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by

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Abstract. This paper presents some recent results on the existence of Nash equilibria in games with a continuum of traders, non-ordered preferences and infinite dimensional strategy sets. The work reported here can also be seen as an application of functional analysis to a particular problem in mathematical economics.

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0. Introduction

This paper is devoted to a problem that occupies a central position in economic theory and whose origins lie in Augustin Cournot's *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. The setting is that of a group of players each of whom maximizes his self-interest as reflected in an individual pay-off function defined on an individual strategy set. What makes the problem interesting is that the optimum choice of any player depends on the actions of all the other players, this dependence being reflected in the pay-off function or in the strategy set or both. As such, this is a problem *par excellence* in what is now termed non-cooperative game theory.

A prototype of such a problem is Cournot's (1838) example of two profit maximizing firms each of whose profits depend not only on their individual output levels but also on that chosen by their rival firm. The solution proposed by Cournot consisted of a pair of output levels (x,y) such that x is the profit maximizing output level for the first firm when the second is restricted to produce y and y is the profit maximizing output level of the second firm when the first is restricted to x. Such an equilibrium notion was formalized by Nash (1950) and shown to exist in a setting with a finite number of players, each with an identical finite dimensional strategy set. Two years later, Debreu (1952) generalized the results of Nash and gave an existence proof based on Kakutani's (1941) fixed point theorem.[1] It is of interest that Debreu referred to his result as a "social existence theorem" and that this theorem constituted an important ingredient in one of the first general proofs of the existence of competitive equilibrium.[2]

Recent work has extended Debreu's result on the existence of Cournot-Nash equilibria along several directions; namely (i) cardinality of the set of players, (ii) nature of the pay-off functions, (iii) dimensionality of the strategy sets. In particular, Cournot-Nash equilibria have been shown to exist in settings with (i) a continuum of players, (ii) non-ordered preferences, and (iii) strategy sets in Banach spaces. The work is far from complete--we have no result, for example, that simultaneously incorporates (i), (ii) and (iii)--but it is clear that substantial progress has been made. We report on this here.

The underlying theme of our presentation is that all these extensions are suitable modifications of Debreu's original argument and essentially follow the guideposts laid out by him. As such, this paper could be seen essentially as underscoring the robustness of Debreu's (1952) proof.

The plan of the paper is to show the existence of Cournot-Nash equilibria in settings which are increasingly generalized. Section 1 presents the Nash-Debreu theorem in the context of a game with a finite number of players, with finite dimensional strategy sets and with pay-offs generated by functions. Section 2 considers an abstract economy à la Shafer-Sonnenschein. Here, the pay-off is generated by non-ordered binary relations and the strategy sets also depend on the choices of the other players. In Section 3, we consider a continuum game where the set of players is a measure space of agents but the strategy sets are identical for all players and the pay-off functions are linear functions on the individual strategy sets. In this, our assumptions on the game are closer to Nash than to Debreu. In Section 4, we remedy this and consider generalized continuum games. These are games as in Debreu but with the added assumption that the set of players is an abstract measure space of agents.[3] In Section 5, we consider an abstract continuum economy which is an abstract economy but with the set of players given by an abstract measure space. Section 6 is the final section of the paper and is devoted to generalized continuum games defined on a Banach space. This is the only section in the paper in which the strategy sets are not subsets of a finite dimensional Euclidean space R^{ℓ}. It is worth pointing out that we have no results to report on abstract continuum economies defined on a Banach space.

There are three major omissions in this paper. First, we do not discuss the recent results of Mas-Colell (1983) whose formulation of the dependence of the optimum choice of one player on the choices of the remaining others is drastically different from ours. His formalization of a continuum game is based on the "distribution" approach of Hart-Hildenbrand-Kohlberg (1974). A second omission is the work of Fan (1966), Browder (1968), Ma (1969), Toussaint (1982), Yannelis and Prabhaker (1983a and b). They do not consider a measure space of agents but an arbitrary denumerable or non-denumerable set of agents. This shows up, in particular, in their continuity assumptions on the pay-off functions and strategy sets. It would be of interest to relate the results presented in the sequel to those of Mas-Colell and to those stemming from the work of Fan-Browder. In this context, we may remark that our formulation of a model consisting of an infinity of agents each of whom makes choices from an infinite dimensional strategy set, is strictly along the lines laid out in Aumann (1964) and Bewley (1973). Our third omission relates to the existence of pure strategy equilibria as considered in Schmeidler (1973). Whereas it is certainly true that a consideration of these equilibria is beyond the pale of a paper devoted to the Cournot-Nash theorem, pure strategy equilibria are a primary motivation for considering games with a continuum of players. However, this topic merits an extended treatment in its own right and, in addition to Schmeidler's basic paper, we refer the reader to Khan (1982b, 1983).

A final word of exposition. This paper is written primarily for economists, and as such we have been cavalier in terms of theorems well-known to them but somewhat pedantic in terms of other results. An example of this is that we do not state Kakutani's fixed point theorem but do state and provide a reference for the proof of the Dunford-Pettis theorem on the characterization of weakly compact sets in the space of integrable functions.[4] Nevertheless, we hope that this exposition will also be useful to mathematicians both in suggesting new problems and in showing how relatively recent theorems in functional analysis seem tailor-made for a problem whose origins lie in political economy.

1. The Nash-Debreu Theorem.

We begin with Debreu's (1952) generalization of Nash's (1950) theorem on the existence of equilibria [subsequently Nash equilibria] in a game Γ with a finite set of players, each with strategy sets in[5] R^{ℓ} and with preference rankings given by pay-off functions defined on these strategy sets. Accordingly, let the set of players T be given by

(1, 2, ..., n), the strategy set of the $t^{\underline{th}}$ player by $X(t) \subseteq R^{\ell}$ and his pay-off function by $u_t : X \to R$ where $X = \prod_{t \in T} X(t)$. We can now present

<u>Definition 1.1</u>. A <u>Nash equilibrium</u> of a game Γ is an element $x^* = (x^*(1), \ldots, x^*(n)) \in X$ such that for all t in T

$$u_t(x^*) \geq u_t(x^*(1), \ldots, x^*(t-1), y, x^*(t+1), \ldots, x^*(n)) \quad \text{for all } y \in X(t).$$

The Nash-Debreu theorem can be stated once we recall that a function $f : D \to R$, D a convex subset of R^{ℓ}, is said to be <u>quasi-concave</u> if for all x, y in D

$$f(\lambda x + (1-\lambda)y) \geq \text{Min } (f(x), f(y)) \text{ for all } \lambda \in (0,1)$$

<u>Theorem 1.1 (Nash-Debreu)</u>: If, for all t in T, X(t) is nonempty, convex and compact, and u_t is a continuous function which is quasi-concave on X(t), there exists a Nash equilibrium for the game Γ.

Theorem 1.1 is a simple consequence of Berge's (1966) maximum theorem and Kakutani's (1941) fixed point theorem. Since proofs of subsequent theorems are elaborations and modifications of Debreu's basic argument, a somewhat leisurely development of the steps of his proof is warranted.

The proof revolves around a mapping $\alpha : X \to X$ where

$$\alpha = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_n,$$

$$\alpha_t : X \to X(t) \text{ with } \alpha_t(x) = \underset{y \in X(t)}{\text{Arg Max}}\; u_t(x(1) \ldots x(t-1), y, x(t+1), \ldots x(n))$$

It is clear that a fixed point of α yields a Nash equilibrium. In order to apply Kakutani's theorem to α, one has to establish the following five claims.

1. Compactness of X.
2. Nonemptiness and convexity of X.
3. For each x in X, nonemptiness and convexity of $\alpha(x)$.
4. For each t in T, upper semicontinuity[6] of α_t
5. Upper semicontinuity of α.

Let us take each claim in turn. (1) follows by hypothesis given that the set of players is finite and that each X(t) is a subset of R^{ℓ}. (2) is trivial. The first assertion in (3) follows from continuity of u_t over the compact set X(t) and the second assertion is a consequence of the quasi-concavity of u_t over X(t). Once we have (4), it is easy to show the validity of (5) and (4) is a straightforward consequence of Berge's theorem. Since we shall have a need for it in the sequel, it is worthwhile to have a general statement of Berge's theorem.

Theorem 1.2 (Berge): Let X and Y be topological spaces. If f is a continuous numerical function of Y and Φ is a continuous (set-valued) mapping of X into Y such that for each x, $\Phi(x) \neq \phi$, then the numerical function $M(x) = \max_{y \in \Phi(x)} f(y)$ is continuous in X and the mapping $\operatorname{Arg\,Max}_{y \in \Phi(x)} f(y)$ is an upper semicontinuous mapping of X into Y.

Proof: See Berge (1966, p. 116)

We need only remind the reader that a continuous set-valued mapping is one which is both upper and lower semicontinuous.[7] The reader has also undoubtedly noticed that in the application of Theorem 1.2 to Claim 4, the analogue of Φ is trivially continuous.

2. The Shafer-Sonnenschein Theorem.

The Nash-Debreu theorem assumes that each player's preference ranking is complete and transitive. Note that a preference ranking of a player t is a binary relation $\succsim_t$ defined on X(t) x X(t) such that for all x, y in X(t)

$$x \succsim_t y \text{ iff } u_t(z(1), \ldots, z(t-1), x, z(t+1), \ldots, z(n)) \geq u_t(z(1), \ldots, z(t-1), y, z(t+1), \ldots, z(n)) \text{ for all } z(i) \in X(i),\ i \neq t.$$

A natural question arises as to whether the Nash-Debreu theorem can be extended to a set-up where each player's preference ranking $\succsim_t$ is non-ordered, i.e., neither complete nor transitive. That this can indeed be accomplished was shown by Shafer-Sonnenschein (1975) once Mas-Colell (1974) had given reason to believe that such a result could indeed be proved.

Before stating the Shafer-Sonnenschein theorem, we note two points of exposition. Firstly, they formalize the preference ranking as a set-valued mapping $P_t : X \to X(t)$ where $P_t(x)$ has the interpretation of being the set of strategies which are preferred (strictly) by player t to x(t). We leave it to the reader to convince himself that it is a simple matter to go from $\succsim_t$ to P_t and vice versa provided the dependence of $\succsim_t$ on the other players' strategies is made explicit. Secondly, Shafer-Sonnenschein allow each player's strategy set to depend on the strategy choices of the other players. They do this by introducing an additional mapping $A_t : X \to X(t)$. Thus, in the Shafer-Sonnenschein set-up, a game Γ consists of a finite set of players T each of whom has a strategy set X(t), a choice (set-valued) mapping $A_t : X \to X(t)$ and a preference mapping $P_t : X \to X(t)$. With these reformulated primitives, we shall follow Shafer-Sonnenschein and refer to Γ as an <u>abstract economy</u>.

The only remaining point concerns the formal definition of a Nash equilibrium for the abstract economy Γ. This is given by

<u>Definition 2.1</u>. A <u>Nash equilibrium</u> of an abstract economy Γ is an element $x^* = (x^*(1), \ldots., x^*(n)) \in X$ such that for all t in T,

$$x^*(t) \in A_t(x^*) \text{ and } P_t(x^*) \cap A_t(x^*) = \phi.$$

A little reflection will convince the reader that Definition 2.1 is a direct generalization of Definition 1.1 to a setting with non-ordered preferences.

We can now state

<u>Theorem 2.1 (Shafer-Sonnenschein)</u>: If, for all t in T,

(i) X(t) is nonempty, convex and compact,

(ii) A_t is a continuous correspondence such that for each x in X, $A_t(x)$ is nonempty and convex,

(iii) P_t has an open graph in $X_t \times X$ such that for each x in X, $x(t) \notin \text{Con } P_t(x)$, [Con(B) is the convex hull of the set B],

there exists a Nash equilibrium for the abstract economy Γ.

It seems fair to say that at first sight, it is difficult to see how the structure of the proof of Theorem 1.1 can be modified to construct a proof of Theorem 2.1. That it can be done follows from an ingenious construction of a pseudo pay-off function corresponding to each of the preference mappings P_t. We turn to this.

Shafer-Sonnenschein [also see Shafer (1974)] define a real valued function $u_t : X(t) \times X \to R$ such that

$$u_t(z,x) = \underset{y \in \text{Graph } P_t^c}{\text{Infimum}} \ \rho[(z,x), y]$$

where Graph $P_t^c = \{(z,x) \in X(t) \times X : z \notin P_t(x)\}$ and ρ is the Euclidean metric. Note that the Graph P_t^c is a closed set by hypothesis[8] and, given the compactness of $X(t)$ and hence of X, $u_t(\cdot)$ is well-defined. Note also that

$$U_t(z,x) > 0 \text{ iff } z \in P_t(x) \qquad (*)$$

This simple and elegant construction (one may be forgiven for recalling Urysohn's lemma) can now be used in tandem with the Debreu argument to prove Theorem 2.1. Let us sketch the basic steps.

As in the proof of Theorem 1.1, let $\alpha : X \to X$, $\alpha = \alpha_1 \times \dots \times \alpha_n$, and $\alpha_t : X \to X(t)$ where $\alpha_t(x) = \text{Con } [\underset{z \in A_t(x)}{\text{Arg Max}} \ U_t(z,x)]$.

As before, we appeal to Kakutani's theorem to guarantee a fixed point for α. Claims 1 to 5 remain as before and are as easy to establish. The only marginal difference lies in the application of Berge's theorem for which we now have to utilize the continuity hypothesis on the correspondence A_t and a subsidiary claim that

6. the convex hull of an upper semicontinuous mapping is upper semicontinuous.

The fact that 6 is true in the setting of R^{ℓ} is well-known.[9]

But, of course, a price has to be paid at some stage for working with the pseudo utility function and this consists in the fact that it is no longer obvious that the fixed point of α is indeed a Nash equilibrium for Γ. This follows from a somewhat delicate argument.

Let x^* be a fixed point of α and let $H_t(x^*) = \underset{z \varepsilon A_t(x^*)}{\text{Arg Max}}\ u_t(z,x^*)$.

Since $A_t(x^*)$ is convex, certainly $x^*(t) \varepsilon A_t(x^*)$ for all t in T. Let us suppose that for some t, $A_t(x^*) \cap P_t(x^*) = z$. Since $z \varepsilon P_t(x^*)$, certainly as a consequence of (*), $u_t(z,x^*) > 0$. But this implies that $u_t(y,x^*) > 0$ for all $y \varepsilon H_t(x^*)$. By a second appeal to (*), this shows that $H_t(x^*) \subseteq P_t(x^*)$. But $x^*(t) \varepsilon \alpha_t(x^*) = \text{con } H_t(x^*) \subseteq \text{con } P_t(x^*)$, and we obtain a contradiction.

1.3. A Theorem of Schmeidler.

In this subsection we abandon non-ordered preferences as in Shafer-Sonnenschein and pursue another generalization of the Nash-Debreu theorem. This is Schmeidler's (1973) theorem on the existence of Nash equilibria in games with a measure space of players.

In Schmeidler's setting, the set of players T consists of the unit interval endowed with Lebesgue measure μ. Each player's strategy set is given by the set $X = \{x \varepsilon R^{\ell} : x_i \geq 0, \sum_{1}^{\ell} x_i = 1\}$ and the pay-off function of the $t^{\underline{th}}$ player is given by $u_t : L_1(\mu,R^{\ell}) \to R$ where $u_t(x) = x(t) \cdot h(t,x)$, $L_1(\mu,R^{\ell})$ denotes the equivalence class of the set of Lebesgue integrable functions taking values in R^{ℓ}, $h(t,\cdot) : L_1(\mu,R^{\ell}) \to R^{\ell}$, and $a \cdot b$ denotes the inner product of a, $b \in R^{\ell}$. We shall refer to games with a measure space of players as continuum games. We can now present

Definition 3.1. A Nash equilibrium for a continuum game Γ is an element $x^* \in L_1(\mu,R^{\ell})$ such that for almost all t in T, $x^*(t) \in X$ and

$$u_t(x^*) \geq y \cdot h(t,x^*) \quad \text{for all } y \in X.$$

Two points need to be noted about Schmeidler's formalization of games with a continuum of players. Definitions 1.1 and 1.2 leave untouched the non-cooperative aspect embodied in a Nash equilibrium. Given the strategy choices of the other players, the only restriction on a given player's choice of strategy is the natural one that it be limited to his strategy set. This is no longer the case in Definition 3.1 where the strategy choices have to obey the further restriction that they be measurable. (Integrability is an obvious consequence of measurability given that the compact set X is the same strategy set for all the players.) This difficulty with the measurability requirement is also brought out in Dubey-Shapley (1977) and is of obvious significance in the modelling of non-cooperative games with a continuum of players.

A second difficulty concerns the definition of the pay-off functions. Since $u_t(\cdot)$ is defined on an equivalence class of Lebesgue integrable functions, it makes sense for almost all players rather than for all players. This is a simple consequence of the fact that perturbations on a set of measure zero do not change an element x of $L_1(\mu,R^{\ell})$

but do change the value of $u_t(x)$ for all players in that set of zero measure. It may be worth remarking that this difficulty does not arise in the literature stemming from Aumann's (1964, 1966) papers. Of course, since the equilibrium concepts one is dealing with also neglect sets of measure zero (even in the Aumann setting), this difficulty is not of any fundamental significance. It will nevertheless stay with us when we consider generalizations of Schmeidler's theorem.

We can now present

Theorem 3.1 (Schmeidler): If, for almost all t in T, $h(t,\cdot)$ is weakly continuous on[10] $L_1(\mu,X)$ and for all $x \in L_1(\mu,X)$, and all $i = 1, \ldots, \ell$, the set $\{t \in T : h_i(t,x) > h_i(t,x)\}$ is measurable, there exists a Nash equilibrium for the continuum game Γ.

It is fair to say that Debreu's basic argument for the proof of the Nash-Debreu theorem continues to have relevance for Schmeidler's theorem. The obvious modification relates to the fact that we are no longer in the confines of an Euclidean space but in the space $L_1(\mu,R^\ell)$. Thus, we have to replace Kakutani's fixed point theorem by

Theorem 3.2 (Fan-Glicksberg): Let C be a non-empty, compact convex set in a locally convex space V. If Φ is an upper semi-continuous mapping of C into C and if, for all x, the set $\Phi(x)$ is convex and non-empty, then there exists a point x_0 such that $x_0 \in \Phi(x_0)$.

Proof: See Ky Fan (1952) or Glicksberg (1952).

The proof of Theorem 3.1 revolves around a mapping $\alpha : L_1(\mu,X)$ into $L_1(\mu,X)$ where[11]

$$\alpha(x) = \{y \in L_1(\mu,X) : y(t) \in \alpha_t(x) \text{ for almost all } t \text{ in } T\}$$

$$\alpha_t : L_1(\mu,X) \to X \text{ where } \alpha_t(x) = \operatorname*{Arg\,Max}_{p\in X} (p\cdot h(t,x))$$

It is clear that a fixed point of α yields a Nash equilibrium. In order to apply the Fan-Glicksberg theorem to α, one has to establish the same five claims one established in the proof of Theorem 1.1 with the only (!) difference that they now pertain to $L_1(\mu,X)$ instead of X.

Let us take each claim in turn after observing that the statement of Theorem 3.1 already makes clear that the locally convex space we shall be working in is $L_1(\mu,R^\ell)$ endowed with its weak topology.[12]

The weak compactness of $L_1(\mu,X)$ is a straightforward consequence of the following classical theorem.

Theorem 3.3 (Dunford-Pettis): A subset K of[13] $L_1(\mu)$ has a weakly compact closure if (and only if) (i) $\sup_{f\in K} \int_T |f(t)|d\lambda < \infty$ and (ii) given $\varepsilon > 0$ there is a $\delta > 0$ such that if $\mu(A) \leq \delta$, then $\int |f(t)|d\lambda \leq \varepsilon$ for all $f \in K$.

Proof: See, for example, Diestel (1984, p. 93).

By viewing $L_1(\mu,R^\ell)$ as the space of integrable functions defined on $\prod_{i=1}^{n} [0,1]$ endowed with the n-fold product measure $\mu \times \mu \times \ldots \times \mu$, it is clear that Theorem 3.3 applies to $L_1(\mu,X)$. Thus all that needs to be established is that $L_1(\mu,X)$ is weakly closed. But this is easy once we recall

<u>Theorem 3.4 (Mazur)</u>: If K is a convex subset of a normed linear space, then the closure of K in the norm topology coincides with the weak closure of K.

<u>Proof</u>: See Diestel (1984; page 11).

In order to finish the proof of Claim 1, all we need is to observe is that

(i) $L_1(\mu,X)$ is convex,

(ii) X is a closed set,

(iii) a sequence of elements in $L_1(\mu,R^\ell)$ tending to a limit in norm has a subsequence tending to that limit almost everywhere.[14]

The nonemptiness of $L_1(\mu,X)$ as required by the first part of Claim 2 is trivial as is the convexity of $\alpha(x)$ for a given x in $L_1(\mu,X)$, as required by Claim 3.

Claim 3 also asserts that $\alpha(x)$ is non-empty for each x. In order to establish this, Schmeidler draws on the special structure of his model. Let $T_i = \{t \in T : h_j(t,x) \leq h_i(t,x),\ j=1, \ldots, n\}$ and observe that $\bigcup_{i=1}^{n} T_i = T$ and that $e_i \in \alpha_t(x)$ where e_i is a vector with one in the i^{th} place and zero everywhere else. Let $S_1 = T_1$ and[15] $S_i = (T_i \,/\, \bigcup_{j=1}^{i} T_j)$, i=2, ..., n. Given the nature of the pay-off functions, $y \in \alpha(x)$ where $y(t) = e_i$ for all $t \in S_i$, for all i.

Since we are working in the compact sets X and $L_1(\mu,X)$, Claim 4 on the upper semicontinuity of $\alpha_t(x)$ reduces to showing that the graph of $\alpha_t(\cdot)$ is closed[16] in $X \times L_1(\mu,X)$. But this is straightforward given the continuity assumption on $h_t(\cdot)$.

The final claim relates to the upper semicontinuity of α. This is clearly the hardest part in the proof of Theorem 3.1. Utilizing again the weak compactness of $L_1(\mu,X)$, we need only show that the graph of α is closed in $L_1(\mu,X) \times L_1(\mu,X)$. However, compactness gives us more. It allows us to establish closedness by considering sequences instead of nets. In his paper, Schmeidler asserts this fact without offering any proof but it is worth pointing out that it is a consequence of the following two results.

Theorem 3.5 (Eberlein-Smulian): A subset of a Banach space is relatively weakly compact iff it is relatively weakly sequentially compact. In particular, a subset of a Banach space is weakly compact iff it is weakly sequentially compact.

Proof: See Diestel (1984; Chapter III).

Theorem 3.6: Suppose that E is a linear space with a vector topology which is metrizable and that A is a subset of E with the property that every sequence of points of A has a weak cluster point in E. Then any point of the weak closure of A is the weak limit of a sequence of points of A.

Proof: See Kelley-Namioka (1963; Problem 17L, page 165).

Now in the context of Claim 5, let (x^ν, y^ν) be a net converging to (x,y) where $y^\nu \in \alpha(x^\nu)$. We have to show that $y \in \alpha(x)$. Since the union of the net (x^ν, y^ν) and (x,y) is a weakly compact subset of $L_1(\mu,X) \times L_1(\mu,X)$, it is weakly sequentially compact by virtue of Theorem 3.5. We can now appeal to Theorem 3.6 to extract a sequence

(x^n, y^n) from (x^ν, y^ν) that tends to (x,y). We now need the following corollary of Theorem 3.4.

Theorem 3.7: If $\{x_n\}$ is a sequence in the normed linear space for which the weak limit of x_n is zero, then there is a sequence $\{\sigma_n\}$ of convex combinations of the x_n such that σ_n converges to zero in norm.

Proof: See Diestel (1984; page 11) or Dunford-Schwartz (1958; V. 3.14).

Theorem 3.7 along with the reasoning used in the proof of Claim 1 above allows us to conclude that $y(t) \in X$ almost everywhere in T.

Suppose, per absurdum, that $y(t) \notin \alpha_t(x)$ for all t in S where $\mu(S) > 0$. For each t, $\alpha_t(x)$ is a convex hull of a subset of $(e_1, \ldots., e_n)$. Thus, there is a nonnull, measurable subset V of S and a strict subset $(e_{i_1}, \ldots., e_{i_k})$ of $(e_1, \ldots., e_n)$ such that for each t in V, $y(t) \notin \text{con}\,(e_{i_1}, \ldots., e_{i_k})$. Hence there is a $p \in P$ such that for all t in V

(i) $py(t) > 0$ and $p \cdot e_{i_j} = 0$, $j = 1, \ldots, k$.

Hence $\int_V py(t)d\mu > 0$ and $\int_V pz(t)d\mu = 0$ for each $z \in L_1(\mu,P)$ such that $z(t) \in \alpha_t(x)$ for all $t \in V$. But weak convergence of y^n to y implies that

(ii) $\int_V y(t)d\mu = \lim_{n\to\infty} \int_V y^n(t)d\mu$,

and hence

(iii) $\int_V y(t)d\mu \in \int_V \limsup (y_n(t))d\mu = \{\int_V z(t)d\mu$: $z(t)$ is a limit point of z^n for almost all t in $V\}$

This clearly gives us a contradiction if we take upper semicontinuity of $\alpha_t(x)$ into account.

Only the validity of (iii) still needs to established and this follows from

Theorem 3.8 (Aumann): If F_1, F_2 .... is a sequence of set-valued functions from T into $\mathcal{P}(R^\ell)$ that are integrably bounded, then $\int \limsup F_k(t)d\mu \subseteq \limsup \int F_k(t)d\mu$ where for any $A : T \to \mathcal{P}(R^\ell)$, $\int A d\mu = \{\int f(t)d\mu : f : T \to R^\ell$ is measurable and such that for almost all t in T, $f(t) \in A(t)\}$.

Proof: See Aumann (1965).

4. A Generalization of Schmeidler's Theorem.

Schmeidler's theorem is cast in the set-up of Nash (1950) rather than that of Debreu (1952). It is thus natural to ask whether it can be generalized along the lines of Debreu by dropping the assumption of an identical strategy set and that of the pay-off function being linear on the player's own strategy set. In the concluding remark in his paper, Schmeidler sketches such a generalization and briefly indicates how it may be proved. In this section, we provide a complete argument, one which uses results not available then to Schmeidler. A principal motivation for such an exercise lies in the fact that it sets the stage for our subsequent results; however, it is also of independent interest.

Let $(T,\mathfrak{J},\mu)$ be a complete, finite measure space, i.e., μ is a real valued, non-negative, countably additive measure defined on a σ-field of subsets of a point set T such that $\mu(T) < \infty$.

We shall say that a set-valued mapping $X : T \to \mathcal{P}(R^\ell)$ is <u>measurable</u> if its graph, Graph $X = \{(t,x) \in T \times R^\ell : x \in X(t)\}$ belongs to the product σ-algebra $\mathfrak{J} \otimes \mathcal{B}(R^\ell)$, where $\mathcal{B}(R^\ell)$ denotes the Borel σ-algebra on R^ℓ and $\mathcal{P}(R^\ell)$ the space of all subsets of R^ℓ. We shall say that P is <u>integrably bounded</u> if the real valued function on T given by $t \to \text{Sup}$ $\{||x|| : x \in P(t)\}$ is integrable.

A <u>generalized continuum game</u> Γ consists of a set of players given by the abstract measure space $(T,\mathfrak{J},\mu)$, a measurable set-valued mapping $X : T \to \mathcal{P}(R^\ell)$ and for each t in T a pay-off function[17] $u(t,.) : X(t) \times L_1(\mu,X(\cdot)) \to R$. We have only a marginal modification in the definition of a Nash equilibrium for a generalized continuum game relative to that of a continuum game.

<u>Definition 4.1</u>. A <u>Nash equilibrium</u> of a generalized continuum game Γ is an element $x^* \in L_1(\mu,X(\cdot))$ such that for almost all t in T

$$u(t,x^*(t),x^*) \geq u(t,y,x^*) \quad \text{for all } y \in X(t).$$

We can now present

<u>Theorem 4.1 (Schmeidler)</u>: Let $\Gamma = [(t, \ ,\mu),X,u]$ be a generalized continuum game which satisfies the following assumptions.

(i) X is a measurable, integrably bounded map such that for all t in T, X(t) is nonempty, convex and weakly compact.

(ii) u is a map such that

(a) for all $x \in L_1(\mu, X(\cdot))$, $u(\cdot,\cdot,x)$ is a measurable function on Graph $X \equiv \{(t,x) \in T x R^{\ell} : x \in X(t)\}$,

(b) for all t in T, for all $x \in L_1(\mu, X(\cdot))$, $u(t,\cdot,x)$ is quasi-concave on $X(t)$,

(c) for all t in T, $u(t,\cdot,\cdot)$ is continuous on $X(t) \times L_1(\mu, X(\cdot))$ where $L_1(\mu, X(\cdot))$ is endowed with the relative weak topology.

Then the generalized continuum game Γ has a Nash equilibrium.

At this stage there is little need to remind the reader that the proof of Theorem 4.1 follows that of Theorem 3.1 with the obvious modifications to the mapping α.

As in the case of Theorem 3.1, it is clear that the Dunford-Pettis theorem can be used to establish the validity of Claim 1 that $L_1(\mu, X(\cdot))$ is weakly compact.

The convexity of $L_1(\mu, X(\cdot))$ is straightforward given the convex valuedness of the mapping X. The non-emptiness of $L_1(\mu, X(\cdot))$ is a consequence of the Von-Neumann-Aumann measurable selection theorem. Since we shall need a more general version of the theorem in the sequel, we give Sainte-Beuve's (1974) version of this result.

<u>Theorem 4.2 (Von-Neumann, Aumann, Sainte-Beuve):</u> Let $(T, \mathfrak{J})$ be a measurable space and S a Suslin space. Let Φ be a set-valued function from T into non-empty subsets of S, and whose graph belongs to $\mathfrak{J} \otimes \mathfrak{B}(S)$. Then there exists a sequence (σ_n) of selections of Φ such that, for every t, $\{\sigma_n(t)\}$ is dense in $\Phi(t)$ and σ_n is measurable for the completion of $\mathfrak{J}$ and $\mathfrak{B}(S)$.

Proof: See proof of Theorem III.22 in Castaing-Valadier (1977).

Note that a Suslin space[18] is a Hausdorff, topological space S such that there is a continuous surjection from a separable, complete and metrizable topological space to S. [For details, see Schwartz (1973), Chapter II.1]

The first part of Claim 3 that $\alpha(x)$ is convex for a given x in $L_1(\mu,X(\cdot))$ is straightforward since the pay-off's are quasi-concave in the relevant argument. Nonemptiness of $\alpha(x)$ is a consequence of Theorem 4.2 once we show that $\alpha_t(x)$ has a measurable graph. The following theorem, which may be looked on as a measure-theoretic cousin of Berge's theorem, seems ideally suited for this.

Theorem 4.3 (Castaing-Valadier): Let $(T,\mathfrak{J})$ be a measurable space, S a Suslin space, $u : T \times S \to R$ a $\mathfrak{J}\otimes\mathcal{B}(S)$ measurable function and Φ a measurable set-valued function from T into $\mathcal{P}(S)$. If for every t, $\Psi(t) \equiv \underset{x\in\Phi(t)}{\text{Arg Max}}\, u(t,x)$ is nonempty, then the graph of Ψ belongs to $\hat{\mathfrak{J}} \times B(S)$ where $\hat{\mathfrak{J}}$ denotes the completion of $\mathfrak{J}$.

Proof: See Lemma III.20 and the Application in Castaing-Valadier (1977).

Claim 4 on the upper semi-continuity of $\alpha_t : L_1(\mu,X(\cdot)) \to X(t)$ is an easy consequence of Berge's theorem; our Theorem 1.2.

The final claim on the upper semi-continuity of $\alpha : L_1(\mu,X(\cdot)) \to L_1(\mu,X(\cdot))$ can be established as a consequence of the following theorem.

Theorem 4.4 (Artstein): If a uniformly integrable sequence (f_n) chosen from $L_1(\mu,R^\ell)$ converges weakly to g, then there exists a set B of measure zero such that[19] $g(t) \in$ con lim sup $(f_n(t))$ for all t in T / B.

Proof: See Proposition C in Artstein (1979).

Recall that a subset K of $L_1(\mu,R^\ell)$ is uniformly integrable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $A \in \mathcal{J}$, $\mu(A) \leq \delta$ implies $\|\int_A f(t)d\mu\| < \varepsilon$ for all f in K.

As in the proof of Claim 5 in the context of Theorem 3.1, let a sequence (x^n,y^n) converge to (x,y) where $y^n \in \alpha(x^n)$. Suppose per absurdum that there exists a set S of nonnull measure such that $y(t) \in \alpha(x)$ for all t in S. But for any t, upper semicontinuity of $\alpha_t(\cdot)$ implies that $\overline{y}(t) \in \alpha_t(x)$ where $\overline{y}(t) \in \lim\sup\,(y_n(t))$. Since $\alpha_t(x)$ is convex valued, certainly con lim sup $(y_n(t)) \in \alpha_t(x)$. We can now apply Artstein's theorem to get a contradiction and complete the proof.

At this stage, a natural question arises as to whether one can furnish a proof of Claim 5 without appealing to Artstein's theorem. In other words, one may ask how Schmeidler could have proved Claim 5 in 1973. A possible answer is given below. In addition to Aumann's theorem, it relies on the following classical theorems.

Theorem 4.5 (Steinhaus-Dunford): If $(T,\mathcal{J},\mu)$ is a σ-finite measure space, there is an isometric isomorphism between $(L_1(\mu))^*$ (the space of continuous, linear functions on $L_1(\mu)$) and $L_\infty(\mu)$ (the space of essentially bounded measurable functions) in which corresponding vectors x^* and g are related by the identity

$$x^*(f) = \int_T g(t)f(t)d\mu \qquad f \in L_1(\mu).$$

Proof: See Dunford-Schwartz (1958, IV.8.5).

Theorem 4.6 (Hahn-Banach): If K_1 and K_2 are disjoint closed convex subsets of a real locally convex linear topological space V, and if K_1 is compact, then there exist constants c and $\varepsilon > 0$ and a continuous linear functional f on V such that

$$f(K_2) \leq c - \varepsilon < c \leq f(K_1).$$

Proof: See Dunford-Schwartz (1957, V.2.10).

Now let (x_n, y_n) converge weakly to (x,y) such that $y_n \in \alpha(x_n)$ for all n. Suppose per absurdum that there exists a nonnull set $S \in \mathfrak{J}$ such that $y(t) \notin \alpha_t(x)$ for all t in S. Now denote the restriction of $\mathfrak{J}$ and μ to S by $\mathfrak{J}_S$, μ_S respectively and let $\alpha_S(x) = \{z \in L_1(\mu_S) : z(t) \in \alpha_t(x)$ for all t in $S\}$. $\alpha_S(x)$ can be shown to be nonempty, convex and weakly closed by using the arguments utilized in the proof of our earlier claims.

Let y_S be the restriction of y to S. By hypothesis, $y_S \notin \alpha_S(x)$. We can now apply Theorem 4.6 to claim the existence of a non-zero, continuous linear functional f such that

(i) $\quad f(y_S) > f(z)$ for all z in $\alpha_S(x)$.

By Theorem 4.5 we can represent f by $g \in L_\infty(\mu, R^\ell)$ and rewrite (i) as

(ii) $\quad \int_S (y(t)\cdot g(t))d\mu > \int_S (z(t)\cdot g(t)) \ \forall \ z \in \alpha_S(x)$

Since y_n converges weakly to y, certainly

(iii) $\quad \lim \int_S (y_n(t)\cdot g(t))d\mu = \int_S (y(t)\cdot g(t))d\mu$

We can now appeal to Aumann's theorem, Theorem 3.6 above, to assert that

(iv) $\int_S (y(t)\cdot g(t))d\mu \subseteq \int_S \limsup (y_n(t)\cdot g(t))d\mu$

Since $\limsup (y_n(t)\cdot g(t)) = (\limsup y_n(t) \cdot (g(t))$ and each limit point of $y_n(t)$ is in $\alpha_t(x)$ from Claim 4, we obtain the required contradiction.

5. The Shafer-Sonnenschein Theorem with a Continuum of Players

In this subsection we present a generalization of the Shafer-Sonnenschein theorem to a setting where the set of players is given by the measure space $(T,\mathcal{J},\mu)$. Our result could be alternatively viewed as a generalization of Schmeidler's theorem to a setting where the pay-off functions are replaced by non-ordered preference rankings.

As in Shafer-Sonnenschein, we work with an abstract economy. However, in our generalized set-up, a precise definition is warranted.

Definition 5.1: An abstract continuum economy Γ is a quadruple $[(T, ,\mu), X, A, P]$ where $(T, ,\mu)$ as in Section 4, $X : T \to \mathcal{P}(R^\ell)$, $A : T \times L_1(\mu,X(\cdot)) \to \mathcal{P}(X(t))$ and $P : T \times L_1(\mu,X(\cdot)) \to \mathcal{P}(X(t))$.

As in Section 2, for each t in T, X(t) is the t^{th} player's strategy set, and for any $x \in L_1(\mu,X(\cdot))$, A(t,x) is the choice set of player t which, given the actions x of all other agents, determines the subset of X(t) from which t chooses his strategy. P is a preference correspondence with the obvious interpretation that P(t,x), $x \in L_1(\mu,X(\cdot))$ is the "better-than-set" of agent t with x(t) as the point of reference. However, it is worth emphasizing that the difficulty about the interpretation of the pay-off functions that we discussed in Section 3, is also present here. For any given x, the interpretation of P(t,x) can be sustained only for almost all agents simply because $P(t,\cdot)$ is defined on a space of equivalence class of functions.

Definition 5.2. A Nash equilibrium of an abstract continuum economy Γ is an element $x^* \in L_1(\mu,X(\cdot))$ such that for almost all t in T

$$x^*(t) \in A(t,x^*) \text{ and } A(t,x^*) \cap P(t,x^*) = \phi.$$

We can now state

Theorem 5.1 (Khan-Vohra): Let an abstract continuum economy Γ given by $[T,\mathfrak{J},\mu), X, A, P]$ satisfy the following assumptions.

1. $(T,\mathfrak{J},\mu)$ is a finite, positive, complete measure space such that $L_1(\mu)$ is separable.
2. X is an integrably bounded measurable map such that for all t in T, X(t) is nonempty, convex and compact.
3. A is a map such that
   (a) for all x in $L_1(\mu,X(\cdot))$, the graph of $A(\cdot,x)$ belongs to $\mathfrak{J} \otimes \mathcal{B}(R^\ell)$,
   (b) for all t in T and for all x in $L_1(\mu,X(\cdot))$, A(t,x) is a non-empty, closed and convex subset of X(t),
   (c) for all t in T, $A(t,\cdot)$ is a continuous correspondence.
4. P is a map such that
   (a) the graph of $P(\cdot,\cdot)$ belongs to $\mathfrak{J} \otimes \mathcal{B}(L_1(\mu,X(\cdot)) \otimes \mathcal{B}(R^\ell)$,
   (b) for all t in T, the graph of $P(t,\cdot)$ is open in the set $X(t) \times L_1(\mu,X(\cdot))$,
   (c) for almost all t in T, for all x in $L_1(\mu,X(\cdot))$, $x(t) \notin \text{con } P(t,x)$.

Then there exists a Nash equilibrium for an abstract continuum economy Γ.

It should be evident that the proof of Theorem 5.1 follows the proof of Theorem 4.1 provided we can construct a pseudo pay-off function. However, this is not as easily accomplished as in Section 2. To begin with, we shall need the following theorems.

Theorem 5.2: The weak topology of a weakly compact subset A of a separable Banach space is a metric topology.

Proof: See the proof of Theorem V.6.3 in Dunford-Schwartz (1958).

Theorem 5.3: Let $(T,\mathcal{T})$ be a measurable space, V a separable metric space, and Φ is a set-valued mapping from T to nonempty, complete subsets of V. Then the following properties are equivalent.

(a) $d(x,\Phi(\cdot))$ is measurable for every $x \in V$ where d is the distance from x to any subset of V.

(b) Φ admits a sequence of measurable selections (σ_n) such that for all t in T, $(\sigma_n(t))$ is dense in $\Phi(t)$.

Proof: See the proof of III.9 in Castaing-Valadier (1977).

Theorem 5.4: Let $(T,\mathcal{T})$ be a measurable space, U a metrizable space, V a separable metrizable space and $u : T \times V \to U$. If u is measurable (respectively $(\mathcal{T},\mathcal{B}(U))$ measurable) in t and continuous in V, then u is measurable (respectively $(\mathcal{T} \otimes \mathcal{B}(V), \mathcal{B}(U))$ measurable).

Proof: See the proof of Lemma III.14 in Castaing-Valadier (1977).

We are now ready to construct our pseudo pay-off function as

$$u(t,y(t),x) = \inf_{z \in G_P^c(t)} \rho((y(t),x),z)$$

where $G_P^c(t)$ refers to the graph of the complement of $P(t,\cdot)$ and ρ is the metric that induces the product topology on $R^\ell \times L_1(\mu,X(\cdot))$. The fact that $u(t,\cdot,\cdot)$ is weakly continuous on $L_1(\mu,X(\cdot))$ and continuous on $X(t)$ follows from an application of Berge's theorem (Theorem 1.2 above) once we observe that $L_1(\mu,X(\cdot))$ is weakly compact. The fact that such a metric exists follows from the weak compactness of $L_1(\mu,X(\cdot))$, the separability of $L_1(\mu)$ and Theorem 5.2 above.

In order to show that $u(\cdot,\cdot,x)$ is jointly measurable on $T \times R^\ell$, we simply apply Theorems 4.2, 5.3 and 5.4 above.

In adapting the proof of Theorem 1.1 to prove Theorem 5.1, we note from the discussion in Section 2 that α_t represents the convex hull of a the best response correspondence of the player t. It was precisely this fact that necessitated the use of the result that the convex hull of an upper semicontinuous map is also upper semicontinuous. We need a similar result for a measurable correspondence. Fortunately, this is available in Hildenbrand (1974, p. 60).

A natural question arises at this stage as to whether Theorem 5.1 can be proved without the requirement on the measure μ that $L_1(\mu)$ be separable. The answer to this question is positive if we are allowed a marginal strengthening of assumption 4 in the statement of Theorem 5.1.

<u>Theorem 5.2 (Khan-Vohra)</u>: Theorem 5.1 is valid iff (1) and 4(a,b) are substituted by

(1)' $(T,\mathcal{J},\mu)$ is a finite, positive, complete measure space, 4(a)' for all x in $L_1(\mu, X(\cdot))$, the graph of $P(\cdot, x)$ belongs to $\mathcal{J} \otimes \mathcal{B}(R^\ell)$,

4(b)' for all t in T, $P^c(t, \cdot) : L_1(\mu, X(\cdot)) \to X(t)$ is a continuous correspondence.

Under these changed hypotheses, the only change required in the proof of Theorem 5.1 pertains to the construction of the pseudo pay-off function. This is now given by

$$u(t,z,x) \equiv d(z, P^c(t,x)) \text{ for any } x \in L_1(\mu, X(\cdot)).$$

Note that $d(z,P^c(t,x)) = \inf_{y \in P^c(t,x)} \rho(x,y)$ where ρ is the Euclidean metric on R^ℓ. Since ρ is a continuous function of its arguments and $P^c(t,\cdot)$ is a continuous correspondence by hypothesis, we can appeal to Theorem 1.2 to assert that $d(\cdot, P^c(t,x))$ is continuous and $d(z, P^c(t,\cdot))$ is weakly continuous. Indeed, Berge's theorem shows that $d(\cdot, P^c(t,\cdot))$ is jointly continuous with respect to these topologies.

All that remains is to show that $d(\cdot, P^c(\cdot, x))$ is jointly measurable on $R^\ell \times T$ for any given $x \in L_1(\mu, X(\cdot))$. But on using assumption 4(a)', this follows from Theorem 4.2, 5.3 and 5.4 above.

6. Games on a Banach Space.

So far, we have restricted our attention to games whose strategy sets lie in a finite dimensional space. In this section, we relax this assumption and allow strategy sets to be subsets of a real Banach space. Recall that a Banach space is a complete normed linear space. It is also worth observing that a separable Banach space is a Suslin space since the identity mapping is trivially surjective.[20]

We shall work in the framework of generalized continuum games discussed in Section 4. A <u>generalized continuum game on a Banach space</u> is defined as a generalized continuum game with the only difference being that the range of X is a Banach space. The definition of a measurable set-valued mapping is now modified to refer to the (norm) Borel σ-algebra of the Banach space rather than R^{ℓ}. The definition of an integrably bounded map remains unchanged from that of Section 4 as indeed does the definition of a Nash equilibrium. We need only specify that for any Banach space E, $L_1(\mu,E)$ now stands for the space of all (equivalence classes of) E-valued Bochner integrable functions f defined on T with $\int_T ||f(t)||d\mu < \infty$. It is well known[21] that $L_1(\mu,E)$ is a Banach space under the norm $||\cdot||_1$ where

$$||f||_1 = \int_T ||f(t)||d\mu$$

The weak topology on $L_1(\mu,E)$ figures prominently in our next result.

<u>Theorem 6.1</u>. Let $\Gamma = [(T,\mathfrak{J},\mu),X,u]$ be a generalized continuum game defined on a Banach Space E and let Γ satisfy the following assumptions.

(i) E is separable,

(ii) X is a measurable, integrably bounded map such that for all t in T, X(t) is nonempty, closed convex subset of a weakly compact set K,

(iii) u is a map such that

(a) for all $x \in L_1(\mu,X(\cdot))$, $u(\cdot,\cdot,x)$ is a measurable function on Graph X,

(b) for all t in T, for all x in $L_1(\mu,X(\cdot))$, $u(t,\cdot,x)$ is quasi-concave on $X(t)$,

(c) for all t in T, $u(t,\cdot,\cdot)$ is continuous on $X(t) \times L_1(\mu,X(\cdot))$ where the latter is endowed with the product of the relative weak topologies.

Then the generalized continuum game Γ has a Nash equilibrium.

A proof of Theorem 6.1 can be based on that of the proof of Theorem 4.1 which itself goes back to the proofs of Theorems 3.1 and 1.1. As before, we consider a mapping $\alpha : L_1(\mu,X(\cdot)) \to L_1(\mu,X(\cdot))$ where

$$\alpha(x) = \{y \in L_1(\mu,X(\cdot)) : y(t) \in \alpha_t(x) \text{ for almost all } t \text{ in } T\}$$

$$\alpha_t(x) : L_1(\mu,X(\cdot)) \to X(t) \text{ where } \alpha_t(x) = \operatorname*{Arg\,Max}_{y \in X(t)} u(t,y,x).$$

It is clear that a fixed point of α yields a Nash equilibrium and thus we need to verify that the Fan-Glicksberg theorem can be applied. Towards this end, let us consider Claim 1 whereby it is asserted that $L_1(\mu,X(\cdot))$ is weakly compact. The fact that this is indeed so is established by the following result.

Theorem 6.2 (Diestel): Let K be a weakly compact subset of a Banach space E. Then $L_1(\mu,K)$ is weakly compact in $L_1(\mu,E)$.

Proof: See Diestel (1977) or, for an alternative proof based on James' theorem, see Khan (1984).

Claim 2 asserts the nonemptiness and convexity of $L_1(\mu,X(\cdot))$. The latter is straightforward given the convex valuedness of X. Nonemptiness follows from the measurable selection theorem (Theorem 4.2 above) and the fact that X is integrably bounded. Note that in this context, one is utilizing the following simple but useful result.

Theorem 6.3: A measurable function $f : T \to E$ is Bochner integrable if and only if $\int_T ||f(t)||d\mu < \infty$.

Proof: See Diestel-Uhl (1977, p. 45).

Claim 3 asserts the nonemptiness and convexity of $\alpha(x)$ for each x in $L_1(\mu,X(\cdot))$. For this we have to show first that for each x in $L_1(\mu,X(\cdot))$ and for almost all t in T, $\alpha_t(x)$ is nonempty and convex. This is straightforward given the weak compactness of X(t), weak continuity of $u(t,\cdot,x)$ and the fact that $u(t,\cdot,x)$ is quasi-concave on X(t). However, to show that $\alpha(x)$ is nonempty, we have yet to establish that $\alpha_t(x)$ is a measurable set valued map. As in Section 4, the Castaing-Valader theorem establishes this. An appeal to the selection theorem completes the proof of the claim.

As in Section 4, Claim 4 on the upper semicontinuity of $\alpha_t(\cdot)$ on $L_1(\mu,X(\cdot))$ is an easy consequence of Berge's theorem (our Theorem 1.2).

The final claim on the upper semicontinuity of α can be established by using the following generalization of Artstein's theorem on weak sequential convergence.

Theorem 6.3 (Khan-Majumdar): Let $\{f_n\}$ be a sequence from $L_1(\mu,X(\cdot))$ such that for all t in T, and for all n, $f_n(t) \in K$, K weakly compact. If $\{f_n\}$ converges weakly to f, then almost everywhere in[22] T, $f(t) \in \overline{co}\, Ls\{f_n(t)\}$.

Proof: See proof of Theorem 1 in Khan-Majumdar (1984).

The argument showing how Theorem 6.3 can be used to prove Claim 5 is identical to the one based on Artstein's theorem and used in Section 4. A proof of Claim 5 completes the proof of Theorem 6.1.

One restrictive aspect of Theorem 6.1 is the requirement that all strategy sets must be subsets of the same weakly compact set K. This is, of course, less restrictive than the Nash-Schmeidler assumption of identical strategy sets for all traders but, as we saw in Sections 1 and 4, in an Euclidean setting this can be viewed as an expositional simplification and relaxed at no additional cost. It is not clear whether this is the case in an infinite dimensional setting. However, before we discuss this observation, it is worth pointing out that such a requirement allows our theorem to apply to some non-separable spaces.

Theorem 6.4: Theorem 6.1 is valid if

(i) is replaced by (i)' or (i)" where

(i)' E is $L_\infty(\nu)$, the space of essentially bounded measurable functions on a finite, measure space $(\Omega, \ ,\nu)$

(i)" E is the dual of a separable Banach space.

Note that if the measure space $(\Omega, \ ,\nu)$ is such that $L_1(\nu)$ is separable,[23] the above theorem need only be proved for (i)".

Theorem 6.4 can be deduced from Theorem 6.1 with the help of the following results.

Theorem 6.5 (Rosenthal): Let $(\Omega, \ ,\nu)$ be a finite measure space. Then every weakly compact subset of $L_\infty(\mu)$ is norm separable.

Proof: See, for example, Diestel-Uhl (1977, p. 252).

Theorem 6.6: Let E be a Banach space which is the dual of a separable space. Then every weakly compact subset of E is norm separable.

Proof: See, for example, Wilansky (1978, problem 9.5.113).

We now return to the question of relaxation of the assumption that each trader's strategy set is a subset of a weakly compact set. Our next result is an answer to this question.

Theorem 6.7: Theorem 6.1 is valid if (ii)' is substituted for (ii) where

(ii)' X is a measurable integrably bounded map with weakly compact convex values and such that for all $\varepsilon > 0$, there exist $T_\varepsilon \in \mathcal{J}$, $\mu(T - T_\varepsilon) < \varepsilon$; a uniformly bounded integrable subset J_ε of $L_1(\mu|_{T_\varepsilon})$, and a weakly compact subset $K_\varepsilon \subseteq E$ such that $x \in L_1(\mu, X(\cdot))$ implies $x(t) = \sum_n \lambda_n f_n(t) x_n$ for almost all t in T, with scalars λ_n such that $\sum_n |\lambda_n| \leq 1$, $f_n \in J_\varepsilon$ and $x_n \in K_\varepsilon$.

The generality of (ii)' can best be appreciated in steps. The first extension of the assumption that all the strategy sets sit in the same weakly compact subset K of E is to allow this compact set to change in a manner which is regulated by an integrable function f, i.e., $X(t) \subseteq f(t)K$. Such an assumption occurs, for example, in Castaing's work; see [Castaing-Valadier (1977, Corollary V.4)]. The next step is to allow this change to be regulated not by one integrable function but by a countably infinite number chosen from a bounded, integrable family J, i.e., $X(t) \subseteq \sum_n \lambda_n f_n(t) x_n$ where λ_n are scalars such that $\sum_n |\lambda_n| \leq 1$, $f_n \in J$ and $x_n \in K$. The final step is to allow for the fact that the above representation does not obtain for the strategy set of every trader and that there is a subset of traders T_ε with $\mu(T_\varepsilon) < \varepsilon$ whose strategy sets do not fit in this mould. Once we allow ε to take on arbitrarily small values and index J and K by ε, we obtain condition (ii)'.

Assumption (ii)' causes the proof of Theorem 6.1 to fail on two counts. First, in the context of Claim 1, we need a generalization of Diestel's theorem to establish weak compactness of $L_1(\mu,X(\cdot))$. Second, we need a corresponding generalization of Theorem 6.3 to establish Claim 5.

Our first difficulty is overcome by another result of Diestel.

Theorem 6.8 (Diestel): Let X be a mapping from T into a Banach space E which satisfies (ii) of Theorem 6.7. Then $L_1(\mu,X(\cdot))$ is weakly compact.

Proof: See Diestel (1977).

Our second difficulty can be handled by providing an alternative proof of Claim 5, a proof that does not utilize Theorem 6.3. We saw the structure of such an alternative proof in Section 4. We need the following theorem in order to apply it in our set-up.

Theorem 6.9 (Dieudonné-Tulcea-Tulcea): If $(T,\mathcal{J},\mu)$ is a complete, finite measure space, and E is a Banach space, there is an isometric isomorphism between $(L_1(\mu,E))^*$ (the space of continuous linear functions on $L_1(\mu,E)$) and $L_\infty^w(\mu,E^*)$ (the space of equivalence classes of essentially bounded, weak* measurable functions on T) in which corresponding vectors x* and g are related by the identity

$$x^*(f) = \int_T \langle g(t), f(t) \rangle \, d\mu \qquad f \in L_1(\mu,E).$$

Proof: See Tulcea-Tulcea (1962).

We can now mimic the argument in the last two paragraphs of Section 4 with the Dieudonné-Tulcea-Tulcea theorem substituted for the Banach-Steinhaus Theorem.

Our next result shows the extent of variability of the strategy sets that is allowed under a restriction on the underlying Banach space. We shall say that a Banach space E has the Radon-Nikodym property (henceforth RNP) with respect to $(T,\mathfrak{J},\mu)$ if for each μ-continuous vector measure $G : \mathfrak{J} \to E$ of bounded variation, there exists $g \in L_1(\mu,E)$ such that $G(A) = \int_A g(t)d\mu$ for all $A \in \mathfrak{J}$. E is said to have RNP if E has RNP with respect to every finite measure space.[24]

We can now state

Theorem 6.10: Theorem 6.1 is valid of (i)' and (ii)' are substituted for (i) and (ii) where

(i)' E is a separable Banach space whose dual has RNP

(ii)' X is a measurable, integrably bounded map such that for all t in T, X(t) is nonempty, convex and weakly compact.

Everything is in place for a proof of Theorem 6.10 other than a proof of Claim 1 on the weak compactness of $L_1(\mu,X(\cdot))$. This is furnished by the following result.

Theorem 6.11: Let E be a separable, Banach space such that E* has RNP. Let $X : T \to \mathcal{P}(E)$ be an integrably bounded, measurable mapping such that for almost all t in T, X(t) is a nonempty, weakly compact convex subset of E. Then, the set $L_1(\mu,X(\cdot))$ is weakly compact.

Proof: See proof of Theorem 1 and Corollary in Khan (1982).

It is by no means clear that Theorem 6.7 and 6.11 extend to $L_\infty(\mu)$ and to spaces which are duals of separable Banach spaces. Thus, it is worthwhile to have a result which pertains primarily to such spaces. It is of interest, though maybe not surprising, that we can present such a result without insisting on weak compactness of the strategy sets but only that they be norm bounded and weak* closed.[25]

Theorem 6.12: Let $\Gamma = [(T,\mathfrak{J},\mu),X,u]$ be a generalized continuum game defined on a Banach space E* and let Γ satisfy the following assumptions.

(i) E* is the dual of a separable Banach space,

(ii) X is a measurable, integrably bounded map such that for all t in T, X(t) is nonempty, convex, weak* closed subset of a norm bounded set K.

(iii) u is a map such that

(a) for all $x \in L_\infty^w(\mu,X(\cdot))$, $u(\cdot,\cdot,x)$ is a measurable function on Graph X,

(b) for all t in T, for all x in $L_\infty^w(\mu,X(\cdot))$, $u(t,\cdot,x)$ is quasi-concave on X(t),

(c) for all t in T, $u(t,\cdot,\cdot)$ is continuous on $X(t) \times L_\infty^w(\mu,X(\cdot))$ where the latter is endowed with the product of the relative weak* topologies on both components,

(iv) the measurability assumptions on X and u are interpreted with respect to the weak* Borel σ-algebra on E*.

Then the generalized continuum game Γ has a Nash equilibrium.

The basic outline of the proof of Theorem 6.12 is the same as that of the proof of Theorem 6.1 other than the fact that we now work with the weak* topology on $L^w_\infty(\mu,X(\cdot))$. Note that $L_1(\mu,E)$ is a predual of $L^w_\infty(\mu,E^*)$ is precisely the Dieudonné-Tulcea-Tulcea theorem. In view of the mathematical machinery accumulated so far, the only difficult part of the proof relates to the weak* compactness of $L^w_\infty(\mu,E^*)$ as required for Claim 1. This follows from

Theorem 6.13 (Castaing-Valadier): For all t in T, let X(t) be nonempty, convex, weak* closed subsets of a norm bounded set $K \subseteq E^*$. Then $L^w_\infty(\mu,X(\cdot))$ is a weak* compact subset of $L^w_\infty(\mu,E^*)$.

Proof: Follows from Theorem V.1 in Castaing-Valadier (1977).

Footnotes

1. It may be noted here that an year earlier Nash (1951) also furnished an existence proof based on the Brouwer fixed point theorem but for his set-up with linear pay-off functions and an identical strategy set.
2. See Arrow-Debreu (1954).
3. Consistency demands that we refer to the Nash set-up with linear pay-off functions and an identical strategy set as a game and the generalized setting studied in Debreu (1952) as a generalized game. However, we follow conventional usage.
4. A general guiding principle has been to state and reference all results not found in Hildenbrand's (1974) book.
5. R^{ℓ} denotes ℓ-dminensional Euclidean space.
6. For definition of an upper semicontinuous multivalued mapping see Berge (1966, Chapter VI). It should be noted that Hildenbrand (1974) and subsequent authors also use the term upper hemi-continuous synonymously.
7. For details, see Berge (1966, Chapter VI) and Hildenbrand (1974, Part I, BIII).
8. Graph P_t^c should really be written as (Graph $P_t)^c$ where A^c is the complement in R^{ℓ} of the set $A \subseteq R^{\ell}$.
9. See, for example, Hildenbrand (1974, page 26). Note that one also needs the qualification that the relevant mapping is compact valued.

10. $L_1(\mu,X)$ is a subset of $L_1(\mu,R^\ell)$ such that $f \varepsilon L_1(\mu,X)$ implies $f(t) \varepsilon X$ for almost all t in T.

11. The reader is invited to compare the mapping α with the corresponding one in Section 1 above.

12. For an introduction to the weak topology of a Banach space, see, for example, Diestel (1984, Chapter II).

13. $L_1(\mu)$ is intended to abbreviate $L_1(\mu,R)$.

14. See, for example, Dunford-Schwartz (1958, III.3.6 and III.6.3).

15. For sets A and B, A/B denotes set-theoretic subtraction.

16. See Berge (1966, Corollory on p. 114).

17. $L_1(\mu,X(\cdot))$ denotes the set of integrable functions f such that almost everywhere in T, $f(t) \varepsilon X(t)$. In terms of this notation, X in $L_1(\mu,X)$ refers to a constant correspondence with value X.

18. Note that a separable Banach space endowed with the weak topology is Suslin even though it is not globally metrizable.

19. Note that lim sup $(f_n(t))$ has already been defined as the set of limit points of $(f_n(t))$ in the steps prior to Theorem 3.6.

20. Footnote 18 is also relevant here.

21. For details on Bachner integration, see Diestel-Uhl (1977, Chapter II).

22. $\overline{co}$ denotes closed convex hull.

23. Conditions on the measure space which imply this can be found, for example, in Dieudonné (1970, theorem 13.11.6) and Dunford-Schwartz (1958, problem III.9.6).

24. For a detailed discussion of RNP, see Diestel-Uhl (1977). In particular, the reader is referred to pages 217-219 of this reference.

25. For an introduction to the weak* topology on Banach spaces, see for example, Diestel (1984, Chapter II).

References

K. J. Arrow and G. Debreu (1954), Existence of equilibrium for a competitive economy, Econometrica 22, 265-290.

Z. Artstein (1979), A note on Fatou's Lemma in several dimensions, Journal of Mathematical Economics 6, 277-282.

R. J. Aumann (1964), Markets with a continuum of traders, Econometrica 32, 39-50.

R. J. Aumann (1965), Integrals of set-valued functions, Journal of Mathematical Analysis and Applications 12, 1-12.

R. J. Aumann (1966), Existence of competitive equilibria in markets with a continuum of agents, Econometrica 34, 1-17.

C. Berge (1963), Topological Spaces, Oliver and Boyd, Edinburgh.

T. F. Bewley (1973), The equality of the core and the set of equilibria with infinitely many commodities and a continuum of agents, International Economic Review 14, 383-396.

F. E. Browder (1968), The fixed point theory of multi-valued mappings in topological vector spaces, Mathematische Annalen, 177, 283-301.

C. Castaing and M. Valadier (1977), Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics No. 480, Springer Verlag, New York.

A. A. Cournot (1838), Recherches sur les Principes Mathematiques de la Theorie des Richesses, librairie des sciences politiques et sociales, Paris. Also translation by Nathaniel T. Bacon (1897), Macmillan, New York.

G. Debreu (1952), A social equilibrium existence theorem, Proceedings of the National Academy of Sciences of the U.S.A. 38, 886-893.

J. Diestel (1977), Remarks on weak compactness in $L_1(\mu,X)$, Glasgow Mathematics Journal 18, 87-91.

J. Diestel (1984), Sequences and Series in Banach Spaces, Springer-Verlag, New York.

J. Diestel and J. J. Uhl, Jr. (1977), Vector Measures, Mathematical Surveys No. 15, American Mathematical Society, Rhode Island.

J. Dieudonné (1970), Treatise on Analysis, Vol. II, Academic Press, San Francisco.

N. Dincleanu (1973), Linear operations on L^p-spaces, in D. H. Tucker and H. B. Maynard (eds.), Vector and Operator Valued Measures and Applications, Academic Press, New York.

P. Dubey and L. Shapley (1977), Noncooperative exchange with a continuum of traders, Cowles Foundation Discussion Paper No. 447.

N. Dunford and J. T. Schwartz (1958), Linear Operators: Part I, John Wiley Publishing Co., New York.

Ky Fan (1952), Fixed points and minimax theorems in locally convex linear spaces, Proceedings of the National Academy of Sciences of the U.S.A. 38, 121-126.

Ky Fan (1966), Applications of a theorem concerning sets with convex sections, Mathematische Annalen 163, 189-203.

I. L. Glicksberg (1952), A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points, Proceedings of the American Mathematical Society 3, 170-174.

S. Hart, W. Hildenbrand and E. Kohlberg (1974), On equilibrium allocations as distributions on the commodity space, Journal of Mathematical Economics 1, 159-166.

W. Hildenbrand (1974), Core and Equilibria of a Large Economy, Princeton University Press, Princeton.

S. Kakutani (1941), A generalization of Brouwer's fixed point theorem, Duke Mathematics Journal 8, 457-458.

J. L. Kelley and I. Namioka (1963), Linear Topological Spaces, Springer-Verlag Graduate Text in Mathematics No. 36, Springer-Verlag, New York.

M. Ali Khan (1982a), On the integration of set-valued mappings in a non-reflexive Banach space, John Hopkins Working Paper No. 98.

M. Ali Khan (1982b), Equilibrium points of nonatomic games over a non-reflexive Banach space, John Hopkins Working Paper No. 100, forthcoming in the Journal of Approximation Theory.

M. Ali Khan (1983), Equilibrium points of nonatomic games over a Banach space, presented at the N.S.F.-N.B.E.R. Conference on Mathematical Economics in April 1983.

M. Ali Khan (1984), An alternative proof of Diestel's theorem, Glasgow Mathematics Journal 25, 45-46.

M. Ali Khan and R. Vohra (1984), Equilibrium in abstract economies without ordered preferences and with a measure space of agents, Journal of Mathematical Economics.

M. Ali Khan and M. Majumdar (1984), Weak sequential convergence in $L_1(\mu,X)$ and an approximate version of Fatou's Lemma, Cornell University preprint.

T. Ma (1969), On sets with convex sections, Journal of Mathematical Analysis and Applications 27, 413-416.

A. Mas-Colell (1974), An equilibrium existence theorem without complete or fransitive preferences, Journal of Mathematical Economics 1, 237-246.

A. Mas-Colell (1975), A model of equilibrium with differentiated commodities, Journal of Mathematical Economics 2, 263-295.

A. Mas-Colell (1983), On a theorem of Schmeidler, Harvard University preprint.

J. F. Nash (1950), Equilibrium points in N-person games, Proceedings of the National Academy of Sciences of the U.S.A. 36, 48-49.

J. F. Nash (1951), Non-cooperative games, Annals of Mathematics 54, 286-295.

H. P. Rosenthal (1970), On injective Banach spaces and the spaces $L_\infty(\mu)$ for finite measures μ, Acta Mathematika 124, 205-248.

M. F. Sainte-Beuve (1974), On the extension of von-Neumann-Aumann's Theorem, Journal of Functional Analysis 17, 112-129.

D. Schmeidler (1973), Equilibrium points of nonatomic games, Journal of Statistical Physics 7, 295-300.

L. Schwartz (1973), Random Measures on Arbitrary Topological Spaces and Cylindrical Measures, Oxford University Press, Bombay.

W. Shafer (1974), The nontransitive consumer, Econometrica 42, 355-381.

W. Shafer and H. Sonnenschein (1975), Equilibrium in abstract economies without ordered preferences, Journal of Mathematical Economics 2, 345-348.

S. Toussaint (1982), On the existence of equilibrium with infinitely many commodities, Mannheim Working Paper. Also forthcoming in Journal of Economic Theory.

A. Ionescu Tulcea and C. Ionescu Tulcea (1962), On the lifting property II, Journal of Mathematics and Mechanics 11, 773-795.

A. Wilansky (1978), Modern Methods in Topological Vector Spaces, McGraw Hill Publishing Co., New York.

N. Yannelis and N. Prabhaker (1983a), Existence of maximal elements and equilibria in linear topological spaces, Journal of Mathematical Economics 12, 233-245.

N. Yannelis and N. Prabhaker (1983b), Equilibrium in abstract economies with an infinite number of agents, an infinite number of commodities and without ordered preferences, Wayne State University preprint.

D/146A

T. Ma (1969), On sets with convex sections, Journal of Mathematical Analysis and Applications 27, 413-416.

A. Mas-Colell (1974), An equilibrium existence theorem without complete or fransitive preferences, Journal of Mathematical Economics 1, 237-246.

A. Mas-Colell (1975), A model of equilibrium with differentiated commodities, Journal of Mathematical Economics 2, 263-295.

A. Mas-Colell (1983), On a theorem of Schmeidler, Harvard University preprint.

J. F. Nash (1950), Equilibrium points in N-person games, Proceedings of the National Academy of Sciences of the U.S.A. 36, 48-49.

J. F. Nash (1951), Non-cooperative games, Annals of Mathematics 54, 286-295.

H. P. Rosenthal (1970), On injective Banach spaces and the spaces $L_\infty(\mu)$ for finite measures μ, Acta Mathematika 124, 205-248.

M. F. Sainte-Beuve (1974), On the extension of von-Neumann-Aumann's Theorem, Journal of Functional Analysis 17, 112-129.

D. Schmeidler (1973), Equilibrium points of nonatomic games, Journal of Statistical Physics 7, 295-300.

L. Schwartz (1973), Random Measures on Arbitrary Topological Spaces and Cylindrical Measures, Oxford University Press, Bombay.

W. Shafer (1974), The nontransitive consumer, Econometrica 42, 355-381.

W. Shafer and H. Sonnenschein (1975), Equilibrium in abstract economies without ordered preferences, Journal of Mathematical Economics 2, 345-348.

S. Toussaint (1982), On the existence of equilibrium with infinitely many commodities, Mannheim Working Paper. Also forthcoming in Journal of Economic Theory.

A. Ionescu Tulcea and C. Ionescu Tulcea (1962), On the lifting property II, Journal of Mathematics and Mechanics 11, 773-795.

A. Wilansky (1978), Modern Methods in Topological Vector Spaces, McGraw Hill Publishing Co., New York.

N. Yannelis and N. Prabhaker (1983a), Existence of maximal elements and equilibria in linear topological spaces, Journal of Mathematical Economics 12, 233-245.

N. Yannelis and N. Prabhaker (1983b), Equilibrium in abstract economies with an infinite number of agents, an infinite number of commodities and without ordered preferences, Wayne State University preprint.

D/146A

T. Ma (1969), On sets with convex sections, Journal of Mathematical Analysis and Applications 27, 413-416.

A. Mas-Colell (1974), An equilibrium existence theorem without complete or fransitive preferences, Journal of Mathematical Economics 1, 237-246.

A. Mas-Colell (1975), A model of equilibrium with differentiated commodities, Journal of Mathematical Economics 2, 263-295.

A. Mas-Colell (1983), On a theorem of Schmeidler, Harvard University preprint.

J. F. Nash (1950), Equilibrium points in N-person games, Proceedings of the National Academy of Sciences of the U.S.A. 36, 48-49.

J. F. Nash (1951), Non-cooperative games, Annals of Mathematics 54, 286-295.

H. P. Rosenthal (1970), On injective Banach spaces and the spaces $L_\infty(\mu)$ for finite measures μ, Acta Mathematika 124, 205-248.

M. F. Sainte-Beuve (1974), On the extension of von-Neumann-Aumann's Theorem, Journal of Functional Analysis 17, 112-129.

D. Schmeidler (1973), Equilibrium points of nonatomic games, Journal of Statistical Physics 7, 295-300.

L. Schwartz (1973), Random Measures on Arbitrary Topological Spaces and Cylindrical Measures, Oxford University Press, Bombay.

W. Shafer (1974), The nontransitive consumer, Econometrica 42, 355-381.

W. Shafer and H. Sonnenschein (1975), Equilibrium in abstract economies without ordered preferences, Journal of Mathematical Economics 2, 345-348.

S. Toussaint (1982), On the existence of equilibrium with infinitely many commodities, Mannheim Working Paper. Also forthcoming in Journal of Economic Theory.

A. Ionescu Tulcea and C. Ionescu Tulcea (1962), On the lifting property II, Journal of Mathematics and Mechanics 11, 773-795.

A. Wilansky (1978), Modern Methods in Topological Vector Spaces, McGraw Hill Publishing Co., New York.

N. Yannelis and N. Prabhaker (1983a), Existence of maximal elements and equilibria in linear topological spaces, Journal of Mathematical Economics 12, 233-245.

N. Yannelis and N. Prabhaker (1983b), Equilibrium in abstract economies with an infinite number of agents, an infinite number of commodities and without ordered preferences, Wayne State University preprint.

D/146A