

GABOR FRAMES WITH TRIGONOMETRIC SPLINE DUAL WINDOWS

BY

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DISSERTATION

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Abstract

A Gabor system is a collection of modulated and translated copies of a window function. If we have a signal in $L^2(\mathbb{R})$, it can be analyzed with a Gabor system generated by a certain window g and then synthesized with a Gabor system generated by another window h . If this leads us to a perfect reconstruction, we say that g and h are dual Gabor windows.

Few explicit examples of dual window pairs are known. This thesis constructs explicit examples of Gabor dual windows with trigonometric form. The windows have fixed support and have an arbitrary smoothness. Also, in the discrete time domain, the trigonometric form allows us to evaluate the Gabor coefficients efficiently using the Discrete Fourier Transform. For the higher dimensional cases, we find window examples for a large class of modulation parameter lattices, including shear lattices. Also, a sufficient condition on the norm of the modulation lattice to have explicit dual Gabor windows is presented, for every dimension.

To Jeonghun

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Chapter 1

Introduction

Fourier analysis is well known and has been widely applied, but it has limitations in representing a function for a local time or frequency space since the representing functions $e^{2\pi i n x}$ ($n \in \mathbb{Z}$) are not localized. If one wants to analyze nonstationary signals such as music or speech, one would need a tool that can handle the signal or data in a localized time and frequency [10]. Multiplying the signal by a window is used for this job. A window, a compactly supported function, can be shifted and applied to a signal to capture all information of the signal. Here, the shifting is not only about time but also about frequency. Gabor analysis in $L^2(\mathbb{R})$ uses two shift operators as follows:

- Translation by a , $(T_a f)(x) = f(x - a)$
- Modulation by b , $(E_b f)(x) = e^{2\pi i b x} f(x)$

Here, $T_a f$ represents the time shift by a and $E_b f$ represents the frequency shift by b . Using these operators, we can reconstruct any function in $L^2(\mathbb{R})$, as follows. For some nice g and h in $L^2(\mathbb{R})$, we can analyze any function f in $L^2(\mathbb{R})$ with translated and modulated copies of g , and synthesize with translated and modulated copies of h . i.e., we have a reconstruction formula :

$$f = \iint \langle f, E_\omega T_x g \rangle E_\omega T_x h \, d\omega \, dx \quad (1.1)$$

which is known as the Inversion formula for the STFT (Short Time Fourier Transform) (Corollary 3.2.3 in [12]). Here, the STFT represents the coefficients $\langle f, E_\omega T_x g \rangle$ inside of the integrals. Notice $\langle f, E_\omega T_x g \rangle$ is the Fourier transform of $f \overline{T_x g}$, so it is the Fourier transform of time-truncated f with a shifted window g (time-truncated because of the finite support of the window g) of different parts. As x is changed, the STFT captures the Fourier transform of the function f . In the real world, however, the above reconstruction formula might be hard to apply because x and w range over all real numbers. So instead of this continuous

representation, we want to consider the discretized time-frequency representation :

$$f = ab \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}h, \quad f \in L^2(\mathbb{R}) \quad (1.2)$$

for certain fixed a and b . Notice (1.2) is a “Riemann sum version” of the STFT inversion formula (1.1). Here the collection $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} = \{g(x - na)e^{2\pi imbx}\}_{m,n \in \mathbb{Z}}$ is called a Gabor system and the function g is called a generator or a window.

It was around 1930 when J. von Neumann, H. Weyl, and E. Wigner started to work on quantum mechanics and D. Gabor set the theoretical foundation of signal analysis in 1946. Those were the origins of the time-frequency analysis. In 1946, D. Gabor studied the window $g = e^{-\pi z^2}$, a Gaussian window, with the translation, modulation parameters $a = b = 1$, using it to express information of a function in a local time-frequency domain [11]. For this reason, the system $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ has been called a Gabor system. Later, around 1980, A.J.E.M. Janssen played an important role in establishing the mathematical field of time-frequency analysis [16]. Then, Ingrid Daubechies’s work on wavelet theory and time-frequency theory brought a huge development of both fields and mutual influences [8].

The systems $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$ can be frames if $ab \leq 1$ (Theorem 9.1.12 in [3]). In other words, given a sufficiently dense sampling in the time frequency plane, there exist windows g and h for which the series in (1.2) converges unconditionally for all $f \in L^2(\mathbb{R})$ (Corollary 3.1.4 in [3]).

For the Gabor frame $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ and the dual Gabor frame $\{E_{mb}T_{na}h\}$, many nice theorems are known, such as the characterization theorem for being a dual Gabor frame, the theorem that gives an alternate form of the frame operator, and so on. However there are not many explicit examples of Gabor dual frames for practical uses.

For example, a construction in literature is the square-root method of Daubechies [9]. Assume that $s(x)$ is nonnegative, bounded, supported on an interval of length L and $\sum_{n \in \mathbb{Z}} s(x - n) = 1$. Then if $0 < b \leq 1/L$, $g = h = \sqrt{s}$ gives a pair of dual Gabor windows. Here the root reduces smoothness and the formula of g and h are not simple. Also, the canonical dual window of $g(x)$ is found as $g(x) / \sum_{n \in \mathbb{Z}} |g(x - n)|^2$ when g is supported in an interval of length $1/b$. Here, the denominator should be bounded below and above (Corollary 9.1.8 in [3]) and this canonical dual is more complicated than the object window g . Another example of construction is made by Janssen [17]. With the scaling chosen as $\varphi(x) := 2^{1/4}e^{-\pi x^2}$, it was shown that for any $\epsilon > 0$, $\epsilon < 1 - ab$, the function

$$\gamma_\epsilon(x) := 2^{-1/4}bK^{-1}e^{\pi x^2} \sum_{k \in \mathbb{Z}} (-1)^k e^{-\pi a(k+1/2)^2/b} \operatorname{erfc}[(x - (k+1/2)a)\sqrt{\pi/\epsilon}]$$

with

$$K := \sum_{k \in \mathbb{Z}} (-1)^k (2k+1) e^{-\pi a(k+1/2)^2}, \quad \text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-s^2} ds$$

generates a dual frame of $\{E_{mb}T_{na}\varphi\}_{m,n \in \mathbb{Z}}$. This dual window is very complicated and is not compactly supported. These three examples of construction of dual windows show that they are very complex and it is not easy to apply them in practice.

So the goal of this thesis is to find explicit dual Gabor windows g, h with simple form which satisfy the identity (1.2), so that we can reconstruct any function f in $L^2(\mathbb{R})$ practically. The features of the constructed windows will be better in some aspects compared to existing constructions of Gabor dual frames in work by O. Christensen and R.Y. Kim [7],[6], O. Christensen [2], O. Christensen, H.O. Kim and R.Y. Kim [4] and R.S. Laugesen [20]. Especially, the paper of R.S. Laugesen [20] inspired this thesis. Christensen, Kim, and Laugesen used polynomial-based constructions for the Gabor dual windows. This thesis deals with trigonometric windows.

Some features of the work in this thesis, compared to above existing works, are as follows. We present examples with nice sized supports, with simple extensions from 1 dimension to higher dimensions, and with accessibility to Fourier analysis because of the trigonometric form. In detail, the Gabor dual windows in 1 dimensional space, in this paper, have support $[-1, 1]$. We can raise the smoothness of the window g without changing the support. The support $[-1, 1]$ is relatively small, which might be useful for practical uses. The figures and the formulas of the Gabor dual window examples in 1-dim. are shown in Figure 3.1 – Figure 3.5. Also, the windows are trigonometric functions in a restricted interval so it is easy to take the Fourier transform, especially the Discrete Fourier transform in the discrete-time case. With this property, we have a formula for the expression of the Gabor coefficients that leads to a relatively simple process of computing the Gabor coefficients, using the windows in this paper (Chapter 4). Also, if one wants to analyze signals in 2 dimensional space, such as images, one can tensorize the 1 dimensional examples. The process is not complicated and even the sufficient condition on constructions (such as the size of the modulation parameter matrix B) is less restrictive than the existing result [6]. These initial examples can be modified to get the shape of the support in 2 dimensional case to be not just a square but a parallelogram. Some explicit examples of Gabor dual windows in 2-dimension are in Figure 5.6 and Figure 5.7. Furthermore we can extend the construction and get examples in 3 and higher dimensional space, retaining some nice features as in 1 and 2 dimensional spaces.

Here is the structure of this thesis. In Chapter 2, we present some important theorems and corollaries which gives the background of Gabor theory. Especially notice the window condition (Theorem 2.2), which gives a necessary and sufficient condition for two functions g and h to give perfect reconstruction. This

condition is fundamental in our construction. The range of the degrees of trigonometric spline dual Gabor windows and the method of constructing the Gabor dual windows in 1 dimensional space are introduced in Chapter 3 (Theorem 3.2). Chapter 4 contains the topic of discrete-time Gabor systems. For the extension from 1 dimensional Gabor dual windows to higher dimensional Gabor dual windows, see Chapter 5 (for the 2-dimensional case) and Chapter 6 (for 3 and higher dimensions). The basic extension is by a product of 1-dimensional windows, but complications arrive when the modulation matrix B is not a multiple of the identity matrix. Chapters 3, 5, and 6 further have comparison sections explaining the relation with existing works in the literature.

Chapter 2

Background in Gabor theory

In Gabor theory, two operators are fundamental. One is the translation operator $(T_a f)(x) = f(x - a)$ and the other is the modulation operator $(E_b f)(x) = e^{2\pi i b x} f(x)$. The modulation operator represents a translation in frequency domain. If one uses a compactly supported function g as a window with sufficiently large discrete system of translations and modulations, one can cover whole time and frequency spaces and so be able to reconstruct certain function in $L^2(\mathbb{R})$.

Before entering to such Gabor analysis, let us see the continuous time-frequency representation. Here, the functional

$$\langle f, E_\omega T_x g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t - x)} e^{2\pi i t \omega} dt$$

is called the STFT (Short Time Fourier Transform) of f with respect to g .

Theorem 2.1 (Inversion Formula for the STFT, [12]). *Suppose $g, h \in L^2(\mathbb{R})$ and $\langle g, h \rangle = 1$. Then $\forall f \in L^2(\mathbb{R})$, perfect reconstruction holds:*

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, E_\omega T_x g \rangle E_\omega T_x h d\omega dx$$

Here, a coefficient $\langle f, E_\omega T_x g \rangle$ holds information of the function f in a local time-frequency domain, around $(x, \omega) \in \mathbb{R} \times \mathbb{R}$ with the size of the window g . The continuous inversion formula has a simple assumption that the inner product of two windows g, h is 1. However the computation of the double integral, integrating with respect to the time variable x and the frequency variable ω , might be hard to do. Also the computation is redundant : we could sample just some of the windowed function f with respect to g to cover all value of f . So we should not need to add (integrate) for every value of the time and frequency variables x and ω .

Hence we try to use a Gabor system $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$, for fixed $a, b > 0$, in a discretized reconstruction formula. The function in the system, g is called the generator or the *window* of this Gabor system. Here, the translation and modulation parameters, a and b respectively, determine the fineness of sampling. Sufficiently dense sampling leads to the almost perfect covering of f by the samples. If $ab \leq 1$, the system $\{E_{mb} T_{na} g\}$

can be a frame (Theorem 9.1.12 in [3]). It means that the Gabor system samples the time-frequency space “sufficiently densely”. For example, if one has a low sampling rate in the time domain (large a) then one must compensate with a high sampling rate in frequency (small b). A system $\{E_{mb}T_{na}g\}$ is a *Gabor frame* (or Weyl-Heisenberg frame) for $L^2(\mathbb{R})$, meaning a frame that has the structure of Gabor system, if there exists $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_{na}g \rangle|^2 \leq B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \quad (2.1)$$

A system $\{E_{mb}T_{na}g\}$ is called a *Bessel sequence* if there exists $B > 0$ such that the right-hand inequality in (2.1) holds. For a Gabor frame $\{E_{mb}T_{na}g\}$ the frame operator S , consisting of analysis followed by synthesis, is well-defined :

$$Sf \stackrel{\text{def}}{=} ab \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}g.$$

From this frame operator, it is possible to get the discrete form of reconstruction formula if we change the window in the synthesis operator from g to h . i.e., we want to reconstruct a function f as :

$$f = S_{g,h}f \stackrel{\text{def}}{=} ab \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}h.$$

Here the systems $\{E_{mb}T_{na}g\}$ and $\{E_{mb}T_{na}h\}$ are called the *Gabor dual frames*.

Next we have a simple equivalent condition for having the discrete reconstruction. The condition is due to Janssen [19] with weaker hypotheses on the windows. Our goal here is to give a simple direct proof for “nice” windows.

Theorem 2.2 (Window Condition). *Assume that $g, h \in L^2(\mathbb{R})$ are bounded and have compact supports. Assume $ab \leq 1$. Then we have*

$$\begin{aligned} f &= S_{g,h}f \quad \forall f \in L^2(\mathbb{R}) \\ \iff a \sum_{n \in \mathbb{Z}} \overline{g(x - m/b - na)} h(x - na) &= \delta_{m,0}, \quad a.e. \ x \in [0, a]. \end{aligned} \quad (2.2)$$

The double sum in the definition of $S_{g,h}f$ converges unconditionally, as the proof below will remark.

Proof. (\Leftarrow) First of all, the identity

$$\sum_{m \in \mathbb{Z}} (\widehat{f \cdot T_{na} \bar{g}})(mb) e^{2\pi i m b x} = \frac{1}{b} \sum_{m \in \mathbb{Z}} (f \cdot T_{na} \bar{g})(x - \frac{m}{b}) \quad (2.3)$$

can be verified as follows. The right hand side of (2.3) is $\frac{1}{b}$ -periodic and square integrable, because $f \in L^2(\mathbb{R})$ and g is bounded and has compact support, so we can express it as the Fourier series $\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n b x}$. By periodization, the Fourier coefficient is $c_k = \int_{\mathbb{R}} f(x) \cdot T_{na} \bar{g}(x) e^{-2\pi i k b x} dx = \widehat{f \cdot T_{na} \bar{g}}(kb)$ and the identity (2.3) is proved.

Then we have

$$S_{g,h}f = ab \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb} T_{na} g \rangle E_{mb} T_{na} h \quad (2.4)$$

$$\begin{aligned} &= ab \sum_{n \in \mathbb{Z}} \left\{ \sum_{m \in \mathbb{Z}} \langle f, E_{mb} T_{na} g \rangle e^{2\pi i m b x} \right\} h(x - na) \\ &= ab \sum_{n \in \mathbb{Z}} \left\{ \sum_{m \in \mathbb{Z}} (\widehat{f \cdot T_{na} \bar{g}})(mb) e^{2\pi i m b x} \right\} h(x - na) \\ &= ab \sum_{n \in \mathbb{Z}} \left\{ \frac{1}{b} \sum_{m \in \mathbb{Z}} (f \cdot T_{na} \bar{g})(x - \frac{m}{b}) \right\} h(x - na) \quad \text{by (2.3)} \\ &= \sum_{m \in \mathbb{Z}} \left\{ a \sum_{n \in \mathbb{Z}} \bar{g}(x - \frac{m}{b} - na) h(x - na) \right\} f(x - \frac{m}{b}) \quad (2.5) \\ &= f(x) \quad \text{a.e. } x \in [0, a], \quad \text{by hypothesis (2.2)} \end{aligned}$$

The summation in (2.4) converges unconditionally since $g, h \in L^2(\mathbb{R})$ are bounded and have compact supports (Corollary 9.1.6 of [3], Theorem 8.4.4 of [1]). Incidentally, (2.5) is called Walnut's representation of the frame operator. See Theorem 2.5 below.

(\Rightarrow) Assume that $S_{g,h}f = f \quad \forall f \in L^2(\mathbb{R})$. Then by (2.5) we have

$$f(x) = \sum_{m \in \mathbb{Z}} c_m(x) f(x - \frac{m}{b}) \quad \forall f \in L^2(\mathbb{R}).$$

where $c_m(x) = a \sum_{n \in \mathbb{Z}} \bar{g}(x - \frac{m}{b} - na) h(x - na)$ where $m \in \mathbb{Z}$. Let us consider f as the L^2 -function $\chi_{[0, \frac{1}{b}]}$. Because of the support of $f(x)$, we have that

$$\begin{aligned} c_m(x) &= 0 \quad \forall x \in (\frac{m}{b}, \frac{m+1}{b}) \quad \forall m \in \mathbb{Z} \setminus \{0\}, \quad \text{and} \\ c_0(x) &= 1 \quad \forall x \in (0, \frac{1}{b}). \end{aligned}$$

Also note that $c_m(x)$ is a -periodic. Since $a \leq \frac{1}{b}$, the interval on which $c_m(x)$ is equal to zero (which has length $\frac{1}{b}$) contains every value of $c_m(x)$. Thus, we can say that

$$c_m(x) = \delta_{m,0} \quad a.e. \quad x \in [0, a],$$

and so we have proved the necessary condition of the window condition. □

From the window condition in Theorem 2.2, we can get another corollary characterizing Gabor dual windows which is called Wexler-Raz Theorem (Theorem 9.3.4 in [3]). We omit the proof, since we will not need this result.

Theorem 2.3 (Wexler-Raz Biorthogonality Relations). *Let $g, h \in L^2(\mathbb{R})$ and $a, b > 0$ be given. If the two Gabor systems $\{E_{mb}T_{na}g\}, \{E_{mb}T_{na}h\}$ are Bessel sequences, then*

$$\begin{aligned} f &= S_{g,h}f \quad \forall f \in L^2(\mathbb{R}) \\ \iff \langle h, E_{m/a}T_{n/b}g \rangle &= \delta_{m,0}\delta_{n,0} \quad \text{for } m, n \in \mathbb{Z}. \end{aligned}$$

Other important theorems about the duality of Gabor systems include Walnut's representation theorem (Theorem 6.3.2 in [12]) and Janssen's representation theorem (Theorem 7.2.1 in [12]). These theorems are important steps in proving a more general version of the window condition (Theorem 9.3.5 in [3]). The assumption on g, h in Theorem 2.2 is that they are bounded functions in $L^2(\mathbb{R})$ with compact supports. Walnut's and Janssen's representation theorems assume only $g, h \in W(\mathbb{R})$ (Definition 2.4 below) and the function pair (g, h) satisfies condition (A') (Definition 2.6) respectively. However Theorem 2.2 is nice to have because of its simpler and clearer proof.

We will not need the Janssen or Walnut representations, but for the sake of providing background information, we will state them anyway. First, we need to see what the Wiener space W means.

Definition 2.4. *A function $g \in L^\infty(\mathbb{R})$ belongs to the Wiener space $W = W(\mathbb{R})$ if*

$$\|g\|_W = \sum_{n \in \mathbb{Z}} \operatorname{esssup}_{x \in \mathbb{Q}} |g(x+n)| < \infty$$

where $\mathbb{Q} = [0, 1]$. (Note $\mathbb{Q} = [0, 1]^d$ in d -dimensional space.)

Now we have Walnut's representation theorem for the Gabor frame operator S . Here the object function f comes out from the coefficient and the new coefficients in the representation consist of inner products of

windows for analysis and synthesis operator, g and h . Notice the independent variable occurs inside f in (2.6) below.

Theorem 2.5 (Walnut's representation). *Let $g, h \in W(\mathbb{R})$ and let $a, b > 0$. Then for $f \in L^2(\mathbb{R})$,*

$$S_{g,h}f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}h$$

can be written as

$$S_{g,h}f = \sum_{m \in \mathbb{Z}} c_m(x) \cdot T_{\frac{m}{b}}f \quad (2.6)$$

Notice (2.6) is just formula (2.5)

Now we give the definition of condition (A') for the convergence of Janssen's representation.

Definition 2.6. *A pair (g, h) of functions in $L^2(\mathbb{R})$ satisfies condition (A') for the parameters $a, b > 0$ if*

$$\sum_{m,n \in \mathbb{Z}} |\langle h, T_{\frac{m}{b}}E_{\frac{n}{a}}g \rangle| < \infty$$

If we use the Poisson summation formula with Walnut's representation (Sec.7.2 in [12]), we have another useful representation theorem for the Gabor frame operator S . The modulation and translation parameters are changed into "adjoint" form, that is, $\frac{1}{b}$ and $\frac{1}{a}$ instead of a and b .

Theorem 2.7 (Janssen's adjoint representation). *Assume that (g, h) satisfies condition (A') for $a, b > 0$. Then*

$$\begin{aligned} S_{g,h}f &\stackrel{def}{=} ab \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}h \\ &= \sum_{m,n \in \mathbb{Z}} \langle h, E_{\frac{m}{a}}T_{\frac{n}{b}}g \rangle E_{\frac{m}{a}}T_{\frac{n}{b}}f \end{aligned}$$

with absolute convergence in the operator norm.

Chapter 3

One dimensional Gabor windows

In this chapter, we construct dual Gabor windows in 1 dimension. Before we start, let us define a class \mathcal{T} of functions.

Definition 3.1. *A function f defined on \mathbb{R} is in class \mathcal{T} if f satisfies :*

- f is supported on $[-1, 1]$ and $f(\pm 1) = 0$
- f on $[-1, 1]$ equals the restriction of a symmetric trigonometric polynomial with period 4
- $f(0) = 1$

We will construct dual Gabor window pairs (g, h) where g, h are in class \mathcal{T} . Here, let us assume by a rescaling that the translation parameter $a = 1$ and assume the modulation parameter b is less than or equal to $\frac{1}{2}$ so that $ab \leq \frac{1}{2}$. For

$$a = 1 \text{ and } b \leq \frac{1}{2},$$

we have much simpler window condition, as we now show. If we plug in $a = 1$ into the window condition in Theorem 2.2, we get

$$\sum_{n \in \mathbb{Z}} \overline{g(x - \frac{m}{b} - n)} h(x - n) = \delta_{m,0} \quad a.e. \quad x \in [0, 1]. \quad (3.1)$$

If $b \leq \frac{1}{2}$, then for every non-zero integer m , we have $|\frac{m}{b}| \geq 2$. So (3.1) is satisfied automatically when $m \neq 0$, because g and h are supported in the interval $[-1, 1]$, which has length 2. When $m = 0$, we have

$$\sum_{n \in \mathbb{Z}} \overline{g(x - n)} h(x - n) = 1 \quad a.e. \quad x \in [0, 1]. \quad (3.2)$$

If g, h are in \mathcal{T} , then we only need to consider $n = 0, 1$ in (3.2) since the support is $[-1, 1]$. So we have the simple window condition as follows :

$$\overline{g(x-1)}h(x-1) + \overline{g(x)}h(x) = 1 \quad a.e. \quad x \in [0, 1].$$

Consider a C^m -smooth trigonometric function $g(x) = \cos^{m+1}(\frac{\pi x}{2})$ restricted on $[-1, 1]$. Then g satisfies all conditions so this function is in class \mathcal{T} and the degree of g is $m + 1$. Now finding h , the dual window of g , is our goal.

3.1 Constructing dual windows

Here is our main result in the one dimensional case, which is about the existence of a dual Gabor window of g . In the proof of this theorem, we show the method of constructing the explicit dual window h .

Theorem 3.2. *Fix $m \geq 0$. There exists $h \in \mathcal{T}$ with $\deg(h) = 3m + 1$ and $h \in C^m(\mathbb{R})$ such that g and h are dual Gabor windows.*

Our methods show that if $h \in C^m(\mathbb{R})$ is any other dual window of g in class \mathcal{T} , then $\deg(h) > 3m + 1$. We leave the verification to the reader.

3.2 Examples

Below are examples of the dual Gabor window pairs g and h in class \mathcal{T} as found using Theorem 3.2. The examples are plotted in Figures 3.1 – 3.5. Notice in the examples that g is a factor of h , so that h automatically satisfies the C^m -smoothness condition at $x = \pm 1$.

Example : $m = 0$

$$\begin{aligned} g(x) &= \cos\left(\frac{\pi x}{2}\right) \\ h(x) &= \cos\left(\frac{\pi x}{2}\right) \end{aligned}$$

Example : $m = 1$

$$\begin{aligned} g(x) &= \cos^2\left(\frac{\pi x}{2}\right) \\ h(x) &= \cos^2\left(\frac{\pi x}{2}\right)(2 - \cos(\pi x)) \end{aligned}$$

Example : $m = 2$

$$\begin{aligned} g(x) &= \cos^3\left(\frac{\pi x}{2}\right) \\ h(x) &= \cos^3\left(\frac{\pi x}{2}\right)[19 - 18 \cos(\pi x) + 3 \cos(2\pi x)]/4 \end{aligned}$$

Example : $m = 3$

$$g(x) = \cos^4\left(\frac{\pi x}{2}\right)$$

$$h(x) = \cos^4\left(\frac{\pi x}{2}\right)[104 - 131 \cos(\pi x) + 40 \cos(2\pi x) - 5 \cos(3\pi x)]/8$$

Example : $m = 4$

$$g(x) = \cos^5\left(\frac{\pi x}{2}\right)$$

$$h(x) = \cos^5\left(\frac{\pi x}{2}\right)[2509 - 3650 \cos(\pi x) + 1520 \cos(2\pi x) - 350 \cos(3\pi x) + 35 \cos(4\pi x)]/64$$

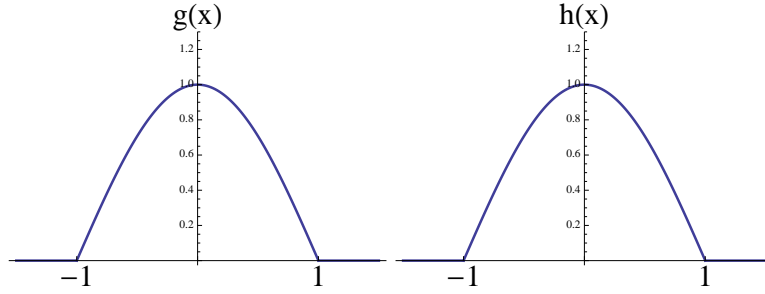


Figure 3.1: g and h for $m = 0$

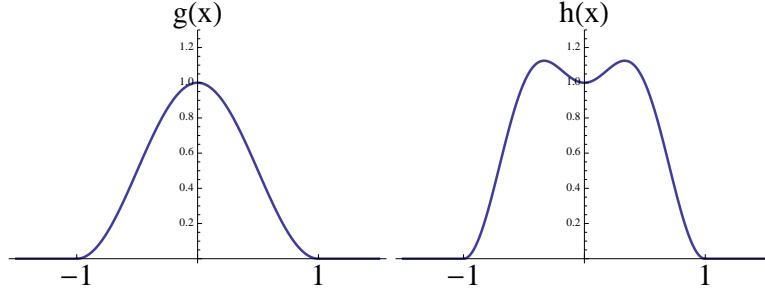


Figure 3.2: g and h for $m = 1$

We can recognize that h has local minimum at $x = 0$ for every $m \geq 1$, as follows. Note $g'(0) = 0$ and $h'(0) = 0$ by symmetry. Assume g and h satisfy the window condition $g(x)\overline{h(x)} + g(x-1)\overline{h(x-1)} \equiv 1$. If we differentiate it twice and put $x = 0$, we have $g''(0)\overline{h(0)} + 2g'(0)\overline{h'(0)} + g(0)\overline{h''(0)} + g''(-1)\overline{h(-1)} + 2g'(-1)\overline{h'(-1)} + g(-1)\overline{h''(-1)} \equiv 0$. Because of the normalization $g(0) = 1$, $h(0) = 1$, the boundary condition $(g(-1) = h(-1) = 0, g'(-1) = h'(-1) = 0$ by \mathbf{C}^m -smoothness with $m \geq 1$) the formula reduces to $g''(0) = -\overline{h''(0)}$. Since g has a negative second derivative at $x = 0$, h has a positive one and so h has a local minimum at $x = 0$.

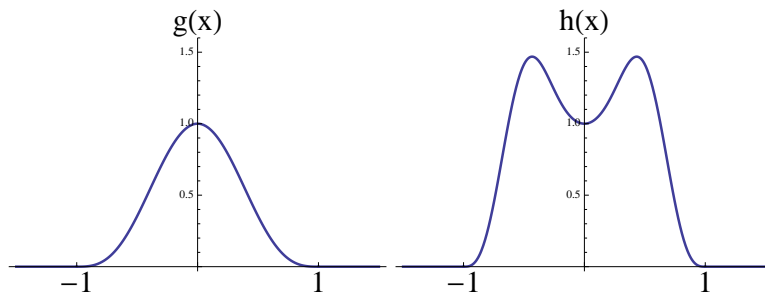


Figure 3.3: g and h for $m = 2$

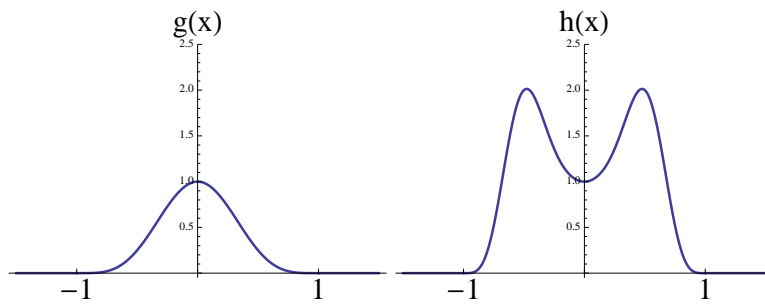


Figure 3.4: g and h for $m = 3$

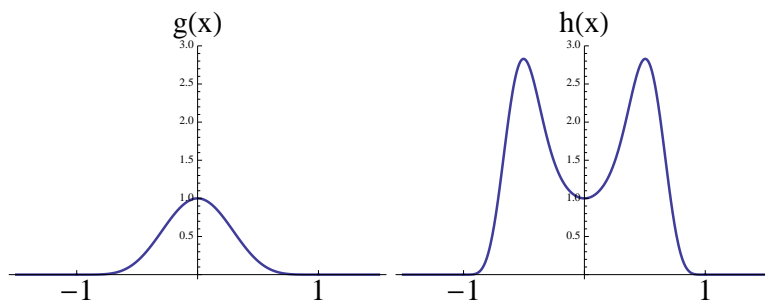


Figure 3.5: g and h for $m = 4$

3.3 Proof of Theorem 3.2

We know that $\deg(g) = m + 1$ and $g(x) = \cos^{m+1}(\frac{\pi x}{2}) = 2^{-(m+1)}(e^{\pi x i/2} + e^{-\pi x i/2})^{m+1}$. Since g and h are to be Gabor dual windows and g is real-valued, they must satisfy the simplified window condition :

$$g(x)h(x) + g(x-1)h(x-1) \equiv 1 \quad \forall x \in [0, 1].$$

Let $h(x) = g(x)w(x)$. Then the window condition becomes :

$$g(x)^2w(x) + g(x-1)^2w(x-1) \equiv 1 \quad \forall x \in [0, 1]$$

We want to know the function $w(x)$. Set $g(x) = G(e^{\pi x i/2}) = G(z)$ and $w(x) = W(e^{\pi x i/2}) = W(z)$ where G, W are Laurent polynomials, $z = e^{\pi x i/2} \in \mathbb{C}$. Then the window condition becomes :

$$G(z)^2W(z) + G(-iz)^2W(-iz) \equiv 1 \quad \forall z \in \mathbb{C}$$

Let $w(x) = \sum_{k=-r}^r c_k e^{\pi i k x/2} = \sum_{k=-r}^r c_k z^k$ since $w(x)$ is 4-periodic. (Note that symmetry leads the fact that $c_k = c_{-k}$. We have

$$\begin{aligned} G(z)^2 &= (z + z^{-1})^{2(m+1)} \cdot 2^{-2(m+1)} \\ &= \sum_{j=0}^{2m+2} \binom{2m+2}{j} z^{2j-2m-2} / 2^{2m+2} \end{aligned}$$

by a binomial identity. Now, if we plug in $W(z)$ and $G(z)^2$ into the window condition, we have

$$\sum_{j=0}^{2m+2} \binom{2m+2}{j} z^{2j-2m-2} \sum_{k=-r}^r c_k z^k + \sum_{j=0}^{2m+2} \binom{2m+2}{j} (-1)^{j-m-1} z^{2j-2m-2} \sum_{k=-r}^r c_k (-iz)^k \equiv 2^{2m+2}.$$

We combine the powers of z to obtain z^t , where $t = 2j - 2m - 2 + k$, and the window condition becomes

$$\sum_{t=-2m-2-r}^{2m+2+r} (1 + (-i)^t) \left\{ \sum_{\substack{k=-r \\ k \equiv t \pmod{2}}}^r \binom{2m+2}{m+1+(t-k)/2} c_k \right\} z^t \equiv 2^{2m+2}. \quad (3.3)$$

Here we use the convention that $\binom{2m+2}{j} = 0$ when $j < 0$ or $j > 2m+2$.

Assume that $r = 2m$ so that h has degree $3m+1$. Then the degree of the left-hand side of the window condition (3.3) is $4m+2$. Now let us divide cases.

Case 1 Let us think about the terms $z^{-4m-1}, z^{-4m+1}, \dots, z^{4m+1}$, i.e., the odd powers or odd values of t .

First, see the coefficient of z^{4m+1} in (3.3). From (3.3), $\sum_{\substack{k=-2m \\ k:\text{odd}}}^{2m} \binom{2m+2}{3m+(3-k)/2} c_k$ is the coefficient of z^{4m+1} . Note that $3m + (3 - k)/2 \leq 2m + 2$ because of the binomial term in the coefficient. Thus, $k = 2m - 1$. i.e., the coefficient of z^{4m+1} is c_{2m-1} .

Since the right-hand side of the window condition is constant, this coefficient vanishes. i.e., $c_{2m-1} = 0$.

If the power of z is odd number, the coefficient of z^t is consist of odd-indexed c . By induction, thus, with the information from the coefficient of the odd power of z 's, $c_k = 0$ for k : odd number.

Case 2 The coefficient of the $(4l + 2)$ -th power of z ($l \in \mathbb{Z}$) is automatically zero, because $(1 + (-i)^t) = 0$ when $t = 4l + 2$.

Case 3 Now we examine the coefficient of the z with form $t = 4l$ ($l \in \mathbb{Z}$).

From (3.3) that coefficient is

$$2 \sum_{k=-m}^m \binom{2m+2}{m+1+2l-k} c_{2k}.$$

Because the right side of (3.3) is constant, the coefficient of z^{4l} must equal 2^{2m+2} if $l = 0$, and 0 if $l \neq 0, |l| \leq m$. We can set this as a matrix form and can get the window condition as follows :

$$\mathbf{M}_m \cdot \begin{pmatrix} c_{-2m} \\ c_{-2m+2} \\ \vdots \\ c_0 \\ \vdots \\ c_{2m-2} \\ c_{2m} \end{pmatrix} = 2^{2m+1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where \mathbf{M}_m is the $(2m+1) \times (2m+1)$ -matrix defined as below

$$\mathbf{M}_m = \begin{bmatrix} \binom{2m+2}{1} & \binom{2m+2}{0} & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \binom{2m+2}{3} & \binom{2m+2}{2} & \binom{2m+2}{1} & \binom{2m+2}{0} & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{2m+2}{2m-1} & \binom{2m+2}{2m-2} & \cdots & \cdots & \cdots & \binom{2m+2}{2} & \binom{2m+2}{1} & \binom{2m+2}{0} & 0 \\ \binom{2m+2}{2m+1} & \binom{2m+2}{2m} & \cdots & \binom{2m+2}{m+2} & \binom{2m+2}{m+1} & \binom{2m+2}{m} & \cdots & \binom{2m+2}{2} & \binom{2m+2}{1} \\ 0 & \binom{2m+2}{2m+2} & \binom{2m+2}{2m+1} & \binom{2m+2}{2m} & \cdots & \cdots & \cdots & \binom{2m+2}{4} & \binom{2m+2}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & \binom{2m+2}{2m+2} & \binom{2m+2}{2m+1} & \binom{2m+2}{2m} & \binom{2m+2}{2m-1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \binom{2m+2}{2m+2} & \binom{2m+2}{2m+1} \end{bmatrix} \quad (3.4)$$

Note $(\mathbf{M}_m)_{i,j} = \binom{2m+2}{2i-j}$, for $1 \leq i, j \leq 2m+1$.

If \mathbf{M}_m is invertible, we can get the even coefficient c_{2k} 's of $w(x)$ and get $h(x) = g(x)w(x)$, a dual window of $g(x)$.

Claim 1. \mathbf{M}_m is invertible, with $\det(\mathbf{M}_m) = 2^{2m^2+3m+1}$.

Proof. The claim is obvious when $m = 0$. When $m = 1$,

$$\mathbf{M}_m = \begin{bmatrix} \binom{4}{1} & \binom{4}{0} & 0 \\ \binom{4}{3} & \binom{4}{2} & \binom{4}{1} \\ 0 & \binom{4}{4} & \binom{4}{3} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

and its determinant is 2^6 which satisfies the Claim.

To prove this claim for $m \geq 2$, we will break the matrix \mathbf{M}_m into some simple matrices and proceed by induction. If we see the general element of \mathbf{M}_m ,

$$\begin{aligned} (\mathbf{M}_m)_{i,j} &= \binom{2m+2}{2i-j} = \sum_{l=0}^2 \binom{2m}{2i-j-l} \binom{2}{l} \text{ by a counting argument} \\ &= \sum_{n=j-1}^{j+1} \binom{2m}{2i-n-1} \binom{2}{n-j+1} \end{aligned}$$

by letting $l = n+1-j$. If $j = 2, 3, \dots, 2m$, then the values $n = j-1, \dots, j+1$ lie in $[1, 2m+1]$ which

is the number of columns and rows of \mathbf{M}_m . Hence if we define matrices \mathbf{A}_m and \mathbf{B}_m with elements :

$$(\mathbf{A}_m)_{i,j} = \binom{2m}{2i-j-1} \quad \text{for } i, j = 1, \dots, 2m+1$$

and

$$(\mathbf{B}_m)_{i,j} = \binom{2}{i-j+1} \quad \text{for } i = 1, \dots, 2m+1, j = 2, 3, \dots, 2m,$$

then from the 2nd column to the $(2m)$ th column of \mathbf{M}_m can be expressed as a product of \mathbf{A}_m and \mathbf{B}_m .

For the first column of B_m , we have a claim as below.

Claim 2. *Let*

$$\{(\mathbf{B}_m)_{i,1}\}_{i=1,\dots,2m+1} = \left\{ \binom{2m}{1} + 2, -\binom{2m}{2} + 1, \binom{2m}{3}, -\binom{2m}{4}, \binom{2m}{5}, \dots, -\binom{2m}{2m}, 0 \right\} \quad (3.5)$$

Then $(A_m B_m)_{i,1} = (M_m)_{i,1}$.

Proof. Let us plug (3.5) in the equation

$$\begin{aligned} A_m(B_m)_{i,1} &= (M_m)_{i,1} \\ \Leftrightarrow \sum_{k=1}^{2m+1} \binom{2m}{2i-k-1} (B_m)_{k,1} &= \binom{2m+2}{2i-1}. \end{aligned} \quad (3.6)$$

And use a binomial property $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ twice to reduce the upper part of binomials in the equation (3.6) from $(2m+2)$ to $2m$. Then the equation (3.6) simplifies to

$$\sum_{k \in \mathbb{Z}} (-1)^{k+1} \binom{2m}{2i-k-1} \binom{2m}{k} = 0. \quad (3.7)$$

The change of summation variable $\ell = 2i - k - 1$ changes the left side of (3.7) to

$$\sum_{\ell \in \mathbb{Z}} (-1)^\ell \binom{2m}{\ell} \binom{2m}{2i-\ell-1},$$

which equals (-1) times the left side of (3.7). Hence the left side of (3.7) equals 0.

□

Similarly, the $(2m+1)$ th column of B_m is a sequence which is opposite order of the first column of B_m .

i.e., we have \mathbf{B}_m as follows :

$$\mathbf{B}_m = \begin{bmatrix} \binom{2m}{1} + 2 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\binom{2m}{2} + 1 & 2 & 1 & 0 & 0 & \cdots & \cdots & 0 & -\binom{2m}{2m} \\ \binom{2m}{3} & 1 & 2 & 1 & 0 & \cdots & \cdots & 0 & \binom{2m}{2m-1} \\ -\binom{2m}{4} & 0 & 1 & 2 & \ddots & \cdots & \cdots & \vdots & -\binom{2m}{2m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & & & \vdots & \vdots \\ \binom{2m}{2m-1} & 0 & 0 & 0 & \cdots & \cdots & \ddots & 1 & \binom{2m}{3} \\ -\binom{2m}{2m} & 0 & 0 & 0 & \cdots & \cdots & \ddots & 2 & -\binom{2m}{2} + 1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 1 & \binom{2m}{1} + 2 \end{bmatrix}$$

The determinant of \mathbf{M}_m is the product of $\det(\mathbf{A}_m)$ and $\det(\mathbf{B}_m)$. Here, note that the $\det(\mathbf{A}_m)$ is :

$$\det(\mathbf{A}_m) = \det(\mathbf{M}_{m-1})$$

since the centered $(2m-1) \times (2m-1)$ sub-matrix of \mathbf{A}_m is the matrix \mathbf{M}_{m-1} , i.e.,

$$\begin{aligned} (\mathbf{M}_{m-1})_{i,j} &= \binom{2(m-1)+2}{2i-j} = \binom{2m}{2i-j} \\ &= \binom{2m}{2(i+1)-(j+1)-1} = (\mathbf{A}_m)_{i+1,j+1} \end{aligned}$$

and the top left element and the bottom right element of \mathbf{A}_m are both 1 which are the only non-zero element of the first and the last row of \mathbf{A}_m .

So the determinant of \mathbf{B}_m is the ratio of the determinant of \mathbf{M}_m and \mathbf{M}_{m-1} . Now to prove the Claim we only need to show that

$$\det(\mathbf{B}_m) = \frac{2^{2m^2+3m+1}}{2^{2(m-1)^2+3(m-1)+1}} = 2^{4m+1}.$$

To show this, let us change the form of \mathbf{B}_m . Move each column to the left, with the first column

moving around to become the last column. If we let the new matrix as \mathbf{B}'_m , it looks like :

$$\mathbf{B}'_m = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & \binom{2m}{1} + 2 \\ 2 & 1 & 0 & 0 & \cdots & \cdots & 0 & -\binom{2m}{2m} & -\binom{2m}{2} + 1 \\ 1 & 2 & 1 & 0 & \cdots & \cdots & 0 & \binom{2m}{2m-1} & \binom{2m}{3} \\ 0 & 1 & 2 & \ddots & \cdots & \cdots & \vdots & -\binom{2m}{2m-2} & -\binom{2m}{4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \ddots & 1 & \binom{2m}{3} & \binom{2m}{2m-1} \\ 0 & 0 & 0 & \cdots & \cdots & \ddots & 2 & -\binom{2m}{2} + 1 & -\binom{2m}{2m} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 1 & \binom{2m}{1} + 2 & 0 \end{bmatrix}$$

So, $\det(\mathbf{B}'_m) = \det(\mathbf{B}_m) \times (-1)^{2m} = \det(\mathbf{B}_m)$. i.e., the determinant is preserved.

Now we want to divide \mathbf{B}'_m into two matrices as follows :

$$\mathbf{B}'_m = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a_1 & b_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 & b_2 \\ 0 & 0 & 1 & \cdots & 0 & a_3 & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{2m-1} & b_{2m-1} \\ 0 & 0 & 0 & \cdots & 0 & a_{2m} & b_{2m} \\ 0 & 0 & 0 & \cdots & 0 & a_{2m+1} & b_{2m+1} \end{bmatrix}$$

where $a_1 = 0, b_1 = \binom{2m+2}{1}$, and for $i \geq 2$ we have

$$a_i = \delta_{i,2m} + (-1)^{i+1} \sum_{k=0}^{i-2} \binom{2m}{2m-k} (i-k-1) \quad (3.8)$$

$$b_i = \sum_{k=0}^{i-2} \binom{2m}{k+2} (-1)^{i+1} (i-k-1) + (-1)^{i+1} ((2m+1)i+1) \quad (3.9)$$

which can be easily showed with simple calculation, by induction on i .

So now we have

$$\det(\mathbf{B}'_m) = \det \begin{bmatrix} a_{2m} & b_{2m} \\ a_{2m+1} & b_{2m+1} \end{bmatrix} = a_{2m} b_{2m+1} - a_{2m+1} b_{2m}.$$

We need to show that $a_{2m}b_{2m+1} - a_{2m+1}b_{2m} = 2^{4m+1}$. If we use some binomial identities such as

$$\sum_{k=0}^n \binom{n}{k} = 2^n, \quad \sum_{k=0}^n k \binom{n}{k} = n2^{n-1}, \quad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

with (3.8) and (3.9), it can be shown easily. So the Claim is verified. \square

We have shown \mathbf{M}_m is invertible. Thus we can get the exact value of the coefficients of $W(z)$ which means we can get $h(x)$, the dual window of $g(x)$. So we have proved the theorem.

3.4 Remarks on the construction

First of all, we can explain our choice of the window g .

Theorem 3.3. *There exist a unique $g(x)$ in class \mathcal{T} with C^m -smoothness such that $\deg(g)$ is the smallest. That degree is $m+1$, and $g(x) = \cos^{m+1}(\frac{\pi x}{2}) = 2^{-(m+1)}(e^{\pi i x/2} + e^{-\pi i x/2})^{m+1}$.*

Proof. Assume that $g \in \mathcal{T}$ with C^m -smoothness. Let $g(x) = G(e^{\pi x i/2}) = G(z)$ with G : a Laurent polynomial where $z \in \mathbb{C}$. Then because of the boundary condition, $g(1) = G(e^{\pi i/2}) = G(i) = 0$ and $g(-1) = G(e^{-\pi i/2}) = G(-i) = 0$ and thus $G(z)$ has a factor $(z-i)(\frac{1}{z}-i)$. Since g is C^m -smooth and the support of g is in $[-1, 1]$, $g^{(m)}(1) = g^{(m)}(-1) = 0$. i.e., $G^{(m)}(i) = G^{(m)}(-i) = 0$. So $G(z)$ has a factor $(z-i)^{m+1}(\frac{1}{z}-i)^{m+1}$ and by the fundamental theorem of algebra, $G(z) = c(z-i)^{m+1}(\frac{1}{z}-i)^{m+1}$ with some constant c is the function with the smallest degree which satisfies assumptions.

Now, the normalization condition in class \mathcal{T} can be used to find out the coefficient c . If we apply $g(0) = G(1) = 1$, we have a unique function $g(x) = 2^{-(m+1)}(e^{\pi i x/2} + e^{-\pi i x/2})^{m+1}$. \square

The following theorem says that if one of the dual Gabor window pairs in class \mathcal{T} has the smallest degree, then that window is a factor of the other window which is a dual window.

Theorem 3.4. *Let $g(x)$ and $h(x)$, both in class \mathcal{T} , be dual Gabor windows and C^m -smooth for $m \in \mathbb{N}$. Suppose that g has the smallest degree. Then $h(x)$ is a multiple of $g(x)$. i.e. there exists trigonometric function $w(x)$ such that $h(x) = g(x)w(x)$.*

Proof. From Theorem 3.3, we know that there exists g unique with smallest degree. Let $g(x) = P(e^{\pi x i/2}) = P(z)$, $h(x) = Q(e^{\pi x i/2}) = Q(z)$ where P, Q are Laurent polynomials, $z \in \mathbb{C}$. From the assumption, g and h are symmetric and have same boundary conditions. If $g(1) = 0$, $P(e^{\pi i/2}) = P(i) = 0$. i.e., P has a factor

$(z - i)$. If $g'(1) = 0$, $P'(i) = 0$. i.e., P has a factor $(z - i)^2$. By symmetry P has $(\frac{1}{z} - i)^2$ also. i.e., P has a factor $(z + i)^2$. Similarly, since $g^{(m)}(1) = 0$, P has $(z + i)^{m+1}$ and $(z - i)^{m+1}$. Since $h(x)$ has the same boundary condition, Q has factors $(z + i)^{m+1}$ and $(z - i)^{m+1}$. And we know that $g(x)$ is defined only from the boundary condition and has the smallest degree. Thus, by the Fundamental theorem of algebra, $h(x)$ is multiple of $g(x)$. \square

So the window function g we use in the first part of this chapter is found. This g could not be chosen any other way, unless the degree is increased. Also, from Theorem 3.4, g has to be a factor of h which is a dual window of g .

3.5 Comparison with work of Christensen and Kim, and Laugesen

In this section we try to compare our work with the known results so that we can see what's new and what aspect of our work is valuable in what sense. The work by Christensen [2], Christensen and Kim [7], and Laugesen [20] will be concerned because their work includes the issue of constructing the Gabor dual windows.

Features of our windows

In the first part of this chapter we assumed some properties of the window and the dual window are satisfied and we proved that such windows provide examples of Gabor dual windows. First of all, we have the fixed support for windows as $[-1, 1]$. The support is fixed for any smoothness of the windows, and it is relatively small support so the examples might be useful. And our windows are trigonometric splines so it is easy to take the Fourier transform. (We will use that feature in the next chapter for the discrete-time Gabor systems.) Also they are 4-periodic and symmetric, and the dual window of a certain window has that certain window as a factor. So if we arrange the main features of our windows, it would be as follows.

- fixed support $[-1, 1]$
- symmetric trigonometric splines with period 4
- one of the dual window is a factor of the other dual window

Main points of Christensen and Kim's work in 1-dim.

In Theorem 3.1 of Christensen and Kim's work [7], the general theorem of the paper, it shows that if we let $N \in \mathbb{N}$, $g \in L^2(\mathbb{R})$, bounded and $\text{supp}(g) \subset [0, N]$, $\sum_{n \in \mathbb{Z}} g(x - n) = 1$ and $b \in]0, \frac{1}{2N-1}]$ then, the dual window $h(x)$ is a finite linear combination of translates of $g(x)$ so the support is larger than the support of $g(x)$ (we can see that $\text{supp}(h) \subset [-N + 1, 2N - 1]$). (In Christensen's other work[2], it has an example of such dual window with a certain choice of the coefficients of the linear combination.)

To apply their result in practice, one needs an explicitly given g with the partition of unity property. Christensen and Kim have the positive result with B-spline as windows. Note that here, one has $N = m + 2$ where N is the degree of g and m is the smoothness of g . The dependence of support region on the smoothness m is due to the choice of g as a B-spline. There exists (at least in theory) windows g of arbitrary smoothness and fixed support $[0, 2]$ that satisfy the partition of unity property, but for the B-splines, the support grows as the smoothness increases.

Now we can arrange the features of the windows in their work as follows.

- g : polynomials restricted to compact intervals
- also they use the bounded function g s.t. $\text{supp}(g) \subseteq [0, N]$ and $\sum_{n \in \mathbb{Z}} g(x - n) = 1$, i.e., a partition of unity or "constant periodization".
- they use B-splines as explicit window examples.
- growing support with respect to the smoothness of the window (length = $O(N)$)
- modulation variable b is restricted by the size of the support of the window (or the smoothness of the window) ($b \leq \frac{1}{2N-1}$, so $b \leq \frac{1}{2m+3}$.)
- h , the dual window of g , is a finite linear combination of translates of g (so the size of the support of h is $3N - 2$ and it is bigger than the size of the support of g which is N).
- due to the property of h that h is a finite linear combination of translates of g , the behavior of h in the frequency domain can be determined from that of g .

In the Christensen and Kim's work [7], they also used window g that is a polynomial (not spline) restricted to compact intervals, i.e., spline with knots only at end points of support. They proved that none of its dual windows can have a similar form (i.e., polynomials with compact supports) but almost all of the dual windows are B-splines.

Comparison with work of Christensen and Kim

Here we can have three main issues in comparing our work and Christensen and Kim's work [7]. Those are the smoothness of windows, the length of support, and the range of modulation parameter b .

At first, we can see that, in our work, if one decides the smoothness of the window g as $m \geq 0$, then one has the explicit function g with smoothness m , which is a trigonometric polynomial in support $[-1, 1]$ with degree $m + 1$. Also one can have a dual window function h with degree $3m + 1$ (or bigger than $3m + 1$) by the construction process we have in the Theorem 3.2. In the Christensen and Kim's work, if one wants g to have a smoothness $m + 1$, then one has g as a B-spline with degree $m + 1$ and h with the same degree $(m + 1)$ since h is a finite linear combination of g . So one can decide the smoothness of window g arbitrarily for both cases but Christensen and Kim's work is better if one wants to use lower degree window for same smoothness. (See Table 3.1)

Second, let us see the length of supports of g and h in each work. In our work, the supports of g and h are fixed as $[-1, 1]$. So the length of the support is 2 and it is not changed due to the smoothness of the window. On the other hand, in the Christensen and Kim's work, the support of g depends on the smoothness of g . As the smoothness grows, the support of g grows. And since h is the linear combination of g , the length of h is even bigger. By the Theorem 3.1 of [7], if the length of support of g is N , then the length of support of h is $3N - 2$ and this variable N corresponds to the smoothness of g . So if one wants to use very smooth dual windows with small support or same support, our examples would be suitable (See Table 3.2).

For the third main issue, if we see Christensen and Kim's work, the range of the modulation parameter b is restricted by the smoothness of g . Since the support of g is also related to the smoothness of g , we can say that the modulation parameter b , the support of g , and the smoothness of g are all related. By the Theorem 3.1 of [7], if the length of support of g is N then $b \leq \frac{1}{2N-1}$. Whereas, in our work, the modulation parameter b is not affected by smoothness or supports of windows. So in this case, one can use any $b \leq \frac{1}{2}$ (See Table 3.2).

Also we can consider other features of windows such as the complexity of computing windows. In Christensen and Kim's paper, they used B-splines which is well known so one can write it up easily. For our examples of windows, we coded it up in Mathematica so we are able to have the explicit examples in a minute. And for higher dimension cases, both methods adapt differently, although new difficulties arises so those can be compared in many ways (See section 5.3 and section 6.2).

In another paper that Christensen wrote with M.S. Jacobsen [5], they have a similar result with the result of this paper but their method is different. They constructed trigonometric polynomial based Gabor dual window pairs for the smoothness $m < 6$ (Theorem 3.2 in [5]). Here, the modulation parameter b is in $(0, \frac{1}{5}]$

m	C & K		this paper	
	deg g	deg h	deg g	deg h
0	1	1	1	1
1	2	2	2	4
2	3	3	3	7
3	4	4	4	10
4	5	5	5	13
m	$m+1$	$m+1$	$m+1$	$3m+1$

Table 3.1: Here, m means the smoothness of g , C & K means Christensen and Kim's work, and "this paper" means our work in this paper.

m	C & K		this paper	
	modulation	support	modulation	support
0	$b < \frac{1}{3}$	4	$b \leq \frac{1}{2}$	2
1	$b < \frac{1}{5}$	7	$b \leq \frac{1}{2}$	2
2	$b < \frac{1}{7}$	10	$b \leq \frac{1}{2}$	2
3	$b < \frac{1}{9}$	13	$b \leq \frac{1}{2}$	2
4	$b < \frac{1}{11}$	16	$b \leq \frac{1}{2}$	2
m	$b < \frac{1}{2m+3}$	$3m+4$	$b \leq \frac{1}{2}$	2

Table 3.2: Here, "modulation" means the modulation parameter b , and "support" means the length of support of h , the dual of g . As m grows, the region of modulation parameter b becomes small in Christensen and Kim's work whereas our work keeps the region of b as $[0, \frac{1}{2}]$. Also the length of support of the dual window of g is growing in C&K but it is fixed as 2 in our work.

and the window g is multiple of $\sin(\frac{1}{3}\pi x)\chi_{[0,3]}(x)$. (In this paper, the window g is multiple of $\cos(\frac{\pi x}{2})\chi_{[-1,1]}$ and the result of this paper can handle the smoothness $m \geq 0$.)

Comparison with work of Laugesen

Since my idea about constructing the Gabor dual windows was derived from Laugesen's work [20], lots of common features can be found in our work and in [20]. Both works deal with windows in $[-1, 1]$ which is the fixed support of the windows, and windows can be taken to have C^m -smoothness and symmetry. Also, the modulation parameter b satisfies $b \in [0, \frac{1}{2}]$ if we assume the translation parameter $a = 1$. So we can see that the three main issues we took as criteria for comparison in section 3.5 have same contents in our work and in Laugesen's work [20].

The main difference is the form of the windows. In [20], he used polynomial splines as the dual Gabor

windows while we use trigonometric polynomials as the windows. That's why in [20], he introduced 3 knots (at $x = 0, -1, 1$) for windows because the window condition (Theorem 2.2 in this paper) does not work for a pair of two 2-knot polynomial splines. However, in our work, one can find the dual Gabor windows with only 2-knots (at $x = -1, 1$) since we are dealing with trigonometric functions. And we have proved that our process of construction works generally, for any smoothness m , while [20] does not include any proof that the polynomial algorithm works in general.

Chapter 4

Discrete-time Gabor systems

When someone wants to use the Gabor system in the real world (for example, in signal processing), it would be helpful if the windows are in discrete form. This naturally makes us to find the relationship between the continuous Gabor windows and the discrete Gabor windows. There are already results about this issue. In [3] and [18], the authors have conditions on the Gabor windows in $L^2(\mathbb{R})$ which imply that we can get the discretized Gabor windows with similar forms. We gather these existing results and prove the characterization of dual Gabor windows in $\ell^2(\mathbb{Z})$ in nice and very similar way with the continuous case (Theorem 4.1 below).

First, we need to define functions in $\ell^2(\mathbb{Z})$ and the translation and modulation operators again, since there are some differences between a continuous time domain and a discrete time domain. Let us change the translation and modulation parameters to

$$a = N \text{ and } b = \frac{1}{M} \quad \text{where } ab = \frac{N}{M} \leq \frac{1}{2}.$$

If we just sample a Gabor frame $\{E_{mb}T_{na}g\}_{m \in \mathbb{Z}, n \in \mathbb{Z}} = \{E_{\frac{m}{M}}T_{nN}g\}_{m \in \mathbb{Z}, n \in \mathbb{Z}}$ in a continuous domain at the integer values of the time variable, giving $\{E_{\frac{m}{M}}T_{nN}g(j)\}_{m \in \mathbb{Z}, n \in \mathbb{Z}}$ where $j \in \mathbb{Z}$, the discretized system is not even a Bessel sequence in a discrete time domain, because the modulated amount when $m = 0, \pm M, \pm 2M, \dots$ are same ($e^{2\pi i \frac{\pm M}{M}j} = 1$ for all integers j , and so on). Thus, in a discrete domain, we restrict the modulation variable m as $m = 0, 1, 2, \dots, M-1$. And, let us define the sampled function of $f(x), x \in \mathbb{R}$, with integer values as $f^D(j), j \in \mathbb{Z}$. It is observed in [12] that it does not matter whether the discretization is performed before or after the Gabor system is formed : $\{E_{\frac{m}{M}}T_{nN}g^D\}_{m=0, \dots, M-1, n \in \mathbb{Z}} = \{(E_{\frac{m}{M}}T_{nN}g)^D\}_{m=0, \dots, M-1, n \in \mathbb{Z}}$.

4.1 Discrete window condition

We find the characterization of the window condition in the discrete-time Gabor system.

Theorem 4.1 (Janssen-type Characterization of dual Gabor frames of $\ell^2(\mathbb{Z})$). *Suppose $g^D, h^D \in \ell^2(\mathbb{Z})$ have*

compact support and $M, N > 0$ are given. Then $\{E_{\frac{m}{M}} T_{nN} g^D\}_{m=0, \dots, M-1, n \in \mathbb{Z}}$ and $\{E_{\frac{m}{M}} T_{nN} h^D\}_{m=0, \dots, M-1, n \in \mathbb{Z}}$ are Gabor dual frames with $N \leq M$ if and only if

$$\sum_{n \in \mathbb{Z}} \overline{g^D(j - mM - nN)} h^D(j - nN) = \frac{1}{N} \delta_{m,0}(j), \quad \forall j \in \mathbb{Z}, m \in \mathbb{Z}$$

Proof. This proof is very similar with that of the continuous-time case, in Theorem 2.2.

(\Leftarrow) We have for all $f^D \in \ell^2(\mathbb{Z})$ that

$$\begin{aligned} f^D(j) &= N \sum_{m \in \mathbb{Z}} \left\{ \sum_{n \in \mathbb{Z}} \overline{g^D(j - mM - nN)} h^D(j - nN) \right\} f^D(j - mM) \quad \text{by assumption} \\ &= N \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \overline{g^D(j - mM - nN)} f^D(j - mM) h^D(j - nN) \\ &= N \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} f^D(l) \overline{g^D(l - nN)} \frac{1}{M} \sum_{k=0}^{M-1} e^{-2\pi i l \frac{k}{M}} e^{2\pi i \frac{k}{M} j} h^D(j - nN) \quad \text{by geometric sum over } k \\ &= \frac{N}{M} \sum_{n \in \mathbb{Z}} \sum_{k=0}^{M-1} \langle f^D, E_{\frac{k}{M}} T_{nN} g^D \rangle_{\ell^2(\mathbb{Z})} E_{\frac{k}{M}} T_{nN} h^D(j) \\ &= S_{g^D, h^D} f^D(j) \end{aligned}$$

So, $\{E_{\frac{m}{M}} T_{nN} g^D\}_{m=0, \dots, M-1, n \in \mathbb{Z}}$ and $\{E_{\frac{m}{M}} T_{nN} h^D\}_{m=0, \dots, M-1, n \in \mathbb{Z}}$ are Gabor dual frames in $\ell^2(\mathbb{Z})$.

(\Rightarrow) Assume that $S_{g^D, h^D} f^D = f^D$ for all $f^D \in \ell^2(\mathbb{Z})$. Then by the first line of the above equations, we have

$$f^D(j) = \sum_{m \in \mathbb{Z}} c_m(j) f^D(j - mM) \quad \forall f^D \in \ell^2(\mathbb{Z})$$

where $c_m(j) = N \sum_{n \in \mathbb{Z}} \overline{g^D(j - mM - nN)} h^D(j - nN)$ for $m \in \mathbb{Z}$. Let us consider f^D as the $\ell^2(\mathbb{Z})$ -function :

$$f^D(j) = \begin{cases} 1 & \text{if } 1 \leq j \leq M \\ 0 & \text{otherwise} \end{cases}$$

Then we have $c_m(j) = 0$ for all $j \in (mM, (m+1)M]$, $\forall m \in \mathbb{Z} \setminus \{0\}$ and $c_0(j) = 1$ for all $j \in (0, M]$. Here, note that $c_m(j)$ is a N -periodic function and that the necessary condition for the discrete Gabor system $\{E_{\frac{m}{M}} T_{nN} g(j)\}$ to be a frame is $\frac{N}{M} \leq 1$. Since $N \leq M$, $c_m(j) = \delta_{m,0}(j)$ and the proof is done. \square

If we have a pair of Gabor dual windows in the continuous time, then they can be Gabor dual windows in the discrete time.

Corollary 4.2. Assume that $g(x)$ and $h(x)$ are continuous, compactly supported Gabor dual windows. Then their discrete forms are dual windows in $\ell^2(\mathbb{Z})$.

Proof. By Theorem 2.2 and Theorem 4.1 with change of translation and modulation parameters a, b to $N, \frac{1}{M}$ and integer valued sampling, it is clear. \square

4.2 Examples

By Corollary 4.2, the examples of dual Gabor windows in discrete domain can be made from continuous dual Gabor windows. We need to decide the number of points by setting the number N . Below figures are the discrete dual Gabor windows in $\ell^2(\mathbb{Z})$ -sense which are from g and h , dual Gabor windows in continuous domain, with different number of support points and smoothness in continuous domain.

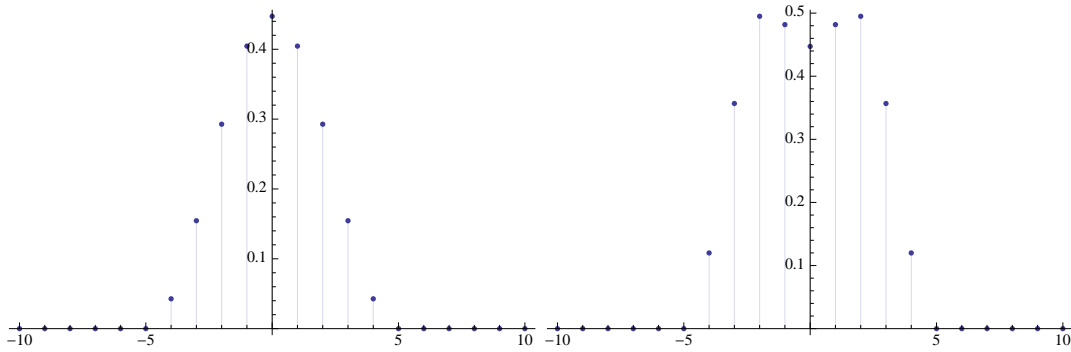


Figure 4.1: When $N = 5$. This g^D, h^D are from g, h in continuous case when $m = 1$.

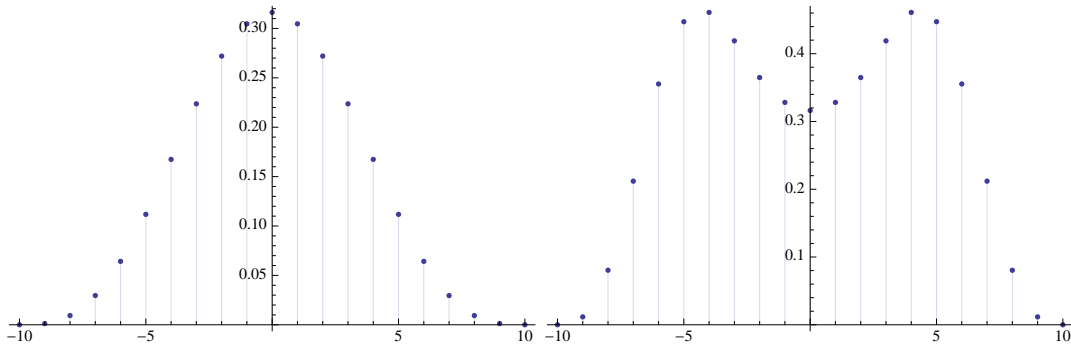


Figure 4.2: When $N = 10$. This g^D, h^D are from g, h in continuous case when $m = 2$.

4.3 Gabor coefficients by the DFT

In Theorem 4.3 we will show that we can express the Gabor coefficient in terms of the discrete Fourier transform of some portion of values of f^D . We can just take a certain portion of f , take the discrete Fourier transform of it, and filter it by summing certain linear combinations of translations of it.

Theorem 4.3. Fix $N \in \mathbb{N}$ and a function g in class \mathcal{T} , say $g(x) = \sum_{k=-p}^p c_k e^{i\pi k x/2} \chi_{[-1,1]}(x)$. Let $G(j) = \frac{1}{\sqrt{N}} g(\frac{j}{N})$ be a discrete form of g with support of $[-N, N]$ ($j \in \mathbb{Z}$). If $M = 2N$, then for any $f^D \in \ell^2(\mathbb{Z})$, the Gabor coefficients of f^D can be expressed as

$$\langle f^D, E_{\frac{m}{M}} T_{nN} G \rangle = \frac{1}{\sqrt{N}} e^{-\pi i m(n-1)} \left\{ \sum_{\substack{k=-p \\ k:\text{even}}}^p \overline{c_k} i^k (T_{-\frac{k}{2}} A_n)(m) + \sum_{\substack{k=-p \\ k:\text{odd}}}^p \overline{c_k} i^k (T_{-\frac{k-1}{2}} B_n)(m) \right\}$$

where $A_n = \mathcal{F} a_n$ (the DFT of a_n) where $a_n(j) = (T_{(1-n)N} f^D)(j) \chi_{[0,2N)}(j)$, i.e., shifted windowed version of data of length $2N$ around location nN (see Figure 4.3), and $B_n = \mathcal{F} b_n$ where $b_n(j) = E_{-\frac{1}{4N}} T_{(1-n)N} f^D(j) \chi_{[0,2N)}(j) = E_{-\frac{1}{4N}} a_n(j)$. (Here \mathcal{F} means the discrete Fourier transform with period $2N$, so that A_n and B_n are $2N$ -periodic.)

Note : $\text{supp}(A_n) = \text{supp}(B_n) = [0, 2N) \subseteq \mathbb{Z}$.

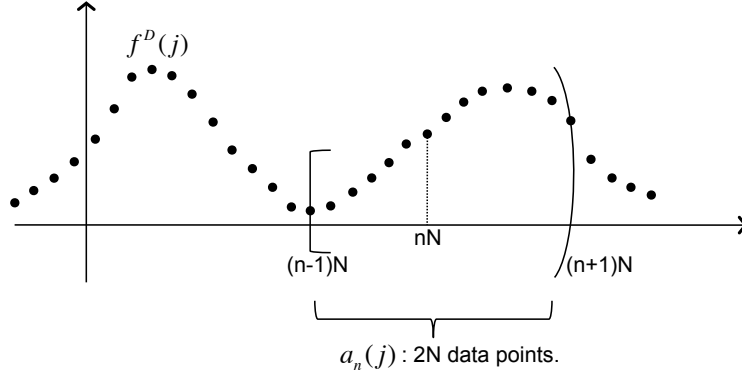


Figure 4.3: a_n is the windowed sample of data of length $2N$ around nN .

Proof. We have $G(j) = \frac{1}{\sqrt{N}} g(\frac{j}{N}) = \frac{1}{\sqrt{N}} \sum_{k=-p}^p c_k e^{i\pi k j/2N} \chi_{[-N,N)}(j)$. If we shift it by nN , $G(j - nN) = \frac{1}{\sqrt{N}} \sum_{k=-p}^p c_k e^{i\pi k j/2N} e^{-i\pi k n/2} \chi_{[(n-1)N, (n+1)N)}(j)$. Because of the support of $G(j - nN)$, we have

$$\langle f, E_{\frac{m}{M}} T_{nN} G \rangle = \sum_{j=(n-1)N}^{(n+1)N-1} f(j) \frac{1}{\sqrt{N}} \left(\sum_{k=-p}^p \overline{c_k} e^{-2\pi i j m/M - i\pi k j/2N} e^{i\pi k n/2} \right).$$

By change of variable,

$$\begin{aligned} \langle f, E_{\frac{m}{M}} T_{nN} G \rangle &= \sum_{j=0}^{2N-1} f(j + (n-1)N) \frac{1}{\sqrt{N}} \left(\sum_{k=-p}^p \overline{c_k} e^{-2\pi i (j+(n-1)N)m/M - i\pi k (j+(n-1)N)/2N} e^{i\pi k n/2} \right) \\ &= \frac{1}{\sqrt{N}} e^{-2\pi i m(n-1)N/M} \sum_{k=-p}^p \overline{c_k} i^k \left(\sum_{j=0}^{2N-1} (E_{-\frac{k}{4N}} T_{(1-n)N} f)(j) e^{-2\pi i j m/M} \right) \end{aligned}$$

Let us let the part $\sum_{j=0}^{2N-1} E_{-\frac{k}{4N}} T_{(1-n)N} f(j) e^{-2\pi i j m / M}$ above as (\star) . If k : even, let us say $k = 2k'$. Then

$$\begin{aligned}
(\star) &= \sum_{j=0}^{2N-1} T_{(1-n)N} f(j) e^{-2\pi i j (m+k') / 2N} \quad \text{since } M = 2N \\
&= A_n(m + k') \\
&\quad \text{if we let } a_n(j) = (T_{(1-n)N} f)(j) \chi_{[0, 2N-1]} \text{ and } A_n = \mathcal{F}a_n \\
&= (T_{-\frac{k}{2}} A_n)(m)
\end{aligned}$$

If k : odd, let us say $k = 2k' + 1$. Then

$$\begin{aligned}
(\star) &= \sum_{j=0}^{2N-1} E_{-\frac{2k'+1}{4N}} T_{(1-n)N} f(j) e^{-2\pi i j m / M} \\
&= \sum_{j=0}^{2N-1} E_{-\frac{1}{4N}} T_{(1-n)N} f(j) e^{-2\pi i j (m+k') / 2N} \quad \text{since } M = 2N
\end{aligned}$$

If we let $E_{-\frac{2k'+1}{4N}} T_{(1-n)N} f(j)$ as $b_n(j)$ and $B_n = \mathcal{F}b_n$, we have the result. □

Chapter 5

Two dimensional Gabor windows

In this chapter we will expand the dimension of domain to 2 dimension. We use the dual windows in 1 dimensional case to construct 2 dimensional dual Gabor windows.

5.1 Constructing dual windows in two dimensional space

In two dimensional case, parameters for the translation lattice and the modulation lattice are 2×2 -sized matrices. In one dimension, the parameters were $a > 0$ (we could assume $a = 1$) and $b > 0$ respectively. Similarly, we can let those lattices in two dimension be $A\mathbb{Z}^2$ and $B\mathbb{Z}^2$ for some 2×2 matrices A and B , respectively, and we can assume $A = I$ by rescaling. That is, the Gabor system generated by g is $\{E_{Bm}T_{In}g\}_{m,n \in \mathbb{Z}^2}$. And, analogous to Theorem 2.2, the characterization of dual Gabor windows in 2 dimension is as follows :

$$\sum_{k \in \mathbb{Z}^2} \overline{g(x - B^{-T}n - k)} h(x - k) = \delta_{n,0}, \quad a.e. \ x \in [0,1]^2, \ n \in \mathbb{Z}^2, \quad (5.1)$$

where we assume throughout this chapter that g and h are bounded with compact support. Throughout this chapter, we assume that 1-dim. functions g_1, g_2, h_1 and h_2 are bounded and compactly supported.

First we give the simplest example of the extension of windows from 1 dimension to 2 dimension domain.

Proposition 5.1 (Product construction for $B = \frac{1}{2}I$). *Assume that g_1, h_1 and g_2, h_2 are Gabor dual window pairs in one dimension, for parameters $a = 1$ and $b = \frac{1}{2}$. Then $g = g_1 \otimes g_2$ and $h = h_1 \otimes h_2$ are 2-dimensional dual Gabor windows for parameters $A = I$ and $B = \frac{1}{2}I$.*

Here $g_1 \otimes g_2$ means simply $(g_1 \otimes g_2)(x_1, x_2) = g_1(x_1)g_2(x_2)$.

Proof. From the assumption, we have $B^{-T} = 2I$. Let $x = (x_1, x_2)^T \in \mathbb{R}^2, n = (n_1, n_2)^T \in \mathbb{Z}^2$, and

$k = (k_1, k_2)^T \in \mathbb{Z}^2$. Then we have

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}^2} \overline{g_1 \otimes g_2}(x - 2n - k) h_1 \otimes h_2(x - k) \\
&= \sum_{k_1 \in \mathbb{Z}} \overline{g_1}(x_1 - 2n_1 - k_1) h_1(x_1 - k_1) \sum_{k_2 \in \mathbb{Z}} \overline{g_2}(x_2 - 2n_2 - k_2) h_2(x_2 - k_2) \\
&= \delta_{n_1, 0} \delta_{n_2, 0} \quad \text{by Theorem 2.2 with } b = \frac{1}{2} \\
&= \delta_{n, 0}
\end{aligned}$$

since (g_1, h_1) and (g_2, h_2) are dual window pairs. By the window condition in 2 dimension (5.1), g and h are dual Gabor windows. \square

For general B , we can get the Gabor dual windows if $\|B\|$ is sufficiently small. Following theorem shows the threshold.

Theorem 5.2 (B small enough). *In 2-dim. if $\|B\| \leq \frac{1}{2\sqrt{2}}$, then we can construct examples of Gabor dual windows with that B .*

Here $\|B\|$ denotes the operator norm of the matrix B .

Proof. Let g_1, h_1 and g_2, h_2 be dual Gabor window pairs in 1-dim. as constructed in Chapter 3. Assume that $g(x) = g_1 \otimes g_2(x)$ and $h(x) = h_1 \otimes h_2(x)$ for $x \in \mathbb{R}^2$. Let us think about the lattice from $B^{-T}n$ for $n = (n_1, n_2)^T \in \mathbb{Z}^2$. Assume that we shift a rectangle $[-1, 1]^2$ along with this lattice and there's no overlaps. Then the window condition for B :

$$\sum_{k \in \mathbb{Z}^2} g_1 \otimes g_2(x - B^{-T}n - k) \overline{h_1 \otimes h_2}(x - k) = \delta_{n, 0} \quad a.e. x \in \mathbb{R}^2$$

will be satisfied automatically when $n = 0$ since g_1, h_1 and g_2, h_2 are dual window pairs in 1 dim. and satisfy

$$\sum_{k_1 \in \mathbb{Z}} g_1(x - k_1) \overline{h_1}(x - k_1) \sum_{k_2 \in \mathbb{Z}} g_2(x - k_2) \overline{h_2}(x - k_2) = 1, \text{ for } x \in \mathbb{R}.$$

Also, when $n \neq 0$, because there does not exist overlaps of a rectangle $[-1, 1]^2$ on this lattice,

$$\sum_{k \in \mathbb{Z}^2} g_1 \otimes g_2(x - B^{-T}n - k) \overline{h_1 \otimes h_2}(x - k) = 0,$$

since the support of g_1, g_2, h_1 and h_2 is $[-1, 1]$. Here, $\sup_{x \in [-1, 1]^2} |B^T x| \leq \frac{1}{2}$ is the sufficient condition for

the window condition with a modulation parameter B . And we have :

$$\frac{1}{\sqrt{2}} \sup_{x \in [-1,1]^2} |B^T x| = \sup_{x \in [-1,1]^2} |B^T(\frac{1}{\sqrt{2}}x)| \leq \sup_{|y|=1} |B^T y| = \|B^T\|.$$

So if $\|B^T\| = \|B\| \leq \frac{1}{2\sqrt{2}}$, then $\sup_{x \in [-1,1]^2} |B^T x| \leq \frac{1}{2}$. Thus the window condition is satisfied. i.e., we can use our g_1, h_1 and g_2, h_2 , dual windows in 1-dim. to make the 2-dimensional dual windows. \square

The above theorem was for the general modulation parameter B . In the next theorem, with a specified modulation parameter B which is a one-way shear matrix, we have a better sufficient condition for existence of windows being dual Gabor windows in 2-dimension. This condition does not require the norm of B to be small.

Theorem 5.3 (Sufficient condition (one-way shear)). *Let $t \in \mathbb{R}$. Assume that (g_1, h_1) and (g_2, h_2) are dual window pairs in 1-dimension for $a = 1$ and $b = \frac{1}{2}$. Then $g_1 \otimes g_2(2B^T x)$ and $h_1 \otimes h_2(2B^T x)$ are dual Gabor window in 2-dim. with $A = I$ and the modulation parameter $B = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$ for any $t \in \mathbb{R}$.*

Proof. Since (g_1, h_1) and (g_2, h_2) are dual window pairs, we have

$$\sum_{k_1 \in \mathbb{Z}} g_1(y_1 - b_1^{-1}n_1 - k_1) \overline{h_1}(y_1 - k_1) = \delta_{n_1,0} \quad a.e. \quad y_1 \in \mathbb{R} \quad for \quad b_1 = \frac{1}{2} \quad and \quad (5.2)$$

$$\sum_{k_2 \in \mathbb{Z}} g_2(y_2 - b_2^{-1}n_2 - k_2) \overline{h_2}(y_2 - k_2) = \delta_{n_2,0} \quad a.e. \quad y_2 \in \mathbb{R} \quad for \quad b_2 = \frac{1}{2} \quad (5.3)$$

Then we have :

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}^2} g(x - B^{-T}n - k) \overline{h(x - k)} \\
&= \sum_{(k_1, k_2) \in \mathbb{Z}^2} g_1 \otimes g_2(2B^T x - 2n - 2B^T k) \overline{h_1 \otimes h_2(2B^T x - 2B^T k)} \\
&= \sum_{(k_1, k_2) \in \mathbb{Z}^2} g_1(y_1 - 2n_1 - k_1 + tk_2) \overline{h_1(y_1 - k_1 + tk_2)} g_2(y_2 - 2n_2 - k_2) \overline{h_2(y_2 - k_2)} \\
&\quad \text{by letting } (y_1, y_2)^T = 2B^T x \\
&= \sum_{k_2 \in \mathbb{Z}} g_2(y_2 - 2n_2 - k_2) \overline{h_2(y_2 - k_2)} \left\{ \sum_{k_1 \in \mathbb{Z}} g_1(y_1 - 2n_1 - k_1 + tk_2) \overline{h_1(y_1 - k_1 + tk_2)} \right\} \\
&= \sum_{k_2 \in \mathbb{Z}} g_2(y_2 - 2n_2 - k_2) \overline{h_2(y_2 - k_2)} \delta_{n_1, 0} \quad a.e. \quad y_1 \in \mathbb{R} \quad \text{by (5.2)} \\
&= \delta_{n_1, 0} \delta_{n_2, 0} \quad a.e. \quad y_1, y_2 \in \mathbb{R} \quad \text{by (5.3)} \\
&= \delta_{n, 0} \quad a.e. \quad x \in \mathbb{R}^2
\end{aligned}$$

i.e., the window condition with B in 2-dim. is satisfied. So, g and h are dual Gabor window in 2-dim. with the modulation parameter matrix B . \square

In one dimension, our dual Gabor window pairs in Chapter 3 work for any modulation parameter b which is less than or equal to $\frac{1}{2}$. With a similar idea, in 2 dimension, we can try to fix a dual windows pair and change the modulation parameter lattice to have a smaller norm and see if the dual windows pair works with the modified modulation parameter B .

Following is the result that if we have dual window pairs for a certain type of modulation parameter B , then we can preserve the windows for “smaller” modulation parameters.

Theorem 5.4 (Smaller shear matrices). *Assume that $g(x) = g_1 \otimes g_2(2B_0^T x)$ and $h(x) = h_1 \otimes h_2(2B_0^T x)$, where g_1, g_2, h_1, h_2 have supports in $[-1, 1]$, are dual Gabor window pairs for*

$$A = I, \quad B_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \quad (5.4)$$

for some $t \in \mathbb{R}$. Then they are also dual windows for

$$A = I, B = \begin{bmatrix} \beta & 0 \\ -t & \gamma \end{bmatrix} \quad \text{where } \beta, \gamma \in (0, \frac{1}{2}]. \quad (5.5)$$

Proof. For simplicity, first assume $\beta = \gamma$. Let us set $B_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$, $B_1 = \beta \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$ where $0 < \beta < \frac{1}{2}$

,and $B_2 = \begin{bmatrix} \beta & 0 \\ -t & \gamma \end{bmatrix}$ where $0 < \beta \leq \frac{1}{2}$, $0 < \gamma \leq \frac{1}{2}$. Since $g(x)$ and $h(x)$ are dual Gabor windows with B_0 , they satisfies the window condition in 2-dim.

$$\sum_{k \in \mathbb{Z}^2} g(x - B_0^{-T}n - k) \overline{h(x - k)} = \delta_{n,0} \quad a.e. x \in \mathbb{R}^2$$

Here, since g_1, g_2, h_1, h_2 have supports in $[-1, 1]$, g and h have supports in $S = B_0^{-T}([-\frac{1}{2}, \frac{1}{2}]^2)$.

In the first case, we need to show that

$$\sum_{k \in \mathbb{Z}^2} g(x - B_1^{-T}n - k) \overline{h(x - k)} = \delta_{n,0} \quad a.e. x \in \mathbb{R}^2.$$

If $n = 0$, it works by the above window condition with B_0 . If $n \neq 0$, (i.e., $n \in \mathbb{Z}^2 \setminus \{(0, 0)\}$)

$$\begin{aligned} \sum_{k \in \mathbb{Z}^2} g(x - B_1^{-T}n - k) \overline{h(x - k)} &= \sum_{k \in \mathbb{Z}^2} g_1 \otimes g_2(2B_0^T(x - B_1^{-T}n - k)) \overline{h_1 \otimes h_2(2B_0^T(x - k))} \\ &= \sum_{k \in \mathbb{Z}^2} g_1 \otimes g_2(2B_0^T x - 2B_0^T B_1^{-T}n - 2B_0^T k) \overline{h_1 \otimes h_2(2B_0^T x - 2B_0^T k)}. \end{aligned} \quad (5.6)$$

We can easily compute that

$$2B_0^T B_1^{-T}n = 2 \cdot \frac{1}{2} \frac{1}{\beta} \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \frac{1}{\beta} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

Here, $2B_0^T x \in [-1, 1]^2$ since $x \in B_0^{-T}([-\frac{1}{2}, \frac{1}{2}]^2) = S$, the supports of g and h . Let $y = (y_1, y_2)^T = 2B_0^T x \in [-1, 1]^2$ then (5.6) becomes like below :

$$\begin{aligned} &\sum_{k \in \mathbb{Z}^2} g_1(y_1 - \frac{1}{\beta}n_1 - k_1 + tk_2) \overline{h_1(y_1 - k_1 + tk_2)} g_2(y_2 - \frac{1}{\beta}n_2 - k_2) \overline{h_2(y_2 - k_2)} \\ &= \sum_{k_2 \in \mathbb{Z}} g_2(y_2 - \frac{1}{\beta}n_2 - k_2) \overline{h_2(y_2 - k_2)} \sum_{k_1 \in \mathbb{Z}} g_1(y_1 - \frac{1}{\beta}n_1 - k_1 + tk_2) \overline{h_1(y_1 - k_1 + tk_2)}. \end{aligned} \quad (5.7)$$

If $|\frac{1}{\beta}n_1|$ or $|\frac{1}{\beta}n_2|$ is bigger than 2, this equation is zero because $y_1, y_2 \in [-1, 1]$ so supports of shifted g_i and shifted h_i ($i = 1, 2$) does not overlap. Here, $|\frac{1}{\beta}n_1|$ and $|\frac{1}{\beta}n_2|$ are strictly bigger than 2 if n_1 and n_2 are not zero respectively since $\frac{1}{\beta} > 2$. Thus the equation (5.7) is zero if $n \neq 0$. Also, when $n = 0$, the equation (5.7)

becomes :

$$\begin{aligned}
& \sum_{k_2 \in \mathbb{Z}} g_2(y_2 - k_2) \overline{h_2}(y_2 - k_2) \sum_{k_1 \in \mathbb{Z}} g_1(y_1 - k_1 + tk_2) \overline{h_1}(y_1 - k_1 + tk_2) \\
&= \sum_{k_2 \in \mathbb{Z}} g_2(y_2 - k_2) \overline{h_2}(y_2 - k_2) \\
&= 1 \quad \text{by the duality of } g_1, h_1 \text{ and } g_2, h_2.
\end{aligned}$$

i.e., the window condition is satisfied for B_1 with the preserved g and h .

The second case, with a modulation parameter B_2 , can be showed similarly. We need to show the new window condition for B_2 , $\sum_{k \in \mathbb{Z}^2} g(x - B_2^{-T}n - k) \overline{h(x - k)} = \delta_{n,0} \quad a.e. x \in \mathbb{R}^2$, is satisfied. In this case, $2B_0^T B_2^{-T}n = \begin{bmatrix} \frac{1}{\beta}n_1 + \frac{1}{\beta\gamma}t(1-\beta)n_2 \\ \frac{1}{\gamma}n_2 \end{bmatrix}$. If $n_2 \neq 0$, $|\frac{1}{\gamma}n_2| \geq 2$ so the window condition is satisfied. If $n_2 = 0$ and $n_1 \neq 0$, then $|\frac{1}{\beta}n_1 + \frac{1}{\beta\gamma}t(1-\beta)n_2| = |\frac{1}{\beta}n_1| \geq 2$. So if $n = (n_1, n_2)^T \neq (0, 0)^T$, supports of shifted g_i and shifted h_i ($i = 1, 2$) does not overlap and the left-hand side of window condition is zero. When $n = 0$, the window condition works with same reason for the case with B_1 . i.e., the window condition is satisfied also. Therefore the window condition is satisfied for B_2 with the preserved g and h . \square

Note that the windows in Theorem 5.3 also work for B in Theorem 5.4. Since Theorem 5.3 works with any $t \in \mathbb{R}$, Theorem 5.4 can have dual windows for any $t \in \mathbb{R}$.

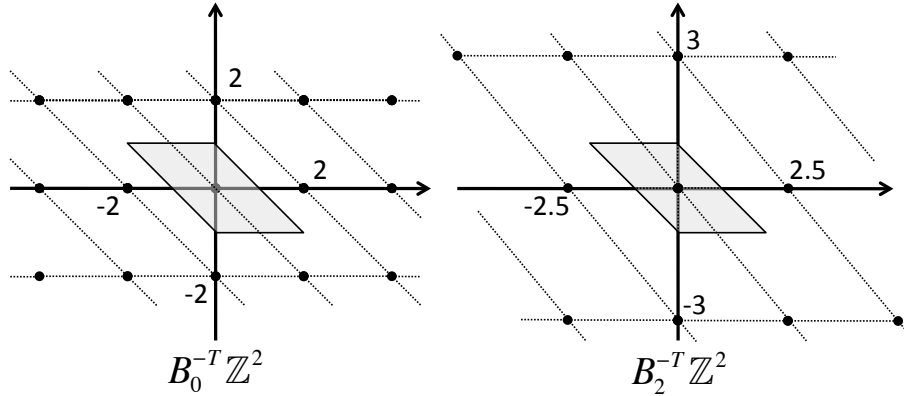


Figure 5.1: Lattices of $B_0^{-T}\mathbb{Z}^2$ and $B_2^{-T}\mathbb{Z}^2$ where B_0 is as in 5.4 with $t = -1$, and B_2 is as in 5.5 with $t = -1, \beta = \frac{1}{3}, \gamma = \frac{2}{5}$. And the shaded region is the support of g and h which is $B_0^{-T}([-\frac{1}{2}, \frac{1}{2}]^2)$.

Figure 5.1 can explain the idea of Theorem 5.4. In the Figure 5.1, the lattice of $B_2^{-T}\mathbb{Z}^2$ is wider vertically and horizontally than the lattice of $B_0^{-T}\mathbb{Z}^2$. So if a certain region does not have any overlap when it is shifted along $B_0^{-T}\mathbb{Z}^2$, it would not have any overlap when it is shifted through $B_2^{-T}\mathbb{Z}^2$ since the lattices become

larger. In the Figure 5.1, the shaded region which is the support of g and h is the certain shifting region. Since the support of g and h is decided to have no overlaps in the lattice $B_0^{-T}\mathbb{Z}^2$, the support region has no overlaps in the lattice $B_2^{-T}\mathbb{Z}^2$. That's why the dual window pairs with the modulation parameter B_0 satisfy the window conditions for B_2 (and also for B_1).

Next we have an equivalent condition for the window condition for two-way shear modulation lattice with integer amount.

Theorem 5.5 (Two-way integer shear matrix). *Fix $s, t, \in \mathbb{Z}$. Let $B_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$ which is vertical-shear matrix, $B_1 = \frac{1}{2} \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$ which is horizontal-shear matrix and $B = 2B_0B_1$. Let $g(x) = g_1 \otimes g_2(2B^T x)$ and $h(x) = h_1 \otimes h_2(2B^T x)$ where g_1, g_2, h_1 and h_2 are supported in $[-1, 1]$ and are bounded. Then g and h are Gabor dual windows for $A = I, B = 2B_0B_1$ if and only if $g_1\overline{h_1}(y_1) + g_1\overline{h_1}(y_1 - 1) = 1$ a.e. $y_1 \in [0, 1]$ and $g_2\overline{h_2}(y_2) + g_2\overline{h_2}(y_2 - 1) = 1$ a.e. $y_2 \in [0, 1]$ i.e., g_1, h_1 and g_2, h_2 are Gabor dual windows in 1-dim. .*

Proof. The window condition in 2-dim. with B is :

$$\sum_{k \in \mathbb{Z}^2} g(x - B^{-T}n - k)\overline{h}(x - k) = \delta_{n,0} \quad \text{a.e. } x \in \mathbb{R}^2.$$

After applying our g and h , we have

$$\sum_{k \in \mathbb{Z}^2} g_1 \otimes g_2(2B^T x - 2n - 2B^T k)\overline{h_1 \otimes h_2}(2B^T x - 2B^T k) = \delta_{n,0} \quad \text{a.e. } x \in \mathbb{R}^2.$$

The window condition is satisfied automatically when $n \neq 0$ because either $|2n_1| \geq 2$ or $|2n_2| \geq 2$ while the supports of g_1, g_2, h_1, h_2 have length 2. So we have to verify the window condition when $n = 0$:

$$\sum_{k \in \mathbb{Z}^2} g_1 \otimes g_2(4B_1^T B_0^T x - 4B_1^T B_0^T k)\overline{h_1 \otimes h_2}(4B_1^T B_0^T x - 4B_1^T B_0^T k) = 1 \quad \text{a.e. } x \in \mathbb{R}^2$$

after applying $B = 2B_0B_1$. Here we can check that $4B_1^T B_0^T$ is a bijection on $\mathbb{Z} \times \mathbb{Z}$ and $\det(4B_1^T B_0^T) = 1$. i.e., we can let $4B_1^T B_0^T k$ as $(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}$. Also we can let $4B_1^T B_0^T x = (y_1, y_2) \in [-1, 1]^2$. Thus the left

hand side of the above equation is same as the product of the window conditions in 1-dim.i.e.,

$$\sum_{k_1 \in \mathbb{Z}} g_1 \overline{h_1}(y_1 - k_1) = 1 \quad a.e. \quad y_1 \in \mathbb{R} \quad \text{and}$$

$$\sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) = 1 \quad a.e. \quad y_2 \in \mathbb{R}.$$

Applying the support $[-1, 1]$ of g_1, g_2, h_1 , and h_2 , we can get the result of the theorem. \square

If one wants the specific direction of the modulation lattice, we can approximate the angle of the direction with 2-way integer amount shearing. i.e., we can give the specifically directed dual windows by the above theorem. Let us say we have a modulation variable matrix as a two-way integer shear matrix

$$B = \frac{1}{2} \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+st & -s \\ -t & 1 \end{bmatrix}. \quad (5.8)$$

Then the support of g and h in Theorem 5.5 which is $B^{-T}([-\frac{1}{2}, \frac{1}{2}]^2)$ can be shown as a shaded region in Figure 5.2 from the computation of the inverse transpose of B as $B^{-T} = 2 \begin{bmatrix} 1 & t \\ s & 1+st \end{bmatrix}$. Every vertex of the support is in terms of s and t , and from those information, we can get the direction of the support as $\frac{1}{t} + s$, which is the slope of the line through the shaded region in Figure 5.2.

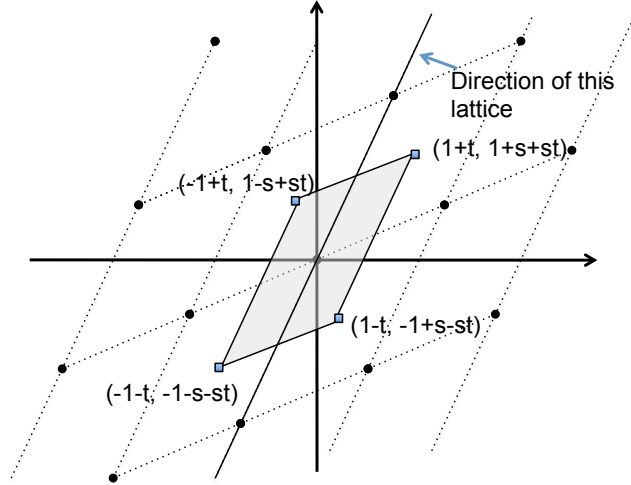


Figure 5.2: Parallelogram region is the support of g (and h) from the modulation parameter matrix B in (5.8) and the slope $\frac{1}{t} + s$ of the line through the region represents the direction of this lattice B^{-T} .

For example, if one wants $\tan^{-1}(1)$ as the angle of a modulation lattice B^{-T} , we can decide s and t as $s = 0, t = 1$ and if one wants $\tan^{-1}(\frac{2}{3})$, then $s = 1, t = -3$ will be the right choice of shearing of the

lattice (see Figure 5.3). The modulation parameter matrices B are $B = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $B = \frac{1}{2} \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}$ respectively. Of course not every numbers can be expressed as $\frac{1}{t} + s$ with integer s and t . In this case we can approximate the direction with some integers s and t .

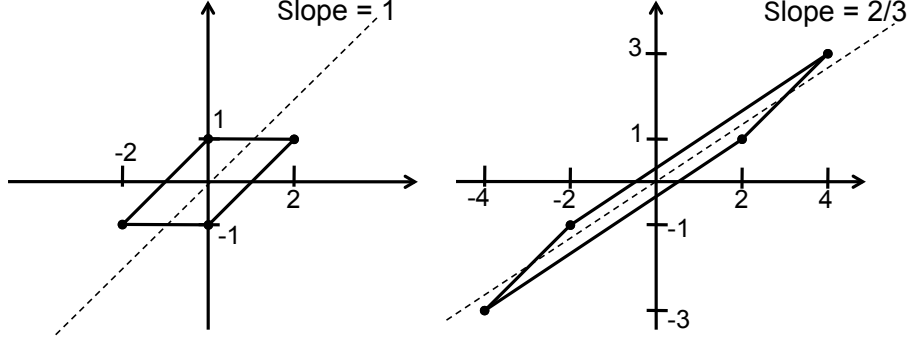


Figure 5.3: modulation lattices with the angle $\tan^{-1}(1)$ (when $s = 0, t = 1$) and $\tan^{-1}(2/3)$ (when $s = 1, t = -3$).

Next we find a necessary condition for the window condition in 2-dim. with $B = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$ when $t \in \mathbb{R}$.

A later theorem gets more conditions when $t \in \mathbb{Q}$.

Theorem 5.6 (One-way real-value shear matrix). *Assume that $g(x) = g_1 \otimes g_2(2B^T x)$ and $h(x) = h_1 \otimes h_2(2B^T x)$ are dual Gabor windows in 2-dim. where g_1, g_2, h_1, h_2 are supported in $[-1, 1]$ and $B = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$ for some $t \in \mathbb{R}$. Then $g_2 \overline{h_2}$ has constant periodization. i.e.,*

$$\sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) = \text{const.} \neq 0 \quad \text{a.e.} \quad y_2 \in \mathbb{R}.$$

i.e., $g_2 \overline{h_2}(y_2) + g_2 \overline{h_2}(y_2 - 1) = \text{const.}$ a.e. $y_2 \in [0, 1]$, so that (after normalizing) g_2 and h_2 are dual windows in 1 dim. for $b \leq \frac{1}{2}$.

The main idea of the proof of Theorem 5.6 can be explained with Figure 5.4. Since the matrix B^{-T} is one-way sheared horizontally, the shape of the support of g and h have fixed width 2, for any values of y_2 in the support, as in Figure 5.4. Let us fix an arbitrary value of y_2 in the support region, say \tilde{y}_2 . And let us consider any one point inside of the support, say a . Window condition in 2-dim. states that if we shift this arbitrary point by all integer pairs and add all of the values of the function $g\overline{h}$ on those points, we will get 1. If we integrate horizontally the sum of values of the function $g\overline{h} = g_1 \otimes g_2(2B^T x) \overline{h_1} \otimes \overline{h_2}(2B^T x)$ on a and on the shifted points on the line $y = \tilde{y}_2$, we will get the area under the function $g\overline{h}$ on the line $y = \tilde{y}_2$. Since the

area is constant, we can put it out of the summation of the window condition in 2 dimension. So only the vertical summation remains and that gives the result of Theorem 5.6 that $g_2\overline{h_2}$ has constant periodization.

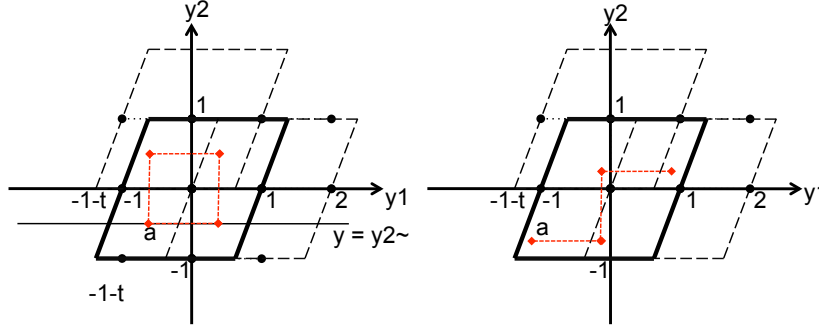


Figure 5.4: Bold-outlined region is the support of g, h corresponding to B in Theorem 5.6 and dotted lines represent shifted supports by $(1, 0)$ and $(0, 1)$. For an arbitrary point a in the support region, we can shift by $(1, 0), (0, 1)$ and $(1, 1)$ (left side) or by $(1, 0), (1, 1)$ and $(2, 1)$ (right side), or another similar three shiftings, to stay in the support region.

Proof. The window condition in 2-dim. with $B^T = \frac{1}{2} \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$ implies :

$$\sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) \left\{ \sum_{k_1 \in \mathbb{Z}} g_1 \overline{h_1}(y_1 + tk_2 - k_1) \right\} = 1 \quad a.e. \quad (y_1, y_2) \in \mathbb{R}^2.$$

If we integrate this equation with respect to y_1 on $[0, 1]$, we have :

$$\begin{aligned} 1 &= \sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) \left(\int_{[0,1]} \sum_{k_1 \in \mathbb{Z}} g_1 \overline{h_1}(y_1 + tk_2 - k_1) dy_1 \right) \\ &= \sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) \left(\int_{-\infty}^{\infty} g_1 \overline{h_1}(y) dy \right) \\ &= \sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) \int_{[-1,1]} g_1 \overline{h_1}(y) dy \quad \text{since } g_1, h_1 \text{ are supported on } [-1, 1] \end{aligned}$$

So the window condition is separated like above. If we do the integration with respect to y_2 on $[0, 1]$, we have :

$$1 = \left(\int_{[-1,1]} g_2 \overline{h_2}(y) dy \right) \left(\int_{[-1,1]} g_1 \overline{h_1}(y) dy \right).$$

With the above equation, thus, we can conclude that :

$$\sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) = \int_{[-1,1]} g_2 \overline{h_2}(y) dy = \text{const.} \quad a.e. \quad y_2 \in \mathbb{R}.$$

□

We have better necessary conditions for the window condition in 2-dim. when the shear parameter t is a rational number. The previous theorem was proved by using integrations, whereas the following theorem will be proved by Fourier coefficients of windows.

Theorem 5.7 (One-way rational-value shear matrix). *Assume that g_1, g_2 and h_1, h_2 are supported in $[-1, 1]$ and $g = (g_1 \otimes g_2)(2B^T x)$, $h = (h_1 \otimes h_2)(2B^T x)$, are dual Gabor windows in 2-dim. where $B = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$, $t = \frac{p}{q} \in \mathbb{Q}$. i.e., t is rational. Then $\sum_{k_1 \in \mathbb{Z}} g_1 \overline{h_1}(y_1 - \frac{k_1}{q}) = \frac{q}{c}$ a.e. $y_1 \in \mathbb{R}$ and $\sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) = c$ a.e. $y_2 \in \mathbb{R}$ where $c = \int_{[-1, 1]} g_2 \overline{h_2}(y) dy \neq 0$.*

The new information here is that $g_1 \overline{h_1}$ has constant periodization with respect to $\frac{1}{q}$.

Proof. The window condition says that the function

$$\sum_{k \in \mathbb{Z}^2} (g_1 \otimes g_2)(2B^T(x - k)) \overline{h_1 \otimes h_2}(2B^T(x - k)) = 1 \quad \text{a.e. } x \in \mathbb{R}^2.$$

So the function $F = (g_1 \otimes g_2)(2B^T \cdot) \overline{h_1 \otimes h_2}(2B^T \cdot)$ has constant periodization with respect to translations in \mathbb{Z}^2 . Hence we have

$$\widehat{F}(\xi_1, \xi_2) = \begin{cases} 1 & \text{if } (\xi_1, \xi_2) = (0, 0) \\ 0 & \text{if } (\xi_1, \xi_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\} \end{cases} \quad (5.9)$$

We know by simple computation that

$$\begin{aligned} \widehat{F}(\xi_1, \xi_2) &= \int F(y_1, y_2) e^{-2\pi i y \cdot \xi} dy \\ &= \int \int f_1(y_1 - ty_2) f_2(y_2) e^{-2\pi i y_1 \xi_1} e^{-2\pi i y_2 \xi_2} dy_1 dy_2 \\ &= \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2 + t\xi_1), \end{aligned}$$

where $f_1 = g_1 \overline{h_1}$ and $f_2 = g_2 \overline{h_2}$. Write $c_1 = \widehat{f}_1(0)$, $c_2 = \widehat{f}_2(0)$. Choosing $\xi_1 = \xi_2 = 0$ in (5.9) shows $c_1 c_2 = 1$.

Choosing $\xi_1 = 0$ and $\xi_2 \in \mathbb{Z} \setminus \{0\}$ shows $\widehat{f}_2(\xi_2) = 0$, so that f_2 has constant periodization, that is ,

$$\sum_{k_2 \in \mathbb{Z}} g_2 \overline{h_2}(y_2 - k_2) = c_2 \quad \text{a.e. } y_2 \in \mathbb{R}.$$

Now let us get the value of $\widehat{f}_1(\xi_1)$ when ξ_1 is non-zero integer. In case t is an integer, if $\xi_2 = -t\xi_1$, $\widehat{f}_1(\xi_1) \widehat{f}_2(0) = 0$ by (5.9) for all $\xi_1 \in \mathbb{Z} \setminus \{0\}$. Since $\widehat{f}_2(0) = c_2$, we can conclude that $\widehat{f}_1(\xi_1) = 0 \quad \forall \xi_1 \in \mathbb{Z} \setminus \{0\}$.

Now we want to see the case when t is not an integer but a rational number. i.e., let $t = \frac{p}{q} \in \mathbb{Q} \setminus \mathbb{Z}$. We know that $\xi_2 + t\xi_1 \in \mathbb{Z}$ if $\xi_1 \equiv 0 \pmod{q}$ i.e., $\xi_1 = sq$ for some $s \in \mathbb{Z}$. Among them, $\xi_2 + t\xi_1 = 0$ if $(\xi_1, \xi_2) = (sq, -sp)$ ($s \in \mathbb{Z}$) and in that case $\hat{f}_2(\xi_2 + t\xi_1) = \hat{f}_2(0) = c_2$. So by (5.9), with $\xi_1 = sq$ and $\xi_2 = -sp$, we find

$$\hat{f}_1(\xi_1) = \begin{cases} c_1 & \text{if } \xi_1 = 0 \\ 0 & \text{if } \xi_1 \in q\mathbb{Z} \setminus \{0\} \end{cases} \quad (5.10)$$

By (5.10) we have that f_1 has constant periodization with respect to translations by $\frac{1}{q}$. So

$$\sum_{k_1 \in \mathbb{Z}} f_1(y_1 - \frac{k_1}{q}) = \text{const.} = qc_1.$$

□

5.1.1 Relationship with the alias-free sampling problem

We can explain our work with the well known alias-free sampling problem. In d -dimensional space, we say a frequency support D allows an *alias-free M -fold sampling*, if shifted copies of D with respect to M^{-T} are disjoint, i.e., $D \cap (D + k) = \emptyset \quad \forall k \in \{M^{-T}l : l \in \mathbb{Z}^d\} \setminus \{0\}$. And we call alias-free sampling lattices with maximum packing density *the optimal sampling lattices*. In [21], an equivalent condition is given for a frequency region D to have an M -fold alias-free sampling as following (Proposition 1 in [21]) :

$$|M| \sum_{n \in \Lambda, \|n\|_\infty \leq r} |\widehat{\chi_D}(n)|^2 \leq m(D)$$

where $\Lambda = \{Mn : n \in \mathbb{Z}^d\}$ and $\widehat{\chi_D}(n)$ is the Fourier transform of the indicator function $\chi_D(w)$. $m(D)$ means the Lebesgue measure (i.e., the area) of D .

In our 2-dimensional work (in Theorem 5.2), when $B_0 = \frac{1}{2}I$ and some modulation matrix B is given, we wanted to get the threshold of $\|B\|$ such that the shifts of the support of $g_1 \otimes g_2$, which is $[-1, 1]^2$, along the lattice $B^{-T}\mathbb{Z}^2$ do not have any overlapping (see Figure 5.5 (a)). In section VII of [21], they propose an algorithm, that gives optimal packing lattice M , when polytope-shaped D is given and the quantization scale is given. Here, we have the polygonal support $D = B^T([-1, 1]^2)$ in \mathbb{Z}^2 -lattice (see Figure 5.5 (b)). If the lattice \mathbb{Z}^2 is shrunk by M^{-T} , i.e., if $\|M^{-T}\| \leq 1$ where $\|M\| = \sup_{|x|=1} |Mx|$ for $|x| = \|x\|_\infty$, then $B^T([-1, 1]^2)$ would not have any overlaps in \mathbb{Z}^2 -lattice. i.e., $[-1, 1]^2$ would not have any overlaps in $B^{-T}\mathbb{Z}^2$ -lattice (See Figure 5.5). So, if the packing matrix M^{-T} with $D = B^T([-1, 1]^2)$ has an infinity operator norm smaller than or equal to 1, then B can be the modulation lattice satisfying the window condition in

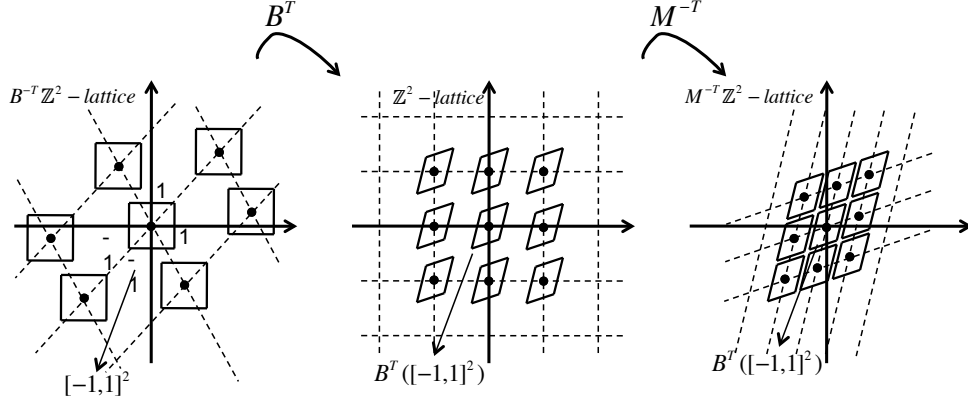


Figure 5.5: (a) : Shifts of support of $g_1 \otimes g_2$ and $h_1 \otimes h_2$, which is $[-1, 1]^2$ along the lattice $B^{-T}\mathbb{Z}^2$. (b) : After applying B^T to (a). (c) : After applying M^{-T} to (b) where M is the alias-free sampling matrix for D .

2-dimension. This can be a method to figure out if a modulation matrix B allows 2-dimensional examples of Gabor dual windows or not.

In another aspect, for the window condition we want that the shifted g does not have any overlaps with h . Assuming g and h have the same support $[-1, 1]^2$, this can be considered as a packing problem. $M = B$ might not be the optimal packing matrix for $D = [-1, 1]^2$ but the critical sampling method would be the sufficient condition of the window condition. Thus, by Proposition 1 in [21], the matrix B satisfies the window condition when $n \neq (0, 0)$ if and only if

$$\begin{aligned}
 & |B| \sum_{n \in \Lambda} |\widehat{\chi_{[-1,1]^2}}(n)|^2 \leq 4 \\
 \iff & |B| \sum_{n \in \Lambda} |\text{sinc}(2n_1) \text{sinc}(2n_2)|^2 \leq \frac{1}{4},
 \end{aligned} \tag{5.11}$$

where $n = (n_1, n_2) \in \mathbb{Z}^2$, $\Lambda = \{B^{-T}n : n \in \mathbb{Z}^2\}$ and $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

In conclusion, the first concern gives the sufficient condition of the window condition. We need to use the algorithm in section VII of [21] to get the matrix M . On the other hand, the second concern gives a practical necessary condition for the window condition, (It is hard to use (5.11) for the sufficient condition, because we need to consider the infinite sum over $n \in \Lambda$.)

5.2 Examples

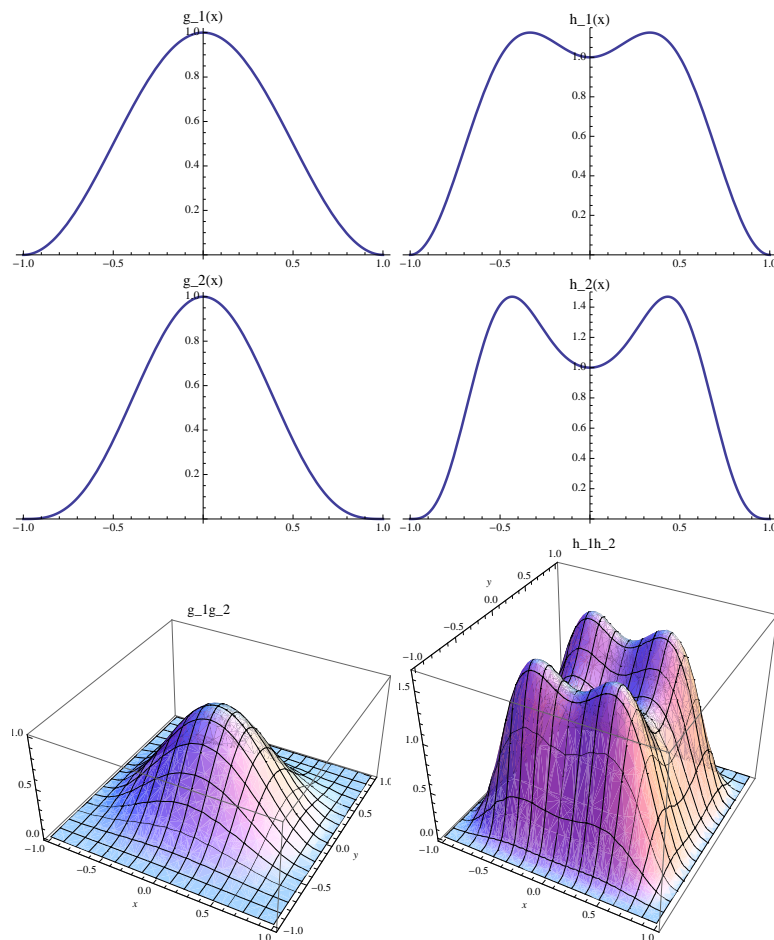


Figure 5.6: When B is half the identity matrix $\frac{1}{2}I$. We use g_1, h_1 for $m = 1$ and g_2, h_2 for $m = 2$ in $1 - dim$. then we have the product functions $g_1 \otimes g_2$ and $h_1 \otimes h_2$ which are dual windows in 2-dim. with $B = \frac{1}{2}I$ (See Proposition 5.1)

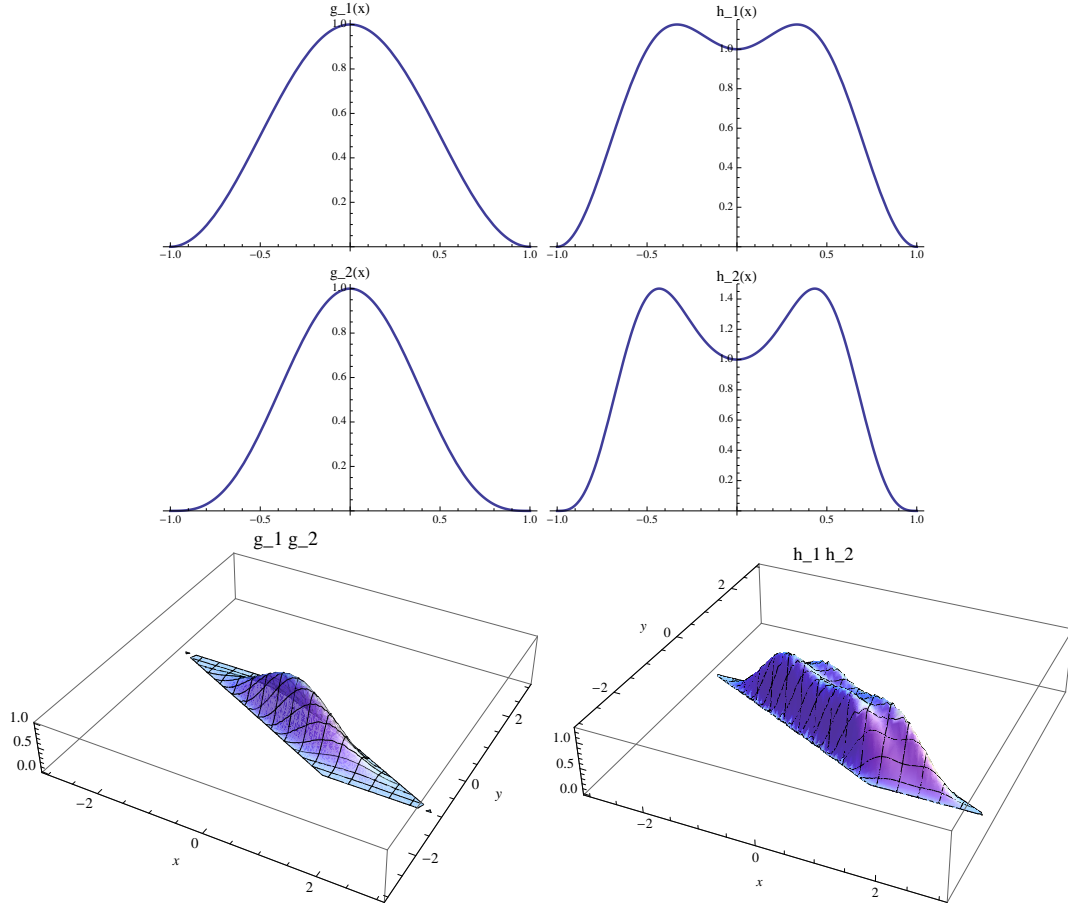


Figure 5.7: When $B =$ shear matrix as in Theorem 5.3 with $t = -1$. We use g_1, h_1 for $m = 1$ and g_2, h_2 for $m = 2$ in 1-dim. then we get product functions $g_1 \otimes g_2(2B^T x)$, $h_1 \otimes h_2(2B^T x)$ which are dual windows in 2-dim.

5.3 Comparison with two dimensional work of Christensen and Kim

Christensen and Kim have work of Gabor window construction in 2-dimension [6]. Similar with one dimensional case, Christensen and Kim have a window $g \in L^2(\mathbb{R}^2)$ whose \mathbb{Z}^2 -translates form a partition of unity, and h , the dual window of g , as a finite linear combination of translates of g . In our work, we make our windows g and h from 1-dimension examples and only changed the variable to have suitable (i.e., alias-free for the lattice B^{-T}) supports for window condition with the modulation parameter B .

First of all, let us see the restriction on the modulation parameter B . In [6], they have $\|B\| \leq \frac{1}{\sqrt{2(2N-1)}}$ where N is the natural number related with the smoothness in B-spline case, whereas we have $\|B\| \leq \frac{1}{2\sqrt{2}}$ in our work. So B in our work is less restricted than their work.

Also we can compare supports of windows. In [6], $\text{supp}(g) \subseteq [0, N]^2$ and $\text{supp}(h) \subseteq [-N + 1, 2N - 1]^2$ and the shape of the supports are non-convex unions of squares (See Example 1. in page 249 in [6]). In our work, we have fixed supports $\text{supp}(g) = \text{supp}(h) \subseteq B^{-T}([-\frac{1}{2}, \frac{1}{2}]^2)$ regardless of the smoothness.

Finally, in [6], the extension from 1 dimension to 2 dimension is nontrivial i.e., if one wants to construct dual windows in 2 dimensional space, it is not clear how to decide them from 1 dimensional case. However, in our work, we can construct 2 dimensional case by change of variables and product of 1 dimensional examples.

Chapter 6

Higher dimensional Gabor windows

In this chapter we consider 3 and higher dimensions. We have some interesting new features, for the modulation matrix B allowed in the sufficient condition (Theorem 6.3).

6.1 Constructing dual windows in 3 and higher dimensions

Throughout this section, the window functions g and h are bounded with compact support. The following is the simplest example of window extension from 1-dimension to d -dimension using products of windows.

Proposition 6.1 (Product construction for $B = I$). *Let $d \in \mathbb{N}$. Assume that each of $(g_1, h_1), (g_2, h_2), \dots, (g_d, h_d)$ are one dimensional Gabor dual window pairs for parameters $a = 1$ and $b \leq \frac{1}{2}$. Then $g_1 \otimes \dots \otimes g_d$ and $h_1 \otimes \dots \otimes h_d$ are d -dimensional Gabor dual windows for parameters $A = I$ and $B = \frac{1}{2}I$ where I is the identity matrix in d -dimension.*

As in the 2-dim. we have the threshold of size of the modulation parameter matrix B to have dual window examples in d -dimension.

Theorem 6.2 (B small enough). *In d -dim. if $\|B\| \leq \frac{1}{2\sqrt{d}}$, then we can construct explicit examples of Gabor dual windows that are bounded and compactly supported.*

Next we want window constructions when $\|B\| > \frac{1}{2\sqrt{d}}$. In 2-dim. we considered one-shear matrix with one non-zero element off-diagonal. The property of the one-shear matrix B that it leads to a nested summation in the window condition made us be able to have the connection between 1-dimensional and 2-dimensional examples (Theorem 5.3). Now in the 3-dimensional case, a more general type of one-way shear matrix would be an upper or a lower triangular matrix. An upper or a lower triangular matrix leads to a nested summation in the window condition.

The next theorem states the collection of modulation matrices we can handle. Notice $\|B\|$ is not restricted to be small.

Theorem 6.3 (Sufficient condition for window condition in 3-dim.). *Let $s, t, u \in \mathbb{R}$. Assume that $(g_1, h_1), (g_2, h_2)$ and (g_3, h_3) are dual window pairs in 1-dimension, all supported in $[-1, 1]$, for $a = 1, b \leq \frac{1}{2}$. Then $g_1 \otimes g_2 \otimes g_3(2B^T x)$ and $h_1 \otimes h_2 \otimes h_3(2B^T x)$ are dual Gabor window in 3-dim. for $A = I$ and for modulation matrix B having the triangular form if $B = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ t & u & 1 \end{bmatrix}$ or $\frac{1}{2} \begin{bmatrix} 1 & s & t \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix}$.*

Sketch of proof. For reducing the 3 dim. window condition to a product of the 1 dim. window conditions, we want nested summations (for example, in 2-dim. see (5.7)). \square

The shape of the lattice $B\mathbb{Z}^3$ with the modulation parameter matrix B which is one of the matrices in Theorem 6.3 is a parallelepiped generated by three vectors such as in Figure 6.1, since one row of B has one non-zero element, another row of B has two non-zero element, and the other row does not have any. We can see that the moves of unit vectors. One does not change direction, one changes direction in a plane, and the last one moves for all of y_1, y_2 and y_3 directions.

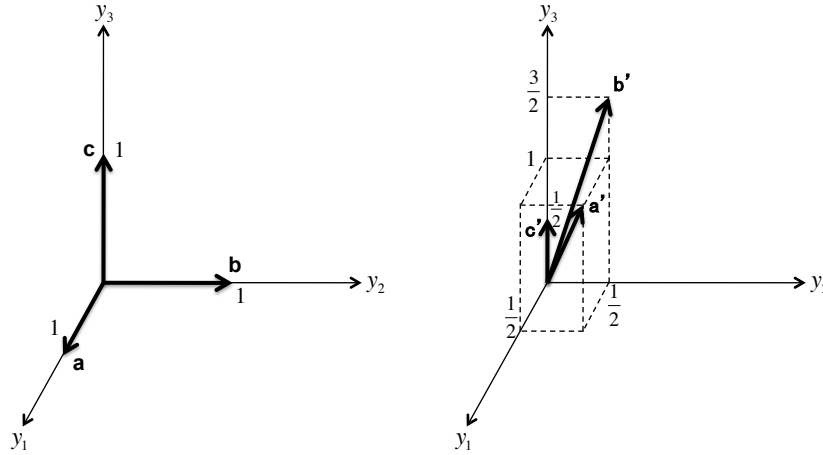


Figure 6.1: The shape of the lattice $B\mathbb{Z}^3$ with the modulation parameter matrix B is a parallelepiped generated by three vectors a', b' and c' when $s = 1, t = 2$ and $u = 3$ for the upper triangular matrix in the Theorem 6.3.

Note that we can extend this theorem to higher dimensions with an upper triangular or a lower triangular matrix as the modulation matrix B , because these types of matrices always lead to a nested summation in the window condition.

Next theorem says that if we have dual window pairs for some lower triangular matrices as the modulation parameter, then the windows are also dual for somewhat modified matrices.

Theorem 6.4 (Modulation parameter with the lower triangular matrix form). *Assume that $g(x) = g_1 \otimes g_2 \otimes g_3(2B_0^T x)$ and $h(x) = h_1 \otimes h_2 \otimes h_3(2B_0^T x)$, where $g_1, g_2, g_3, h_1, h_2, h_3$ are supported in $[-1, 1]$, are dual*

Gabor windows for

$$A = I, \quad B_0 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ s & \frac{1}{2} & 0 \\ t & u & \frac{1}{2} \end{bmatrix} \quad (6.1)$$

for some $s, t, u \in \mathbb{R}$. Then they are also dual windows for $A = I$, $B = \begin{bmatrix} \alpha & 0 & 0 \\ s & \beta & 0 \\ t & u & \gamma \end{bmatrix}$ when $0 < \alpha \leq \frac{1}{2}$, $0 < \beta \leq \frac{1}{2}$ and $0 < \gamma \leq \frac{1}{2}$.

Sketch of proof. We can prove Theorem 6.4 in similar way with the 2-dimensional case. We need to show that the window condition for B :

$$\begin{aligned} & \sum_{k \in \mathbb{Z}^3} g(x - B^{-T}n - k) \overline{h(x - k)} \\ &= \sum_{k \in \mathbb{Z}^3} g_1 \otimes g_2 \otimes g_3 (2B_0^T x - 2B_0^T B^{-T}n - 2B_0^T k) \overline{h_1 \otimes h_2 \otimes h_3 (2B_0^T x - 2B_0^T k)} \\ &= \delta_{n,0} \quad a.e. x \in \mathbb{R}^3 \end{aligned}$$

is satisfied. And here,

$$2B_0^T B^{-T}n = \begin{bmatrix} \frac{1}{\alpha}n_1 + (\frac{s}{\beta} - \frac{s}{\alpha\beta})n_2 + (\frac{t}{\gamma} - \frac{su}{\beta\gamma} + \frac{su-\beta t}{\alpha\beta\gamma})n_3 \\ \frac{1}{\beta}n_2 + (\frac{u}{\gamma} - \frac{u}{\beta\gamma})n_3 \\ \frac{1}{\gamma}n_3 \end{bmatrix}$$

for $n_1, n_2, n_3 \in \mathbb{Z}$. Thus, for any non-zero element $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$, at least one of the elements of $2B_0^T B^{-T}n$ has an absolute value which is bigger than 2 and so $2B_0^T B^{-T}n$ is an alias-free sampling lattice for $2B_0^T x \in [-1, 1]^3$. i.e., the window condition works.

Note that if we use the upper triangular matrix instead of the lower one, it works as well. Also, we can extend Theorem 6.4 to d -dimensional space because of the property of the matrix algebra that the product of upper triangular matrices is an upper triangular matrix in any dimension space.

Next we have an equivalent condition of window condition in 3-dim. when the modulation parameter matrix B is a product of an upper triangular matrix and a lower triangular matrix with integer elements.

Theorem 6.5 (B , the product of an upper and a lower triangular matrix with integer elements). *Fix*

$s, t, u, p, q, r \in \mathbb{Z}$. Let $B_0^T = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ t & u & 1 \end{bmatrix}$, $B_1^T = \frac{1}{2} \begin{bmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix}$ and $B = 2B_0B_1$. Let $g(x) = g_1 \otimes g_2 \otimes g_3(2B^Tx)$ and $h(x) = h_1 \otimes h_2 \otimes h_3(2B^Tx)$ where $g_1, g_2, g_3, h_1, h_2, h_3$ are supported in $[-1, 1]$. Then g and h are Gabor dual windows for $A = I, B = 2B_0B_1$ if and only if (g_1, h_1) , (g_2, h_2) and (g_3, h_3) are Gabor dual window pairs in 1-dim. .

Note that if the upper triangular matrix comes before the lower triangular matrix, one still has the same result. Also we can extend this theorem to d -dimension since the matrices $2B_0$ and $2B_1$ have 1 as diagonal elements so $2B_0B_1$ is bijective in \mathbb{Z}^d .

6.2 Comparison with higher dimensional work of Christensen and Kim

Christensen and Kim [6] also have work of Gabor window construction in \mathbb{R}^d . If we see the modulation parameter matrix B with size $d \times d$, in [6] they have the region of B as $\|B\| \leq \frac{1}{\sqrt{d(2N-1)}}$. In our work, we have $\|B\| \leq \frac{1}{2\sqrt{d}}$ and it does not depend on the smoothness of windows or the size of the supports of windows and the region is larger than the region of B in [6].

Almost every other comparisons is the same with 2-dimensional case. The assumptions on windows for construction such as the condition of g and h are same with 2-dim. case. Also, the trivial property of expansibility is same as 2-dim. while in [6], the expansibility is not trivial in higher dimensions. Supports of windows are similar with 2-dimensional case. In [6], $\text{supp}(g) \subseteq [0, N]^d$ and $\text{supp}(h) \subseteq [-N+1, 2N-1]^d$ which is non-convex unions of cubes. In our work, $\text{supp}(g) = \text{supp}(h) = B^{-T}([-\frac{1}{2}, \frac{1}{2}]^d)$ which is a parallelepiped.

Chapter 7

Open Problems

So far we constructed compactly supported Gabor dual windows for large class of lattices. However the existence of such windows for general translation-modulation lattices and for general dimensions is an open problem.

Open problems

One of the well known theorems about Gabor analysis is the density theorem which states the necessary condition of being basis and frames with respect to the modulation and translation parameters [15]. Let us define a Gabor system $G(\Lambda, g) = \{e^{2\pi i \langle l, x \rangle} g(x - k) | (k, l) \in \Lambda\}$ where Λ be a full rank lattice in \mathbb{R}^{2d} and $g \in L^2(\mathbb{R}^d)$. Here, any full rank lattice Λ can be written as $\Lambda = R\mathbb{Z}^{2d}$ with a non-singular $2d \times 2d$ matrix R . So far in thesis, we used product form lattices. For example, in 1 dimension case, if we let $\Lambda = R\mathbb{Z}^2$ with

$R = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, we have a product form lattice $a\mathbb{Z} \times b\mathbb{Z}$ that we used. Also if we have a 4×4 block matrix $R = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ with $A, B : 2 \times 2$ matrix, we get the lattice $A\mathbb{Z}^2 \times B\mathbb{Z}^2$ that we used in 2 dimension space.

The Density Theorem for Gabor systems is stated as follows in D.Han and Y.Wang [14].

Theorem 7.1 (The Density Theorem). *Let Λ be a full rank lattice in \mathbb{R}^{2d} .*

1. *If there exists $g \in L^2(\mathbb{R}^d)$ such that $G(\Lambda, g)$ is complete in $L^2(\mathbb{R}^d)$, then $D(\Lambda) \geq 1$.*
2. *If there exists $g \in L^2(\mathbb{R}^d)$ such that $G(\Lambda, g)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then $D(\Lambda) = 1$.*

Here $D(\Lambda)$ means the density of Λ and defined as $D(\Lambda) = |\det(R)|^{-1}$ where $\Lambda = R\mathbb{Z}^{2d}$ with R : non-singular $2d \times 2d$ matrix. The Density theorem is proved in the work of J.Ramanathan and T.Steger [22]. Considering the converse of this theorem and removing some results already proved lead us to some open problems stated in [14] as following :

- If $D(\Lambda) \geq 1$, is there $g \in L^2(\mathbb{R}^d)$ such that $G(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$ when Λ is non-separable lattice and $d > 1$?
- If $D(\Lambda) \geq 1$, is there compactly supported $g \in L^2(\mathbb{R}^d)$ such that $G(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$ when Λ is a separable lattice and $d > 1$?
- If $D(\Lambda) \geq 1$, is there compactly supported $g \in L^2(\mathbb{R}^d)$ such that $G(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$ when Λ is a non-separable lattice and $d \geq 1$?

In particular, [14] states the conjecture about irrational matrix lattice when $d = 1$ as following :

- For $\Lambda = R\mathbb{Z}^2$ with $R = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and $b \notin \mathbb{Q}$, there does not exist compactly supported $g \in L^2(\mathbb{R})$ such that $G(\Lambda, g)$ is an orthonormal basis for $L^2(\mathbb{R})$.

Known results

The exact converse of the Density theorem is almost proved for a general $g \in L^2(\mathbb{R}^d)$. If we state the frame completion part of the converse of the Density theorem, we have the question :

- If $D(\Lambda) \geq 1$, is there $g \in L^2(\mathbb{R}^d)$ such that $G(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$?

The existence in this question is proved for separable lattices Λ in $d \geq 1$ case by D.Han and Y.Wang in [13]. For non-separable lattices, the existence is proved for $d = 1$ case in [12] and the remaining case, when $d > 1$, is on the list in the open problems section above.

If we consider a stronger property of the window g having a compact support, we will meet a question like this :

- If $D(\Lambda) \geq 1$, is there a compactly supported $g \in L^2(\mathbb{R}^d)$ such that $G(\Lambda, g)$ is a tight frame for $L^2(\mathbb{R}^d)$?

For $d = 1$, if we have $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, we can choose the window $g = \frac{1}{\sqrt{|a|}}\chi_{[0, |a|]}$ and then $G(\Lambda, g)$ is a tight frame [14]. Also this question is proved for the rational matrix Λ (i.e., the matrix R has real elements) and for some specific lattice family in [14]. Yet, for separable lattices with $d > 1$ and for non-separable lattices with $d \geq 1$ are not proved yet. However, this question is proved for the rational matrix Λ and for some specific lattice family in [14].

Relation between open problems and our work

This thesis only treats separable lattices such as $\Lambda = R\mathbb{Z}^{2d}$ where R is a block matrix which is mentioned in the previous section. However we prove more than just existence, we found explicit constructions of dual windows. Since we fixed the translation parameter matrix A as the identity matrix I , the condition $|\det(B)| \leq 1$ is equivalent to the condition $D(\Lambda) \geq 1$. We found explicit examples of Gabor windows (even the dual Gabor windows) in d -dimension for $\|B\| \leq \frac{1}{2\sqrt{d}}$, and that could be one way of showing the existence of compactly supported Gabor windows for sufficiently large density of the lattice Λ .

References

- [1] O. Christensen, *An Introduction to Frames and Riesz Bases*, Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2003
- [2] O. Christensen, *Pairs of dual Gabor frame generators with compact support and desired frequency localization*, Appl. Comput. Harmon. Anal. 20:403-410, 2006.
- [3] O. Christensen, *Frames and Bases: An Introductory Course*, Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2008
- [4] O. Christensen, H.O. Kim and R.Y. Kim, *Gabor windows supported on $[-1, 1]$ and compactly supported dual windows*, Appl. Comput. Harmon. Anal. 28, no. 1, 89-103, 2010
- [5] O. Christensen and M.S. Jakobsen, *Dual pairs of Gabor frames for generators without the partition of unity property*, preprint, Dec. 2010
- [6] O. Christensen and R.Y. Kim, *Pairs of explicitly given dual Gabor frames in $L^2(\mathbb{R}^d)$* , J. Fourier Anal. Appl. 12, no. 3, 243-255, 2006
- [7] O. Christensen and R.Y. Kim, *On dual Gabor frame pairs generated by polynomials*, J. Fourier Anal. Appl. 16, no. 1, 1-16, 2010
- [8] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory 39, 961-1005
- [9] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. 27:1271-1283, 1986
- [10] M. Dörfler, *Time-frequency analysis for music signals : A mathematical approach*, J. New Music Research, 30, no. 1, 3-12, 2001
- [11] D. Gabor, *Theory of communication*, J. IEE (London), 93(III) : 429-457, 1946
- [12] K. Gröchenig, *Foundations of time-frequency analysis*, Birkhäuser Boston, Inc., Boston, MA, 2001
- [13] D. Han and Y. Wang, *Lattice tiling and the Weyl-Heisenberg frames*, Geom. Funct. Anal. 11, no. 4, 742-758, 2001
- [14] D. Han and Y. Wang, *The existence of Gabor bases and frames*, Contemp. Math., 345, Amer. Math. Soc., Providence, RI, 2004
- [15] C. Heil, *History and evolution of the density theorem for Gabor frames*, J. Fourier Anal. Appl. 13, no. 2, 113-166, 2007
- [16] A.J.E.M. Janssen, *Gabor representation of generalized functions*, J. Math. Anal. Appl. 83, 377-394, 1981
- [17] A. J. E. M. Janssen, *Signal analytical proofs of two basic results on lattice expansions*, Appl. Comp. Harm. Anal. 1: 350-354, 1994

- [18] A.J.E.M. Janssen, *From continuous to discrete Weyl-Heisenberg frames through sampling*, J. Fourier Anal. Appl. 3, no. 5, 583-596, 1997
- [19] A.J.E.M. Janssen, *The duality condition for Weyl-Heisenberg frames*, Appl. Numer. Harmon. Anal. 33-84, 1998
- [20] R.S. Laugesen, *Gabor dual spline windows*, Appl. Comput. Harmon. Anal. 27, no. 2, 180-194, 2009.
- [21] Y.M. Lu, M.N. Do and R.S. Laugesen, *A computable Fourier condition generating alias-free sampling lattices*, Signal Processing, IEEE Transactions on, vol. 57, no. 5, 2009
- [22] J. Ramanathan and T. Steger, *Incompleteness of sparse coherent states*, Appl. Comp. Harm. Anal. 2 , 148-153, 1995