






UNIVERSITY OF  
ILLINOIS LIBRARY  
**AT URBANA-CHAMPAIGN**  
BOOKSTACKS





Digitized by the Internet Archive  
in 2011 with funding from  
University of Illinois Urbana-Champaign

330  
B385  
1992:151 COPY 2

STX

#20-166

# Convergence Theorems on the Core of a Public Goods Economy: Sufficient Conditions

*John P. Conley*  
*Department of Economics*

The Library of the  
SEP 22 1992

University of Illinois  
at Urbana-Champaign



# BEBR

FACULTY WORKING PAPER NO. 92-0151

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

August 1992

Convergence Theorems on the Core of a Public  
Goods Economy: Sufficient Conditions

John P. Conley

Department of Economics





Convergence Theorems on the Core of a Public Goods Economy:  
Sufficient Conditions

John P. Conley†

Department of Economics  
University of Illinois  
Champaign, IL 61820

Revised: August 1992

---

† I would like to thank Marcus Berliant, Dimitrios Diamantaras, Mamoru Kaneko, William Thomson, Shlomo Weber, Simon Wilkie, and Myrna Wooders for their help and comments. All remaining errors are my own.



## Abstract

In 1963, Debreu and Scarf proved Edgeworth's conjecture that the core of an exchange economy converges to the competitive equilibria as the number of traders gets large. Unfortunately, Muench (1973), and others, have demonstrated that this result cannot be generalized to a public goods economy. This is due to the increasing returns to coalitional size that are embedded in the public goods technology. The purpose of this paper is to investigate the conditions under which core convergence can be regained when public goods are present. We show that if all consumers become asymptotically satiated in public good, then the core of a replica economy converges to the set of Lindahl equilibria. *Asymptotic satiation* means that the value to consumers, in terms of private goods, of receiving some multiple of their current level of public goods consumption goes to zero as the level goes to infinity. This assumption is trivially implied if consumers become satiated in public good. We also show that strict nonsatiation, which is almost the inverse of asymptotic satiation, is a sufficient condition for core convergence. *Strict nonsatiation* means that consumers' marginal rates of substitution of public for private goods is strictly bounded from zero. We take up question of the necessity and genericity of these conditions a companion paper.



## 1. Introduction

In 1963, Debreu and Scarf proved Edgeworth's conjecture that the core of an exchange economy converges to the set of competitive equilibrium allocations as the number of traders gets large. Economists have traditionally been interested in the core because the basic hypothesis that consumers always do as well as they can given their constraints implies that it is unlikely that allocations outside the core will be stable in market economies. Thus, the knowledge that the competitive allocations are the only ones that remain in the core as the economy gets large is of great practical importance. It confirms the central position of competitive equilibrium in economic theory, and provides a justification for the study of such subjects as comparative statics, stability, and computation general equilibria.

It is natural to wonder whether or not this result can be extended to an economy with public goods. Here the question becomes: does the core of a public goods economy converge to the set of Lindahl equilibrium allocations as the economy gets large? It has long been known that it is impossible to give an affirmative answer in general. Muench (1973), Champsaur, Roberts, and Rosenthal (1975), Milleron (1972), and Kaneko (1976), among others, provide compelling counterexamples to this hypothesis. The conventional wisdom seems to be that these counterexamples are robust. The core of a public goods economy is thought either never to converge,<sup>1</sup> or perhaps to converge only in the uninteresting knife edge case in which the increasing returns to coalitional size inherent in this type of economy are precisely offset by crowding,<sup>2</sup> diminishing marginal returns in production or something similar.

---

<sup>1</sup> For example, Kaneko (1975) shows that for finite, one public good, one private good, transferable utility economies, the core never coincides with the (unique) Lindahl equilibrium if any public good is produced. This is a nice result; however, Kaneko's consumption sets are not bounded below in private goods. This can lead to nonexistence of Lindahl equilibria. Strict nonsatiation does not imply core convergence in this cases. Also, Kaneko is interested in the non-equivalence of the core and the Lindahl equilibrium. Core convergence in a large finite economy is quite a different matter.

<sup>2</sup> For example, see Wooders (1991a).



The existing literature focuses on providing examples and counterexamples to convergence rather than addressing the general question of when the core converges and when it stays larger than the set of Lindahl equilibria. Here, and in related paper we make an attempt to address the problem in general. In this paper, we give two conditions on consumers' preferences which are independently sufficient to guarantee that the core converges to the set of Lindahl equilibrium. These conditions are restrictive, of course, but have very natural economic interpretations.

The theorems are proved for a replica rather than a continuum economy. This is because it is difficult to interpret allocations in a public goods economy with a continuum of consumers. Public and private goods are measured in fundamentally different, and irreconcilable ways. Public goods consumption is given as a finite number, while each the allocation of private goods is given as a distribution function. It is not clear what to make the fact that public goods consumption level for each consumer is infinitely higher than (infinitesimal) private goods consumption level. Even if the consumption levels of public goods happen to be infinitesimal as well (this would mean that the public good level is unmeasurable in the model), it is difficult to figure out exactly what sort of large finite economy allocation the continuum allocation is supposed to reflect as a limiting case. This is because the *ratio* of public to private goods consumption for each consumer may be bounded, or may still be undefined, despite the fact that the public goods level is unmeasurable. Distinguishing these two cases is important, and only possible in a finite economy.

The first theorem says that if all consumers become asymptotic satiated in public goods, then the core converges. A consumer's preferences are said to exhibit *asymptotic satiation* if the value to the consumer, in terms of private good, of receiving some multiple of his current public goods consumption level goes to zero as the level of public good goes to infinity. This does not seem to be a terribly

---

unreasonable assumption to make about consumers' preferences for many types of public goods. For example, if a consumer had ten channels on his cable system, he might be willing to pay a lot for ten more. But if he already had one hundred, would he care that much about another hundred? Certainly, if he already had one thousand, adding another thousand would hardly improve his welfare at all. Or take a public park as an example. While a larger park is always better, a consumer would probably care less and less about each successive doubling of the park's size. Eventually, the park gets so big that the consumer can never hope to see the whole thing in any event. Obviously, asymptotic satiation is trivially satisfied if consumers actually become satiated in public goods.

The second theorem says that if all consumers are strictly nonsatiated in public goods then the core converges. A consumer's preferences are said to exhibit *strict nonsatiation* if the desire for public goods is so strong that the marginal rate of substitution of public for private goods is everywhere strictly bounded from zero. For example, this assumption is satisfied if public and private goods are perfect substitutes. This assumption has much less economic appeal; however, we argue that there is some theoretical interest in this case.

This work is probably most closely related to Wooders (1991a, 1991b). She studies transferable utility games derived from market economies. She provides necessary and sufficient conditions on the pre-game under which the  $\epsilon$ -core of the game derived from the market shrinks to Walrasian payoffs of the market. However, the conditions described in these papers are only of the derived game form. No attempt is made to take things a step back and discover the primitive assumptions that would be required on underlying market economy. Also, Wooders restricts her attention to transferable utility economies. In contrast, this paper concentrates on developing conditions on the economy itself that are sufficient to give core convergence in the general NTU case.

Wooders' theoretical framework is quite powerful. Extensions from the private

goods market game to a local public goods game seem to be fairly straightforward. However, the generalization to a pure public good economy is somewhat problematic. One reason is the notion of convergence used by Wooders. Roughly speaking, convergence in her model means the core of the game contracts to the equilibrium payoffs which are either stationary as the economy is replicated (as in the case of a private goods economy) or go to some definite limit (as in a local public goods economy). In a pure public goods economy, on the other hand, the typical case is for the Lindahl equilibrium to move as the economy gets large, and it is optimal to provide ever increasing levels of public goods. Only when all consumers are satiated in public goods will the Lindahl equilibrium ever converge to a definite limit. This is only in this case that convergence is possible in Wooders' sense. Thus, it seems to us that this is probably not the appropriate notion of convergence for a public goods economy.

We suggest a different notion in this paper. Roughly stated, the core is said to converge if for a sufficiently large economy, any allocation that is significantly different<sup>3</sup> from a Lindahl equilibrium cannot be an element of the core. In essence, this requires that the core contracts to the moving target of the Lindahl allocations as the economy gets large.

Another limitation to the application of Wooders' line of research to pure public goods economies is that the requirement of *bounded per capita payoffs* seems to play an important role in her theorems<sup>4</sup> This is a fairly innocuous requirement for private and local public goods economies. However, in a pure public good economy, in which the public goods level may get arbitrarily large, it is quite limiting. It

---

<sup>3</sup> "Significantly different" means that as the economy grows without bound, a nonvanishing fraction of the consumers pay an amount that is at least a nonvanishing  $\epsilon$  different from their Lindahl taxes for a Pareto optimal level of public goods

<sup>4</sup> For example, this assumption shows up in Wooders (1983, 1991c, 1990), Scotchmer and Wooders (1988), and Shubik and Wooders (1983).

requires that in some sense consumers eventually become satiated in public goods, which is certainly not a trivial assumption. We show here that even when per capita payoffs are not bounded, the core of a public goods economy may still converge.

Another paper that is related to this work is Weber and Weismeth (1992). They prove a very nice theorem that says that core of a one public, one private good NTU economy<sup>5</sup> with strictly convex technology is equivalent to the cost share equilibrium with respect to a strictly monotone cost share method. However, the cost share equilibria represent a superset of the Lindahl-Foley equilibria. They also consider the relationship between the core and the Lindahl-Foley equilibrium, which is the closer to the topic of this paper. Here their results are somewhat weaker. They don't show equivalence or convergence, but rather provide a characterization of set of equilibria. Their proposition 4.2 essentially says that the Lindahl-Foley equilibrium are in one to one correspondence with the core allocations that *also satisfy an additional restriction*. The point of this proposition seems to be to provide a means of identifying the elements of the core that can be supported by linear cost share methods<sup>6</sup> rather than to prove that the core converges the Lindahl equilibria in any sense. In addition, *strictly* convex production technology is required for these theorems to hold. Thus, in a formal sense, there is no overlap between these results and the current paper.

In a related, paper Conley (1992), we examine the question of when the core of a public goods economy does not converge. We show that the conditions given in the current paper are almost, but not quite necessary for core convergence. However, if slight changes in the model that reduce the domain of economies considered are made, then these conditions become necessary and sufficient for core convergence. Finally, we show that in any event these conditions generically do not hold, and so

---

<sup>5</sup> Assumption 2.1 of their paper excludes the TU case.

<sup>6</sup> Which Mas-Colell and Silvestre (1989) showed to be identical to the Lindahl-Foley equilibria.

generically the core does not converge.



## 2. The Model

We consider a simple replica economy,  $\mathcal{E}^R$ , with  $T$  types of consumers indexed by  $t \in \mathcal{T} \equiv \{1, \dots, T\}$ , and  $R$  consumers of each type indexed by  $r \in \mathcal{R} \equiv \{1, \dots, R\}$ . Each consumer type has complete and transitive preferences  $\succeq_t$  defined over  $X \equiv \mathbb{R}_+^{M+1}$ , a one private good,  $M$  public good consumption set where public goods are indexed by  $m \in \mathcal{M} \equiv \{1, \dots, M\}$ . Each consumer type is endowed with a strictly positive amount of private good  $\omega^t \in \mathbb{R}_+$ . A typical consumption bundle in  $X$  will be written  $(x; y)$  where  $x$  is a level of private good, and  $y$  is a vector of public goods. Consumers are referred to by their number and type. Thus  $\{r, t\}$  is consumer number  $r$  of type  $t$ . Superscripts are used to represent consumers and subscripts to represent types of public goods. Superscripts and subscripts will also sometimes be used to distinguish different vectors of goods from one another, but this will be done a way that prevents confusion.

The following assumptions are made on  $\succeq_t$  for all  $t \in \mathcal{T}$ .

CONT.  $\succeq_t$  is continuous.

CONV. If  $(x; y) \succeq_t (x'; y')$ , then for all  $\lambda \in [0, 1]$ ,  $\lambda(x; y) + (1 - \lambda)(x'; y') \succeq_t (x'; y')$ .

(Weak convexity)

MONO. If  $(x; y) \geq (x'; y')$ , then<sup>7</sup>  $(x; y) \succeq_t (x'; y')$ ; also, if  $x > x'$ , and  $y \geq y'$ , then  $(x; y) \succ_t (x'; y')$ .

(Monotonicity in all goods, and strict monotonicity in the private good.)

Define the  $K$  dimensional simplex  $\Delta^K$  as:

$$\Delta^K \equiv \left\{ p \in \mathbb{R}_+^{K+1} \mid \sum_{i=1}^{K+1} p_i = 1 \right\}. \quad (1)$$

---

<sup>7</sup> The three kinds of vector inequalities are represented by  $\geq$ ,  $>$ , and  $\gg$ .

Note that monotonicity implies that in equilibrium all prices are non-negative, and in addition, the price of the private good is positive.

Let  $Y \subset \mathbb{R}_- \times \mathbb{R}_+^M$  denote the production set. A typical feasible production plan is written  $(z; y)$  where  $z$  is a negative number which is interpreted as the input of private good, and  $y$  is a positive output vector of public goods. Define the *marginal cost correspondence* for  $Y$ ,  $MC : Y \rightarrow \Delta^M$ , as follows:

$$MC(z; y) \equiv \{(p; q) \in \Delta^M \mid (p; q)(z; y) \geq (p; q)(z'; y') \forall (z'; y') \in Y\}. \quad (2)$$

The production set is assumed to satisfy:

CLCC.  $Y$  is a Closed Convex Cone.

NFPR.  $Y \cap \mathbb{R}_+^{M+1} = \{0\}$ . (No Free Production.)

BAMC. There exists  $\phi > 0$  such that for all  $m \in \mathcal{M}$ , all  $(z; y) \in Y$ , and all  $(p; q) \in MC(z; y)$ ,  $q_m/p \leq \phi$ . (All public goods are producible with Bounded Above Marginal Cost.)

Let  $G^R$  denote the grand coalition for the economy  $\mathcal{E}^R$ . A *Lindahl equilibrium* for a coalition  $S \subseteq G^R$  is a vector  $(x; y; p; q) \in \mathbb{R}_+^{|S|} \times \mathbb{R}_+^M \times \Delta^{|S| \times M}$ , such that:

$$\forall \{r, t\} \in S, (p; q^{r,t})(x^{r,t}; y) \leq p\omega^t, \quad (A)$$

$$(x^{r,t}; y) \succeq_t (\hat{x}; \hat{y}) \forall (\hat{x}; \hat{y}) \in X \text{ s.t. } (p; q^{r,t})(\hat{x}; \hat{y}) \leq p\omega^t, \quad (B)$$

$$\left( \sum_{\{r,t\} \in S} (x^{r,t} - \omega^t); y \right) \in Y, \quad (C)$$

and

$$\forall (\hat{z}; \hat{y}) \in Y,$$

$$\left( p; \sum_{\{r,t\} \in S} q^{r,t} \right) \left( \sum_{\{r,t\} \in S} [x^{r,t} - \omega^t]; y \right) \geq \left( p; \sum_{\{r,t\} \in S} q^{r,t} \right) (\hat{z}; \hat{y}). \quad (D)$$

Now define the set of Lindahl allocations for a coalition  $S \subseteq G^R$  as:

$$L(S) \equiv \left\{ (x; y) \in \mathbb{R}_+^{|S|} \times \mathbb{R}_+^M \mid \right. \quad (3)$$

$$\left. \exists (p; q) \in \Delta^{|S| \times M} \text{ s.t. } (x; y; p; q) \text{ is a Lindahl equilibrium for } S \right\}$$

The *separating price correspondence* for  $\succeq_t$ ,  $H^t : X \rightarrow \Delta^M$  is defined as:

$$H^t(x; y) \equiv \left\{ (p; q) \in \Delta^M \mid \right. \quad (4)$$

$$\left. (p; q)(x; y) < (p; q)(x'; y') \forall (x'; y') \in X \text{ s.t. } (x'; y') \succ_t (x; y) \right\}.$$

Note that the range of  $H^t$  is  $\Delta^M$ . This means that the prices given by  $H^t$  are not the same prices faced by the consumer at a Lindahl equilibrium for a coalition  $S$  since these prices are elements of  $\Delta^{|S| \times M}$ . However, if  $(x; y)$  is a Lindahl allocation, then there must exist normalization factors  $k^{r,t} \in (0, 1]$  for each consumer that can be used to put the prices  $(p; q) \in H^t(x; y)$  back into  $\Delta^{|S| \times M}$ . This normalization is described in detail below.

An allocation  $(x; y)$  for a coalition  $S$  is said to be *S-optimal* if it is feasible for  $S$ , and there does not exist a feasible Pareto dominant allocation  $(\tilde{x}; \tilde{y})$  for  $S$ . Formally,  $(x; y)$  is *S-optimal* if

$$\left( \sum_{\{r,t\} \in S} [x^{r,t} - \omega^t]; y \right) \in Y,$$

$$\nexists (\tilde{x}; \tilde{y}) \in \mathbb{R}_+^{|S|} \times \mathbb{R}_+^M \text{ s.t. } \left( \sum_{\{r,t\} \in S} [\tilde{x}^{r,t} - \omega^t]; \tilde{y} \right) \in Y,$$

$$\forall \{r, t\} \in S, (\tilde{x}^{r,t}; \tilde{y}) \succeq_t (x^{r,t}; y),$$

and

$$\exists \{r, t\} \in S \text{ s.t. } (\tilde{x}^{r,t}; \tilde{y}) \succ_t (x^{r,t}; y).$$

It will frequently be convenient to take advantage of the fact that the well-known *Samuelson conditions* must be satisfied by any  $S$ -optimal allocation. In the notation of this paper we say: if an allocation  $(x; y)$  for a coalition  $S \in G^R$  is  $S$ -optimal, then

$$\forall \{r, t\} \in S, \exists (p^{r,t}; q^{r,t}) \in H(x^{r,t}; y), \exists \text{ and } \exists k^{r,t} \in (0, 1]$$

$$\text{s.t. } k^{r,t} \times p^{r,t} = p, (p; k^{1,1} \times q^{1,1}, \dots, k^{R,T} \times q^{R,T}) \in \Delta^{|S| \times M},$$

and

$$\exists (p; q) \in MC \left( \sum_{\{r,t\} \in G^R} [x^{r,t} - \omega^t]; y \right) \text{ s.t. } \forall m \in \mathcal{M}, \sum_{\{r,t\} \in S} k^{r,t} \times q_m^{r,t} = q_m.$$

Note that the normalization factor cancels out in the last line.

The *offer correspondence*,  $OC^t : \mathbb{R}_+^M \rightarrow \mathbb{R}_+$ , is defined as follows:

$$OC^t(y) = \{x \in \mathbb{R}_+ \mid \exists (p; q) \in \Delta^M \text{ s.t.} \quad (5)$$

$$(p; q)(x; y) \leq p\omega^t \text{ and } (x; y) \succeq_t (x'; y') \forall (x'; y') \in X \text{ s.t. } (p; q)(x'; y') \leq p\omega^t\}$$

The graph of this correspondence is the offer curve for a consumer of type  $t$ . The reader may verify that  $x \in OC^t(y)$  if and only if for some  $(p; q) \in H^t(x; y)$ ,  $(p; q)(x; y) = p\omega^t$ .<sup>8</sup>

An allocation  $(x; y)$  is in the *core* for the grand coalition  $G^R$  if,

$$\left( \sum_{\{r,t\} \in G^R} [x^{r,t} - \omega^t]; y \right) \in Y,$$

$$\nexists S \subseteq G^R, \text{ and } (\tilde{x}; \tilde{y}) \in \mathbb{R}_+^{|S|} \times \mathbb{R}_+^M \text{ s.t. } \left( \sum_{\{r,t\} \in S} [\tilde{x}^{r,t} - \omega^t]; \tilde{y} \right) \in Y,$$

---

<sup>8</sup> Also, see equation 8, below.

and

$$\forall \{r, t\} \in S, (\tilde{x}^{r,t}; \tilde{y}) \succeq_t (x^{r,t}; y),$$

and in addition:

$$\exists \{r, t\} \in S \text{ s.t. } (\tilde{x}^{r,t}; \tilde{y}) \succ_t (x^{r,t}; y).$$

This is just the usual requirement that no coalition can block any core allocation. Notice that blocking coalitions can only consume the public goods produced using their own resources. They cannot free ride off the public goods produced by the rest of the consumers. This follows Foley, and others. The set of core allocations for the grand coalition  $G^R$  will be denoted by  $C(G^R)$ .

In order to be assured that Lindahl equilibria exist, and are in the core, we take advantage of Foley's (1970) method of modeling public goods as jointly produced private goods. If it can be shown that competitive equilibria exist, and are in the core for the associated private goods economy, then it may be concluded that Lindahl equilibria exist and are in the core of the original public goods economy. It is easy to verify that the six assumptions on consumers and producers given above imply all of the assumptions used by McKenzie (1981) to show existence.<sup>9</sup> We may then use Milleron's (1972) theorem 4.1 to conclude that the Lindahl equilibrium is always in the core. The exercise of explicitly carrying out the translation of this model into a private goods economy is not performed here because it is long, and it would essentially be a duplication of Foley's contribution.

Lemma 1, we show a useful relationship between the core, consumers' offer curves and the Lindahl equilibria.

**Lemma 1.** *Let  $\mathcal{E}^R$  satisfy CONT, CONV, MONO, CLCC, NFPR, and BAMC, then if  $(x; y) \in C(G^R)$ , and for all  $\{r, t\} \in G^R$ ,  $x^{r,t} \in OC^t(y)$ , then  $(x; y) \in L(G^R)$ .*

---

<sup>9</sup> We cannot use Milleron's (1972) existence theorem directly as he assumes that the endowments are in the interior of the consumption sets. In this model, public goods levels are initially zero. Also, Foley assumes strict monotonicity, so his theorems cannot be used either.



Proof/

1. If for all  $\{r, t\} \in G^R$ ,  $x^{r,t} \in OC^t(y)$ , then by the definitions of  $H^t$  and  $OC^t$

$$\exists (p^{r,t}; q^{r,t}) \in H^t(x^{r,t}; y) \text{ s.t. } (p^{r,t}; q^{r,t})(x^{r,t}; y) \leq p\omega^t$$

and

$$(x^{r,t}; y) \succeq_t (x'; y') \forall (x'; y') \in X \text{ s.t. } (p^{r,t}; q^{r,t})(x'; y') \leq p^{r,t}\omega^t.$$

2. Also, since core allocations are feasible by definition,

$$\left( \sum_{\{r,t\} \in G^R} (x^{r,t} - \omega^t); y \right) \in Y.$$

3. Finally, suppose

$$\forall \{r, t\} \in G^R, \text{ and } \forall (p^{r,t}; q^{r,t}) \in \Delta^M \text{ s.t. :}$$

- a.  $(p^{r,t}; q^{r,t}) \in H^t(x^{r,t}; y)$ ,
- b.  $(p^{r,t}; q^{r,t})(x^{r,t}; y) \leq p^{r,t}\omega^t$ ,
- c.  $(x^{r,t}; y) \succeq_t (x'; y') \forall (\hat{x}; \hat{y}) \in X, \text{ s.t. } (p^{r,t}; q^{r,t})(\hat{x}; \hat{y}) \leq p^{r,t}\omega^t$ ,

it is not the case that

$$\exists (p^{r,t}; q^{r,t}) \in H(x^{r,t}; y), \text{ and } \exists k^{r,t} \in (0, 1] \text{ s.t.}$$

$$k^{r,t} \times p^{r,t} = p,$$

$$(p; k^{1,1} \times q^{1,1}, \dots, k^{R,T} \times q^{R,T}) \in \Delta^{|S| \times M},$$

and,

$$\forall (\hat{z}; \hat{y}) \in Y,$$

$$\left( p; \sum_{\{r,t\} \in G^R} k^{r,t} \times q^{r,t} \right) \left( \sum_{\{r,t\} \in G^R} [x^{r,t} - \omega^t]; y \right) \geq$$

$$\left( p; \sum_{\{r,t\} \in G^R} k^{r,t} \times q^{r,t} \right) (\hat{z}; \hat{y}).$$

But this would violate the Samuelson conditions. This in turn contradicts the hypothesis that  $(x; y) \in C(G^R)$  and is therefore Pareto optimal.

•

Lemma 1 says that if a core allocation is also on each of the consumers' offer curves, then it is a Lindahl allocation. This suggests the notion of core convergence used in this paper. We say the core converges to the set of Lindahl allocations if for any  $\epsilon > 0$  and for all sequences of core allocations  $\{(x^R; y^R)\}$ , the proportion of the consumers in  $G^R$  who remain off their offer curves by at least  $\epsilon$  goes to zero as  $R$  grows without bound. In other words, the core converges if in the limit, all consumers contribute exactly their Lindahl taxes to support an allocation that is in the core.

More formally, define the coalition  $S(\epsilon, x; y)$  to be the set of consumers who are off their offer curves by at least  $\epsilon$  at the allocation  $(x; y)$ :

$$S(\epsilon, x; y) \equiv \tag{6}$$

$$\{ \{r, t\} \in G \mid \forall (x', y') \text{ s.t. } x' \in OC^t(y'), \| (x'; y'), (x^{r,t}; y) \| \geq \epsilon \}.$$

Then let the function  $NOC : \mathbb{R}_+ \times \mathbb{R}_+^{RT} \times \mathbb{R}_+^M \rightarrow [0, 1]$ , give the proportion of displaced consumers in  $\mathcal{E}^R$ :

$$NOC(\epsilon, x; y) \equiv \frac{|S(\epsilon, x; y)|}{RT}. \tag{7}$$

The core is said to *converge* to the set of Lindahl allocations if for any sequence of core allocations  $\{(x^R; y^R)\}$ , and for any  $\epsilon > 0$ ,

$$\lim_{R \rightarrow \infty} NOC(\epsilon, x^R; y^R) = 0.$$

It is essential to the lemmata in sections three and four that the correspondence  $OC^t$  be well defined. The purpose of lemmata 2 through 6 is to show that under the assumptions of CONT, CONV, and MONO,  $OC^t(y)$  is not empty for any  $y \in \mathbb{R}_+^M$ .

**Lemma 2.** *Let  $\succeq_t$  satisfy CONT, CONV, and MONO, then for all bundles  $(x; y) \in X$ ,  $H^t(x, y)$  is not empty.*

Proof/

Consider the upper-contour set of  $(x; y) \in X$  for  $\succ_t$ :

$$U^t(x; y) \equiv \{(x'; y') \in X \mid (x'; y') \succeq_t (x; y)\}.$$

By CONT, CONV, and MONO,  $U^t(x; y) \neq \emptyset$  and is also closed and convex. Then by the Hahn-Banach theorem, for all  $(x; y) \in X$ ,

$$\exists (p; q) \in H^t(x, y).$$

•

**Lemma 3.** *Let  $\succeq_t$  satisfy CONT, CONV, and MONO. Then  $H^t$  is an upper hemi-continuous correspondence.*

Proof/

First we show  $H^t$  is a closed-valued correspondence. Let  $\{(x^\nu; y^\nu)\}$  be a convergent sequence in  $X$ , let  $\lim_{\nu \rightarrow \infty} (x^\nu; y^\nu) = (\tilde{x}; \tilde{y})$ , and let  $(p^\nu; q^\nu) \in H^t(x^\nu; y^\nu)$  for all  $\nu$ . Note that  $(p^\nu; q^\nu)$  exists by lemma 2, and since  $(p^\nu; q^\nu) \in \Delta^M$  for all  $\nu$ , there exists a subsequence  $\{(p^{\nu'}; q^{\nu'})\}$  such that  $\lim_{\nu' \rightarrow \infty} (p^{\nu'}; q^{\nu'}) = (\tilde{p}; \tilde{q}) \in \Delta^M$ . It must be shown that

$$(\tilde{p}; \tilde{q})(\tilde{x}; \tilde{y}) < (\tilde{p}; \tilde{q})(x'; y') \forall (x'; y') \in X \text{ s.t. } (x'; y') \succ_t (\tilde{x}; \tilde{y}).$$

Suppose not, then

$$\exists (x'; y') \in X \text{ s.t. } (x'; y') \succ_t (\tilde{x}; \tilde{y})$$

and

$$(\tilde{p}; \tilde{q})(\tilde{x}; \tilde{y}) \geq (\tilde{p}; \tilde{q})(x'; y').$$

By CONT,

$$\exists \epsilon > 0 \text{ s.t. } (x' - \epsilon; y') \succ_t (\tilde{x}; \tilde{y})$$

$$\text{and } (\tilde{p}; \tilde{q})(\tilde{x}; \tilde{y}) > (\tilde{p}; \tilde{q})(x' - \epsilon; y').$$

But then by CONT, and MONO, for sufficiently large  $\nu'$ ,

$$(x' - \epsilon; y') \succ_t (x^{\nu'}; y^{\nu'}).$$

By the definition of  $H^t$ , for sufficiently large  $\nu'$ ,

$$(p^{\nu'}; q^{\nu'})(x^{\nu'}; y^{\nu'}) < (p^{\nu'}; q^{\nu'})(x' - \epsilon; y').$$

But,

$$(p^{\nu'}; q^{\nu'}) \rightarrow (\tilde{p}; \tilde{q}), \text{ and } (x^{\nu'}; y^{\nu'}) \rightarrow (\tilde{x}; \tilde{y})$$

so

$$(\tilde{p}; \tilde{q})(\tilde{x}; \tilde{y}) \leq (\tilde{p}; \tilde{q})(x' - \epsilon; y'),$$

a contradiction.

Thus,  $H^t$  is a closed valued correspondence. But since  $\Delta^M$  is compact, by Hildenbrand (1974) proposition B.2,  $H^t$  is UHC.

•

**Lemma 4.** *Let  $\succeq_t$  satisfy CONT, CONV, and MONO, then  $H^t$  is convex-valued.*

Proof/

Suppose not. Then

$$\exists (x; y) \in X, \exists (p; q), (p'; q') \in H^t(x; y) \text{ and } \exists \lambda \in [0, 1] \text{ s.t.}$$

$$\lambda(p; q) + (1 - \lambda)(p'; q') \equiv (p''; q'') \notin H^t(x; y).$$

Then,

$$\exists (\tilde{x}; \tilde{y}) \in X \text{ s.t. } (\tilde{x}; \tilde{y}) \succ_t (x; y) \text{ and } (p''; q'')(x; y) \geq (p''; q'')(\tilde{x}; \tilde{y}),$$

which implies,

$$\lambda \{(p; q)[(x; y) - (\tilde{x}; \tilde{y})]\} + (1 - \lambda) \{(p'; q')[(x; y) - (\tilde{x}; \tilde{y})]\} \geq 0.$$

However, this is impossible since

$$(p; q) \in H^t(x; y) \text{ and } (p'; q') \in H^t(x; y).$$

•

Now define the map  $f_y^t : [0, \omega^t] \subset \mathbb{R}_+ \rightarrow \mathbb{R}$  by:

$$f_y^t(x) \equiv \{\gamma \in \mathbb{R} \mid \exists (p; q) \in H^t(x; y) \text{ and } \gamma = (p; q)(x - \omega^t; y)\}. \quad (8)$$

This correspondence  $f_y^t$  gives the values of excess demand at each of the prices supporting  $(x; y)$ . More exactly, it is the differences between the possible values of the endowment that a consumer of type  $t$  would have to have, and the value of the one he actually has, if  $(x; y)$  were to be on his offer curve. Thus, for a particular value of  $y$ , if  $0 \in f_y^t(x)$ , then  $(x; y)$  is on the offer curve of consumer of type  $t$ .<sup>10</sup>

**Lemma 5.** *Let  $\succeq_t$  satisfy CONT, CONV, and MONO, then for all  $(\omega^t; y) \in X$ , there exists  $x \in [0, \omega^t]$  such that  $0 \in f_y^t(x)$ .*

Proof/

Clearly  $f_y^t$  is UHC, convex-valued, and non-empty valued since it is the product of correspondences with these properties. Consider the following subsets of the domain:

$$D^+ \equiv \{x \in [0, \omega^t] \mid \exists z \in f_y^t(x) \text{ and } z \geq 0\}$$

$$D^- \equiv \{x \in [0, \omega^t] \mid \exists z \in f_y^t(x) \text{ and } z \leq 0\}$$

---

<sup>10</sup> Lemma 5 is This is similar to lemma 3.8 in Diamantaras (1992).



1. Note first that  $D^- \neq \emptyset$ . Let  $(p; q) \in H^t(0; y)$ . Then

$$\forall (x'; y') \in X \text{ s.t. } (x'; y') \succ_t (0; y), (p; q)(0; y) < (p; q)(x'; y').$$

Then since  $qy < px' + qy'$ ,

$$\forall \lambda \in (0, 1], \lambda qy < \lambda px' + \lambda qy'.$$

And since  $\lambda p < p + (1 - \lambda)(1 - p)$ ,

$$\lambda qy < (p + (1 - \lambda)(1 - p))x' + \lambda qy'.$$

But,

$$p + (1 - \lambda)(1 - p) + \lambda \sum_{i=1}^M q_i = p + (1 - \lambda)(1 - p) + \lambda(1 - p) = 1.$$

Therefore,

$$(p + (1 - \lambda)(1 - p); \lambda q) \in H^t(0; y).$$

However,

$$\lim_{\lambda \rightarrow 0} p + (1 - \lambda)(1 - p) = 1.$$

But since  $\omega^t > 0$ , for small enough  $\lambda$ ,

$$(p + (1 - \lambda)(1 - p))(0 - \omega^t) + \lambda qy \leq 0.$$

Thus  $0 \in D^-$ , and so  $D^- \neq \emptyset$ .

2. Also note that  $D^+ \neq \emptyset$ . By MONO

$$\forall (x; y) \in X \text{ and } \forall (p; q) \in H^t(x; y), (p; q) > 0.$$

So

$$p(\omega^t - \omega^t) + qy \geq 0.$$

Thus  $\omega^t \in D^+$ , and so  $D^+ \neq \emptyset$ .

3. Finally note that  $0 \notin f_y^t(x)$  for all  $x \in [0, \omega^t]$  implies  $D^+ \cap D^- = \emptyset$ . This follows from the fact that  $f_y^t$  is convex-valued which implies that for any  $x \in [0, \omega^t]$ , if  $x \in D^+$  and  $x \in D^-$ , then  $0 \in f_y^t(x)$ .
4. It only remains to show that  $D^+ \cap D^- \neq \emptyset$ . Suppose not. Then since both sets are non-empty, at most one of them can be closed. Without loss of generality, suppose  $D^+$  is not closed. Then

$$\exists \{x^\nu\} \mid \forall \nu x^\nu \in D^+, x^\nu \rightarrow x, 0 \leq f_y^t(x^\nu) \text{ and } 0 > f_y^t(x).$$

But this contradicts that  $f_y^t$  is UHC on  $[0, \omega^t]$ . Therefore, there is some  $x \in \mathbb{R}_+^N$  such that  $0 \in f_y^t(x)$ .

•

**Lemma 6.** *Let  $\succeq_t$  satisfy CONT, CONV, and MONO, then for all  $y \in \mathbb{R}_+^M$ ,  $OC^t(y)$  is not empty.*

Proof/

From lemma 5,

$$\forall (\omega^t; y) \in X, \exists x \in [0, \omega^t] \text{ s.t. } 0 \in f_y^t(x).$$

But then

$$\exists (p; q) \in H^t(x; y) \text{ s.t. } (p; q)(x - \omega^t; y) = 0.$$

Therefore, by definition of  $H^t$  :

$$\exists (p; q) \in H^t(x; y) \text{ s.t. } (p; q)(x; y) \leq p\omega^t$$

and

$$\forall (x'; y') \in X \text{ s.t. } (x'; y') \succeq_t (x; y), (p; q)(x'; y') > p\omega^t.$$

Thus  $x \in OC^t(y)$

•

### 3. Two Convergence Theorems

We prove two separate convergence theorems in this section. The first shows that if for all consumers, the value in terms of private good of receiving a given multiple of their current public goods consumption level goes to zero as the level of public goods goes to infinity, then the core converges to the set of Lindahl allocations. This assumption is called asymptotic satiation, and it is easy to see why it drives the result.

The reason that the core of a public goods economy is typically quite large is that the grand coalition is able to spread the costs of public goods production more widely than any subcoalition. As a consequence, subcoalitions find it difficult to block allocations offered by the grand coalition even when all the consumers in the subcoalition are charged more than their Lindahl taxes. When all consumers become asymptotically satiated in public good, this ceases to be true. In this case, consumers do not care very much about extremely high levels of public goods consumption. This puts a limit on the degree to which the grand coalition can “exploit” its cost advantage. Suppose, for example, that the grand coalition tried to give even a small proportion of the consumers a level of private good consumption that was less than the level at a Lindahl allocation. As the economy got large, the members of the subcoalition would find that they are sufficiently numerous that, at lower cost, they could easily produce a bundle of public goods that is almost as good as the one offered by the grand coalition. The subcoalition would then be able to block the effort to give them less private good than they would get at a Lindahl allocation. In short, asymptotic satiation removes the increasing returns to coalitional size that are embedded in the public goods technology. As a result, even small coalitions are able to block attempts on the part of the grand coalition to deviate from the Lindahl allocations.

Before giving a formal definition of asymptotic satiation, consider the notion of *compensating variation*. This will be used to measure the difference in desirability between two given bundles of public and private goods. Specifically, the compensating variation is the largest amount of the private good that could be taken away from bundle 2 while still leaving the consumer at least as well off at bundle 1. The compensating variation can be positive or negative. It is always well defined if the second bundle is better than the first. Formally, for a consumer of type  $t$ :

$$CV^t(x; y; x'; y') \equiv \{\max z \in \mathbb{R} \mid (x; y) \preceq_t (x' - z; y')\}. \quad (9)$$

Now asymptotic satiation can be defined as follows:

AS.SAT.  $\forall x \in \mathbb{R}_+, \forall \sigma \in (0, 1], \lim_{\|y\| \rightarrow \infty} CV^t(x; \sigma y; x; y) = 0.$

Examples of preferences that satisfy asymptotic satiation include those for which consumers are eventually satiated in public good. An example of preferences which satisfy AS.SAT, but which do not exhibit satiation in public goods can be found in Wooders(1981):

$$u(x; y) = x - e^{-y}.$$

In both of these examples, the offer curves are asymptotic to  $\omega^t$  as  $y$  goes to infinity. In other words, the Lindahl taxes go to zero as the bundle of public goods gets large. Lemma 7 shows that this property of the offer curves is an implication of asymptotic satiation. This means, incidentally, that Cobb-Douglas preferences do not satisfy asymptotic satiation. It is not hard to confirm this by noticing that the compensating variation between two bundles as described in the statement of AS.SAT remains constant in the Cobb-Douglas case. Thus, asymptotic satiation is weaker than satiation in public goods, but stronger than assuming that the marginal rate of substitution goes to zero.

**Lemma 7.** *Let  $\succ_t$  satisfy CONT, CONV, and MONO, then if in addition  $\succ_t$  satisfies AS.SAT,*

$$\lim_{\|y\| \rightarrow \infty} OC^t(y) = \omega.$$

Proof/

Suppose that asymptotic satiation is satisfied but that the offer curve does not converge as stated. Then

$$\exists \{y^\nu\} \text{ s.t. } \|y^\nu\| \rightarrow \infty \text{ and } \exists \epsilon > 0 \text{ s.t. } \forall \nu', \exists \nu \geq \nu' \text{ and } \exists x^\nu \in OC^t(y^\nu)$$

$$\text{s.t. } x^\nu \leq \omega^t - \epsilon.$$

Note that the possibility that  $x^\nu \geq \omega^t + \epsilon$  need not be considered since by MONO, all prices are non-negative and so  $x^\nu \leq \omega^t$ . Also, by definition of  $OC^t$ , for all  $\nu$ , and for all  $x^\nu \in OC^t(y^\nu)$

$$\exists (p^\nu; q^\nu) \in H(x^\nu; y^\nu) \text{ s.t. } q^\nu y^\nu = p^\nu(\omega^t - x^\nu).$$

But since,

$$(p^\nu; q^\nu)(\tfrac{1}{2}\omega^t + \tfrac{1}{2}x^\nu; \tfrac{1}{2}y^\nu) = p^\nu\omega,$$

it follows that

$$(\tfrac{1}{2}\omega + \tfrac{1}{2}x^\nu; \tfrac{1}{2}y^\nu) \preceq_t (x^\nu; y^\nu).$$

Recall that AS.SAT. implies that

$$\forall x \in \mathbb{R}_+, \lim_{\|y^\nu\| \rightarrow \infty} CV^t(x; \tfrac{1}{2}y^\nu; x; y^\nu) = 0.$$

So by MONO, for large  $\nu$

$$(\tfrac{1}{2}\omega + \tfrac{1}{2}x^\nu; \tfrac{1}{2}y^\nu) \succ_t (\tfrac{1}{2}\omega + \tfrac{1}{2}x^\nu - \tfrac{1}{2}\epsilon; y^\nu).$$

But by hypothesis,

$$\omega - \epsilon \geq x^\nu,$$

and therefore,

$$\frac{1}{2}\omega + \frac{1}{2}x^\nu - \frac{1}{2}\epsilon \geq x^\nu.$$

So by MONO, for large  $\nu$ ,

$$(\frac{1}{2}\omega + \frac{1}{2}x^\nu - \frac{1}{2}\epsilon; y^\nu) \succeq_t (x^\nu; y^\nu).$$

But then by transitivity,

$$(\frac{1}{2}\omega + \frac{1}{2}x^\nu; \frac{1}{2}y^\nu) \succ_t (x^\nu; y^\nu),$$

which is a contradiction.

•

If we are willing to place a fairly mild additional “uniform continuity” assumption on the separating price correspondence, then this result can be strengthened to prove that AS.SAT is equivalent to the offer curve converging to the endowment.

U.CONT. There exists  $\theta > 0$  such that for all  $(x; y) \in X$ , and all  $(p; q) \in H^t(x; y)$ , if  $\hat{x} \leq x$  then for some  $(\hat{p}; \hat{q}) \in H^t(\hat{x}; y)$ ,  $\frac{\hat{q}_m}{\hat{p}} \leq \theta \frac{q_m}{p}$ .

This assumption says that Marginal Rate of Substitution (MRS) between any public good type  $m$  and private good at an allocation  $(x; y)$  can go up by at most a bounded proportion as the level of private good consumption goes down. This may seem like an odd requirement, but in reality it is only a weakening of better known assumptions. For example, if preferences are quasi-linear in the private good, then  $\theta = 1$ . Alternatively if the ordinary assumption of diminishing MRS in the private good is made, then  $\theta \leq 1$ . The assumption of U.CONT allows the MRS to actually increase as private goods consumption decreases. It just places a upper bound on the rate of increase. Thus, U.CONT is not very restrictive. It only excludes preferences for which the MRS changes in the “wrong” direction in such to an extreme degree that no uniform bound can be found.



**Lemma 8.** *Let  $\succ_t$  satisfy CONT, CONV, MONO, and U.CONT, then in addition  $\succ_t$  satisfies AS.SAT if and only if*

$$\lim_{\|y\| \rightarrow \infty} OC^t(y) = \omega^t.$$

Proof/

From lemma 7, if  $\succ_t$  satisfies AS.SAT, the offer curve converges to the endowment.

Now suppose that the offer curve converges, but  $\succ_t$  does not satisfy AS.SAT. Consider what happens to compensating variation along the offer curve. We know that

$$\forall \{y^\nu\} \text{ s.t. } \|y^\nu\| \rightarrow \infty \text{ and } \forall x^\nu \in OC^t(y^\nu), x^\nu \rightarrow \omega^t.$$

But this implies that

$$\forall (p^\nu; q^\nu) \in H^t(x^\nu; y^\nu), q^\nu y^\nu = p^\nu(\omega^t - x^\nu) \rightarrow 0.$$

Let  $\sigma \in (0, 1]$ . Then at these supporting prices a bundle of public goods  $\frac{1}{\sigma}$  times as big as  $y^\nu$  could have been chosen in exchange for  $\frac{1}{\sigma} \frac{y^\nu q^\nu}{p^\nu}$  private goods but was not. Then by CONV, this puts an upper bound on the compensating variation. In particular,

$$\forall \sigma \in (0, 1], \frac{q^\nu y^\nu}{\sigma p^\nu} \geq CV^t(x^\nu; y^\nu; x^\nu; \frac{1}{\sigma} y^\nu).$$

Therefore,

$$CV^t(x^\nu; y^\nu; x^\nu; \frac{1}{\sigma} y^\nu) \rightarrow 0$$

By hypothesis  $x^\nu \rightarrow \omega^t$ . Thus it only remains to show that the compensating variation goes to zero for sequences of allocations  $\{(\hat{x}^\nu; y^\nu)\}$  such that  $\hat{x}^\nu \not\rightarrow \omega^t$ . That is, to show the AS.SAT. holds for arbitrary consumption bundles, not just bundles on the offer curve. So suppose that for large  $\nu$ ,  $\hat{x}^\nu < x^\nu$ . But by U.CONT,

$$\exists \theta > 0, \text{ and } (\hat{p}^\nu; \hat{q}^\nu) \in H^t(\hat{x}^\nu; y^\nu) \mid \frac{\hat{q}_m^\nu}{\hat{p}^\nu} \leq \theta \frac{q_m^\nu}{p^\nu}.$$

But then

$$CV(\hat{x}^\nu; y^\nu; \hat{x}^\nu; \frac{1}{\sigma} y^\nu) \leq \frac{\hat{q}^\nu y^\nu}{\sigma \hat{p}^\nu} \leq \frac{\theta q^\nu y^\nu}{\sigma p^\nu} \rightarrow 0.$$

•

We are now ready to give the first convergence theorem.

**Theorem 1.** *Let  $\{\mathcal{E}^R\}$  be any sequence of economies satisfying CONT, CONV, MONO, CLCC, NFPR, and BAMC. Then if for all  $t \in \mathcal{T}$ ,  $\succeq_t$  satisfies AS.SAT, then the core converges to the set of Lindahl allocations.*

Proof/

Two cases are considered.

1. First suppose that  $y^R \rightarrow \infty$ . It is sufficient to show that

$$\exists \{(x^R; y^R)\}, \text{ s.t. } \forall R, (x^R; y^R) \in C(G^R) \text{ and}$$

$$\exists \epsilon > 0 \text{ and } \exists \alpha \in (0, 1] \text{ s.t. } \forall R', \exists R > R', \text{ s.t. } NOC(\epsilon, x^R; y^R) \geq \alpha.$$

Let  $S$  be coalition of consumers who are off their offer curves by at least  $\epsilon$ . Divide  $S$  into two subcoalitions:  $S_\epsilon^{R+}(x^R; y^R)$ , consisting of the consumers who are above their offer curves by  $\epsilon$  or more (and are therefore paying less than their Lindahl taxes), and  $S_\epsilon^{R-}(x^R; y^R)$ , consisting of the consumers who are below their offer curves by  $\epsilon$  or more (and are therefore paying more than their Lindahl taxes). Formally,

$$S_\epsilon^{R+}(x^R; y^R) \equiv \{\{r, t\} \in G^R \mid \forall \tilde{x}^{r,t} \in OC^t(y^R), x^{r,t} \geq \tilde{x}^{r,t} + \epsilon\} \quad (10)$$

and

$$S_\epsilon^{R-}(x^R; y^R) \equiv \{\{r, t\} \in G^R \mid \forall \tilde{x}^{r,t} \in OC^t(y^R), x^{r,t} \leq \tilde{x}^{r,t} - \epsilon\}. \quad (11)$$

Also define the sequence of allocations  $\{(x^{R-}; y^{R-})\}$  for the sequence of coalitions  $\{S_\epsilon^{R-}(x^R; y^R)\}$  as follows:  $x^{R-,r,t} \equiv \omega^t - \frac{1}{2}\epsilon$  for  $\{r, t\} \in S_\epsilon^{R-}(x^R; y^R)$ ,

and  $y^{R^-}$  is the largest feasible vector of public goods when all the tax revenue collected by  $S_\epsilon^{R^-}(x^R; y^R)$  is spent on a bundle proportional to  $y^R$ .

Consider two subcases:

a. First, suppose

$$\|y^R\| \rightarrow \infty \text{ and } \forall R', \exists R > R' \text{ s.t. } \frac{|S_\epsilon^{R^-}(x^R; y^R)|}{RT} \geq \frac{1}{2}\alpha.$$

Then there exists an arbitrarily large  $R$  such that  $S_\epsilon^{R^-}(x^R; y^R)$  can block  $(x^R; y^R)$  with the allocation  $(x^{R^-}; y^{R^-})$ . To see this note that the most that  $G^R$  can collect in taxes is  $TR\omega^{\max}$  where  $\omega^{\max}$  is the largest endowment held by any consumer of any type. On the other hand, for all  $R'$ , there exists  $R > R'$  such that the tax revenue collected by  $S_\epsilon^{R^-}(x^R; y^R)$  is at least  $\frac{1}{2}\alpha TR\frac{1}{2}\epsilon$ . But by CLCC,  $Y$  is a convex cone with zero origin, and the tax revenue of the coalition  $S_\epsilon^{R^-}(x^R; y^R)$  is always at least a fraction,  $\sigma$ , of  $G^R$ 's for arbitrarily large  $R$  where

$$\sigma \equiv \frac{\frac{1}{2}\alpha RT\frac{1}{2}\epsilon}{TR\omega^{\max}} = \frac{\frac{1}{4}\alpha\epsilon}{\omega^{\max}},$$

and so  $y^{R^-} \geq \sigma y^R$ . Thus,

$$\forall \{r, t\} \in S_\epsilon^{R^-}(x^R; y^R), (x^{R^-}, r, t; y^{R^-}) \succeq_t (\omega^t - \frac{1}{2}\epsilon; \sigma y^R). \quad (12)$$

Also by MONO, all prices are non-negative, and so

$$\forall \{r, t\} \in G^R, \omega^t \geq x^{R, r, t}$$

Then since agents in  $S_\epsilon^{R^-}(x^R; y^R)$  are below their offer curves by at least  $\epsilon$ ,

$$\forall \{r, t\} \in S_\epsilon^{R^-}(x^R; y^R), \omega^t - \epsilon \geq x^{R, r, t}. \quad (13)$$

Also by AS.SAT. for large  $R$ ,

$$\forall \{r, t\} \in G^R, \lim_{\|y^R\| \rightarrow \infty} CV^t(x^{R, r, t}; \sigma y^R; x^{R, r, t}; y^R) = 0.$$

So for large  $R$ :

$$\forall \{r, t\} \in S_\epsilon^{R-}(x^R; y^R), (\omega^t - \frac{1}{2}\epsilon; \sigma y^R) \succ_t (\omega^t - \epsilon; y^R). \quad (14)$$

So by (12), (13), and (14),

$$(x^{R-}, r, t; y^{R-}) \succeq_t (\omega^t - \frac{1}{2}\epsilon; \sigma y^R) \succ_t (\omega^t - \epsilon; y^R) \succeq_t (x^{R, r, t}; y^R). \quad (15)$$

Therefore,  $S_\epsilon^{R-}(x^R; y^R)$  blocks  $(x^R; y^R)$  with  $(x^{R-}; y^{R-})$  for large  $R$

b. Now suppose

$$\|y^R\| \rightarrow \infty \text{ and } \forall R', \exists R > R' \text{ s.t. } \frac{|S_\epsilon^{R+}(x^R; y^R)|}{RT} \geq \frac{1}{2}\alpha.$$

But by lemma 7,

$$\forall t \in \mathcal{T} \quad \lim_{\|y\| \rightarrow \infty} OC^t(y) = \omega^t.$$

Therefore, for large  $R$ ,

$$\forall \{r, t\} \in S_\epsilon^{R+}(x^R; y^R), x^{R, r, t} > \omega^t.$$

But then consumers  $S_\epsilon^{R+}(x^R; y^R)$  consume more private good than they are endowed with. This could not be a core allocation since the remaining agents,  $G^R \setminus S_\epsilon^{R+}(x^R; y^R)$ , could block  $(x^R; y^R)$  by ejecting the consumers in  $S_\epsilon^{R+}(x^R; y^R)$  from  $G^R$  and dividing among themselves the net payment of private goods that would have gone to these ejected consumers.

2. Finally suppose that for some sequence of core allocations  $(x^R; y^R)$ ,

$$\exists y' \in \mathbb{R}_+^M \text{ and } R' \text{ s.t. } \forall R \geq R', \|y^R\| \leq y'.$$

Since attention is now restricted to a compact subset of the consumption set, and we are interested only in the limit allocations it is sufficient to restrict attention to sequences of core allocations for which the level of public goods converges to a limit. To save notation let  $y^R \rightarrow y'$ .

First we show that for all but a vanishing fraction of agents, the consumption of private good converges to endowment. Let  $S_\epsilon^R(x^R; y^R)$  be the coalition of agents who consume at least  $\epsilon$  less than  $\omega^t$  at the core allocation.

$$S_\epsilon^R(x^R; y^R) \equiv \{ \{r, t\} \in G^R \mid \omega^t - x^{R,r,t} \geq \epsilon \}.$$

Note again that we need not consider the case in which  $\omega^t < x^{R,r,t}$ . This would be a situation in which some agents receive a net subsidy of private good. But this is impossible for a core allocation by the argument given in (1. b.) above. So to prove the claim then we must show:

$$\forall \epsilon > 0 \quad \nexists \alpha \in (0, 1] \text{ s.t. } \forall R' \exists R > R' \text{ and } \frac{|S_\epsilon^R|}{RT} \geq \alpha.$$

Suppose not, then we claim that  $S_\epsilon^R(x^R; y^R)$  can block  $(x^R; y^R)$  for large  $R$ . This is because for all  $R'$ , there exist  $R > R'$  such that the coalition  $S_\epsilon^R(x^R; y^R)$  pays at least  $\alpha RT \epsilon \rightarrow \infty$  in taxes. But by CLCC, and NFPR, this must exceed the cost of producing any finite bundle of public goods  $y'$ . Thus for large  $R$ ,  $S_\epsilon^R(x^R; y^R)$  could produce  $y'$  on its own at lower cost. This allocation clearly blocks  $(x^R; y^R)$ .

Now we show

$$\forall \{r, t\} \in G^R \setminus S_\epsilon^R(x^R; y^R), \omega^t \in OC^t(y').$$

Suppose not, recall that  $(x^R; y^R)$  is in the core for all  $R$ . Thus  $(x^R; y^R)$  is Pareto optimal and satisfies the Samuelson conditions. Then

$$\forall \epsilon' > 0, \exists R' \text{ s.t. } \forall R > R' \text{ and } \forall \{r, t\} \in G^R \setminus S_\epsilon^R(x^R; y^R),$$

$$\exists (p^R; q^{R,r,t}) \in H^t(x^{R,r,t}, y^R) \text{ and } q^{R,r,t} \leq \epsilon'.$$

Then since by lemma 3,  $H^t$  is UHC,

$$(1; 0) \in \lim_{R \rightarrow \infty} H^t(x^{R,r,t}; y^R).$$

But we already know that

$$\forall \{r, t\} \in G^R \setminus S_\epsilon^R(x^R; y^R), (x^{R,r,t}; y^R) \rightarrow (\omega^t; y').$$

Thus  $(\omega^t; y')$  is supported by prices  $(1; 0)$  and is therefore on the offer curve.

This proves the claim.

Finally, it is clear that

$$\lim_{R \rightarrow \infty} S(\epsilon, x^R; y^R) = 0.$$

This is because for large for all but a vanishing fraction of agents,  $(x^{R,r,t}, y^R) \rightarrow (\omega^t; y')$ .

•

The second convergence theorem says that if the MRS of public for private goods does not go to zero as the quantity of public goods goes to infinity, then core converges. This assumption is called *strict nonsatiation* and it is even easier to see why it drives the result. For coalitions of sufficiently large size, strict nonsatiation implies that every core allocation is one in which essentially all of the private good is used to produce a Pareto optimal quantity of public goods. This obviously coincides with the Lindahl Equilibria.

As we noted in the introduction, strict nonsatiation is of less practical interest than asymptotic satiation. It implies that consumers have such a strong preference for public activity that even for economies of moderate size, consumers choose to devote all of their resources to public goods. This might reflect the tastes of members of utopian communities, but these preferences are clearly not shared by the majority of consumers. We also note that it is a very strong assumption. It could probably be weakened (eg. strict nonsatiation in a least one good per agent type.) without changing the conclusions. However, since this condition is mainly of



technical interest, we do not attempt to prove the most general theorem possible.

Formally, strict nonsatiation is defined as:

S.N.SAT. There exists  $\beta > 0$  such that for all  $(x; y) \in X$ , for some  $(p; q) \in H^t(x; y)$ , and for all  $m \in \mathcal{M}$ ,  $\frac{q_m}{p} \geq \beta$ .

An example of a class of preferences that satisfy strict nonsatiation is any preference relation in which public goods and private goods are perfect substitutes.

Remark: Notice that S.N.SAT, and CONV. imply that for all  $(x; y) \in X$  such that  $x > 0$ , for all  $(p; q) \in H^t(x; y)$ , and all  $m \in \mathcal{M}$ ,  $\frac{q_m}{p} \geq \beta$ . That is, the marginal rate of substitution of each public good for the private good is bounded below by  $\beta$  in the interior of the consumption set.

**Lemma 9.** Let  $\{\mathcal{E}^R\}$  be any sequence of economies satisfying CONT, CONV, MONO, S.N.SAT, CLCC, NFPR, and BAMC. Let  $\{(x^R; y^R)\}$  be a sequence of allocations such that for all  $R$ ,  $(x^R; y^R) \in C(G^R)$ . Finally, given any  $\epsilon > 0$ , let  $S_\epsilon^R(x^R; y^R) \equiv \{\{r, t\} \in G^R \mid x^{R,r,t} \geq \epsilon\}$ . Then,

$$\forall \epsilon > 0, \lim_{R \rightarrow \infty} \frac{|S_\epsilon^R(x^R; y^R)|}{RT} \equiv \alpha_\epsilon^R(x^R; y^R) = 0.$$

Proof/

Suppose not. Then

$$\exists \{(x^R; y^R)\} \text{ s.t. } \forall R, (x^R; y^R) \in C(G^R), \exists \epsilon > 0, \text{ and } \exists \bar{\alpha} \in (0, 1] \text{ s.t.}$$

$$\forall R', \exists R \geq R' \text{ s.t. } \alpha_\epsilon^R(x^R; y^R) \geq \bar{\alpha}.$$

Let

$$S_\epsilon^{R,t}(x^R; y^R) \equiv \{\{r, j\} \in S_\epsilon^R(x^R; y^R) \mid j = t\}.$$

Then, since there is only a finite number of consumer types,

$$\exists t' \in \mathcal{T} \text{ and } \exists \hat{\alpha} \in (0, 1] \text{ s.t. } \forall R', \exists R \geq R', \text{ s.t.}$$

$$\frac{|S_{\epsilon}^{R,t'}(x^R; y^R)|}{RT} \equiv \alpha_{\epsilon}^{R,t'}(x^R; y^R) \geq \hat{\alpha}.$$

In words, at least  $\hat{\alpha} \times RT$  consumers of some type  $t'$  consume more than  $\epsilon$  of the private good at some core allocation for an economy of arbitrarily large size.

Then since  $x^{R,r,t} > 0$ , by S.N.SAT, and CONV (see remark above),

$$\forall (p^{r,t'}; q^{r,t'}) \in H^t(x^{R,r,t}; y^R), \forall m \in \mathcal{M} \frac{q_m^{r,t'}}{p^{r,t'}} \geq \beta > 0.$$

But since by monotonicity all prices are non-negative:

$$\forall R', \exists R \geq R' \text{ s.t. } \forall (p^{r,t}; q^{r,t}) \in H^t(x^{R,r,t}; y^R) \quad \forall m \in \mathcal{M}$$

$$\sum_{r \in \mathcal{R}} \sum_{t \in T} \frac{q_m^{r,t}}{p^{r,t}} \geq \hat{\alpha} RT \beta^{t'}.$$

However, by assumption BAMC,

$$\forall (z; y) \in Y, \text{ and } \forall (\tilde{p}; \tilde{q}) \in MC(z; y), \quad \frac{\tilde{q}_m}{\tilde{p}} \leq \phi.$$

So clearly, for large enough  $R$ ,  $\phi < \hat{\alpha} RT \beta$ . Thus, for large enough  $R$ , the Samuelson conditions are not satisfied at  $(x^R; y^R)$ . This contradicts the hypothesis that  $\{(x^R; y^R)\}$  is a sequence of core allocations for  $G^R$ .

•

In words, it is impossible for a fixed fraction of consumers to consume a positive quantity of the private good because, eventually, these consumers become so numerous that even among themselves, the marginal benefit of spending their private good on public goods production exceeds the cost. Thus, no such allocation could be Pareto optimal, much less in the core.

**Lemma 10.** *Let  $\{\mathcal{E}^R\}$  be any sequence of economies satisfying CONT, CONV, MONO, S.N.SAT, CLCC, NFPR, and BAMC. Let  $\{(x^R; y^R)\}$  be such that for all  $R$ ,  $(x^R; y^R) \in C(G^R)$ . Then*

$$\lim_{R \rightarrow \infty} OC^t(y^R) = 0.$$

Proof/

By lemma 9,

$$\sum_{r \in \mathcal{R}} \sum_{t \in \mathcal{T}} (\omega^t - x^{R,r,t}) \rightarrow \infty.$$

Thus, by BAMC

$$\exists m \in \mathcal{M} \text{ s.t. } y_m^R \rightarrow \infty.$$

Then suppose

$$\forall R', \exists R \geq R' \text{ s.t. } \exists t \in \mathcal{T}, \text{ and } \exists \epsilon \geq 0 \text{ and } \exists x \in OC^t(y^R) \text{ s.t. } x \geq \epsilon.$$

But since  $y^R \rightarrow \infty$ , while the maximum level of Lindahl taxes paid by an agent of type  $t$  is bounded above by  $\omega^t$ ,

$$\forall t \in \mathcal{T}, \exists m \in \mathcal{M} \text{ s.t. } \forall \left\{ \frac{q_m^{R,r,t}}{p^{R,r,t}} \right\} \text{ s.t. } \forall R, (p^{R,r,t}, q^{R,r,t}) \in H^t(x^{R,r,t}; y^R),$$

$$\frac{q_m^{R,r,t}}{p^{R,r,t}} \rightarrow 0.$$

However, by S.N.SAT, and CONV, (see remark above)

$$\forall (x; y) \in X \text{ s.t. } x > 0 \forall (p; q) \in H^t(x; y), \text{ and } \forall m \in \mathcal{M}, \frac{q_m}{p} \geq \beta,$$

a contradiction. Thus for large  $R$ ,

$$\forall t \in \mathcal{T}, \nexists \epsilon \geq 0 \text{ and } \exists x \in OC^t(y^R) \text{ s.t. } x \geq \epsilon.$$

It only remains to show that  $0 \in OC^t(y^R)$  for all  $t \in \mathcal{T}$ . But this is immediate since by lemma 6,  $OC^t(y^R) \neq \emptyset$ , and we have just shown that no other  $x$  is an element of the offer curve for large  $R$ .

•

Note that the two assumptions discussed in this section have exactly opposite implications for the behavior of the offer correspondence. Strict nonsatiation implies that the offer correspondence converges to zero, while asymptotic satiation implies that it converges to the consumer's endowment. We now show that strict nonsatiation is a sufficient condition for core convergence.

**Theorem 2.** *Let  $\{\mathcal{E}^R\}$  be any sequence of economies satisfying CONT, CONV, MONO, CLCC, NFPR, and BAMC. Then if for all  $t \in T$ ,  $\succeq_t$  satisfies S.N.SAT, then the core converges to the set of Lindahl allocations.*

Proof/

By lemma 10,

$$\forall \{(x^R; y^R)\} \text{ s.t. } \forall R, (x^R; y^R) \in C(G^R), \lim_{R \rightarrow \infty} OC^t(y^R) = 0.$$

By lemma 9,

$$\forall \epsilon \geq 0 \lim_{R \rightarrow \infty} \alpha_\epsilon^R \equiv \frac{|S_\epsilon^R(x^R; y^R)|}{RT} = 0.$$

Then since consumers can't consume negative quantities of private goods it follows immediately that,

$$\forall \epsilon > 0 \lim_{R \rightarrow \infty} NOC(\epsilon, x^R; y^R) = 0.$$

•

This theorem is of interest mainly because it illustrates the difficulty in generalizing the intuition gained from studying private goods economies to ones with public goods. The basic idea behind existing results is that the core converges if (or sometimes, if and only if) the returns to coalitional size are constant in the limit. This makes the public goods economy “private like” in the limit, and causes any derived game to be linearly additive. In contrast, theorem 2 shows the core also converges if the returns to coalitional size are sufficiently large. We know of no corresponding result for a private goods economy.

The reader may object that this is merely a technical artifact of our assumption that the consumption sets are bounded from below. There are two responses to this criticism. First, this assumption is very reasonable from an economic standpoint. It does not make much real world sense to imagine agents consuming unboundedly negative quantities of goods. Second, although it is true that dispensing with the

assumption (as in Kaneko (1975)) makes it possible to prove a theorem like “the core converges if and only if there are constant returns to coalitional size in the limit”, this comes at the expense of existence of Lindahl equilibria. For example, if consumers are strictly nonsatiated in Kaneko’s model, the only Pareto efficient allocations involve consumers getting infinite utility<sup>11</sup> by consuming a positive infinity of public goods and a negative infinity of private goods. Thus, bounding the consumption sets is not just a technical detail.

It is also worth noting that core convergence also holds if some types of consumers in the economy satisfy asymptotic satiation, and others satisfy strict nonsatiation. Theorem 2 says that “almost all” consumers who are strictly nonsatiated are almost exactly on their offer curves for any core allocation of a large enough economy. Theorem 1 says the same for consumers who are asymptotically satiated. Thus, if both types are in a single economy, then almost all consumers of both types are almost exactly on their offer curves for every core allocation of a large enough economy. Therefore, the core converges.

---

<sup>11</sup> Or arbitrarily close to the supremum if the utility function is bounded above.

## 4. Conclusion

Muench showed that core of a public goods continuum economy is not necessarily equivalent to the set of Lindahl allocations. The question of the generality of this result was left open. This paper goes part way to giving an answer. Both asymptotic satiation and strict nonsatiation are shown to be sufficient to guarantee core convergence for a class of convex and monotone public goods economies. Asymptotic satiation has a natural economic interpretation. Moreover, preferences for many, though probably not all, types of public goods are likely to satisfy this assumption. Strict nonsatiation is less appealing economically but serves to point out the differences between the conditions required for convergence in public and private goods economies.

The two convergence conditions discussed in this paper seem to describe extreme situations. Asymptotic satiation implies that the offer correspondences converge to the endowment, and that in the limit, only a negligible part of each consumer's endowment of the private good is devoted to public goods production at any core allocation. Strict nonsatiation, on the other hand, implies that the offer curves converge to zero, and that essentially all of the private goods are devoted to public goods production at any core allocation. This provides support for the feeling among public goods economists that the core is generally larger than the Lindahl equilibria. In a related paper, we study a much more restricted model (a one public good, one private good, transferable utility economy) and show that under certain conditions, asymptotic satiation is necessary and sufficient for core convergence. We also show that under slightly more general conditions, generically the core does not converge to the set of Lindahl allocations.



## References

- Anderson, Robert M. (1979): "An Elementary Core Equivalence Theorem," *Econometrica*, Vol. 46, No 6 pp. 1483-7.
- Anderson, Robert M. (1981): "Core Theory with Strongly Convex Preferences," *Econometrica*, Vol. 49, No 6 pp. 1457-68.
- Anderson, Robert M. (1986): "Notions of Core Convergence," in *Contributions to Mathematical Economics in Honor of Gerard Debreu*, by Werner Hildenbrand and Andrue Mas-Colell, Eds., North-Holland, Amsterdam, 25-46.
- Champsaur, Paul, Donald J. Roberts, and Robert W. Rosenthal (1975): "On Core in Economies with Public Goods," *International Economic Review*, Vol. 18, No. 3, 751-64.
- Debreu, Gerard (1959): *Theory of Value*. Yale University Press, New Haven.
- Debreu, Gerard, and Herbert Scarf (1963): "A Limit Theorem on the Core of an Economy," *International Economic Review*, Vol. 4, No. 3, 235-46.
- Diamantaras, Dimitrios (1992): "On Equity with Public Goods," *Social Choice and Welfare*, Forthcoming.
- Foley, Duncan K. (1970): "Lindahl's Solution and the Core of an Economy with Public Goods," *Econometrica*, Vol. 38, No.1, 66-72.
- Hildenbrand, Werner (1974): *Core and Equilibria of a Large Economy*. Princeton University Press, Princeton, New Jersey.
- Kaneko, Mamoru (1975): "On Lindahl Equilibria and the Core of an Economy with Public Goods," *Mimeo*, .
- Kaneko, Mamoru (1976): "Examples that Lindahl Equilibria May not Belong to the Core," *Mimeo*, .
- Mas-Colell, Andrue, and Joaquim Silvestre (1989): "Cost Share Equilibria: a Lindahlian Approach," *Journal of Economic Theory*, Vol. 47, No. 42, 239-56.
- McKenzie, Lionel, W. (1981): "The Classical Theorem on Existence of Competitive Equilibrium," *Econometrica*, Vol. 49, 819-841.
- Milleron, Jean-Claude (1972): "Theory of Value with Public Goods: A Survey Article," *Journal of Economic Theory*, Vol. 5, 419-77.
- Muench, Thomas J. (1972): "The Core and the Lindahl Equilibrium of an Economy with a Public Good: An Example," *Journal of Economic Theory*, Vol. 4, 241-255.
- Scotchmer, Suzanne, and Myrna Wooders (1988): "Monotonicity in Games that Exhaust Gains to Scale," *Mimeo*, .

- Shubik, Martin, and Myrna Wooders** (1983): "Approximate Cores of Replica Games and Economies. Part I: Replica games, externalities, and approximate cores," *Mathematical Social Sciences*, Vol. 6 27-48.
- Richter Donald K.** (1974): "The Core of a Public Goods Economy," *International Economic Review*, Vol. 15, No. 1, 131-142.
- Roberts, Donald J.** (1974): "A Note on Returns to Group Size and the Core with Public Goods," *Journal of Economic Theory*, Vol. 9, 350-6.
- Samuelson, Paul A.** (1954): "The Pure Theory of Public Expenditure," *The Review of Economics and Statistics*, Vol. XXXVI, 387-9.
- Weber, Shlomo, and Hans Weismeth** (1992): "The Equivalence of Core and Cost Share Equilibria in an Economy with a Public Good," *Journal of Economic Theory*, Vol. 54, No.1 180-197.
- Wooders, Myrna** (1983): "Epsilon Core of a large Replica Game," *Journal of Mathematical Economics*, Vol. 11, 277-300.
- Wooders, Myrna** (1990): "Inessentiality of Large Coalitions and the Approximate Core Property: Two Equivalence Theorems," *University of Bonn Sonderforschungsbereich 303*, Discussion Paper No. B-151.
- Wooders, Myrna** (1990): "Inessentiality of Large Coalitions and the Approximate Core Property: Two Equivalence Theorems," *University of Bonn Sonderforschungsbereich 303*, Discussion Paper No. B-151.
- Wooders, Myrna** (1990): "Inessentiality of Large Coalitions and the Approximate Core Property: Two Equivalence Theorems," *University of Bonn Sonderforschungsbereich 303*, Discussion Paper No. B-151.
- Wooders, Myrna** (1991a): "On Large Games and Competitive Markets 2: Applications," *University of Bonn Sonderforschungsbereich 303*, Discussion Paper No. B-196.
- Wooders, Myrna** (1991b): "On Large Games and Competitive Markets 1: Theory," *University of Bonn Sonderforschungsbereich 303*, Discussion Paper No. B-195.
- Wooders, Myrna** (1991c): "Necessary and Sufficient Conditions for the Approximate Core Property," *University of Bonn Sonderforschungsbereich 303*, Discussion Paper No. B-178.





HECKMAN  
BINDERY INC.

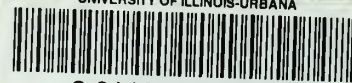


**JUN 95**

Bound-To-Please® N. MANCHESTER,  
INDIANA 46962



UNIVERSITY OF ILLINOIS-URBANA



3 0112 060295851