac{1}{P^2} \beta^b & s^{ab} + \frac{1}{P^2} \beta^a \beta^b \end{pmatrix} . \quad (\text{A.19})$$

The Christoffel components $\gamma^\lambda_{\mu\nu}$ of the induced metric $h_{\mu\nu}$ (A.19) are

$$\begin{aligned} \gamma^\rho_{\rho\rho} &= \frac{1}{2} P^{-2} \partial_\rho P^2 + \frac{1}{2} P^{-2} (\partial_\rho s^{fc}) \beta_c \beta_f + \frac{1}{2} P^{-2} \beta^f \partial_f (P^2 + \beta_c \beta^c) \\ \gamma^\rho_{\rho b} &= \frac{1}{2} P^{-2} \partial_b (P^2 + \beta_f \beta^f) - \frac{1}{2} (P^{-2} \beta^f) (\partial_\rho s_{bf} + \partial_b \beta_f - \partial_f \beta_b) \\ \gamma^\rho_{ab} &= \frac{1}{2} P^{-2} (\partial_a \beta_b + \partial_b \beta_a - \partial_\rho s_{ab}) - \frac{1}{2} P^{-2} \beta^f (\partial_a s_{fb} + \partial_b s_{fa} - \partial_f s_{ab}) \\ \gamma^a_{\rho\rho} &= -\beta^a \partial_\rho \ln P - \frac{1}{2} P^{-2} \beta^a (\beta_f (\partial_\rho s^{fc}) \beta_c + s^{ab} \partial_\rho \beta_b - \frac{1}{2} (s^{ab} + P^{-2} \beta^a \beta^b) \partial_b (P^2 + \beta_f \beta^f)) \\ \gamma^a_{\rho b} &= -\frac{1}{2} P^{-2} \beta^a \partial_b (P^2 + \beta_f \beta^f) + \frac{1}{2} (s^{ac} + P^{-2} \beta^a \beta^c) (\partial_\rho s_{cb} + \partial_b \beta_c - \partial_c \beta_b) \\ \gamma^a_{bc} &= -P^{-2} \beta^a \partial_{(b} \beta_{c)} + \frac{1}{2} P^{-2} \beta^a \partial_\rho s_{bc} + \frac{1}{2} (s^{af} + P^{-2} \beta^a \beta^f) (\partial_b s_{fc} + \partial_c s_{fb} - \partial_f s_{bc}) , \end{aligned} \quad (\text{A.20})$$

where $\beta_a := s_{ab} \beta^b$.

From (A.17) the ambient metric (A.15) becomes

$$ds^2 = (\sigma N^2 + h_{\mu\nu} N^\mu N^\nu) d\tau^2 + 2(P^2 + s_{ab}\beta^a\beta^b) N^\rho d\rho d\tau + 2s_{ab}\beta^b N^a d\rho d\tau + 2s_{ab}\beta^b N^\rho dx^a d\tau + 2s_{ab} N^a dx^b d\tau + (P^2 + s_{ab}\beta^a\beta^b) d\rho^2 + s_{ab} dx^a dx^b + 2s_{ab}\beta^b dx^a d\rho. \quad (\text{A.21})$$

In matrix form the metric (A.21) (the columns/rows are arranged in the order of $\{\tau, \rho, a\}$) is

$$g_{MN} = \begin{pmatrix} \sigma N^2 + h_{\mu\nu} N^\mu N^\nu & (P^2 + s_{ab}\beta^a\beta^b) N^\rho + s_{ab}\beta^b N^a & s_{bc}\beta^c N^\rho + s_{bc} N^c \\ (P^2 + s_{ab}\beta^a\beta^b) N^\rho + s_{ab}\beta^b N^a & P^2 + s_{ab}\beta^a\beta^b & s_{bc}\beta^c \\ s_{ac}\beta^c N^\rho + s_{ac} N^c & s_{ac}\beta^c & s_{ab} \end{pmatrix}, \quad (\text{A.22})$$

where $h_{\mu\nu} N^\mu N^\nu = (P^2 + s_{ab}\beta^a\beta^b) (N^\rho)^2 + 2s_{ab}\beta^b N^a N^\rho + s_{ab} N^a N^b$.

Using (A.18) the inverse metric (A.16) in matrix form is

$$g^{MN} = \begin{pmatrix} \frac{\sigma}{N^2} & -\frac{\sigma}{N^2} N^\rho & -\frac{\sigma}{N^2} N^b \\ -\frac{\sigma}{N^2} N^\rho & \frac{\sigma}{N^2} (N^\rho)^2 + \frac{1}{P^2} & \frac{\sigma}{N^2} N^\rho N^b - \frac{1}{P^2} \beta^b \\ -\frac{\sigma}{N^2} N^a & \frac{\sigma}{N^2} N^\rho N^a - \frac{1}{P^2} \beta^a & s^{ab} + \frac{\sigma}{N^2} N^a N^b + \frac{1}{P^2} \beta^a \beta^b \end{pmatrix}. \quad (\text{A.23})$$

The orthogonality condition assumption (see Section 2.2) $\mathcal{T} \perp \Sigma_\tau$ means $\underline{r} \cdot \underline{n}|_{\mathcal{T}} = 0$ which implies

$$N^\rho|_{\mathcal{T}} = 0. \quad (\text{A.24})$$

After imposing (A.24) the metric (A.22) becomes

$$g_{MN} \Big|_{\mathcal{T}, N^\rho|_{\mathcal{T}}=0} = \begin{pmatrix} \sigma N^2 + s_{ab} N^a N^b & s_{ab}\beta^b N^a & s_{bc} N^c \\ s_{ab}\beta^b N^a & P^2 + s_{ab}\beta^a\beta^b & s_{bc}\beta^c \\ s_{ac} N^c & s_{ac}\beta^c & s_{ab} \end{pmatrix}, \quad (\text{A.25})$$

with $\sqrt{|g|} = NP\sqrt{s}$ and inverse

$$g^{MN} \Big|_{\mathcal{T}, N^\rho|_{\mathcal{T}}=0} = \begin{pmatrix} \frac{\sigma}{N^2} & 0 & -\frac{\sigma}{N^2} N^b \\ 0 & \frac{1}{P^2} & -\frac{1}{P^2} \beta^b \\ -\frac{\sigma}{N^2} N^a & -\frac{1}{P^2} \beta^a & s^{ab} + \frac{\sigma}{N^2} N^a N^b + \frac{1}{P^2} \beta^a \beta^b \end{pmatrix}. \quad (\text{A.26})$$

The unit normal one form r_μ (see Eq. (2.41)) and dual vector $\underline{r}$ normal to ρ - surfaces on Σ_τ are

$$r = P d\rho, \quad \underline{r} = P^{-1} \underline{\partial}_\rho - P^{-1} \underline{\beta}, \quad (\text{A.27})$$

where $\underline{\beta} = \beta^a \underline{\partial}_a$ and $r_\mu r^\mu = 1$. The radial extrinsic curvature tensor (2.42) of $\mathcal{S}_\tau$ is defined to be

$$k_{\mu\nu} = r^\sigma r^\lambda{}_\nu D_\sigma r_\lambda, \quad (\text{A.28})$$

where $r_{\mu\nu} = h_{\mu\nu} - r_\mu r_\nu$ is the ρ - projector. A very useful formula is

$$D_\mu r_\nu = k_{\mu\nu} + r_\mu a_\nu, \quad (\text{A.29})$$

where $a_\mu := r^\sigma D_\sigma r_\mu$ is the radial acceleration that satisfies (analog of Eq. (A.5))

$$a_\mu = -^{(2)}D_\mu \ln P, \quad (\text{A.30})$$

where $^{(2)}D_\mu$ is the cod-2 induced covariant derivative at constant ρ surfaces (as embedded in Σ_τ). The cod-2 Gauss-Codazzi equations (which are the analogs of Eqs. (2.22) proven in Appendix A.1) are

$$\begin{aligned} r^\sigma_\alpha r^\kappa_\beta r^\rho_\gamma r^\tau_\delta R_{\sigma\kappa\rho\tau} &= (k_{\alpha\delta} k_{\gamma\beta} - k_{\alpha\gamma} k_{\beta\delta}) + ^{(2)}R_{\alpha\beta\gamma\delta} \\ r^\sigma_\alpha r^\kappa_\beta r^\rho_\gamma R_{\sigma\kappa\rho\tau} r^\tau &= ^{(D-2)}D_\alpha k_{\beta\gamma} - ^{(2)}D_\beta k_{\alpha\gamma} \\ r^\lambda_\mu r^\kappa_\nu R_{\lambda\rho\kappa\sigma} r^\rho r^\sigma &= k_\mu^\sigma k_{\sigma\nu} - \frac{1}{P} ^{(2)}D_\mu ^{(2)}D_\nu P - \frac{1}{P} \mathcal{L}_{P\bar{r}} k_{\mu\nu}, \end{aligned} \quad (\text{A.31})$$

If we express the lie derivative in terms of induced covariant derivative $\mathcal{L}_{P\bar{r}} k_{\mu\nu} := Pr^\lambda D_\lambda k_{\mu\nu} + k_{\lambda\nu} D_\mu (Pr^\lambda) + k_{\mu\lambda} D_\nu (Pr^\lambda)$ and use (A.29) we find

$$\mathcal{L}_{P\bar{r}} k_{\mu\nu} = PD_\tau k_{\mu\nu} + 2Pr_{(\mu} k_{\nu)\lambda} a^\lambda + 2Pk_{\mu\lambda} k^\lambda_{\nu}. \quad (\text{A.32})$$

Using (A.32) we find that (A.31) become the Gauss-Codazzi-Ricci equations given in (2.43), namely

$$\begin{aligned} r^\sigma_\alpha r^\kappa_\beta r^\rho_\gamma r^\tau_\delta R_{\sigma\kappa\rho\tau} &= (k_{\alpha\delta} k_{\gamma\beta} - k_{\alpha\gamma} k_{\beta\delta}) + ^{(2)}R_{\alpha\beta\gamma\delta} \\ r^\sigma_\alpha r^\kappa_\beta r^\rho_\gamma R_{\sigma\kappa\rho\tau} r^\tau &= ^{(2)}D_\alpha k_{\beta\gamma} - ^{(2)}D_\beta k_{\alpha\gamma} \\ r^\lambda_\mu r^\kappa_\nu R_{\lambda\rho\kappa\sigma} r^\rho r^\sigma &= -k_\mu^\sigma k_{\sigma\nu} + ^{(2)}D_\mu a_\nu - a_\mu a_\nu - D_\tau k_{\mu\nu} - 2r_{(\mu} k_{\nu)\lambda} a^\lambda. \end{aligned} \quad (\text{A.33})$$

A.3 Reduction to General Relativity

In this appendix we reduce the path integral with the auxiliary fields to the gravitational path integral of pure General Relativity (GR). There are two different approaches for showing this, based on the Lagrangian or canonical path integral. We will present both.¹

In the first approach we use the Lagrangian formalism defined by the action (2.10) and path integral (2.11) and reduce to the GR Lagrangian by integrating out the auxiliary fields. We find the effective Lagrangian (A.38) is indeed the EH Lagrangian with the GHY boundary term. Similarly we repeat the procedure starting from the canonical form of the action (2.36) and path integral (2.39). In this second approach we find the system indeed reduces to the usual GR canonical action (A.49) with GHY boundary term. We will present both approaches.

For GR the function $f(\varrho_{MNP\Sigma})$ appearing in (2.10) is

$$f^{\text{GR}}(\varrho^{MN}_{K\Lambda}) = -\frac{\sigma}{8\pi G_N} (g^{MP} g^{N\Sigma} \varrho_{MNP\Sigma} - 2\Lambda). \quad (\text{A.34})$$

Note that in GR given the function $f^{GR}(\varrho^{MN}_{K\Lambda})$ defined in (A.34) above, the fields $\rho_{MNK\Lambda}$ appear linearly in the action (2.10) and thus act as Lagrange multipliers. Integrating them out along the imaginary contour in field space inserts a Dirac delta functional in the path integral (2.11). After integrating out the fields

¹We note that our manipulations do not keep track of determinants that should be present when integrating out fields but do not affect leading semiclassics.

$\varphi^{MNK\Lambda}$, the Dirac delta functional imposes the following constraint

$$\varphi_{\text{GR}}^{MNK\Lambda} = -\frac{\sigma}{16\pi G_N} (g^{MK} g^{N\Lambda} - g^{NK} g^{M\Lambda}) . \quad (\text{A.35})$$

Enforcing the constraint (A.35) to the generalized GHY term (2.12) we obtain the usual GHY boundary term of GR,

$$\mathcal{S}_{\text{GHY}}^{\text{GR}} = \frac{\sigma}{8\pi G_N} \epsilon \int_{\partial\mathcal{M}} d^{D-1}y \sqrt{|r|} \mathcal{K} , \quad (\text{A.36})$$

where $\mathcal{K} := g^{MN} \mathcal{K}_{MN}$ is the extrinsic curvature trace. In deriving (A.36) we used $\Theta_{\text{GR}}^{MN} = \frac{\sigma}{8\pi G_N} \epsilon r^{MN}$ which can be proven from the definition (2.13) if we use the constraint (A.35). The effective action for the metric degrees of freedom can be found by integrating out $\varrho_{MNP\Sigma}$ and then $\varphi^{MNK\Lambda}$ in the partition function (2.10)

$$Z = \int \mathcal{D}g_{MN} e^{-I_{\mathcal{L}}^{\text{eff}}[g_{MN}]} . \quad (\text{A.37})$$

Enforcing the constraint (A.35), we find the effective action $I_{\mathcal{L}}^{\text{eff}}[g_{MN}]$ to be the Einstein-Hilbert action with GHY boundary term

$$I_{\mathcal{L}}^{\text{eff}}[g_{MN}] = -\frac{\sigma}{16\pi G_N} \int_{\mathcal{M}} d^Dx \sqrt{|g|} (\mathcal{R} - 2\Lambda) - \frac{\sigma}{8\pi G_N} \epsilon \int_{\partial\mathcal{M}} \sqrt{|r|} d^{D-1}y \mathcal{K} . \quad (\text{A.38})$$

For Euclidean metrics we have $\sigma = \epsilon = 1$ which matches the action in equation (2) of [18].²

We now repeat the same procedure, starting from the ADM form of the action (2.36) with corresponding path integral (2.39). The function $f^{\text{GR}}(\varrho_{MNK\Lambda})$ in (A.34) can be expressed in terms of the spatial fields

$$f^{\text{GR}}(\varrho^{MN}{}_{K\Lambda}) = -\frac{\sigma}{8\pi G_N} (h^{\mu\lambda} h^{\nu\sigma} \rho_{\mu\nu\lambda\sigma} + 2\sigma h^{\mu\nu} \Omega_{\mu\nu} - 2\Lambda) , \quad (\text{A.39})$$

where $h_{\mu\nu}$ is the induced metric on Σ_τ , and the fields $\rho_{\mu\nu\lambda\sigma}$, $\Omega_{\mu\nu}$ are defined in (2.26). Note that from (A.39) and the lapse constraint $\mathcal{C}$ given in (2.35) the fields $\Omega_{\mu\nu}$, $\rho_{\mu\nu\kappa\lambda}$, $\rho_{\mu\nu\kappa}$ appear linearly in the ADM action (2.36). Integrating them out along the imaginary contour inserts the following delta functionals to the path integral³

$$\delta \left(\sigma N \left(\Psi^{\mu\nu} + \frac{\sigma}{8\pi G_N} h^{\mu\nu} \right) \right) , \quad \delta \left(\frac{N}{2} \left(\frac{\sigma}{16\pi G_N} (h^{\mu\kappa} h^{\nu\lambda} - h^{\nu\kappa} h^{\mu\lambda}) + \phi^{\mu\nu\kappa\lambda} \right) \right) , \quad (\text{A.40})$$

$$\delta (2\sigma N \phi^{\mu\nu\kappa}) .$$

If we integrate out $\Psi^{\mu\nu}$, $\phi^{\mu\nu\kappa\lambda}$, $\phi^{\mu\nu\kappa}$ the delta functional arguments are imposed as constraints and reduce the path integral (2.39) to

$$Z = \int \mathcal{D}N \mathcal{D}N^\mu \mathcal{D}h_{\mu\nu} \mathcal{D}p^{\mu\nu} \mathcal{D}\Pi_{\mu\nu} e^{-\tilde{I}_{\text{ADM}}} , \quad (\text{A.41})$$

where

$$\tilde{I}_{\text{ADM}} = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \left((p^{\mu\nu} + \frac{\sigma}{8\pi G_N} \Pi^{\mu\nu}) \partial_\tau h_{\mu\nu} - N\tilde{\mathcal{C}} - N^\mu \tilde{\mathcal{C}}_\mu \right) + \tilde{B}_N + \tilde{B}_{\underline{N}} , \quad (\text{A.42})$$

²The GR path integral (A.37) can be expressed for both Lorentzian/ Euclidean signature $\sigma = \mp 1$ as $Z = \int [\mathcal{D}g_{MN}] e^{-\sigma \sqrt{\sigma} I_{\mathcal{L}}^{\text{eff}}[g_{MN}]}$, where $i = \sqrt{-1}$ and $I_{\mathcal{L}}^{\text{eff}}[g_{MN}]$ is given in (A.38).

³We inserted appropriate factors $1/\sqrt{h}$ in the delta functionals using the inner product convention $(A, B) = \int d^{D-1}x \sqrt{h} A \cdot B$ for tensors A, B .

and $\tilde{\mathcal{C}}, \tilde{\mathcal{C}}_\mu, \tilde{B}_N, \tilde{B}_{\underline{N}}$ are

$$\begin{aligned}\tilde{\mathcal{C}} &= \frac{1}{\sqrt{h}} \frac{1}{16\pi G_N} (3\Pi \cdot \Pi + \Pi^2) + \frac{\sigma}{16\pi G_N} \sqrt{h} (R - 2\Lambda) + \frac{1}{\sqrt{h}} 2\sigma p \cdot \Pi, \\ \tilde{\mathcal{C}}_\mu &= -\frac{\sigma}{8\pi G_N} \sqrt{h} h_{\mu\lambda} D_\nu \left(2 \frac{\Pi^{\lambda\nu}}{\sqrt{h}} \right) - 2\sqrt{h} h_{\mu\lambda} D_\nu \left(\frac{p^{\lambda\nu}}{\sqrt{h}} \right), \\ \tilde{B}_N &= -\frac{\sigma}{8\pi G_N} \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2} x \sqrt{|s|} N k, \\ \tilde{B}_{\underline{N}} &= -2 \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2} x \sqrt{s} N^\mu h_{\mu\sigma} \left(\frac{p^{\sigma\nu}}{\sqrt{h}} + \frac{\sigma}{8\pi G_N} \frac{\Pi^{\sigma\nu}}{\sqrt{h}} \right) r_\nu.\end{aligned}\tag{A.43}$$

Note that there is one extra degree of freedom left in the path integral (A.41) compared to the GR canonical path integral. One way to proceed is to integrate out $p^{\mu\nu}$, which yields the GR path integral in Lagrangian form. Let us take a different route and obtain it in the phase space form. We define the new momenta $\tilde{p}^{\mu\nu}, \tilde{\Pi}^{\mu\nu}$

$$\tilde{p}^{\mu\nu} = p^{\mu\nu} + \frac{\sigma}{8\pi G_N} \Pi^{\mu\nu}, \quad \tilde{\Pi}^{\mu\nu} = p^{\mu\nu} - \frac{\sigma}{8\pi G_N} \Pi^{\mu\nu},\tag{A.44}$$

after which the $\tilde{I}_{\text{ADM}}$ takes the more familiar form

$$\tilde{I}_{\text{ADM}} = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \left(\tilde{p}^{\mu\nu} \partial_\tau h_{\mu\nu} - N \tilde{\mathcal{C}} - N^\mu \tilde{\mathcal{C}}_\mu \right) + \tilde{B}_N + \tilde{B}_{\underline{N}},\tag{A.45}$$

where $\tilde{\mathcal{C}}, \tilde{\mathcal{C}}_\mu$ and the boundary terms $\tilde{B}_{\underline{N}}, \tilde{B}_N$ of Eq. (A.43) expressed in terms of $\tilde{p}^{\mu\nu}$ and $\tilde{\Pi}^{\mu\nu}$ are

$$\begin{aligned}\tilde{\mathcal{C}} &= \frac{\pi G_N}{\sqrt{h}} \left(7\tilde{p} \cdot \tilde{p} - 6\tilde{p} \cdot \tilde{\Pi} - 2\tilde{p} \tilde{\Pi} - \tilde{\Pi} \cdot \tilde{\Pi} + \tilde{p}^2 + \tilde{\Pi}^2 \right) + \frac{\sigma}{16\pi G_N} \sqrt{h} (R - 2\Lambda), \\ \tilde{\mathcal{C}}_\mu &= -2\sqrt{h} h_{\mu\lambda} D_\nu \left(\frac{\tilde{p}^{\lambda\nu}}{\sqrt{h}} \right), \\ \tilde{B}_N &= -\frac{\sigma}{8\pi G_N} \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2} x \sqrt{s} N k, \\ \tilde{B}_{\underline{N}} &= -2 \int_{\tau_1}^{\tau_2} d\tau \int_{S_\tau} d^{D-2} x \sqrt{s} N^\mu h_{\mu\sigma} \left(\frac{\tilde{p}^{\sigma\nu}}{\sqrt{h}} \right) r_\nu.\end{aligned}\tag{A.46}$$

Note that $\tilde{\Pi}^{\mu\nu}$ appears quadratically in (A.45). Integrating out $\tilde{\Pi}^{\mu\nu}$ is equivalent to substituting the $\tilde{\Pi}^{\mu\nu}$ variation equations of motion

$$\tilde{\Pi}^{\mu\nu} = -3\tilde{p}^{\mu\nu} + \frac{4}{D-2} \tilde{p} h^{\mu\nu}, \quad \tilde{\Pi} = \frac{D+2}{D-2} \tilde{p},\tag{A.47}$$

into (A.45). The path integral (A.41) then reduces to the GR canonical path integral

$$Z = \int \mathcal{D}N \mathcal{D}N^\mu \mathcal{D}h_{\mu\nu} \mathcal{D}\tilde{p}^{\mu\nu} e^{-I_{\text{ADM}}^{\text{eff}}},\tag{A.48}$$

where

$$I_{\text{ADM}}^{\text{eff}} = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \left(\tilde{p}^{\mu\nu} \partial_\tau h_{\mu\nu} - N \tilde{\mathcal{C}}_{\text{eff}} - N^\mu \tilde{\mathcal{C}}_\mu^{\text{eff}} \right) + \tilde{B}_N + \tilde{B}_{\underline{N}},\tag{A.49}$$

and

$$\begin{aligned}
\tilde{\mathcal{C}}_{\text{eff}} &= \frac{16\pi G_N}{\sqrt{h}} \left(\tilde{p}^{\mu\nu} \tilde{p}_{\mu\nu} - \frac{1}{D-2} \tilde{p}^2 \right) + \frac{\sigma}{16\pi G_N} \sqrt{h} (R - 2\Lambda) , \\
\tilde{\mathcal{C}}_{\mu}^{\text{eff}} &= -2\sqrt{h} h_{\mu\lambda} D_{\nu} \left(\frac{\tilde{p}^{\lambda\nu}}{\sqrt{h}} \right) , \\
\tilde{B}_N &= -\frac{\sigma}{8\pi G_N} \int_{\tau_1}^{\tau_2} d\tau \int_{S_{\tau}} d^{D-2} x \sqrt{s} N k , \\
\tilde{B}_{\underline{N}} &= -2 \int_{\tau_1}^{\tau_2} d\tau \int_{S_{\tau}} d^{D-2} x \sqrt{s} N^{\mu} h_{\mu\lambda} \left(\frac{\tilde{p}^{\lambda\nu}}{\sqrt{h}} \right) r_{\nu} .
\end{aligned} \tag{A.50}$$

The action (A.49) together with the lapse and shift constraints (A.50) is the well known canonical form of the EH action with the GHY boundary term in D dimensions.

A.4 Derivation of purely bulk form of I_{ADM}

In this appendix we derive the complete form of the purely bulk form of both the canonical action given in Eq. (2.47) and microcanonical action given in Eq. (2.57).

We start with the canonical form of the action given in Eq. (2.36). By definition the purely bulk form should have no boundary terms so our goal is to cancel B_N , $B_{\underline{N}}$ given in Eqs. (2.37)-(2.38). As we will see shortly, some of the bulk terms after integration by parts will generate boundary terms that cancel B_N , $B_{\underline{N}}$.

The first step is to integrate by parts the first two terms in the shift constraint $\mathcal{C}_{\mu}$ in Eq. (2.35). The covariant derivative for these two terms will get shifted to N^{μ} and the boundary term generated will cancel the $B_{\underline{N}}$ boundary term. The next step is to integrate by parts the first term of the lapse constraint $\mathcal{C}$ in Eq. (2.35). The boundary term generated will cancel the first term in B_N given in Eq. (2.37). After these two integrations by parts the canonical action (2.36) becomes

$$\begin{aligned}
I_{\text{ADM}} &= \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{\tau}} d^{D-1} x \left[p^{\mu\nu} \partial_{\tau} h_{\mu\nu} + \Pi_{\mu\nu} \partial_{\tau} \Psi^{\mu\nu} \right. \\
&\quad + \sqrt{h} D_{\nu} N^{\mu} \left(2 \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} - 2 h_{\mu\sigma} \frac{p^{\sigma\nu}}{\sqrt{h}} \right) - \sqrt{h} \left(\frac{\Pi_{\alpha\beta}}{\sqrt{h}} N^{\lambda} D_{\lambda} \Psi^{\alpha\beta} \right) + \sqrt{h} (D_{\mu} N) D_{\lambda} \Psi^{\mu\lambda} \\
&\quad + \sqrt{h} N \left(\frac{1}{2} \phi^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda} \right) + 4\sqrt{h} N \phi^{\mu\nu\kappa} D_{\mu} \left(\frac{\Pi_{\nu\kappa}}{\sqrt{h}} \right) \\
&\quad - \sqrt{h} N \left(-\frac{1}{2} f(\varrho^{MN}{}_{K\Lambda}) + \sigma \Psi \cdot \Omega + 2\sigma \sqrt{h} \phi^{\mu\nu\kappa} \rho_{\mu\nu\kappa} \right) \\
&\quad - N \left(\frac{1}{\sqrt{h}} (2\sigma p^{\mu\nu} \Pi_{\mu\nu} - \sigma \Psi \cdot \Pi \cdot \Pi - \sigma \Pi(\Psi \cdot \Pi)) - \frac{\sqrt{h}}{2} \phi^{\mu\nu\kappa\lambda} \left(\frac{2\sigma}{h} \Pi_{\mu\lambda} \Pi_{\nu\kappa} - \rho_{\mu\nu\kappa\lambda} \right) \right) \Big] \\
&\quad + \int_{\tau_1}^{\tau_2} d\tau \int_{S_{\tau}} d^{D-2} x \sqrt{s} \left[-N r_{\nu} \Psi^{\nu\lambda} D_{\lambda} \ln N + 2N k_{\mu\nu} \phi^{\mu\gamma\nu\delta} r_{\gamma} r_{\delta} + 4N r^{\beta} \left(\frac{\Pi_{\alpha\beta}}{\sqrt{h}} \right) \phi^{\alpha\gamma\delta} r_{\gamma} r_{\delta} \right. \\
&\quad \left. + N (D_{\underline{r}} \ln N) (r_{\alpha} r_{\beta} \Psi^{\alpha\beta}) \right] .
\end{aligned} \tag{A.51}$$

Note there are boundary terms remaining in the above expression so the action is not in purely bulk form yet. However the terms in the second and third line of Eq. (A.51) above contain terms which after integration by parts will cancel the remaining boundary terms. To see this explicitly we use Eq. (2.45) and find the

following expressions

$$\begin{aligned}
D_\nu N^\mu & \left(2 \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} - 2h_{\mu\sigma} \frac{p^{\sigma\nu}}{\sqrt{h}} \right) = D_{\underline{r}} N_{|}^\mu \left(2 \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2h_{\mu\sigma} \frac{p^{\sigma\nu}}{\sqrt{h}} r_\nu \right) \\
& + D_{\underline{r}} N_0 \left(2r^\mu \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2r_\mu \frac{p^{\mu\nu}}{\sqrt{h}} r_\nu \right) \\
& + \left({}^{(2)}D_\nu N_{|}^\mu + (k^\mu{}_\nu + r_\nu a^\mu) N_0 + r^\mu ({}^{(2)}D_\nu N_0 - N_{|}^\sigma k_{\sigma\nu}) \right) \left(2 \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} - 2h_{\mu\sigma} \frac{p^{\sigma\nu}}{\sqrt{h}} \right) \\
\\
\frac{\Pi_{\alpha\beta}}{\sqrt{h}} N^\lambda D_\lambda \Psi^{\alpha\beta} & = \frac{\Pi_{\alpha\beta}^{|}}{\sqrt{h}} N^\lambda {}^{(2)}D_\lambda \Psi_{|}^{\alpha\beta} + 2 \frac{\Pi_{\alpha\beta}^{|}}{\sqrt{h}} N_{|}^\lambda k_{\lambda}{}^\alpha \Psi_{|}^\beta - 2 \frac{\Pi_{\beta}^{|}}{\sqrt{h}} N_{|}^\lambda k_{\lambda\alpha} \Psi_{|}^{\alpha\beta} + 2 \frac{\Pi_{\beta}^{|}}{\sqrt{h}} N_{|}^\lambda {}^{(2)}D_\lambda \Psi_{|}^\beta \\
& + 2 \frac{\Pi_{\beta}^{|}}{\sqrt{h}} N_{|}^\lambda k_{\lambda}{}^\beta \Psi_0 - 2 \frac{\Pi_0}{\sqrt{h}} N_{|}^\lambda k_{\lambda\beta} \Psi_{|}^\beta + \frac{\Pi_0}{\sqrt{h}} N_{|}^\lambda {}^{(2)}D_\lambda \Psi_0 - 2 \frac{\Pi_0}{\sqrt{h}} N_{0\alpha\beta} \Psi_{|}^\beta + 2 \frac{\Pi_{\beta}^{|}}{\sqrt{h}} N_{0\alpha} \Psi_{|}^\beta \\
& + \frac{\Pi_{\alpha\beta}^{|}}{\sqrt{h}} N_0 D_{\underline{r}} \Psi_{|}^{\alpha\beta} + 2 \frac{\Pi_{\alpha\beta}^{|}}{\sqrt{h}} N_0 a^\alpha \Psi_{|}^\beta + 2 N_0 \frac{\Pi_{\beta}^{|}}{\sqrt{h}} D_{\underline{r}} \Psi_{|}^\beta + \frac{\Pi_0}{\sqrt{h}} N_0 D_{\underline{r}} \Psi_0 - 2 N_0 \frac{\Pi_{\beta}^{|}}{\sqrt{h}} a_\lambda \Psi_{|}^{\lambda\beta} \\
\\
D_\lambda \Psi^{\mu\lambda} & = {}^{(2)}D_\lambda \Psi_{|}^{\mu\lambda} - \Psi_{|}^{\mu\lambda} a_\lambda + \Psi_{|}^\sigma k_{\sigma}{}^\mu + k \Psi_{|}^\mu + a^\mu \Psi_0 + r^\mu{}_\sigma D_{\underline{r}} \Psi_{|}^\sigma \\
& - r^\mu \left(k_{\alpha\beta} \Psi_{|}^{\alpha\beta} - k \Psi_0 - D_{\underline{r}} \Psi_0 - {}^{(2)}D_\lambda \Psi_{|}^\lambda + 2 \Psi_{|}^\sigma a_\sigma \right), \\
\\
(D_\mu N) D_\lambda \Psi^{\mu\lambda} & = {}^{(2)}D_\mu N \left({}^{(2)}D_\lambda \Psi_{|}^{\mu\lambda} - \Psi_{|}^{\mu\lambda} a_\lambda + \Psi_{|}^\sigma k_{\sigma}{}^\mu + a^\mu \Psi_0 \right) \\
& - (D_{\underline{r}} N) \left(k_{\alpha\beta} \Psi_{|}^{\alpha\beta} - 2 {}^{(2)}D_\lambda \Psi_{|}^\lambda - k \Psi_0 + 2 \Psi_{|}^\lambda a_\lambda - D_{\underline{r}} \Psi_0 \right) \\
& + D_\mu \left(r^\mu \Psi_{|}^\lambda {}^{(2)}D_\lambda N \right) - D_\lambda \left(\Psi_{|}^\lambda D_{\underline{r}} N \right) + \Psi_{|}^\lambda k_{\lambda}{}^\sigma {}^{(2)}D_\sigma N, \\
\\
\phi^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda} & = \theta_{|}^{\mu\nu\kappa\lambda} {}^{(2)}R_{\mu\nu\kappa\lambda} + \theta_{|}^{\mu\nu\kappa\lambda} (k_{\mu\lambda} k_{\nu\kappa} - k_{\mu\kappa} k_{\nu\lambda}) + 4\theta_{|}^{\mu\nu\lambda} ({}^{(2)}D_\mu k_{\lambda\nu} - {}^{(2)}D_\nu k_{\lambda\mu}) \\
& + 4\theta_{|}^{\mu\nu} (-k_{\mu}{}^\sigma k_{\sigma\nu} + {}^{(2)}D_\mu a_\nu - a_\mu a_\nu) + 4k \theta_{|}^{\mu\nu} k_{\mu\nu} + 4k_{\mu\nu} D_{\underline{r}} \theta_{|}^{\mu\nu} - 4D_\mu \left(r^\mu \theta_{|}^{\alpha\beta} k_{\alpha\beta} \right), \\
\\
\phi^{\mu\nu\kappa} D_\mu \left(\frac{\Pi_{\nu\kappa}}{\sqrt{h}} \right) & = \phi_{|}^{\mu\nu\kappa} {}^{(2)}D_\mu \left(\frac{\Pi_{\nu\kappa}^{|}}{\sqrt{h}} \right) + \phi_{|}^{\mu\nu\kappa} k_{\mu\kappa} \frac{\Pi_{\nu}^{|}}{\sqrt{h}} - \phi_{2,|}^{\mu\kappa} k_{\mu}{}^\sigma \frac{\Pi_{\sigma\kappa}^{|}}{\sqrt{h}} + \phi_{2,|}^{\mu\kappa} {}^{(2)}D_\mu \left(\frac{\Pi_{\kappa}^{|}}{\sqrt{h}} \right) \\
& - 2\phi_{2,|}^{\mu\nu} \frac{a_{(\mu} \Pi_{\nu)}^{|}}{\sqrt{h}} - \phi_{1,|}^{\mu\nu} k_{\mu}{}^\kappa \frac{\Pi_{\nu\kappa}^{|}}{\sqrt{h}} + \phi_{1,|}^{\mu\nu} {}^{(2)}D_\mu \left(\frac{\Pi_{\nu}^{|}}{\sqrt{h}} \right) + \phi_{|}^\nu a^\mu \frac{\Pi_{\mu\nu}^{|}}{\sqrt{h}} \\
& - 2\phi_{|}^\mu k_{\mu}{}^\sigma \frac{\Pi_{\sigma}^{|}}{\sqrt{h}} + \phi_{|}^\mu {}^{(2)}D_\mu \left(\frac{\Pi_0}{\sqrt{h}} \right) - \phi_{|}^\mu a_\mu \frac{\Pi_0}{\sqrt{h}} + k \phi_{|}^\mu \frac{\Pi_{\mu}^{|}}{\sqrt{h}} \\
& - D_\mu \left(r^\mu \phi_{|}^\lambda \frac{\Pi_{\lambda}^{|}}{\sqrt{h}} \right) + \frac{\Pi_{\mu}^{|}}{\sqrt{h}} D_{\underline{r}} \phi_{|}^\mu - \phi_{2,|}^{\nu\kappa} D_{\underline{r}} \left(\frac{\Pi_{\nu\kappa}^{|}}{\sqrt{h}} \right).
\end{aligned} \tag{A.52}$$

For the definition of the projected fields (with $|$ subscript) see Eq. (2.46). If we substitute expressions (A.52) into Eq. (A.51) the boundary terms cancel from the total derivative terms in the last three equations of

Eq. (A.52), and we finally find the purely bulk form of the canonical action

$$\begin{aligned}
I_{\text{ADM}} = & \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x (p^{\mu\nu} \partial_\tau h_{\mu\nu} + \Pi_{\mu\nu} \partial_\tau \Psi^{\mu\nu}) \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} (D_{\underline{r}} N) \left[2 {}^{(2)}D_\lambda \Psi_{\mid}^\lambda - k_{\alpha\beta} \Psi_{\mid}^{\alpha\beta} + k \Psi_0 - 2a_\lambda \Psi_{\mid}^\lambda + D_{\underline{r}} \Psi_0 + 2k_{\alpha\beta} \theta_{\mid}^{\alpha\beta} + 4\phi_{\mid}^\lambda \frac{\Pi_{\lambda}^{\mid}}{\sqrt{h}} \right] \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N \left[(D_{\underline{r}} N_{\mid}^\mu) \left(2 \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2h_{\mu\sigma} \frac{p^{\sigma\nu}}{\sqrt{h}} r_\nu \right) + D_{\underline{r}} N_0 \left(2r^\mu \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2r_\mu \frac{p^{\mu\nu}}{\sqrt{h}} r_\nu \right) \right] \\
& - 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N \phi_{2,\mid}^{\nu\kappa} D_{\underline{r}} \left(\frac{\Pi_{\nu\kappa}^{\mid}}{\sqrt{h}} \right) + 2 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N k_{\mu\nu} D_{\underline{r}} \theta_{\mid}^{\mu\nu} \\
& + 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N (D_{\underline{r}} \phi_{\mid}^\mu) \frac{\Pi_\mu^{\mid}}{\sqrt{h}} \\
& - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} N_0 \left(\frac{\Pi_{\alpha\beta}^{\mid}}{\sqrt{h}} D_{\underline{r}} \Psi_{\mid}^{\alpha\beta} + 2 \frac{\Pi_\beta^{\mid}}{\sqrt{h}} D_{\underline{r}} \Psi_{\mid}^\beta + \frac{\Pi_0}{\sqrt{h}} D_{\underline{r}} \Psi_0 \right) \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1}x \sqrt{h} \mathcal{G} (k_{\mu\nu}, a_\mu, \text{other field combinations with no radial derivatives}) ,
\end{aligned} \tag{A.53}$$

where

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \mathcal{G} (k_{\mu\nu}, a_\mu, \text{other field combinations with no radial derivatives}) = \\
& = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \left({}^{(2)}D_\nu N^\mu + (k^\mu{}_\nu + r_\nu a^\mu) N_0 + r^\mu ({}^{(2)}D_\nu N_0 - N^\sigma_\parallel k_{\sigma\nu}) \right) \left(2 \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} - 2 h_{\mu\sigma} \frac{p^{\sigma\nu}}{\sqrt{h}} \right) \\
& - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \left(\frac{\Pi_{\alpha\beta}}{\sqrt{h}} N^\lambda {}^{(2)}D_\lambda \Psi^\alpha_\parallel \Psi^\beta + 2 \frac{\Pi_{\alpha\beta}}{\sqrt{h}} N^\lambda_\parallel k_\lambda{}^\alpha \Psi^\beta - 2 \frac{\Pi_\beta}{\sqrt{h}} N^\lambda_\parallel k_{\lambda\alpha} \Psi^\alpha_\parallel \Psi^\beta + 2 \frac{\Pi_\beta}{\sqrt{h}} N^\lambda_\parallel {}^{(2)}D_\lambda \Psi^\beta \right. \\
& + 2 \frac{\Pi_\beta}{\sqrt{h}} N^\lambda_\parallel k_\lambda{}^\beta \Psi_0 - 2 \frac{\Pi_0}{\sqrt{h}} N^\lambda_\parallel k_{\lambda\beta} \Psi^\beta + \frac{\Pi_0}{\sqrt{h}} N^\lambda_\parallel {}^{(2)}D_\lambda \Psi_0 - 2 \frac{\Pi_0}{\sqrt{h}} N_0 a_\beta \Psi^\beta + 2 \frac{\Pi_\beta}{\sqrt{h}} N_0 a^\beta \Psi_0 \\
& + 2 \frac{\Pi_{\alpha\beta}}{\sqrt{h}} N_0 a^\alpha \Psi^\beta - 2 N_0 \frac{\Pi_\beta}{\sqrt{h}} a_\lambda \Psi^{\lambda\beta} \Big) \\
& + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} {}^{(2)}D_\mu N \left({}^{(2)}D_\lambda \Psi^\mu_\parallel - \Psi^\mu_\parallel a_\lambda + 2 \Psi^\sigma_\parallel k_{\sigma}{}^\mu + a^\mu \Psi_0 \right) \\
& + \frac{1}{2} \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \left(\theta^{\mu\nu\kappa\lambda} {}^{(2)}R_{\mu\nu\kappa\lambda} + \theta^{\mu\nu\kappa\lambda} (k_{\mu\lambda} k_{\nu\kappa} - k_{\mu\kappa} k_{\nu\lambda}) \right. \\
& + 4 \theta^{\mu\nu\lambda} {}^{(2)}D_\mu k_{\lambda\nu} - {}^{(2)}D_\nu k_{\lambda\mu} + 4 \theta^{\mu\nu} (-k_\mu{}^\sigma k_{\sigma\nu} + {}^{(2)}D_\mu a_\nu - a_\mu a_\nu) + 4 k \theta^{\mu\nu} k_{\mu\nu} \Big) \\
& + 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} N \left[\phi^{\mu\nu\kappa} {}^{(2)}D_\mu \left(\frac{\Pi_{\nu\kappa}}{\sqrt{h}} \right) + \phi^{\mu\nu\kappa} k_{\mu\kappa} \frac{\Pi_\nu}{\sqrt{h}} - \phi^{\mu\kappa}{}_\parallel k_\mu{}^\sigma \frac{\Pi_{\sigma\kappa}}{\sqrt{h}} + \phi^{\mu\kappa}{}_\parallel {}^{(2)}D_\mu \left(\frac{\Pi_\kappa}{\sqrt{h}} \right) \right. \\
& - 2 \phi^{\mu\nu}{}_\parallel \frac{a_{(\mu} \Pi_{\nu)}}{\sqrt{h}} - \phi^{\mu\nu}{}_\parallel k_\mu{}^\kappa \frac{\Pi_{\nu\kappa}}{\sqrt{h}} + \phi^{\mu\nu}{}_\parallel {}^{(2)}D_\mu \left(\frac{\Pi_\nu}{\sqrt{h}} \right) + \phi^\nu{}_\parallel a^\mu \frac{\Pi_{\mu\nu}}{\sqrt{h}} - 2 \phi^\mu{}_\parallel k_\mu{}^\sigma \frac{\Pi_\sigma}{\sqrt{h}} + \phi^\mu{}_\parallel {}^{(2)}D_\mu \left(\frac{\Pi_0}{\sqrt{h}} \right) \\
& - \phi^\mu{}_\parallel a_\mu \frac{\Pi_0}{\sqrt{h}} + k \phi^\mu{}_\parallel \frac{\Pi_\mu}{\sqrt{h}} \Big] - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} N \left(-\frac{1}{2} f(\varrho^{MN}{}_{K\Lambda}) + \sigma \Psi \cdot \Omega + 2 \sigma \sqrt{h} \phi^{\mu\nu\kappa} \rho_{\mu\nu\kappa} \right) \\
& - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} N \left(\frac{1}{h} (2 \sigma p \cdot \Pi - \sigma \Psi \cdot \Pi \cdot \Pi - \sigma \Pi(\Psi \cdot \Pi)) - \frac{1}{2} \phi^{\mu\nu\kappa\lambda} \left(\frac{2 \sigma}{h} \Pi_{\mu\lambda} \Pi_{\nu\kappa} - \rho_{\mu\nu\kappa\lambda} \right) \right)
\end{aligned} \tag{A.54}$$

All the terms in Eq. (A.53) that contain radial derivatives are colored with blue. We thus got Eq. (2.47) presented in Subsection 2.5.1.

To derive the purely bulk form of the microcanonical action (2.57) we consider the boundary terms in (2.52) together with the boundary terms $B_{\underline{N}}$ given in Eq. (2.38) and B_N given in Eq. (2.37) in formula (2.36) for I_{ADM} . We first rewrite B_N in Eq. (2.37) in terms of the projected fields (2.45) using (A.52)

$$\begin{aligned}
B_N &= \int_{\tau_1}^{\tau_2} d\tau \int_{\mathcal{S}_\tau} d^{D-2} x \sqrt{s} N \left[2 {}^{(2)}D_\lambda \Psi^\lambda_\parallel + k \Psi_0 + D_\perp \Psi_0 - k_{\alpha\beta} \Psi^\alpha_\parallel \Psi^\beta - 2 \Psi^\sigma_\parallel a_\sigma + 2 k_{\alpha\beta} \theta^{\alpha\beta} \right. \\
& \quad \left. + 4 \phi^\lambda_\parallel \frac{\Pi_\lambda}{\sqrt{h}} \right] + \int_{\tau_1}^{\tau_2} d\tau \int_{\mathcal{S}_\tau} d^{D-2} x \sqrt{s} ({}^{(2)}D_\lambda (N \Psi^\lambda_\parallel)),
\end{aligned} \tag{A.55}$$

where the last term above can be dropped since it is a total derivative on $\mathcal{S}_\tau$. From (2.51), (A.55) and (2.37)

we obtain

$$\begin{aligned} \int_{\tau_1}^{\tau_2} d\tau \int_{\mathcal{S}_\tau} d^{D-2} x \sqrt{s} \left(N \epsilon_{\text{BY}} - J_a N^a \right) + B_N + B_{\underline{N}} = \\ = - \int_{\tau_1}^{\tau_2} d\tau \int_{\mathcal{S}_\tau} d^{D-2} x \sqrt{s} \left[4N \phi_{|}^{ab} \frac{\Pi_{ab}^{|}}{\sqrt{h}} + 4\sigma s_{af} \phi_{2,|}^{(bf)(2)} D_b N^a \right], \end{aligned} \quad (\text{A.56})$$

where we dropped total derivative terms on $\mathcal{S}_\tau$. Substituting (A.56) into (2.52) we obtain after a long calculation the final expression for Eq. (2.57)

$$\begin{aligned} I_{\text{micro}} = & \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x (p^{\mu\nu} \partial_\tau h_{\mu\nu} + \Pi_{\mu\nu} \partial_\tau \Psi^{\mu\nu}) \\ & + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \sqrt{h} N \underline{D}_\tau \epsilon_{\text{BY}} - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} N_{|}^a \underline{D}_\tau J_a + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} N k \epsilon_{\text{BY}} \\ & - 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \phi_{2,|}^{(\nu\kappa)} \underline{D}_\tau \left(N (\Pi_{\nu\kappa}^{|}/\sqrt{h}) + \sigma^{(2)} D_\nu N_{\kappa}^{|} \right) \\ & + 2 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} N k_{\mu\nu} \underline{D}_\tau \theta^{\mu\nu} + 4 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} N (\underline{D}_\tau \phi_{|}^\mu) \frac{\Pi_\mu^{|}}{\sqrt{h}} \\ & + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} (\underline{D}_\tau N_0) \left(2r^\mu \frac{\Pi_{\mu\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2r_\mu \frac{p^{\mu\nu}}{\sqrt{h}} r_\nu \right) \\ & - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} N_0 \left(\frac{\Pi_{\alpha\beta}^{|}}{\sqrt{h}} \underline{D}_\tau \Psi_{|}^{\alpha\beta} + 2 \frac{\Pi_\beta^{|}}{\sqrt{h}} \underline{D}_\tau \Psi_{|}^\beta + \frac{\Pi_0}{\sqrt{h}} \underline{D}_\tau \Psi_0 \right) \\ & + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \mathcal{G}' (k_{\mu\nu}, a_\mu, \text{other field combinations with no radial derivatives}), \end{aligned} \quad (\text{A.57})$$

where

$$\begin{aligned} \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \mathcal{G}' = & \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \mathcal{G} \\ & - \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} k N_{|}^\sigma \left(2 \frac{\Pi_{\sigma\lambda}}{\sqrt{h}} \Psi^{\lambda\nu} r_\nu - 2h_{\sigma\lambda} \frac{p^{\lambda\nu}}{\sqrt{h}} r_\nu \right) - 4\sigma \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} k \phi_{2,|}^{(\alpha\beta)(2)} D_\alpha N_{\beta}^{|} \\ & + \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} d^{D-1} x \sqrt{h} \left[4\sigma N_{\lambda}^{|} k_{\alpha}^{\lambda} a_{\beta} \phi_{2,|}^{(\alpha\beta)} + 4\sigma N_{\beta}^{|} k_{\alpha}^{\sigma(2)} D_\sigma \phi_{2,|}^{(\alpha\beta)} - 4\sigma N_{\beta'}^{|} a^{\beta'} k_{\alpha\beta} \phi_{2,|}^{(\alpha\beta)} \right. \\ & \left. - 4\sigma N_{\beta}^{|} a^{\lambda} k_{\lambda\alpha} \phi_{2,|}^{(\alpha\beta)} + 4\sigma N_{\beta}^{|} ({}^{(2)}D_{\lambda} k_{\kappa}^{\lambda} - {}^{(2)}D_{\kappa} k) \phi_{2,|}^{(\kappa\beta)} + 4\sigma N_{|}^{\beta} ({}^{(2)}D_{\beta} k_{\kappa\alpha} - {}^{(2)}D_{\kappa} k_{\beta\alpha}) \phi_{2,|}^{(\kappa\alpha)} \right], \end{aligned} \quad (\text{A.58})$$

where the first term on the right hand side is given in Eq. (A.54).

A.5 SdS entropy/microcanonical data

In this appendix we present details of the computation of the microcanonical boundary data $\underline{B}_i^{\text{micro}}$ in Eqs. (2.66) and SdS entropy from Wald's formula given in Eq. (2.68).

We are going to use some geometric facts about the SdS geometry given in Eq. (2.65)

$$ds^2 = \mathcal{N}^2 f(\rho) d\tau^2 + f(\rho)^{-1} d\rho^2 + \rho^2 d\Omega^2, \quad f(\rho) = 1 - 2M/\rho - \rho^2/L^2, \quad (\text{A.59})$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 := t_{ab}dx^a dx^b$ and $x^a = \{\theta, \varphi\}$ are the usual polar coordinates on the sphere. We note that the quantities computed hold for points except at $\rho = r$ where the lapse is discontinuous.⁴ The normal unit form n and vector $\underline{n}$ defined in Eq. (2.19) in $(\tau, \rho, \theta, \varphi)$ coordinates are

$$n|_{\text{SdS}} = \mathcal{N}f(\rho)^{1/2}d\rho \quad , \quad \underline{n}|_{\text{SdS}} = \mathcal{N}^{-1}f(\rho)^{-1/2}\underline{\partial}_\tau . \quad (\text{A.60})$$

The induced metric (2.40) on Σ_τ time slices in $x^\mu = \{\rho, \theta, \varphi\}$ is

$$h_{\mu\nu}dx^\mu dx^\nu = f(\rho)^{-1}d\rho^2 + \rho^2 t_{ab}dx^a dx^b . \quad (\text{A.61})$$

The radial unit one form r and vector $\underline{r}$ defined in Eq. (2.41) are $r|_{\text{SdS}} = f(\rho)^{-1/2}d\rho$, $\underline{r}|_{\text{SdS}} = f(\rho)^{1/2}\underline{\partial}_\rho$, the radial acceleration $a_\mu|_{\text{SdS}} = 0$, and the radial extrinsic curvature tensor (2.42) components $k_{\mu\nu}$ are

$$\begin{aligned} k_{ab}|_{\text{SdS}} &= \rho f(\rho)^{1/2}t_{ab} \quad , \quad k_{\rho\rho} = k_{\rho a} = 0 , \\ k|_{\text{SdS}} &:= h^{ab}k_{ab} = 2\rho^{-1}f(\rho)^{1/2} . \end{aligned} \quad (\text{A.62})$$

It is convenient to use the Gauss Codazzi equations (2.22), (2.43) and the fact that for SdS $K_{MN} = 0$ to find the following Riemann tensor components for SdS

$$\begin{aligned} \mathcal{R}_{\rho\tau\rho\tau}|_{\text{SdS}} &= -\frac{1}{2}\mathcal{N}^2 f''(\rho) \quad , \quad \mathcal{R}_{a\tau b\tau}|_{\text{SdS}} = -\frac{1}{2}\mathcal{N}^2 \rho f(\rho) f'(\rho) t_{ab} \quad , \quad \mathcal{R}_{\mu\nu\lambda\tau}|_{\text{SdS}} = 0 , \\ \mathcal{R}_{a\rho b\rho}|_{\text{SdS}} &= -\frac{\rho}{2} \frac{f'(\rho)}{f(\rho)} t_{ab} \quad , \quad \mathcal{R}_{abc\rho}|_{\text{SdS}} = 0 \quad , \quad \mathcal{R}_{abcd}|_{\text{SdS}} = \rho^2 (f(\rho) - 1) (t_{ad}t_{cb} - t_{ac}t_{bd}) . \end{aligned} \quad (\text{A.63})$$

The formulas above will be useful for computing the boundary data $\underline{B}_i^{\text{micro}}$. First we compute $\phi^{MNK\Lambda}$ for SdS using the formula in Eq. (2.63). We find

$$\varphi^{MNK\Lambda}|_{\text{SdS}_4} = \left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) (g^{MK}g^{N\Lambda} - g^{NK}g^{M\Lambda})|_{\text{SdS}} + 2b_1 \mathcal{R}^{MNK\Lambda}|_{\text{SdS}} , \quad (\text{A.64})$$

where $\Lambda = 3/L^2$. We use the above formula (A.64) to compute (some) of the spatial fields defined in Eq. (2.26) for SdS using the formulas (A.63). We find

$$\begin{aligned} \phi^{a\rho b\rho}|_{\text{SdS}} &= \rho^{-2}f(\rho) \left[\left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) - b_1 \rho^{-1}f'(\rho) \right] t^{ab} , \\ \phi^{abc\rho}|_{\text{SdS}} &= 0 , \\ \phi^{abcd}|_{\text{SdS}} &= \rho^{-4} \left[\left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) - 2b_1 \rho^{-2}(f(\rho) - 1) \right] (t^{ac}t^{bd} - t^{bc}t^{ad}) , \\ \phi^{abcd}|_{\text{SdS}} &= \rho^{-4} \left[\left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) - 2b_1 \rho^{-2}(f(\rho) - 1) \right] (t^{ac}t^{bd} - t^{bc}t^{ad}) , \\ \phi^{\mu\nu\rho}|_{\text{SdS}} &= 0 , \\ \Psi^{ab}|_{\text{SdS}} &= 2\rho^{-2} \left[-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) - b_1 \rho^{-1}f'(\rho) \right] t^{ab} , \\ \Psi^{\rho a}|_{\text{SdS}} &= 0 , \\ \Psi^{\rho\rho}|_{\text{SdS}} &= 2f(\rho) \left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) - b_1 f''(\rho) \right) . \end{aligned} \quad (\text{A.65})$$

⁴For the continuous quantities that we will compute, the value at $\rho = r$ can be defined by taking the limit.

Having found the spatial fields components above we can compute the radial projected field defined in (2.46) using (A.65),

$$\begin{aligned}
N^\mu|_{\text{SdS}} &= 0 \quad , \quad N_0|_{\text{SdS}} = 0, \\
\Pi^\mu_{\mu\nu}|_{\text{SdS}} &= 0 \quad , \quad \Pi^\mu_\mu|_{\text{SdS}} = 0 \quad , \quad \Pi_0|_{\text{SdS}} = 0, \\
\Psi^\mu|_{\text{SdS}} &= 2\rho^{-2} \left[-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) - b_1\rho^{-1}f'(\rho) \right] t^{ab} \quad , \quad \Psi^\mu|_{\text{SdS}} = 0 \quad , \\
\Psi_0|_{\text{SdS}} &= -\frac{1}{8\pi G_N} + 2\Lambda(4b_3 + b_2) - 2b_1f''(\rho) \quad , \quad D_r\Psi_0 = -2b_1f(\rho)^{1/2}f'''(\rho), \\
\theta^{abcd}|_{\text{SdS}} &= \rho^{-4} \left[\left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) - 2b_1\rho^{-2}(f(\rho) - 1) \right] (t^{ac}t^{bd} - t^{bc}t^{ad}) \quad , \quad \theta^{abc}|_{\text{SdS}} = 0, \\
\theta^{ab}|_{\text{SdS}} &= \rho^{-2} \left[\left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) - b_1\rho^{-1}f'(\rho) \right] t^{ab} \quad , \\
\phi^{abc}|_{\text{SdS}} &= 0, \quad \phi^{ab}_{1,|}|_{\text{SdS}} = 0, \quad \phi^{ab}_{2,|}|_{\text{SdS}} = 0, \quad \phi^a|_{\text{SdS}} = 0.
\end{aligned} \tag{A.66}$$

We can now compute the $\underline{B}_i^{\text{micro}}$, namely the fields in Eq. (2.58) for SdS₄, namely

$$\begin{aligned}
\epsilon_{\text{BY}}|_{\text{SdS}} &= -2\rho^{-1}f(\rho)^{1/2} \left(-\frac{1}{8\pi G_N} + 2\Lambda(4b_3 + b_2) \right) + 2b_1f(\rho)^{1/2} (2\rho^{-1}f''(\rho) + f'''(\rho)) \quad , \\
J_a|_{\text{SdS}} &= 0 \quad , \quad s_{ab}|_{\text{SdS}} = \rho^2 t_{ab} \quad , \quad \Psi_0|_{\text{SdS}} = -\frac{1}{8\pi G_N} + 2\Lambda(4b_3 + b_2) - 2b_1f''(\rho) \quad , \\
\theta^{ab}|_{\text{SdS}} &= \rho^{-2} \left[\left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) - b_1\rho^{-1}f'(\rho) \right] t^{ab} \quad , \quad \phi^a|_{\text{SdS}} = 0 \quad , \\
N(\Pi^\mu_{ab}/\sqrt{h}) + {}^{(2)}D_{(a}N_{b)}|_{\text{SdS}} &= 0.
\end{aligned} \tag{A.67}$$

Evaluating the formulas in Eq. (A.67) at the location $\rho = r$ of the factorization surface $\mathcal{T}_r$ we arrive at Eq. (2.66) given in Section 2.6.

We end this appendix by presenting some details of the calculation SdS entropy from Wald's formula given in Eq. (2.68). The first thing to note is that Wal'ds formulas uses Lorentzian signature metrics. This amounts to rotating $\mathcal{N} = i$ and decompactifying the time coordinate $\tau \rightarrow t$ in Eq. (2.65). For $f(\text{Riemann})$ type theories the formula reduces to Eq. (2.68). The minus sign in $\mathcal{L} = -(1/2)f(\mathcal{R}^{MN}_{K\Lambda})$ also comes from the continuation to Lorentzian signature. There will be two contributions to the entropy coming from the two killing horizons for the Lorentzian SdS geometry. If we use Eq. (2.16) and its value for SdS (A.64) into the first line of (2.68) for each horizon $\rho = r_H$ we get

$$\begin{aligned}
S_{\text{W}}^{\text{SdS}}|_{r_H} &= -2\pi \oint_{\text{horzn.}} d\theta d\varphi \sqrt{s} \hat{\epsilon}^{MN} \hat{\epsilon}_{K\Lambda} \frac{\partial \mathcal{L}}{\partial \mathcal{R}^{MN}_{K\Lambda}} \\
&= \pi \oint_{\text{horzn.}} d\theta d\varphi \sqrt{s} \hat{\epsilon}^{MN} \hat{\epsilon}_{K\Lambda} \varphi_{MN}{}^{K\Lambda} \\
&= \pi \oint_{\text{horzn.}} d\theta d\varphi \sqrt{s} \left[\left(-\frac{1}{16\pi G_N} + \Lambda(4b_3 + b_2) \right) 2\hat{\epsilon}^{MN} \hat{\epsilon}_{MN} + 2b_1\hat{\epsilon}^{MN} \hat{\epsilon}^{K\Lambda} \mathcal{R}^{\text{SdS}}_{MNK\Lambda} \right] \\
&= \pi r_H^2 - 48\pi^2 \frac{r_H^2}{L^2} (4b_3 + b_2) + 8b_1\pi^2 r_H^2 \hat{\epsilon}^{MN} \hat{\epsilon}^{K\Lambda} \mathcal{R}^{\text{SdS}}_{MNK\Lambda}
\end{aligned} \tag{A.68}$$

where we used $\Lambda = 3/L^2$, $\hat{\epsilon}^{MN}\hat{\epsilon}_{MN} = -2$, since $\tilde{\xi}$ is normalized to unit surface gravity. The quantity $\hat{\epsilon}^{MN}\hat{\epsilon}^{K\Lambda}\mathcal{R}_{MNK\Lambda}^{\text{SdS}}$ can be computed at the static coordinate system at some $\rho = r$ and then take the horizon limit $r \rightarrow r_H$. Being a scalar quantity we expect to get the correct value at horizon by taking the limit even though the static coordinates break down at the horizon. We find using (A.63) that $\lim_{\rho \rightarrow r_H} \hat{\epsilon}^{MN}\hat{\epsilon}^{K\Lambda}\mathcal{R}_{MNK\Lambda}^{\text{SdS}} = 4\mathcal{R}_{\tau\rho\tau\rho}^{\text{SdS}}|_{r_H} = -4/r_H^2$, and if we substitute to the last line of Eq. (A.68) we finally find

$$S_W^{\text{SdS}}|_{r_H} = \pi r_H^2 - 48\pi^2 \frac{r_H^2}{L^2} (4b_3 + b_2) - 32\pi^2 b_1. \quad (\text{A.69})$$

If we add the results for both black hole and cosmological horizons $r_H = r_{b,c}$ we find the result presented in Eqs. (2.67)-(2.68).

Appendix B

Appendices for Chapter 5

B.1 Weyl transformations

For $\tilde{g}_{ij} = \mathcal{B}(x)^{-2}g_{ij}$ we have

$$\mathring{\Gamma}^i_{jk} = \mathring{\Gamma}^i_{jk} - \delta^i_k \partial_j \ln \mathcal{B}(x) - \delta^i_j \partial_k \ln \mathcal{B}(x) + g_{jk} g^{im} \partial_m \ln \mathcal{B}(x) \quad (\text{B.1})$$

$$\mathring{P}_{ij} = \mathring{P}_{ij} + \mathring{\nabla}_i \mathring{\nabla}_j \ln \mathcal{B}(x) + \mathring{\nabla}_i \ln \mathcal{B}(x) D_j \ln \mathcal{B}(x) - \frac{1}{2} (\mathring{\nabla}_k \ln \mathcal{B}(x) \mathring{\nabla}^k \ln \mathcal{B}(x)) g_{ij} \quad (\text{B.2})$$

$$\mathring{C}_{ijk} = \mathring{C}_{ijk} - \partial_m \ln \mathcal{B}(x) \mathring{W}_{jki}{}^m \quad (\text{B.3})$$

$$\mathring{B}_{ij} = \mathcal{B}(x)^2 \left(\mathring{B}_{ij} - 2(d-4) \mathring{\nabla}_k \ln \mathcal{B}(x) \mathring{C}_{(ij)}{}^k - (d-4) \mathring{\nabla}_k \ln \mathcal{B}(x) \mathring{\nabla}_l \ln \mathcal{B}(x) \mathring{W}_{ij}{}^{kl} \right) \quad (\text{B.4})$$

B.2 Metric variations

We provide metric variations of various geometric quantities used in Subsection 5.3.2.

The definitions of various tensors (see Section 5.1)

$$\mathring{R}^i_{jkl} = \partial_k \mathring{\Gamma}^i_{jl} - \partial_l \mathring{\Gamma}^i_{jk} + \mathring{\Gamma}^i_{km} \mathring{\Gamma}^m_{lj} - \mathring{\Gamma}^i_{lm} \mathring{\Gamma}^m_{kj} \quad (\text{B.5})$$

$$\mathring{P}_{ij} = \frac{1}{d-2} \left(\mathring{R}_{ij} - \frac{1}{2(d-1)} \mathring{R} g_{ij} \right) \quad (\text{B.6})$$

$$\mathring{W}_{ikjl} = \mathring{R}_{ikjl} + g_{il} \mathring{P}_{kj} + g_{kj} \mathring{P}_{il} - g_{ij} \mathring{P}_{kl} - g_{kl} \mathring{P}_{ij} \quad (\text{B.7})$$

$$\mathring{C}_{ijk} = \mathring{\nabla}_k \mathring{P}_{ij} - \mathring{\nabla}_j \mathring{P}_{ik} \quad (\text{B.8})$$

$$\mathring{B}_{ij} = \mathring{\nabla}^k \mathring{C}_{ijk} + \mathring{P}^{kl} \mathring{W}_{ikjl} . \quad (\text{B.9})$$

Variations of metric, LC-connection, Riemann, Ricci and Weyl

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{ij}\delta g_{ij} \quad (\text{B.10})$$

$$\delta g^{ij} = -g^{im}g^{jn}\delta g_{mn}, \quad \delta g_{ij} = -g_{im}g_{jn}\delta g^{mn} \quad (\text{B.11})$$

$$\delta\overset{\circ}{\Gamma}_{jk}^i = \frac{1}{2}g^{im}\left(\overset{\circ}{\nabla}_j\delta g_{mk} + \overset{\circ}{\nabla}_k\delta g_{mj} - \overset{\circ}{\nabla}_m\delta g_{jk}\right) \quad (\text{B.12})$$

$$\delta\overset{\circ}{\Gamma}_{jk}^k = \frac{1}{2}g^{km}\overset{\circ}{\nabla}_j\delta g_{mk} \quad (\text{B.13})$$

$$\delta\overset{\circ}{R}_{jkl}^i = g^{im}\overset{\circ}{\nabla}_{[k}\overset{\circ}{\nabla}_{l]}\delta g_{mj} + \frac{1}{2}g^{im}\overset{\circ}{\nabla}_k\overset{\circ}{\nabla}_j\delta g_{ml} + \frac{1}{2}g^{im}\overset{\circ}{\nabla}_l\overset{\circ}{\nabla}_m\delta g_{kj} - \frac{1}{2}g^{im}\overset{\circ}{\nabla}_k\overset{\circ}{\nabla}_m\delta g_{lj} - \frac{1}{2}g^{im}\overset{\circ}{\nabla}_l\overset{\circ}{\nabla}_j\delta g_{mk} \quad (\text{B.14})$$

$$\delta\overset{\circ}{R}_{ikjl} = \overset{\circ}{R}_{kjl}^m\delta g_{im} + \overset{\circ}{\nabla}_{[j}\overset{\circ}{\nabla}_{l]}\delta g_{ik} + \frac{1}{2}\overset{\circ}{\nabla}_j\overset{\circ}{\nabla}_k\delta g_{il} + \frac{1}{2}\overset{\circ}{\nabla}_l\overset{\circ}{\nabla}_i\delta g_{kj} - \frac{1}{2}\overset{\circ}{\nabla}_j\overset{\circ}{\nabla}_i\delta g_{kl} - \frac{1}{2}\overset{\circ}{\nabla}_l\overset{\circ}{\nabla}_k\delta g_{ij} \quad (\text{B.15})$$

$$\delta\overset{\circ}{R}_{jl} = \frac{1}{2}g^{km}\left(\overset{\circ}{\nabla}_k\overset{\circ}{\nabla}_l\delta g_{mj} + \overset{\circ}{\nabla}_k\overset{\circ}{\nabla}_j\delta g_{ml} - \overset{\circ}{\nabla}_k\overset{\circ}{\nabla}_m\delta g_{lj}\right) - \frac{1}{2}g^{mn}\overset{\circ}{\nabla}_l\overset{\circ}{\nabla}_j\delta g_{mn} \quad (\text{B.16})$$

$$\delta\overset{\circ}{W}_{ikjl} = \delta\overset{\circ}{R}_{ikjl} + (\delta g_{il})\overset{\circ}{P}_{kj} + g_{il}\delta\overset{\circ}{P}_{kj} + (\delta g_{kj})\overset{\circ}{P}_{il} + g_{kj}\delta\overset{\circ}{P}_{il} - (\delta g_{ij})\overset{\circ}{P}_{kl} - g_{ij}\delta\overset{\circ}{P}_{kl} - (\delta g_{kl})\overset{\circ}{P}_{ij} - g_{kl}\delta\overset{\circ}{P}_{ij}, \quad (\text{B.17})$$

where $\overset{\circ}{\nabla}_{[i}\overset{\circ}{\nabla}_{j]} = \frac{1}{2}(\overset{\circ}{\nabla}_i\overset{\circ}{\nabla}_j - \overset{\circ}{\nabla}_j\overset{\circ}{\nabla}_i)$.

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$$\begin{aligned} \delta\overset{\circ}{P}_{ij} = & \frac{1}{2(d-2)}\left(\overset{\circ}{\nabla}^m\overset{\circ}{\nabla}_j(\delta g_{mi}) + \overset{\circ}{\nabla}^m\overset{\circ}{\nabla}_i(\delta g_{mj}) - \overset{\circ}{\nabla}^m\overset{\circ}{\nabla}_m(\delta g_{ij}) - g^{mn}\overset{\circ}{\nabla}_i\overset{\circ}{\nabla}_j(\delta g_{mn})\right. \\ & \left. - \frac{1}{d-1}(\overset{\circ}{\nabla}^m\overset{\circ}{\nabla}^n\delta g_{mn} - g^{mn}\overset{\circ}{\nabla}^k\overset{\circ}{\nabla}_k\delta g_{mn})g_{ij}\right) \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned} & + \frac{1}{2(d-2)(d-1)}\left((d-2)\overset{\circ}{P}^{mn}\delta g_{mn} + \overset{\circ}{P}g^{mn}\delta g_{mn}\right)g_{ij} - \frac{1}{(d-2)}\overset{\circ}{P}\delta g_{ij} \\ \delta\overset{\circ}{P} = & -\frac{1}{2(d-1)}\left((d-2)\overset{\circ}{P}^{mn} + \overset{\circ}{P}g^{mn}\right)\delta g_{mn} - \frac{1}{2(d-1)}(g^{mn}\overset{\circ}{\nabla}^k\overset{\circ}{\nabla}_k - \overset{\circ}{\nabla}^m\overset{\circ}{\nabla}^n)\delta g_{mn} \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} \delta\overset{\circ}{C}_{ijk} = & \overset{\circ}{\nabla}_k\delta\overset{\circ}{P}_{ij} - \overset{\circ}{\nabla}_j\delta\overset{\circ}{P}_{ik} + \frac{1}{2}g^{mr}\overset{\circ}{P}_{mk}\overset{\circ}{\nabla}_j\delta g_{ri} + \frac{1}{2}g^{rm}\overset{\circ}{P}_{mk}\overset{\circ}{\nabla}_i\delta g_{rj} - \frac{1}{2}g^{rm}\overset{\circ}{P}_{mk}\overset{\circ}{\nabla}_r\delta g_{ij} \\ & - \frac{1}{2}g^{rm}\overset{\circ}{P}_{mj}\overset{\circ}{\nabla}_k\delta g_{ri} - \frac{1}{2}g^{rm}\overset{\circ}{P}_{mj}\overset{\circ}{\nabla}_i\delta g_{rk} + \frac{1}{2}g^{rm}\overset{\circ}{P}_{mj}\overset{\circ}{\nabla}_r\delta g_{ki}, \end{aligned} \quad (\text{B.20})$$

$$\begin{aligned}
\delta \hat{B}_{ij} = & \hat{\nabla}^k \hat{\nabla}_k \delta \hat{P}_{ij} - \hat{\nabla}^k \hat{\nabla}_j \delta \hat{P}_{ik} - \hat{\nabla}_k (\hat{C}_{ij}{}^r g^{km} \delta g_{rm}) + \hat{P}^{kl} \hat{\nabla}_{[j} \hat{\nabla}_{l]} \delta g_{ik} + \hat{P}^{kl} \hat{\nabla}_j \hat{\nabla}_{[k} \delta g_{i]l} \\
& + \hat{P}^{kl} \hat{\nabla}_l \hat{\nabla}_{[i} \delta g_{k]j} + \hat{P}_i^k \delta \hat{P}_{kj} + \hat{P}_j^k \delta \hat{P}_{ki} - g_{ij} \hat{P}^{kl} \delta \hat{P}_{kl} - \hat{P} \delta \hat{P}_{ij} + \hat{W}_{ikjl} g^{km} g^{ln} \delta \hat{P}_{mn} \\
& - \frac{1}{2} g^{mr} (\hat{\nabla}_k \delta g_{ri} + \hat{\nabla}_i \delta g_{rk} - \hat{\nabla}_r \delta g_{ki}) \hat{C}_{mj}{}^k - \frac{1}{2} g^{mr} (\hat{\nabla}_k \delta g_{rj} + \hat{\nabla}_j \delta g_{rk} - \hat{\nabla}_r \delta g_{kj}) \hat{C}_{im}{}^k \\
& + \frac{1}{2} (g^{rs} \hat{\nabla}_m \delta g_{rs}) \hat{C}_{ij}{}^m \\
& + \frac{1}{2} \hat{P}_{mk} g^{ms} \hat{\nabla}^k \hat{\nabla}_i \delta g_{sj} + \frac{1}{2} \hat{P}_{mk} g^{ms} \hat{\nabla}^k \hat{\nabla}_j \delta g_{si} - \frac{1}{2} \hat{P}_{mk} g^{ms} \hat{\nabla}^k \hat{\nabla}_s \delta g_{ij} \\
& - \frac{1}{2} \hat{P}_{mj} g^{ms} \hat{\nabla}^k \hat{\nabla}_i \delta g_{sk} - \frac{1}{2} \hat{P}_{mj} g^{ms} \hat{\nabla}^k \hat{\nabla}_k \delta g_{si} + \frac{1}{2} \hat{P}_{mj} g^{ms} \hat{\nabla}^k \hat{\nabla}_s \delta g_{ik} \\
& + \frac{1}{2} g^{ms} \hat{\nabla}_m \hat{P} \hat{\nabla}_i \delta g_{sj} + \frac{1}{2} g^{ms} \hat{\nabla}_m \hat{P} \hat{\nabla}_j \delta g_{si} - \frac{1}{2} g^{ms} \hat{\nabla}_m \hat{P} \hat{\nabla}_s \delta g_{ij} \\
& - \frac{1}{2} g^{ms} \hat{\nabla}^k \hat{P}_{mj} \hat{\nabla}_i \delta g_{sk} - \frac{1}{2} g^{ms} \hat{\nabla}^k \hat{P}_{mj} \hat{\nabla}_k \delta g_{si} + \frac{1}{2} g^{ms} \hat{\nabla}^k \hat{P}_{mj} \hat{\nabla}_s \delta g_{ik} \\
& + \hat{P}^{kl} \hat{R}^m{}_{kjl} \delta g_{im} - 2 \hat{W}_{ikjl} \hat{P}^{kn} g^{lm} \delta g_{mn} \\
& + \hat{P}_{jk} \hat{P}^{kl} \delta g_{il} + \hat{P}_{ik} \hat{P}^{kl} (\delta g_{jl}) - (\hat{P}^{mn} \hat{P}_{mn}) \delta g_{ij} - \hat{P}_{ij} \hat{P}^{mn} \delta g_{mn} .
\end{aligned} \tag{B.21}$$

B.3 Holographic energy-momentum tensor in WFG gauge

In this appendix we derive the energy momentum tensor vev of the boundary theory using the prescription of [149] (see also [202]) in the WFG gauge. We explicitly prove the formula for the energy-momentum tensor given in Eq. (5.100) for $d = 3$ boundary dimension. We note that the method of [149] uses a cut-off hypersurface at $z = \epsilon$ with $\epsilon \rightarrow 0^+$, which shows that the method of cutoff regularization is applicable in the WFG gauge. This is consistent with the point of view that the properties of the dual holographic theories should be independent of the regularization procedure after the regulator has been removed.¹ It would be interesting to use the same prescription but instead of a z -foliation work with the non-integrable distribution (5.92).

We briefly describe the prescription of [149] for computing the holographic energy-momentum tensor. First one computes the regularized bulk on-shell action

$$S_{\text{EH-GH}}^{\text{reg}} = \frac{1}{2\kappa^2} \int_{z=\epsilon} d^{d+1}x \sqrt{g_{\mu\nu}} (R(g_{\mu\nu}) - 2\Lambda) + \frac{1}{\kappa^2} \int_{z=\epsilon} d^d x \sqrt{h} K, \tag{B.22}$$

where $\kappa^2 = 8\pi G_N$, $\Lambda = -d(d-1)/(2L^2)$, h_{ij} is the induced metric on the hypersurface $z = \epsilon$ and $K = h^{ij} K_{ij}$ is the trace of the extrinsic curvature tensor K_{ij} of the hypersurface $z = \epsilon$, defined in terms of outward pointing normal vector $\underline{n}$ shown in Figure B.1. The regularized energy-momentum tensor vev in the boundary theory is the boundary metric variation of the on-shell action (B.22)

$$\langle T_{ij}^{\text{reg}} \rangle = -\frac{2}{\sqrt{\det h}} \frac{\delta S_{\text{EH-GH}}^{\text{o.s., reg}}}{\delta h^{ij}} = \frac{1}{\kappa^2} \left(\frac{1}{\epsilon^{d-2}} (K h_{ij} - K_{ij}) \right), \tag{B.23}$$

where to derive the second equality above we used Eq. (5.65). The regularized energy-momentum tensor $T_{ij}^{\text{o.s., reg}}$ is divergent when $\epsilon \rightarrow 0^+$ and one needs to add counterterms that cancel the divergences and then remove the regulator. The counterterms are of the form $S^{\text{c.t.}} = \int_{z=\epsilon} d^d x \sqrt{\det h} \mathcal{L}[h]$ where $\mathcal{L}[h]$ is a local diff invariant quantity. The precise form depends on d and usually increases in complexity as the dimension increases. We will explicitly find the counterterms needed in $d = 3$ for the WFG gauge.

¹In this appendix we follow the conventions of Chapter 5, namely we assume the boundary metric to be Euclidean.

Taken into account the counterterms the prescription of the holographic energy-momentum tensor is

$$\langle T_{ij} \rangle = \lim_{\epsilon \rightarrow 0^+} (\langle T_{ij}^{\text{reg}} \rangle + \langle T_{ij}^{\text{c.t.}} \rangle) \quad (\text{B.24})$$

where $\langle T_{ij}^{\text{c.t.}} \rangle = -\frac{2}{\sqrt{\det h}} \frac{\delta S^{\text{c.t.}}}{\delta h^{ij}}$. We will verify the formula for $\langle T_{ij} \rangle$ given in Eq. (5.100)

$$\langle \kappa^2 T_{ij} \rangle = \frac{d}{2L^2} \pi_{ij}^{(0)} + \hat{\nabla}_{(i}^{(0)} p_{j)}^{(0)}, \quad (\text{B.25})$$

for $d = 3$ using the prescription of Eq. (B.24).

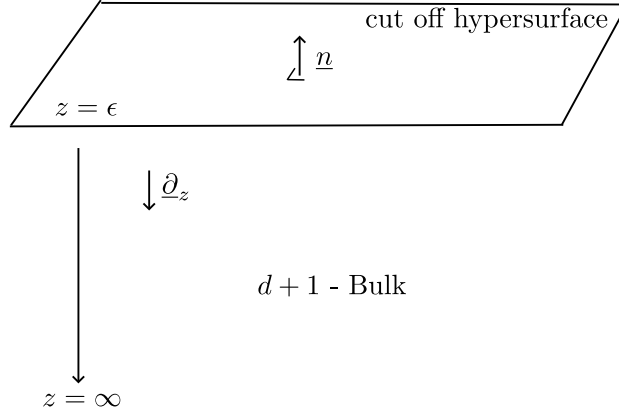


Figure B.1: The cut-off hypersurface at $z = \epsilon$ and the outwards pointing normal vector $\underline{n}$. The extrinsic curvature is defined with respect to $\underline{n}$ with bulk components $K_{\mu\nu} \equiv \gamma^\lambda_\mu \gamma^\rho_\nu \hat{\nabla}_\rho n_\lambda$ and $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ the projector.

To compute formula (B.23) for the regularized energy momentum tensor in the WFG gauge (5.84) it is convenient to use the ADM formalism [157]. In matrix form the WFG metric and inverse matrix components (with column/row order (z, x^i)) are

$$g_{\mu\nu} = \begin{pmatrix} \frac{L^2}{z^2} & -\frac{L^2}{z} a_j(z; x) \\ -\frac{L^2}{z} a_i(z; x) & g_{ij}(z; x) + L^2 a_i(z; x) a_j(z; x) \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \frac{z^2}{L^2} + z^2 a_k(z; x) a^k(z; x) & z a^j(z; x) \\ z a^i(z; x) & g^{ij}(z; x) \end{pmatrix},$$

where $a^i(z; x) = g^{ij}(z; x) a_j(z; x)$.

From the ADM form of the metric

$$ds^2 = N^2 dz^2 + h_{ij}(z; x) (dx^i + N^i dz) (dx^j + N^j dz), \quad (\text{B.26})$$

we can express the ADM data (h_{ij}, N, N^i) in terms of the WFG data (g_{ij}, a_i) .² We find

$$\begin{aligned} h_{ij}(z; x) &= g_{ij}(z; x) + L^2 a_i(z; x) a_j(z; x), \quad N^i = -\left(\frac{L^2}{z}\right) \frac{a^i(z; x)}{1 + L^2 a_k(z; x) a^k(z; x)}, \\ N^2 &= \frac{L^2}{z^2} \left(\frac{1}{1 + L^2 a_k(z; x) a^k(z; x)} \right), \quad h^{ij}(z; x) = g^{ij}(z; x) - \frac{L^2 a^i(z; x) a^j(z; x)}{1 + L^2 a_k(z; x) a^k(z; x)}, \quad N_i = -\frac{L^2}{z} a_i(z; x). \end{aligned} \quad (\text{B.27})$$

We raise/lower indices for the WFG data using the $g_{ij}(z; x)$ metric, namely $a^i(z; x) = g^{ij}(z; x) a_j$, while we

²Note that $h_{ij}(z; x)$ is the induced metric on constant z hypersurfaces.

use $h_{ij}(z; x)$ to raise/lower the ADM data, namely $N_i = h_{ij}(z; x)N^j$.

The outward pointing normal vector $\underline{n}$ (see Figure B.1), the radial evolution vector $\underline{z} = \frac{\partial}{\partial z}$ and the shift vector $\underline{N} = N^i \partial_i$ are related by

$$\underline{z} = -N\underline{n} + \underline{N}. \quad (\text{B.28})$$

The extrinsic curvature tensor is defined as $K_{\mu\nu} \equiv \gamma^\lambda{}_\mu \gamma^\rho{}_\nu \dot{\nabla}_\rho n_\lambda$ where $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the the projector on constant z hypersurface. It can be shown that $K_{ij} = \frac{1}{2N} \mathcal{L}_{N\underline{n}} h_{ij}$ and using Eq. (B.28) one finds that

$$K_{ij} = \frac{1}{2N} \left(\dot{D}_i N_j + \dot{D}_j N_i - \partial_z h_{ij} \right), \quad (\text{B.29})$$

where $\dot{D}_i$ is the LC connection of the induced metric $h_{ij}(z; x)$.

The extrinsic curvature formula (B.29) is important since it allows us to compute $\langle T_{ij}^{\text{reg}} \rangle$ from (B.23) as follows: From the series expansions of the WFG data given in Eqs. (5.87),(5.88) we can use Eq. (B.27) to find the series expansion of the ADM data h_{ij}, N, N^i up to the order required to get finite piece at the end of the calculation. Having the ADM data expansions we can use formula (B.29) to compute K_{ij} as a series expansion in terms of the WFG data and thus the regularized energy momentum tensor in Eq. (B.23). Next we find the appropriate counterterms $S^{\text{c.t}}$ that cancel the divergent parts of the regularized energy-momentum tensor and compute $\langle T_{ij}^{\text{c.t}} \rangle = -\frac{2}{\sqrt{\det h}} \frac{\delta S^{\text{c.t}}}{\delta h^{ij}}$. We finally substitute these results into (B.24) and verify that it matches the right hand side of (B.25).

We now consider the case of $d = 3$. The WFG expansions (5.87),(5.88) are

$$\begin{aligned} g_{ij}(z; x) &= \left[\frac{L^2}{z^2} g_{ij}^{(0)}(x) + g_{ij}^{(2)}(x) + \frac{z^2}{L^2} g_{ij}^{(4)} + \mathcal{O}(z^4) \right] + \left[\frac{z}{L} \pi_{ij}^{(0)} + \frac{z^3}{L^3} \pi_{ij}^{(2)} + \mathcal{O}(z^5) \right], \\ a_i(z; x) &= \left[a_i^{(0)}(x) + \frac{z^2}{L^2} a_i^{(2)}(x) + \mathcal{O}(z^4) \right] + \left[\frac{z}{L} p_i^{(0)} + \mathcal{O}(z^3) \right]. \end{aligned} \quad (\text{B.30})$$

From (B.30) and Eq. (B.27) we find the following expansions (we will set $d = 3$ later in the calculation)

$$\begin{aligned}
g^{ij} &= \frac{z^2}{L^2} g_{(0)}^{ij} - \frac{z^4}{L^4} g_{(2)}^{ij} - \frac{z^5}{L^5} \pi_{(0)}^{ij} + \mathcal{O}(z^6) \\
a_i(z; x) a_j(z; x) &= a_i^{(0)}(x) a_j^{(0)}(x) + \frac{z}{L} a_i^{(0)} p_j^{(0)} + \frac{z}{L} p_i^{(0)} a_j^{(0)} + \mathcal{O}(z^3) \\
h_{ij} &= \frac{L^2}{z^2} g_{ij}^{(0)}(x) + g_{ij}^{(2)} + L^2 a_i^{(0)} a_j^{(0)} + \frac{z}{L} \left(\pi_{ij}^{(0)} + L^2 a_i^{(0)} p_j^{(0)} + L^2 a_j^{(0)} p_i^{(0)} \right) + \frac{z^2}{L^2} g_{ij}^{(4)} + \mathcal{O}(z^3) \\
a_k(z; x) a^k(z; x) &= \frac{z^2}{L^2} g_{(0)}^{ij} a_i^{(0)} a_j^{(0)} + \frac{2z^3}{L^3} a_{(0)} \cdot p_{(0)} + \mathcal{O}(z^4) \\
N_i &= -\frac{L^2}{z} a_i^{(0)} - L p_i^{(0)} - z a_i^{(2)} + \mathcal{O}(z^2) \\
\mathring{\Gamma}_{ij}^k(z; x) &= \mathring{\Gamma}_{(0)}^k{}_{ij} + \mathcal{O}(z^2) \\
\mathring{G}_{ij}[h_{ij}] &= \mathring{G}_{ij}^{(0)} + \mathcal{O}(z^2) \\
\frac{L^2 a^i(z; x) a^j(z; x)}{1 + L^2 a_k(z; x) a^k(z; x)} &= \frac{z^4}{L^2} a_{(0)}^i a_{(0)}^j + \frac{z^5}{L^3} 2a_{(0)}^{(i} p_{(0)}^{j)} + \mathcal{O}(z^6) \\
h^{ij}(z; x) &= \frac{z^2}{L^2} g_{(0)}^{ij} - \frac{z^4}{L^4} \left(g_{(2)}^{ij} + L^2 a_{(0)}^i a_{(0)}^j \right) - \frac{z^5}{L^5} \left(\pi_{(0)}^{ij} + 2L^2 a_{(0)}^{(i} p_{(0)}^{j)} \right) + \mathcal{O}(z^6) \\
\frac{1}{N} &= \frac{z}{L} + \frac{z^3}{2L} a_{(0)}^2 + \frac{z^4}{L^2} a_{(0)} \cdot p_{(0)} + \mathcal{O}(z^5) \\
\mathring{D}_i N_j &= -\frac{L^2}{z} \mathring{\nabla}_i^{(0)} a_j^{(0)} - L \mathring{\nabla}_i p_j^{(0)} + \mathcal{O}(z) \\
-\frac{1}{2N} \partial_z h_{ij}(z; x) &= \frac{L}{z^2} g_{ij}^{(0)} - \frac{z}{2L^2} \left(\pi_{ij}^{(0)} + 2L^2 a_{(0)}^{(i} p_{(0)}^{j)} - 2L^2 (a^{(0)} \cdot p^{(0)}) g_{ij}^{(0)} \right) + \frac{L}{2} a_{(0)}^2 g_{ij}^{(0)} + \mathcal{O}(z^2)
\end{aligned} \tag{B.31}$$

where $\mathring{\Gamma}_{ij}^k = \frac{1}{2} h^{km}(z; x) (\partial_i h_{mj}(z; x) + \partial_j h_{im}(z; x) - \partial_m h_{ij}(z; x))$ is the LC connection of the induced metric $h_{ij}(z; x)$, $\mathring{\nabla}^{(0)}$ is the LC connection of $g_{ij}^{(0)}$ and $a_{(0)}^2 = g_{(0)}^{ij} a_i^{(0)} a_j^{(0)}$, $a^{(0)} \cdot p^{(0)} = g_{(0)}^{ij} a_i^{(0)} p_j^{(0)}$.

From Eq. (B.31) and Eq. (B.29) we find

$$\begin{aligned}
K_{ij} &= \frac{L}{z^2} g_{ij}^{(0)} + \left(\frac{L}{2} a_{(0)}^2 g_{ij}^{(0)} - L \mathring{\nabla}_i^{(0)} a_j^{(0)} \right) - \frac{z}{2L^2} \left(\pi_{ij}^{(0)} + 2L^2 a_{(0)}^{(i} p_{(0)}^{j)} - 2L^2 (a^{(0)} \cdot p^{(0)}) g_{ij}^{(0)} + 2L^2 \mathring{\nabla}_i^{(0)} p_j^{(0)} \right) \\
&\quad + \mathcal{O}(z^2) \\
K &= \frac{d}{L} + \frac{z^2}{L^2} \left(\frac{d-2}{2} L a_{(0)}^2 - \frac{1}{L} \text{tr} (g_{(2)}) - L \mathring{\nabla}^{(0)} \cdot a_{(0)} \right) \\
&\quad - \frac{z^3}{2L^4} \left(3\pi_{(0)} - 2L^2 (d-3) a_{(0)} \cdot p_{(0)} + 2L^2 \mathring{\nabla}^{(0)} \cdot p_{(0)} \right) + \mathcal{O}(z^4) \\
K h_{ij}(z; x) &= \frac{L^2}{z^2} \frac{d}{L} g_{ij}^{(0)} + \left(\frac{d}{L} g_{ij}^{(2)} + d L a_i^{(0)} a_j^{(0)} + g_{ij}^{(0)} \left(\frac{d-2}{2} L a_{(0)}^2 - \frac{1}{L} \text{tr} (g_{(2)}) - L \mathring{\nabla}^{(0)} \cdot a_{(0)} \right) \right) \\
&\quad + \frac{z}{L^2} \left(d \pi_{ij}^{(0)} + 2d L^2 a_{(0)}^{(i} p_{(0)}^{j)} - \frac{1}{2} g_{ij}^{(0)} \left(3\pi_{(0)} - 2L^2 (d-3) a_{(0)} \cdot p_{(0)} + 2L^2 \mathring{\nabla}^{(0)} \cdot p_{(0)} \right) \right) + \mathcal{O}(z^2),
\end{aligned} \tag{B.32}$$

where $\pi_{(0)} = g_{(0)}^{ij} \pi_{ij}^{(0)}$, $\text{tr} g_{(2)} = g_{ij}^{(0)} g_{ij}^{(2)}$, $\mathring{\nabla}^{(0)} \cdot p_{(0)} = g_{(0)}^{ij} \mathring{\nabla}_i^{(0)} p_j^{(0)}$.

We substitute expansions (B.32) into (B.23) and after setting $z = \epsilon$ we obtain

$$\begin{aligned} \langle T_{ij}^{\text{reg}} \rangle &= \frac{d-1}{\kappa^2} \frac{L}{\epsilon^d} g_{ij}^{(0)} + \frac{1}{\kappa^2 \epsilon^{d-2}} \left(\frac{d}{L} g_{ij}^{(2)} + d L a_i^{(0)} a_j^{(0)} + L \overset{\circ}{\nabla}_{(i}^{(0)} a_{j)}^{(0)} + g_{ij}^{(0)} \left(\frac{d-3}{2} L a_{(0)}^2 - \frac{1}{L} \text{tr}(g_{(2)}) - L \overset{\circ}{\nabla}^{(0)} \cdot a_{(0)} \right) \right) \\ &+ \frac{1}{2\kappa^2 L^2} \left((1+2d) \pi_{ij}^{(0)} + (2+4d) L^2 a_{(i}^{(0)} p_{j)}^{(0)} + 2L^2 \overset{\circ}{\nabla}_{(i}^{(0)} p_{j)}^{(0)} - g_{ij}^{(0)} \left(3\pi_{(0)} + 2L^2 (4-d) a_{(0)} \cdot p_{(0)} + 2L^2 \overset{\circ}{\nabla}^{(0)} \cdot p_{(0)} \right) \right) \\ &+ \mathcal{O}(\epsilon^2). \end{aligned} \quad (\text{B.33})$$

Note that for $d = 3$ the first two terms in the expression above are divergent in the $\epsilon \rightarrow 0^+$ limit. These terms will be cancelled from the adding the following two counterterms

$$\begin{aligned} S^{c.t.1} &= \frac{1-d}{L\kappa^2} \int_{z=\epsilon} d^d x \sqrt{\det h}, \\ S^{c.t.2} &= -\frac{2L}{\kappa^2} \int_{z=\epsilon} d^d x \sqrt{\det \gamma} R[h]. \end{aligned} \quad (\text{B.34})$$

The above counterterms one can compute $\langle T_{ij}^{c.t.1,2} \rangle = -\frac{2}{\sqrt{\det h}} \frac{\delta S^{c.t.1,2}}{\delta h^{ij}}$. We find

$$\begin{aligned} \langle T_{ij}^{c.t.1} \rangle &= \frac{1}{\epsilon^{d-2}} \frac{1-d}{L\kappa^2} h_{ij}(\epsilon; x), \\ \langle T_{ij}^{c.t.2} \rangle &= \frac{1}{\epsilon^{d-2}} \frac{2L}{2\kappa^2} \overset{\circ}{G}_{ij}[h_{ij}(\epsilon; x)], \end{aligned} \quad (\text{B.35})$$

where $\overset{\circ}{G}_{ij}[h]$ is the LC Einstein tensor of the induced metric $h_{ij}(z = \epsilon; x)$. From the expansions in Eq. (B.31) we can expand (B.35)

$$\begin{aligned} \langle T_{ij}^{c.t.1} \rangle &= \frac{(1-d)}{L\kappa^2} \frac{L^2}{\epsilon^d} g_{ij}^{(0)}(x) + \frac{1-d}{L\kappa^2} \frac{1}{\epsilon^{d-2}} \left(g_{ij}^{(2)} + L^2 a_i^{(0)} a_j^{(0)} \right) + \frac{1-d}{L^2 \kappa} \frac{1}{\epsilon^{d-3}} \left(\pi_{ij}^{(0)} + L^2 a_i^{(0)} p_j^{(0)} + L^2 a_j^{(0)} p_i^{(0)} \right) \\ &+ \mathcal{O}(\epsilon), \\ \langle T_{ij}^{c.t.2} \rangle &= \frac{1}{\epsilon^{d-2}} \frac{L}{\kappa^2} \overset{\circ}{G}_{ij}^{(0)} + \mathcal{O}(\epsilon), \end{aligned} \quad (\text{B.36})$$

Adding $\langle T_{ij}^{\text{reg}} \rangle$, $\langle T_{ij}^{c.t.1} \rangle$, $\langle T_{ij}^{c.t.2} \rangle$ from (B.33), (B.36) we find

$$\begin{aligned} \langle T_{ij}^{\text{reg}} \rangle + \langle T_{ij}^{c.t.1} \rangle + \langle T_{ij}^{c.t.2} \rangle &= \\ &= \frac{1}{\kappa^2 \epsilon} \left(\frac{1}{L} g_{ij}^{(2)} - \frac{1}{L} g_{ij}^{(0)} \text{tr}(g_{(2)}) + L a_i^{(0)} a_j^{(0)} + L \overset{\circ}{\nabla}_{(i}^{(0)} a_{j)}^{(0)} - L g_{ij}^{(0)} \overset{\circ}{\nabla}^{(0)} \cdot a_{(0)} + L \overset{\circ}{G}_{ij}^{(0)} \right) \\ &+ \frac{1}{\kappa^2 L^2} \left(\frac{3}{2} \pi_{ij}^{(0)} + 3L^2 a_{(i}^{(0)} p_{j)}^{(0)} + L^2 \overset{\circ}{\nabla}_{(i}^{(0)} p_{j)}^{(0)} - \gamma_{ij}^{(0)} \left(\frac{3}{2} \pi_{(0)} + L^2 (4-d) a_{(0)} \cdot p_{(0)} + L^2 \overset{\circ}{\nabla}^{(0)} \cdot p_{(0)} \right) \right). \end{aligned} \quad (\text{B.37})$$

The second line above would be divergent when $\epsilon \rightarrow 0^+$ but it turns out that the term in the parenthesis vanishes identically in $d = 3$ if we use the on-shell expression (5.95) of $g_{ij}^{(2)}$ and set $d = 3$

$$g_{ij}^{(2)} = -L^2 \left(\hat{R}_{(ij)}^{(0)} - \frac{1}{4} \hat{R}^{(0)} g_{ij}^{(0)} \right), \quad (\text{B.38})$$

and express Weyl-Ricci tensor in terms of the LC tensor and Weyl connection using Eq. (5.21) and Eq. (5.19).³

From (B.37) and (B.24) we finally find

$$\begin{aligned}\langle T_{ij} \rangle &= \frac{1}{\kappa^2 L^2} \left(\frac{3}{2} \pi_{ij}^{(0)} + L^2 \hat{\nabla}_{(i}^{(0)} p_{j)}^{(0)} + 3L^2 a_{(i}^{(0)} p_{j)}^{(0)} - g_{ij}^{(0)} L^2 a_{(0)} \cdot p_{(0)} \right) \\ &= \frac{1}{\kappa^2} \left(\frac{3}{2L^2} \pi_{ij}^{(0)} + L^2 \hat{\nabla}_{(i}^{(0)} p_{j)}^{(0)} \right) \quad \text{equal to (B.25) for } d = 3!.\end{aligned}\tag{B.39}$$

To get from the first line to the second line we used the fact that in $d = 3$ the Weyl weight is $w_{p_i^{(0)}} = 1$ and relation (5.15) between the connection coefficients. Thus we have proven that the holographic energy momentum tensor prescription of [149] gives Eq. (5.100) for $d = 3$.

B.4 Proof of “WFG cohomology *is* the FG cohomology” in 2d and 4d

Proof of 5.108. 2d Case: In $2d$ we start from the left hand side of (5.108) using (5.106)

$$\begin{aligned}\mathcal{A}_{2d}^{\text{WFG}} &= -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} \ln \mathcal{B} \hat{R}_{(0)} \quad \text{use (5.21) and (5.20)} \\ &= -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} \left(\ln \mathcal{B} \hat{R}_{(0)} + 2 \ln \mathcal{B} \hat{\nabla} \cdot a_{(0)} \right) \\ &= -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} \left(\ln \mathcal{B} \hat{\square} \ln \mathcal{B} + \hat{R}_{(0)} \ln \mathcal{B} + 2 \ln \mathcal{B} \hat{\nabla} \cdot a_{(0)} - \ln \mathcal{B} \hat{\square} \ln \mathcal{B} \right) \quad \text{use (5.60)} \\ &= \mathcal{A}_{2d}^{\text{FG}} - \frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} \left(2 \ln \mathcal{B} \hat{\nabla} \cdot a_{(0)} - \ln \mathcal{B} \hat{\square} \ln \mathcal{B} \right) \\ &\equiv \mathcal{A}_{2d}^{\text{FG}} + \Xi[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}].\end{aligned}\tag{B.40}$$

What remains to be shown to complete the proof is that $\Xi[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}]$ defined above is exact, namely we will show that

$$\Xi[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}] = \Delta_{\mathcal{B}} \circ S_{2d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}],\tag{B.41}$$

where

$$S_{2d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}] = -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} g_{(0)}^{ij} a_i^{(0)} a_j^{(0)}.\tag{B.42}$$

³See also Appendix A of [126] for relevant formulas.

Let us compute the left hand side of (B.41) using (5.47)

$$\begin{aligned}
\Delta_{\mathcal{B}} \circ S_{2d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}] &= -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} g_{(0)}^{ij} \left((a_i^{(0)} - \partial_i \ln \mathcal{B})(a_j - \partial_j \ln \mathcal{B}) - a_i^{(0)} a_j^{(0)} \right) \\
&= -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} \left(-2a_i^{(0)} \partial_j \ln \mathcal{B} + \partial_i \ln \mathcal{B} \partial_j \ln \mathcal{B} \right) \quad (\text{integrate by parts}) \\
&= -\frac{L}{16\pi G_N} \int d^2x \sqrt{\det g_{(0)}} \left(2 \ln \mathcal{B} \overset{\circ}{\nabla} \cdot a_{(0)} - \ln \mathcal{B} \overset{\circ}{\square} \ln \mathcal{B} \right) \\
&= \Xi[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}],
\end{aligned} \tag{B.43}$$

where in the last step we used the definition of $\Xi[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}]$ given in (B.40). We have thus shown (B.41) and the proof of Eq. (5.108) for 2d is complete.

4d Case: The proof of Eq. (5.108) for 4d is significantly more involved but similar in spirit with the 2d case. We first express (5.107) in terms of LC connection

$$\begin{aligned}
\mathcal{A}_{4d}^{\text{WFG}} &= -\frac{L}{8\pi G_N} \int d^4x \sqrt{\det g_{(0)}} \ln \mathcal{B} \left[\frac{L^2}{8} \left(\hat{R}_{ij}^{(0)} \hat{R}_{(0)}^{ij} - \frac{1}{3} \hat{R}_{(0)}^2 \right) + \hat{\nabla} \cdot a^{(2)} \right] \\
&= -\frac{L^3}{16\pi G_N} \int d^4x \sqrt{\det g_{(0)}} \ln \mathcal{B} \left[\frac{1}{4} \left(\hat{R}_{ij}^{(0)} R_{(0)}^{ij} - \frac{1}{3} \hat{R}^2 \right) + \overset{\circ}{\nabla}^i \left(\overset{\circ}{G}_{ij}^{(0)} a_{(0)}^j \right) + \overset{\circ}{\nabla}^j \left(\left(\overset{\circ}{\nabla}_i a_j^{(0)} \right) a_{(0)}^i \right) \right. \\
&\quad \left. - \overset{\circ}{\nabla}_i \left(a_{(0)}^i \overset{\circ}{\nabla}_j a_{(0)}^j \right) + \overset{\circ}{\nabla}_i \left((a^{(0)} \cdot a^{(0)}) a_{(0)}^i \right) + 2L^{-2} \overset{\circ}{\nabla} \cdot a^{(2)} \right] \quad (\text{integrate by parts}) \\
&= -\frac{L^3}{16\pi G_N} \int d^4x \sqrt{\det g_{(0)}} \left[\frac{1}{4} (\ln \mathcal{B}) \left(\hat{R}_{ij}^{(0)} R_{(0)}^{ij} - \frac{1}{3} \hat{R}^2 \right) - (\overset{\circ}{\nabla}_i \ln \mathcal{B}) \left(\overset{\circ}{G}_{(0)}^{ij} a_j^{(0)} \right) \right. \\
&\quad - (\overset{\circ}{\nabla}^j \ln \mathcal{B}) \left(\left(\overset{\circ}{\nabla}_i a_j^{(0)} \right) a_{(0)}^i \right) + (\overset{\circ}{\nabla}_i \ln \mathcal{B}) \left(a_{(0)}^i \overset{\circ}{\nabla}_j a_{(0)}^j \right) - (\overset{\circ}{\nabla}_i \ln \mathcal{B}) \left((a^{(0)} \cdot a^{(0)}) a_{(0)}^i \right) \\
&\quad \left. - 2L^{-2} (\overset{\circ}{\nabla}_i \ln \mathcal{B}) a_{(2)}^i \right],
\end{aligned} \tag{B.44}$$

where to get to the second line above we used Eq. (5.19)-(5.21). We also used the fact that $\overset{\circ}{\nabla} \cdot a^{(2)}$ is equivalent to $\hat{\nabla} \cdot a^{(2)}$ in 4d, since in 2k-dimension $\hat{\nabla}$ and $\overset{\circ}{\nabla}$ give the same result when acting on a vector with Weyl weight 2k.

We move to the computation of $\Delta_{\mathcal{B}} \circ S_{4d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}]$ where $S_{4d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}]$ was defined in (5.110). We define

$$\begin{aligned}
S_1^{4d} &= \int d^4x \sqrt{\det g_{(0)}} \overset{\circ}{G}_{(0)}^{ij} a_i^{(0)} a_j^{(0)}, \\
S_2^{4d} &= \int d^4x \sqrt{\det g_{(0)}} a_{(0)}^i a_{(0)}^j \overset{\circ}{\nabla}_i^{(0)} a_j^{(0)}, \\
S_3^{4d} &= \int d^4x \sqrt{\det g_{(0)}} (g_{(0)}^{ij} a_i^{(0)} a_j^{(0)}) (g_{(0)}^{mn} \overset{\circ}{\nabla}_m^{(0)} a_n^{(0)}), \\
S_4^{4d} &= \int d^4x \sqrt{\det g_{(0)}} (g_{(0)}^{ij} a_i^{(0)} a_j^{(0)})^2, \\
S_5^{4d} &= \int d^4x \sqrt{\det g_{(0)}} g_{(0)}^{ij} a_i^{(0)} a_j^{(2)},
\end{aligned} \tag{B.45}$$

and we obtain

$$\begin{aligned}\Delta_{\mathcal{B}} \circ S_1^{4d} = & \sqrt{g} \left(- \underline{2\dot{G}^{\mu\nu} a_\mu \dot{\nabla}_\nu \ln \mathcal{B}} - 2a^2 \dot{\nabla}^2 \ln \mathcal{B} + 2a_\mu a_\nu \dot{\nabla}^\mu \dot{\nabla}^\nu \ln \mathcal{B} + 2(a \cdot \dot{\nabla} \ln \mathcal{B})^2 + a^2 (\dot{\nabla}_\sigma \ln \mathcal{B} \dot{\nabla}^\sigma \ln \mathcal{B}) \right. \\ & + 4(a \cdot \dot{\nabla} \ln \mathcal{B}) \dot{\nabla}^2 \ln \mathcal{B} - 4(a_\mu \dot{\nabla}_\nu \ln \mathcal{B}) \dot{\nabla}^\mu \dot{\nabla}^\nu \ln \mathcal{B} - 6(a \cdot \dot{\nabla} \ln \mathcal{B}) (\dot{\nabla}_\sigma \ln \mathcal{B} \dot{\nabla}^\sigma \ln \mathcal{B}) \\ & \left. + \dot{G}^{\mu\nu} \dot{\nabla}_\mu \ln \mathcal{B} \dot{\nabla}_\nu \ln \mathcal{B} - 2(\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) \dot{\nabla}^2 \ln \mathcal{B} + 2\dot{\nabla}_\mu \ln \mathcal{B} \dot{\nabla}_\nu \ln \mathcal{B} \dot{\nabla}^\mu \dot{\nabla}^\nu \ln \mathcal{B} + 3(\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B})^2 \right),\end{aligned}$$

$$\begin{aligned}\Delta_{\mathcal{B}} \circ S_2^{4d} = & \sqrt{\gamma} \left(- \underline{\dot{\nabla}_\mu \ln \mathcal{B} (a^\sigma \dot{\nabla}_\sigma a^\mu)} + a^2 (a \cdot \dot{\nabla} \ln \mathcal{B}) - a^\mu a^\sigma \dot{\nabla}_\sigma \dot{\nabla}_\mu \ln \mathcal{B} - 2(a \cdot \dot{\nabla} \ln \mathcal{B})^2 - a^2 (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) \right. \\ & - a^\mu (\dot{\nabla}^\sigma \ln \mathcal{B}) \dot{\nabla}_\sigma a_\mu + 2a^\mu (\dot{\nabla}^\sigma \ln \mathcal{B}) \dot{\nabla}_\sigma \dot{\nabla}_\mu \ln \mathcal{B} + 3(a \cdot \dot{\nabla} \ln \mathcal{B}) (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) \\ & \left. + \dot{\nabla}_\sigma a_\mu \dot{\nabla}^\mu \ln \mathcal{B} \dot{\nabla}^\sigma \ln \mathcal{B} - (\dot{\nabla}^\mu \ln \mathcal{B}) (\dot{\nabla}^\sigma \ln \mathcal{B}) \dot{\nabla}_\sigma \dot{\nabla}_\mu \ln \mathcal{B} - (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B})^2 \right),\end{aligned}$$

$$\begin{aligned}\Delta_{\mathcal{B}} \circ S_3^{4d} = & \sqrt{\gamma} \left(- \underline{2(a \cdot \dot{\nabla} \ln \mathcal{B}) \dot{\nabla} \cdot a} - 2a^2 (a \cdot \dot{\nabla} \ln \mathcal{B}) - a^2 \dot{\nabla}^2 \ln \mathcal{B} + 2a^2 (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) \right. \\ & + 2(a \cdot \dot{\nabla} \ln \mathcal{B}) \dot{\nabla}^2 \ln \mathcal{B} + 4(a \cdot \dot{\nabla} \ln \mathcal{B})^2 - 6(a \cdot \dot{\nabla} \ln \mathcal{B}) (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) \\ & \left. + (\dot{\nabla} \cdot a) (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) - (\dot{\nabla}^2 \ln \mathcal{B}) (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) + 2(\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B})^2 \right),\end{aligned}$$

$$\begin{aligned}\Delta_{\mathcal{B}} \circ S_4^{4d} = & \sqrt{\gamma} \left(- \underline{4a^2 (a \cdot \dot{\nabla} \ln \mathcal{B})} + 2a^2 (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) + 4(a \cdot \dot{\nabla} \ln \mathcal{B})^2 - 4(a \cdot \dot{\nabla} \ln \mathcal{B}) (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) \right. \\ & \left. + (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B})^2 \right),\end{aligned}$$

$$\Delta \circ S_5^{4d} = - \int d^4x \sqrt{\det g_{(0)}} \underline{g_{(0)}^{ij} (\dot{\nabla}_i \ln \mathcal{B}) a_j^{(2)}} \quad (\text{B.46})$$

Note that the underlined terms above appear in (B.44). If we add all the terms above with specific coefficients and after some manipulations we obtain

$$\begin{aligned}\Delta_{\mathcal{B}} \circ S_{4d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}] \equiv & - \frac{L^3}{16\pi G_N} \left(\frac{1}{2} \Delta_{\mathcal{B}} \circ S_1^{4d} + \Delta_{\mathcal{B}} \circ S_2^{4d} - \frac{1}{2} \Delta_{\mathcal{B}} \circ S_3^{4d} + \frac{3}{4} \Delta_{\mathcal{B}} \circ S_4^{4d} + 2L^{-2} \Delta_{\mathcal{B}} \circ S_5^{4d} \right) \\ = & - \frac{L^3}{16\pi G_N} \int d^4x \sqrt{\det g_{(0)}} \left[- (\dot{\nabla}_i \ln \mathcal{B}) \left(\dot{G}_{(0)}^{ij} a_j^{(0)} \right) - (\dot{\nabla}^j \ln \mathcal{B}) \left((\dot{\nabla}_i a_j^{(0)}) a_{(0)}^i \right) \right. \\ & + (\dot{\nabla}_i \ln \mathcal{B}) \left(a_{(0)}^i \dot{\nabla}_j a_{(0)}^j \right) - (\dot{\nabla}_i \ln \mathcal{B}) \left((a^{(0)} \cdot a^{(0)}) a_{(0)}^i \right) - 2L^{-2} (\dot{\nabla}_i \ln \mathcal{B}) a_{(2)}^i \Big] \\ & - \frac{L^3}{16\pi G_N} \int d^4x \sqrt{\det g_{(0)}} \left[\frac{1}{2} \dot{G}^{\mu\nu} \dot{\nabla}_\mu \ln \mathcal{B} \dot{\nabla}_\nu \ln \mathcal{B} - \frac{1}{2} (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B}) \dot{\nabla}^2 \ln \mathcal{B} \right. \\ & \left. + \frac{1}{4} (\dot{\nabla} \ln \mathcal{B} \cdot \dot{\nabla} \ln \mathcal{B})^2 \right], \quad (\text{B.47})\end{aligned}$$

where again we dropped total derivative terms.

From (B.47) and (B.44) we arrive at

$$\begin{aligned} \mathcal{A}_{4d}^{\text{WFG}} = & -\frac{L^3}{16\pi G_N} \int d^4x \sqrt{\det g_{(0)}} \left[\frac{1}{4} (\ln \mathcal{B}) \left(\mathring{R}_{ij}^{(0)} R_{(0)}^{ij} - \frac{1}{3} \mathring{R}^2 \right) - \frac{1}{2} \mathring{G}^{\mu\nu} \mathring{\nabla}_\mu \ln \mathcal{B} \mathring{\nabla}_\nu \ln \mathcal{B} \right. \\ & \left. + \frac{1}{2} (\mathring{\nabla} \ln \mathcal{B} \cdot \mathring{\nabla} \ln \mathcal{B}) \mathring{\nabla}^2 \ln \mathcal{B} - \frac{1}{4} (\mathring{\nabla} \ln \mathcal{B} \cdot \mathring{\nabla} \ln \mathcal{B})^2 \right] + \Delta_{\mathcal{B}} \circ S_{4d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}], \end{aligned} \quad (\text{B.48})$$

and from (5.61) the above becomes Eq. (5.108)

$$\mathcal{A}_{4d}^{\text{WFG}}[\mathcal{B}; g_{ij}^{(0)}, a_i^{(0)}] = \mathcal{A}_{4d}^{\text{FG}} + \Delta_{\mathcal{B}} \circ S_{4d}^{\text{local}}[g_{ij}^{(0)}, a_i^{(0)}]. \quad (\text{B.49})$$

□

Appendix C

Appendices for Chapter 6

C.1 Coordinate Systems of the Flat Ambient Space

In this appendix we demonstrate the transformation between the flat ambient metric in different coordinate systems introduced in Section 6.2.

Start with Minkowski spacetime $\mathbb{R}^{1,d+1}$ in Lorentzian coordinates $\{X^0, X^i\}$ with $i = 1, \dots, d+1$:

$$\eta = -(\mathrm{d}X^0)^2 + \sum_{i=1}^{d+1} (\mathrm{d}X^i)^2. \quad (\text{C.1})$$

First, we can define a stereographic coordinate system $\{\ell, r, x^i\}$ as follows:

$$X^0 = \ell \frac{L^2 + r^2}{L^2 - r^2}, \quad X^i = \ell \frac{2L}{L^2 - r^2} x^i, \quad i = 1, \dots, d+1, \quad (\text{C.2})$$

where $r^2 = \sum_{i=1}^{d+1} (x^i)^2$ and L is a positive constant, and note that $(X^0)^2 - \sum_{i=1}^{d+1} (X^i)^2 = \ell^2$. In this system, the Minkowski metric (C.1) becomes

$$\eta = -\mathrm{d}\ell^2 + \frac{\ell^2}{L^2} \frac{4}{(1 - (r/L)^2)^2} \sum_{i=1}^{d+1} (\mathrm{d}x^i)^2 = -\mathrm{d}\ell^2 + \frac{\ell^2}{L^2} \frac{4}{(1 - (r/L)^2)^2} (\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_d^2), \quad (\text{C.3})$$

where in the second equality we expressed $\{x^i\}$ in the spherical coordinates. The coordinate patch is $\ell > 0$, $0 \leq r < L$, which covers the interior of the future light cone. Notice that in these coordinates the metric has a “cone” form (6.6), with g^+ given in (6.7), which is the $(d+1)$ -dimensional Euclidean AdS metric g_G^+ in global coordinates. This AdS metric can be converted into the FG form by transforming the coordinate r to a coordinate z

$$r = L \left(\frac{2L - z}{2L + z} \right). \quad (\text{C.4})$$

Then, the metric (C.3) takes the form

$$\eta = -d\ell^2 + \frac{\ell^2}{z^2} \left(dz^2 + L^2 \left(1 - \frac{1}{4} (z/L)^2 \right)^2 d\Omega_d^2 \right), \quad (\text{C.5})$$

and the interior of the future light cone is now covered by $\ell > 0$, $0 < z < 2L$. We can further convert (C.5) into the ambient form (6.15) by setting

$$\ell = zt, \quad z^2 = -2\rho, \quad (\text{C.6})$$

and the metric turns into the form shown in (6.9):

$$\eta = 2\rho dt^2 + 2t dt d\rho + t^2 \left(1 + \frac{\rho}{2L^2} \right)^2 L^2 d\Omega_d^2. \quad (\text{C.7})$$

Plugging (C.6) and (C.4) into (C.2) we find that

$$X^0 + R = 2Lt, \quad \tan \alpha \equiv \frac{R}{X^0} = \frac{1 + \frac{\rho}{2L^2}}{1 - \frac{\rho}{2L^2}}, \quad (\text{C.8})$$

where $R^2 = \sum_{i=1}^{d+1} (X^i)^2$. From the above equation one can see that the constant- t and constant- ρ surfaces are indeed the cones depicted in Figure 6.1, with α the angle of the constant- ρ cone with respect to the X^0 -axis.

The Minkowski metric (C.1) can also be written in the cone form with $g^+ = g_P^+$ the Euclidean AdS metric in Poincaré coordinates given in (6.8). Introduce another coordinate system $\{\ell, x^i, z\}$ as follows:

$$X^0 = \frac{\ell}{2Lz} \left(L^2 + \sum_{i=1}^d (x^i)^2 + z^2 \right), \quad X^{d+1} = \frac{\ell}{2Lz} \left(L^2 - \sum_{i=1}^d (x^i)^2 - z^2 \right), \quad X^i = \frac{\ell x^i}{z}, \quad (\text{C.9})$$

where $(X^0)^2 - \sum_{i=1}^{d+1} (X^i)^2 = \ell^2$. The metric (C.1) becomes

$$\eta = -d\ell^2 + \frac{\ell^2}{z^2} \left(dz^2 + \delta_{ij} dx^i dx^j \right), \quad i = 1, \dots, d, \quad z > 0. \quad (\text{C.10})$$

Define the ambient coordinate system $\{t, x^i, \rho\}$ as

$$\ell = zt, \quad z^2 = -2\rho, \quad (\text{C.11})$$

then the metric (C.10) will have the form shown in (6.10)

$$\eta = 2\rho dt^2 + 2t dt d\rho + t^2 \delta_{ij} dx^i dx^j, \quad i = 1, \dots, d. \quad (\text{C.12})$$

C.2 Proofs

C.2.1 Proof of Theorem 6.1

To prove Theorem 6.1, we first need to introduce a (g, a) -transversal vector (generalized from the concept of a g -transversal vector in [44]), where the horizontal subspace H_p defined by the Weyl connection plays an important role. Once we pick a representative (g, a) in the Weyl class, g induces an isomorphism between $\mathcal{P}_W$ and $\mathcal{G}$ through (6.32), which determines the fibre coordinate t of $\mathcal{P}_W$; a defines for any $p \in \mathcal{P}_W$ a horizontal subspace $H_p \subset T_p \mathcal{P}_W$ given in (6.28), which can also be viewed as a subspace of $T_{(p,0)}(\mathcal{P}_W \times \mathbb{R})$ via the inclusion map $\iota : \mathcal{P}_W \rightarrow \mathcal{P}_W \times \mathbb{R}$. We define a vector $\underline{\mathcal{V}} \in T_{(p,0)}(\mathcal{P}_W \times \mathbb{R})$ to be a (g, a) -transversal vector for $\tilde{g}$ if it satisfies the following three conditions at $(p, 0)$:

$$\textcircled{1} \tilde{g}(\underline{\mathcal{V}}, \underline{\mathcal{T}}) = t^2, \quad \textcircled{2} \tilde{g}(\underline{\mathcal{V}}, \underline{\mathcal{H}}) = 0 \quad \forall \underline{\mathcal{H}} \in H_p, \quad \textcircled{3} \tilde{g}(\underline{\mathcal{V}}, \underline{\mathcal{V}}) = 0. \quad (\text{C.13})$$

When $a_i(x) = 0$ in (6.30), i.e., $\mathbf{a} = \partial_t \otimes dt$, the (g, a) -transversal vector for $\tilde{g}$ goes back to the g -transversal vector for $\tilde{g}$ defined in [44]. From (6.37) one can see that for $(\tilde{\mathcal{M}}, \tilde{g})$ in Weyl-normal form, ∂_ρ is (g, a) -transversal for $\tilde{g}$ at $(p, 0)$. Following the proof of Lemma 2.10 in [44], it is straightforward to show that the (g, a) -transversal vector is unique and dilatation-invariant (i.e. $\delta_{s*} V_p = V_{\delta_s(p)}$) for $\tilde{g}$ at $(p, 0)$.

The proof of Theorem 6.1 proceeds similar to the proof of Proposition 2.8 in [44]; one only has to let the g -transversal vector $\underline{\mathcal{V}}$ to be a (g, a) -transversal vector. Here we will not repeat all the details but only outline the proof and elaborate on the steps when the Weyl connection a is relevant.

Proof of Theorem 6.1. Suppose $p \in \mathcal{P}_W$ and let $\underline{\mathcal{V}}_p$ be the (g, a) -transversal vector for $\tilde{g}$ at $(p, 0)$. One can parametrize the (non-geodesic) curve $C_p : \lambda \mapsto \phi(p, \lambda) \in \tilde{\mathcal{M}}$ with initial conditions

$$\phi(p, 0) = (p, 0), \quad \partial_\lambda \phi(p, \lambda)|_{\lambda=0} = \underline{\mathcal{V}}_p, \quad (\text{C.14})$$

with the “equation of motion” $\nabla_{\underline{\mathcal{U}}} \underline{\mathcal{U}} = \underline{\mathcal{A}}$, where $\underline{\mathcal{U}} = \frac{d}{d\lambda}$ is the tangent vector to the accelerated curve C_p , and the acceleration vector $\underline{\mathcal{A}}$ satisfies $\tilde{g}(\underline{\mathcal{T}}, \underline{\mathcal{A}}) = 0$. Suppose the domain of ϕ is $\tilde{U}_0 \subset \mathcal{P}_W \times \mathbb{R}$, which is dilatation-invariant. Then $\phi : \tilde{U}_0 \rightarrow \tilde{\mathcal{M}}$ is a smooth map commuting with dilatation, and it can be proved that there exists $\tilde{U}_1 \subset \tilde{U}_0$ as a dilatation-invariant neighborhood of $\mathcal{P}_W \times \{0\}$ such that $\phi : \tilde{U}_1 \rightarrow \tilde{\mathcal{M}}$ is a diffeomorphism (see [44]).

Furthermore, one can define $\tilde{\mathcal{M}}' = \{(p, \lambda) \in \tilde{U}_1 \mid (p, \mu) \in \tilde{U}_1, \forall \mu \in \mathbb{R} \text{ satisfying } |\mu| \leq |\lambda|\}$. It is easy to verify that $(\tilde{\mathcal{M}}', \phi^* \tilde{g})$ satisfies the conditions of Definition 6.3 and thus is a Weyl pre-ambient space for $(M, [g, a])$. It follows that for each $p \in \mathcal{P}_W$, the set for λ such that $(p, \lambda) \in \tilde{\mathcal{M}}'$ is an open interval I_p containing 0, and the parametrized curve $C'_p : \lambda \mapsto (p, \lambda)$ with tangent vector $\underline{\mathcal{U}}'$ and the acceleration $\underline{\mathcal{A}}' = \nabla'_{\underline{\mathcal{U}}} \underline{\mathcal{U}}'$ satisfies $\phi^* \tilde{g}(\underline{\mathcal{T}}', \underline{\mathcal{A}}') = 0$, where $\underline{\mathcal{T}}' \equiv \phi^* \underline{\mathcal{T}}$, and ∇' is the Levi-Civita connection associated with $\phi^* \tilde{g}$. Hence, conditions (5.1) and (5.2) of Definition 6.4 are satisfied by $(\tilde{\mathcal{M}}', \phi^* \tilde{g})$.

Finally let us verify condition (5.3) of Definition 6.4. Since $\underline{\mathcal{V}}$ satisfies the conditions in (C.13) and ϕ satisfies

(C.14), under the identification $\mathbb{R}_+ \times \mathcal{M} \times \mathbb{R} \simeq \mathcal{P}_W \times \mathbb{R}$ induced by g we have at $(\lambda = 0, p)$:

$$\begin{aligned} (\phi^* \tilde{g})(\partial_\lambda, \underline{\mathcal{I}}) &= t^2 \\ (\phi^* \tilde{g})(\partial_\lambda, \underline{\mathcal{H}}) &= 0 \quad \forall \underline{\mathcal{H}} \in \mathcal{H}_p, \\ (\phi^* \tilde{g})(\partial_\lambda, \partial_\lambda) &= 0. \end{aligned} \tag{C.15}$$

For a given connection $\mathbf{a} = t\partial_t \otimes (t^{-1}dt + a_i(x)dx^i)$ on $\mathcal{P}_W$, the horizontal subspace $\mathcal{H}_p$ at $(p, 0)$ is spanned by $\underline{D}_i = \partial_i - ta_i\partial_t$. Since $(\tilde{\mathcal{M}}, \phi^*\tilde{g})$ is a Weyl pre-ambient space for $(\mathcal{M}, [g, a])$, $\iota^*(\phi^*g)$ is the tautological tensor $\mathbf{g}_0$ on $\mathcal{P}_W$. Then, the above equations give that $\phi^*\tilde{g}|_{\lambda=0} = t^2\mathbf{g}_0 + 2t(dt + ta_i(x)dx^i)d\lambda$. Therefore, all the conditions in Definition 6.4 are satisfied by $(\mathcal{M}', \phi^*g)$, which completes the existence part of the Proposition. The uniqueness part follows from the fact that the above construction of ϕ is forced. Suppose $\phi : \mathcal{M} \rightarrow \mathcal{M}'$ is a diffeomorphism such that $(\mathcal{M}', \phi^*g)$ is a pre-ambient space in Weyl-normal form, then $\underline{\mathcal{V}}_p$ must be (g, a) -transversal for $\tilde{g}$ at $(p, 0)$, and the curve $C'_p : \lambda \mapsto \phi(z, \lambda)$ must be the unique curve satisfying the initial conditions (C.14) and having the acceleration $\underline{\mathcal{A}}$, which determines $\phi : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}'$ uniquely. $\square$

C.2.2 Proof of Theorem 6.2

Proof of Theorem 6.2. The proof of this theorem has two main parts. First, from $\tilde{Ric}(\tilde{g}) = 0$ and the initial value of $\tilde{g}$ at $\rho = 0$ we will determine the first ρ -derivative of the metric components at $\rho = 0$. Then, using an inductive argument we will show that all higher derivatives (to infinite order) at $\rho = 0$ can also be determined from the Ricci-flatness condition. Let us write the unknown components of $\tilde{g}$ as

$$\tilde{g}_{00} = c(x, \rho), \quad \tilde{g}_{0i} = tb_i(x, \rho), \quad \tilde{g}_{ij} = t^2g_{ij}(x, \rho), \tag{C.16}$$

where $g_{ij}(x, \rho)$ can be considered as a one-parameter family of metrics on $\mathcal{M}$. From property (2) we have the initial values $c(x, 0) = 0$ and $b_i(x, 0) = 0$. The general metric has the form

$$\tilde{g}_{IJ} = \begin{matrix} & 0 & j & \infty \\ \begin{matrix} 0 \\ i \\ \infty \end{matrix} & \begin{pmatrix} c(x, \rho) & tb_i(x, \rho) & t \\ tb_i(x, \rho) & t^2g_{ij}(x, \rho) & t^2a_i(x, \rho) \\ t & t^2a_i(x, \rho) & 0 \end{pmatrix} \end{matrix}, \tag{C.17}$$

and the inverse metric is

$$\tilde{g}^{IJ} = \begin{pmatrix} \frac{a^2}{\chi} & -\frac{(1-a \cdot b)a^j + a^2b^j}{t\chi} & \frac{1-a \cdot b}{t\chi} \\ -\frac{(1-a \cdot b)a^i + a^2b^i}{t\chi} & \frac{g^{ij}}{t^2} + \frac{(1-a \cdot b)(a^ib^j + a^jb^i) + a^2b^ib^j - (c-b^2)a^ia^j}{t^2\chi} & \frac{(c-b^2)a^i - (1-a \cdot b)b^i}{t^2\chi} \\ \frac{1-a \cdot b}{t\chi} & \frac{(c-b^2)a^j - (1-a \cdot b)b^j}{t^2\chi} & \frac{b^2 - c}{t^2\chi} \end{pmatrix}, \tag{C.18}$$

where $a^i \equiv g^{im}a_m$, $b^i \equiv g^{im}b_m$ and $\chi = a^2(c - b^2) + (1 - a \cdot b)^2$, with $a^2 = a_k a^k$, $b^2 = b_k b^k$ and $a \cdot b = a_k b^k$. The Christoffel symbols $\tilde{\Gamma}_{IJK} \equiv \tilde{g}_{KM}\tilde{\Gamma}^M_{IJ}$ are

$$\begin{aligned}
2\tilde{\Gamma}_{IJ0} &= \begin{pmatrix} 0 & \partial_j c & \partial_\rho c \\ \partial_i c & t(\partial_i b_j + \partial_j b_i - 2g_{ij}) & t(\partial_\rho b_i - 2a_i) \\ \partial_\rho c & t(\partial_\rho b_j - 2a_j) & 0 \end{pmatrix}, \\
2\tilde{\Gamma}_{IJk} &= \begin{pmatrix} 2b_k - \partial_k c & t(2g_{jk} + \partial_j b_k - \partial_k b_j) & t(2a_k + \partial_\rho b_k) \\ t(2g_{ik} + \partial_i b_k - \partial_k b_i) & 2t^2 \gamma_{ijk} & t^2(\partial_\rho g_{ik} + F_{ik}) \\ t(2a_k + \partial_\rho b_k) & t^2(\partial_\rho g_{jk} + F_{jk}) & 2t^2 \partial_\rho a_k \end{pmatrix}, \\
2\tilde{\Gamma}_{IJ\infty} &= \begin{pmatrix} 2 - \partial_\rho c & t(2a_j - \partial_\rho b_j) & 0 \\ t(2a_i - \partial_\rho b_i) & t^2(\partial_i a_j + \partial_j a_i - \partial_\rho g_{ij}) & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{C.19}$$

where $\gamma_{ijk} = g_{km}\gamma^m_{ij}$ with $\gamma^m_{ij} = \frac{1}{2}g^{mk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ and $F_{jk} = \partial_j a_k - \partial_k a_j$. Calculating the components $\tilde{R}_{IJ}$ of $\tilde{Ric}(\tilde{g})$ to the leading order in ρ -expansion from

$$\tilde{R}_{IJ} = \frac{1}{2}\tilde{g}^{KL}(\partial_{IL}^2 \tilde{g}_{JK} + \partial_{JK}^2 \tilde{g}_{IL} - \partial_{KL}^2 \tilde{g}_{IJ} - \partial_{IJ}^2 \tilde{g}_{KL}) + \tilde{g}^{KL}\tilde{g}^{PQ}(\tilde{\Gamma}_{ILP}\tilde{\Gamma}_{JKQ} - \tilde{\Gamma}_{IJP}\tilde{\Gamma}_{KLQ}), \tag{C.20}$$

and setting them to zero as the Ricci-flatness condition demands, we obtain

$$\begin{aligned}
c(x, \rho) &= 2\rho + O(\rho^2), & b_i(x, \rho) &= O(\rho^2), \\
g_{ij}(x, \rho) &= g_{ij}(x) + \rho(2\hat{P}_{(ij)} - 2a_i(x)a_j(x)) + O(\rho^2),
\end{aligned} \tag{C.21}$$

where $\hat{P}_{ij}$ is the Weyl-Schouten tensor (5.22). One can observe that this agrees with (6.17), where $g_{ij}(x)$ corresponds to $\gamma_{ij}^{(0)}$ in the expansion (6.20), and the order $O(\rho)$ matches $\gamma_{ij}^{(1)}$ given in (6.22). Note that the above components of a Weyl-ambient metric reduce to the components of an ambient metric in [44] when the Weyl connection a_i is turned off.

The next stage of the proof is to carry out an inductive perturbation calculation for higher orders in ρ . The purpose of this calculation is to prove (inductively) that the Ricci-flatness condition can be used to determine the unknown components of $\tilde{g}$ in Weyl-normal form to infinite order in ρ .

Let $\tilde{g}^{[k]}$ represent a metric that includes the terms of the ρ -expansion of $\tilde{g}$ up to (including) order $O(\rho^k)$, i.e., $\tilde{g} = \tilde{g}^{[k]} + O(\rho^{k+1})$. Then, the Ricci-flatness condition of $\tilde{g}$ implies that the components $\tilde{R}_{IJ}^{[k]}$ of $Ric(\tilde{g}^{[k]})$ satisfy

$$\tilde{R}_{IJ}(\tilde{g}^{[k]}) = O(\rho^k) \quad I, J \neq \infty, \quad \tilde{R}_{I\infty}(\tilde{g}^{[k]}) = O(\rho^{k-1}). \tag{C.22}$$

To carry out the induction, we assume that $\tilde{g}^{[m-1]}$ has been uniquely determined from the condition (C.22) with $k = m - 1$. We have seen this is true for $m = 2$ above by explicit calculation. Now we want to show that $\tilde{g}^{[m]}$ then can be uniquely determined from the condition (C.22) with $k = m$. Set $\tilde{g}_{IJ}^{[m]} = \tilde{g}_{IJ}^{[m-1]} + \Phi_{IJ}$, with

$$\Phi_{IJ} := \begin{pmatrix} \Phi_{00} & \Phi_{0j} & 0 \\ \Phi_{i0} & \Phi_{ij} & \Phi_{i\infty} \\ 0 & \Phi_{j\infty} & 0 \end{pmatrix} = \rho^m \begin{pmatrix} \phi_{00}(x) & t\phi_{0j}(x) & 0 \\ t\phi_{0i}(x) & t^2\phi_{ij}(x) & t^2a_i^{(m)}(x) \\ 0 & t^2a_j^{(m)}(x) & 0 \end{pmatrix}, \tag{C.23}$$

where $a_i^{(m)}(x)$ is the m^{th} order term of $a_i(x, \rho)$ [see (6.21)], and we have considered the fact that $\tilde{g}_{IJ}^{[m]}$ satisfies (6.38). All we have to show is that ϕ_{00} , ϕ_{0i} and ϕ_{ij} can all be uniquely determined. From (C.20) one finds that

$$\begin{aligned}\tilde{R}_{IJ}^{[m]} &= \tilde{R}_{IJ}^{[m-1]} + \frac{1}{2}\tilde{g}_{[m]}^{KL} (\partial_{IL}^2 \Phi_{JK} + \partial_{JK}^2 \Phi_{IL} - \partial_{KL}^2 \Phi_{IJ} - \partial_{IJ}^2 \Phi_{KL}) \\ &\quad + \tilde{g}_{[m]}^{KL} \tilde{g}_{[m]}^{PQ} \left(\tilde{\Gamma}_{ILP}^{[m]} \Gamma_{JKQ}^\Phi + \Gamma_{ILP}^\Phi \tilde{\Gamma}_{JKQ}^{[m]} - \tilde{\Gamma}_{IJP}^{[m]} \Gamma_{KLQ}^\Phi - \Gamma_{IJP}^\Phi \tilde{\Gamma}_{KLQ}^{[m]} \right) + O(\rho^m),\end{aligned}\tag{C.24}$$

where $\tilde{g}_{[m]}^{KL}$ and $\tilde{\Gamma}_{IJK}^{[m]}$ are the inverse and Christoffel symbols of $\tilde{g}_{KL}^{[m]}$, respectively, and $\Gamma_{IJK}^\Phi \equiv \frac{1}{2}(\partial_J \Phi_{IK} + \partial_I \Phi_{JK} - \partial_K \Phi_{IJ})$. The components of Γ_{IJK}^Φ can be expressed as follows:

$$\begin{aligned}2\Gamma_{IJO}^\Phi &= \begin{pmatrix} 0 & 0 & \partial_\rho \Phi_{00} \\ 0 & 0 & \partial_\rho \Phi_{i0} \\ \partial_\rho \Phi_{00} & \partial_\rho \Phi_{0j} & 0 \end{pmatrix} + O(\rho^m), \\ 2\Gamma_{IJK}^\Phi &= \begin{pmatrix} 0 & 0 & \partial_\rho \Phi_{0k} \\ 0 & 0 & \partial_\rho \Phi_{ik} \\ \partial_\rho \Phi_{0k} & \partial_\rho \Phi_{jk} & 2\partial_\rho \Phi_{\infty k} \end{pmatrix} + O(\rho^m), \\ 2\Gamma_{IJ\infty}^\Phi &= \begin{pmatrix} -\partial_\rho \Phi_{00} & -\partial_\rho \Phi_{0j} & 0 \\ -\partial_\rho \Phi_{i0} & -\partial_\rho \Phi_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\rho^m).\end{aligned}\tag{C.25}$$

Substituting (C.25) and the leading order of $\tilde{\Gamma}_{IJK}^{[m]}$ and $\tilde{g}_{[m]}^{IJ}$ [i.e., the leading order of $\tilde{g}^{IJ}$, $\tilde{\Gamma}_{IJK}$ in (C.18), (C.19)] into (C.24), one finds

$$\begin{aligned}t^2 \tilde{R}_{00}^{[m]} &= t^2 \tilde{R}_{00}^{[m-1]} + m\rho^{m-1} \left(m - 1 - \frac{d}{2} \right) \phi_{00} + O(\rho^m), \\ t\tilde{R}_{0i}^{[m]} &= t\tilde{R}_{0i}^{[m-1]} + m\rho^{m-1} \left[\frac{1}{2}\partial_i \phi_{00} + \left(m - 1 - \frac{d}{2} \right) \phi_{0i} \right] + O(\rho^m), \\ \tilde{R}_{ij}^{[m]} &= \tilde{R}_{ij}^{[m-1]} + m\rho^{m-1} \left[\left(m - \frac{d}{2} \right) \phi_{ij} - \frac{1}{2}g_{ij}g^{km}\phi_{km} + \mathring{\nabla}_{(i}\phi_{j)0} + \mathring{P}_{ij}\phi_{00} \right] + O(\rho^m), \\ t\tilde{R}_{0\infty}^{[m]} &= t\tilde{R}_{0\infty}^{[m-1]} + \frac{1}{2}m(m-1)\rho^{m-2}\phi_{00} + O(\rho^{m-1}), \\ \tilde{R}_{i\infty}^{[m]} &= \tilde{R}_{i\infty}^{[m-1]} + \frac{1}{2}m(m-1)\rho^{m-2}\phi_{i0} + O(\rho^{m-1}), \\ \tilde{R}_{\infty\infty}^{[m]} &= \tilde{R}_{\infty\infty}^{[m-1]} - m(m-1)\rho^{m-2} \left(\frac{1}{2}a^2\phi_{00} - a^k\phi_{k0} + \frac{1}{2}g^{km}\phi_{km} \right) + O(\rho^{m-1}),\end{aligned}\tag{C.26}$$

where $\mathring{P}_{ij}$, $\mathring{\nabla}$ are the LC Schouten tensor and LC connection associated with the metric $g_{ij}(x)$. Although the Weyl connection $a_i^{(0)}(x)$ appears throughout the calculation, it cancels itself out rather unexpectedly, except for the terms in $\tilde{R}_{\infty\infty}^{[m]}$. The inductive argument then proceeds in the same way as [44]. First we consider the Ricci components with $I, J \neq \infty$. From the first two equations in (C.26) one can uniquely determine ϕ_{00} and ϕ_{0i} such that $\tilde{R}_{00}^{[m]}$ and $\tilde{R}_{0i}^{[m]}$ both vanish up to order $O(\rho^m)$. Then, from the third equation in (C.26) one can uniquely solve for ϕ_{ij} such that the order $O(\rho^{m-1})$ of $\tilde{R}_{ij}^{[m]}$ vanishes. Therefore, $\tilde{g}_{IJ}^{[m]}$ will be uniquely determined by $\tilde{R}_{IJ}^{[m]} = O(\rho^m)$ for $I, J \neq \infty$ once $\tilde{g}^{[m-1]}$ is determined, and hence the unknown components of $\tilde{g}_{IJ}$ can be determined to infinite order.

Note that when $d = 2m$, the situation becomes subtle because the term ϕ_{ij} vanishes in $\tilde{R}^{[m]}$. In [44], this is attributed to the obstruction of the Ricci-flatness condition at $\mathcal{O}(\rho^{d/2-1})$ when d is an even integer, and one has to carefully consider even and odd d separately. Nevertheless, since we consider the dimension d as a continuous parameter, we can always solve for ϕ_{ij} from the Ricci-flatness condition for any d , and the information regarding these obstructions is not lost but takes the form of poles in ϕ_{ij} at $d = 2m$.

So far we have proved that the unknown components of $\tilde{g}$ are determined to infinite order by the Ricci-flatness condition for $I, J \neq \infty$. To finish the analysis we also need to show that the remaining Ricci components $\tilde{R}_{I\infty}$ also vanish to infinite order when we plug in the solution for $\tilde{g}$ obtained from $\tilde{R}_{IJ} = 0$ for $I, J \neq \infty$. Consider the Bianchi identity $\tilde{g}^{JK}\nabla_I\tilde{R}_{JK} = 2\tilde{g}^{JK}\nabla_J\tilde{R}_{IK}$. Expanding the covariant derivative in terms of the Christoffel symbols we get

$$2\tilde{g}^{JK}\partial_J\tilde{R}_{IK} - \tilde{g}^{JK}\partial_I\tilde{R}_{JK} - 2\tilde{g}^{JK}\tilde{g}^{PQ}\tilde{\Gamma}_{JKP}\tilde{R}_{QI} = 0. \quad (\text{C.27})$$

Since $\tilde{R}_{I\infty} = \mathcal{O}(\rho^{m-2})$ is trivially true for $m = 2$, now we want to show that $\tilde{R}_{I\infty} = \mathcal{O}(\rho^{m-2})$ leads to $\tilde{R}_{I\infty} = \mathcal{O}(\rho^{m-1})$ by means of the Bianchi identity. Expanding (C.27) for $I = 0, i, \infty$ and making use of the homogeneity property of the metric we get

$$\begin{aligned} (d-2-2\rho\partial_\rho)\tilde{R}_{0\infty} &= \mathcal{O}(\rho^{m-1}) \\ (d-2-2\rho\partial_\rho)\tilde{R}_{i\infty} - t\partial_i\tilde{R}_{0\infty} &= \mathcal{O}(\rho^{m-1}) \\ a^2\left(t^{-1}d\tilde{R}_{\infty 0} + 2\partial_0\tilde{R}_{\infty 0}\right) - 2t^{-1}a^m\left(\partial_m\tilde{R}_{\infty 0} - (2-d)t^{-1}\tilde{R}_{\infty m}\right) \\ &\quad + 2t^{-2}(d-2-\rho\partial_\rho)\tilde{R}_{\infty\infty} + 2t^{-2}g^{mk}\tilde{\nabla}_m\tilde{R}_{\infty k} + 2t^{-1}\tilde{P}\tilde{R}_{\infty 0} = \mathcal{O}(\rho^{m-1}). \end{aligned} \quad (\text{C.28})$$

We can see that the Weyl connection appears only in the last equation of (C.28). Note that all the Ricci terms $\tilde{R}_{IJ}$ with $I, J \neq \infty$ has been dropped from (C.28) since they vanish to infinite order. Suppose $\tilde{R}_{I\infty} = \gamma_I\rho^{m-2}$. The first equation in (C.28) gives $(d+2-2m)\gamma_0 = \mathcal{O}(\rho)$, and thus $\tilde{R}_{0\infty} = \mathcal{O}(\rho^{m-1})$. The second equation in (C.28) gives $(d+2-2m)\gamma_i = \mathcal{O}(\rho)$, and thus $\tilde{R}_{i\infty} = \mathcal{O}(\rho^{m-1})$. The last equation then gives $(d-m)\gamma_\infty = \mathcal{O}(\rho)$, so $\tilde{R}_{\infty\infty} = \mathcal{O}(\rho^{m-1})$. This completes the inductive argument and thus $\tilde{R}_{I\infty}$ can also be made to vanish to infinite order.

To summarize, we have shown by an inductive argument that there exists a Weyl-ambient space $(\tilde{\mathcal{M}}, \tilde{g})$ for $(\mathcal{M}, [g, a])$ in Weyl-normal form with acceleration $\underline{\mathcal{A}}$. Some components of $\tilde{g}$ have the form in (6.38), and all the unknown components are determined uniquely to infinite order of ρ at $\mathcal{P}_W \times \{0\}$ by the Ricci-flatness condition. $\square$