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Tests for Normality with Stable Alternatives

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TESTS FOR NORMALITY WITH STABLE ALTERNATIVES

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April, 1985

ABSTRACT: This paper is concerned with testing normality of observations and regression disturbances when the alternative belongs to the stable family.

KEYWORDS: Lagrange multiplier test, simulated critical value, simulation study, stable family, test for normality.

1. INTRODUCTION

Numerous tests are available for testing the normality of observations (see Mardia, 1980), and many of those tests can be modified to test the normality of regression disturbances by using regression residuals (see Pierce and Kopecky, 1979). Most of these tests have been constructed by exploiting some unique properties of the normal distribution that are not shared by other distributions. It is not known whether they possess any "optimal" properties. Recently, Bera and Jarque (1981), Bera (1982), Bera and John (1983), and Poirier et al. (1984) have suggested tests employing the Lagrange multiplier (LM) or Rao's (1948) score principle when the alternatives belong to certain families of distributions which reduce to the normal distribution under certain parametric restrictions. Since these tests are based on the LM principle they can be expected to possess good power properties. Simulation results reported in Bera and Jarque (1981) reveal that the test based on the Pearson alternatives, henceforth denoted by LM_N , performs as well as the available tests in finite samples. Statistic LM_N is very easy to compute (some of the tests which have been found to have good power properties are not very easy to apply, e.g., for the Shapiro and Wilk (1965) test we require certain coefficients which are not available for all sample sizes).

One drawback of the Pearson family is that it does not permit the distributions to have infinite variance. We try to tackle this problem by considering the stable family of distributions. Various investigations (see for instance, Mandelbrot, 1963 and Carlson, 1975) have shown that certain economic variables such as stock price changes and price

expectations can be better described by stable distributions. Another notable aspect of this family is that the distribution of the regression disturbance (which is assumed to be the sum of a large number of independent and identically distributed random variables whose individual effects are very small) may belong to this family as argued by Bartels (1977), Koenker and Bassett (1978) and others. This follows from the generalized central limit theorem which states that for a distribution function to be a limit distribution of a sum of independent and identically distributed random variables, it is necessary and sufficient that it be stable (see Gnedenko and Kolmogorov, 1954, p. 162). In Section 2, we attempt to derive a LM test for normality by considering the stable family as the alternative. It seems that no computationally simple test can be developed by following this approach. Therefore, in Section 3, we investigate whether the LM_N and other available tests can detect non-normality when the alternative belongs to the stable family. This is done through a simulation study which closely follows the design described in Bera and Jarque (1981). We also provide a table from which critical points of LM_N for different sample sizes and significance levels can easily be calculated. Simulations in the paper were carried out on the Australian National University's UNIVAC 1100 Computer (Tables I, II and III), and Cyber 175 (Table IV) at the University of Illinois. The paper is concluded in Section 4 with some remarks.

2. LM TEST WITH STABLE ALTERNATIVES

Let us consider having a set of n independent observations u_1, u_2, \dots, u_n , on a random variable u . Assume the probability density

function (PDF) of u is a member of the stable family. Therefore, we can write (see Gnedenko and Kolmogorov, 1954, p. 164)

$$f(u_j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(itu_j) \cdot \exp[i\mu t - \gamma |t|^\alpha \{1 + \frac{i\delta t}{|t|} \omega(t, \alpha)\}] dt \quad (1)$$

$$-\infty < u_j < \infty$$

$$j = 1, 2, \dots, n$$

where $i = \sqrt{-1}$, t is any real number and

$$\omega(t, \alpha) = \tan\left(\frac{\pi\alpha}{2}\right), \quad \text{if } \alpha \neq 1$$

$$= \frac{2}{\pi} \ln |t|, \quad \text{if } \alpha = 1.$$

There are four parameters: the location parameter $\mu \in (-\infty, \infty)$, the scale parameter $\gamma \in (0, \infty)$, the skewness parameter $\delta \in [-1, 1]$ and the characteristic exponent or the index parameter $\alpha \in (0, 2]$. The stable family includes some important distributions as special cases, for example, the Cauchy distribution when $\alpha = 1$ and $\delta = 0$; the normal distribution with mean μ and variance σ^2 when $\alpha = 2$, $\delta = 0$ and $\gamma = \sigma^2/2$. The distribution has finite variance only when $\alpha = 2$, otherwise the variance is infinite.

Since we are interested only in the shape of the distribution, without loss of generality we can assume $\mu = 0$ and $\gamma = \frac{1}{2}$. Suppose we also assume that the population mean exists, i.e., $\alpha > 1$ then we can write (1) as

$$f(u_j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(itu_j) \cdot \exp\left[-\frac{1}{2} |t|^\alpha \left\{1 + \frac{i\delta t}{|t|} \tan\left(\frac{\pi\alpha}{2}\right)\right\}\right] dt. \quad (2)$$

A test for normality can be achieved by testing $H_0: \alpha = 2, \delta = 0$ in (2). However, there are a few potential difficulties in doing so.

Under H_0 , the value of α lies on the boundary of the parameter space, and in addition, at $\alpha = 2$, the parameter δ disappears from (2). For simplicity let us consider the symmetric stable distribution by assuming $\delta = 0$. Then $f(u_j)$ reduces to

$$f(u_j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(u_j t) \exp\left[-\frac{|t|^\alpha}{2}\right] dt.$$

We can obtain a test for normality by testing $H_0: \alpha = 2$ vs $\alpha < 2$, based on the maximum likelihood estimate (MLE) of α , say $\hat{\alpha}$. This is suggested in DuMouchel (1971) and performance of this test has been investigated by Saniga et al. (1975). However, in a recent paper, DuMouchel (1981) points out two drawbacks of this procedure. First, for $\alpha < 2$, $\hat{\alpha}$ is asymptotically normal with mean α and variance equal to the inverse of the Fisher's information; but this asymptotic theory fails when $H_0: \alpha = 2$ is true. Second, the test based on $\hat{\alpha}$ is not robust to the assumption of stability, e.g., the test may diagnose the exponential distribution as having infinite variance. (Of course, most of the tests will have this defect.) Moreover, the test suffers from computational complexity.

Let us now explore the possibility of constructing a test using the Rao's score test principle. The test may be based on the score value $\partial \ell(\alpha)/\partial \alpha$ evaluated at $\alpha = 2$ where $\ell(\alpha) = \sum_{j=1}^n \ln(f(u_j))$. More specifically, we examine whether the following quantity

$$\frac{\partial \tilde{\ell}(\alpha)}{\partial \alpha} = \frac{\partial \ell(\alpha)}{\partial \alpha} \bigg|_{\alpha=2} = \sum_{j=1}^n \frac{\tilde{f}'(u_j)}{\tilde{f}(u_j)}$$

where

$$\tilde{f}'(u_j) = \left. \frac{\partial f(u_j)}{\partial \alpha} \right|_{\alpha=2} = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \cos(u_j t) \exp\left(-\frac{t^2}{2}\right) t^2 \ln|t| dt$$

and

$$\tilde{f}(u_j) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u_j^2}{2}\right],$$

is significantly different from zero using a one-sided normal test.

By numerical integration, $\partial \tilde{\ell}(\alpha)/\partial \alpha$ and an estimate of its variance (under H_0)

$$-\frac{\partial^2 \tilde{\ell}(\alpha)}{\partial \alpha^2} = -\left. \frac{\partial^2 \ell(\alpha)}{\partial \alpha^2} \right|_{\alpha=2} = -\frac{n}{\sum_{j=1}^n} \frac{\tilde{f}''(u_j) \tilde{f}(u_j) - [\tilde{f}'(u_j)]^2}{[\tilde{f}(u_j)]^2}$$

where

$$\tilde{f}''(u_j) = \left. \frac{\partial^2 f(u_j)}{\partial \alpha^2} \right|_{\alpha=2} = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \cos(u_j t) \exp\left(-\frac{t^2}{2}\right) (\ln|t|)^2 t^2 \left(-\frac{t^2}{2} + 1\right) dt,$$

can be calculated. However, for practical purposes the calculations seem to be quite tedious. One reason for all these difficulties is that no simple expression is available for the density function of the stable distribution.

It might be easier to construct a test using its characteristic function (CF) which has a simpler expression. For example, Epps and Pulley (1982) suggested a test based on a weighted integral of $\phi_n(t) - \phi_0(t)$ where $\phi_n(t)$ and $\phi_0(t)$ are respectively the empirical CF and CF of a normal distribution. To obtain high power against the stable alternatives we can choose the value of t to maximize the power instead of taking a weighted integral over t . But this procedure also involves considerable amount of computation.

3. SIMULATION STUDY

Given the above complexities of the stable distribution, we follow an ad hoc approach. In particular, we examine the performances of the conventional tests against the stable alternatives. Many of these tests are based on the second, third and fourth sample moments, but for the stable distribution the population counterparts of these moments do not exist. Therefore, we cannot be sure about the distribution of these statistics when the stable alternative is true. However, if we find that these tests can detect non-normality when the alternatives come from stable family, for practical purposes we can use them irrespective of their analytical properties. This is what we set out to investigate in this section.

Many simulation studies have been done on assessing the performances of the conventional tests for normality against stable alternatives. For instance, Fama and Roll (1971), Saniga et al. (1975), Smith (1975) and Saniga and Hayya (1977) have examined the powers of various tests for the normality of observations against the symmetric stable distributions, and recently Saniga and Miles (1979) have extended those studies to the asymmetric stable family. To our knowledge there is no similar systematic investigation on testing for the normality of regression disturbances. Therefore, for our simulation experiment, we consider both the symmetric and asymmetric stable family and study the powers of some of the available tests for the normality of both observations and regression disturbances.

We consider the following tests: $\sqrt{b_1}$, b_2 , D'Agostino (1971) D^* , Pearson et al. (1977) R , Shapiro and Wilk (1965) W , Shapiro and Francia

(1972) W' and Bera and Jarque (1981) LM_N . White and MacDonald (1980) describe the first six tests and LM_N takes the following form

$$LM_N = n \left[\frac{(\sqrt{b_1})^2}{6} + \frac{(b_2 - 3)^2}{24} \right]. \quad (3)$$

A short derivation of this statistic is given in the Appendix; for details see Bera and Jarque (1981). Under normality (H_0), LM_N is asymptotically distributed as a χ^2_2 . Distributions under the alternative hypothesis are generated from the stable family using the density function given in (2) with $\alpha = 1.0(.3)1.9$ and $\delta = 0.0(.25)1.0$. To do this we make use of the computer program described in Chambers et al. (1976). Observations under normality ($\alpha = 2.0$, $\delta = 0.0$) are generated using the IBM subroutines RANDO and RANDU.

To calculate powers for $n = 20, 35, 50$ and 100 and tests $\sqrt{b_1}$, b_2 , D^* and R , we use the critical points given in White and MacDonald (1980, p. 20). For $n = 200$, significance points for $\sqrt{b_1}$, b_2 and D^* are obtained respectively from Pearson and Hartley (1962, p. 183), D'Agostino and Pearson (1973, p. 615) and D'Agostino (1971, p. 343); and for the R test we extrapolated the points. For W , W' and LM_N , we compute the significance points by simulation using 500 replications, in order to obtain an empirical significance level equal to .10. For example, for a given n , the significance point for W is $W(50)$ - the 50th largest of the values of W in the 500 replications under normal observations and similarly for W' . For LM_N , the point is set as $LM_N(450)$. With the empirical significance level set at .10, easier power comparisons among the one-sided tests W , W' and LM_N can be made. Since $\sqrt{b_1}$, b_2 and D^* are two-sided tests and R is a four-sided test, it is difficult to adjust the significance points to achieve an empirical significance level equal to .10.

Estimated powers of each test (obtained by dividing the number of times H_0 is rejected by 500) for testing for the normality of observations are reported in Table I for $n = 20, 35, 50, 100$ and 200 , except for W which cannot be computed for $n > 50$. The maximum standard error of each entry in the table is $\sqrt{.5(1-.5)/500} \approx .022$. In the table, the highest power is underlined (for each alternative and sample size) except when three or more tests have this power. To save space, results corresponding to only nine combinations of α and δ namely, $\alpha = 1.0, 1.6$ and 1.9 and $\delta = 0.0, 0.5$ and 1.0 are given. They are representative of the overall results.

For each n , the first row gives the estimated sizes of the tests: $\sqrt{b_1}$, b_2 , D^* and R ; they are, in most cases, around .10. Therefore, power comparison of these tests with W , W' , and LM_N is valid. From the table we note that for a given value of α the powers are not sensitive to changes in the value of the skewness parameter δ . However, when δ is kept fixed and α is varied, powers change considerably. The overall behavior of the first five tests, $\sqrt{b_1}$, b_2 , D^* , R and W is very similar to those reported in Saniga and Miles (1979). Two additional tests we consider are W' and LM_N , and it is seen that they outperform the remaining ones. In particular, LM_N does very well and it has the highest power in 33 cases out of 45 cases considered; when it does not, its power is within less than 2 percent of the maximum power. It has very good power for smaller sample sizes and also when α is near to 2, i.e., when the alternative is close to the normal distribution.

We now study the power of the tests for the normality of unobserved regression disturbances. The tests we consider are the same as discussed

TABLE I

Estimated powers of the tests against stable alternatives

| | α | δ | $\sqrt{b_1}$ | b_2 | D^* | R | W | W' | LM_N |
|---------|----------|----------|--------------|-------|-------|-------|-------|-------|--------|
| n = 20 | 2.0 | 0.0 | .076 | .062 | .096 | .080 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .804 | .872 | .902 | .872 | .896 | .912 | .898 |
| | 1.0 | 0.5 | .832 | .842 | .890 | .886 | .902 | .922 | .910 |
| | 1.0 | 1.0 | .956 | .806 | .894 | .938 | .968 | .966 | .968 |
| | 1.6 | 0.0 | .486 | .462 | .466 | .482 | .472 | .518 | .538 |
| | 1.6 | 0.5 | .504 | .466 | .476 | .500 | .496 | .536 | .542 |
| | 1.6 | 1.0 | .558 | .422 | .436 | .518 | .536 | .550 | .578 |
| | 1.9 | 0.0 | .234 | .196 | .192 | .232 | .204 | .222 | .244 |
| | 1.9 | 0.5 | .220 | .190 | .204 | .216 | .206 | .222 | .224 |
| | 1.9 | 1.0 | .212 | .172 | .188 | .190 | .204 | .222 | .246 |
| n = 35 | 2.0 | 0.0 | .082 | .100 | .090 | .086 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .880 | .982 | .996 | .982 | .982 | .994 | .986 |
| | 1.0 | 0.5 | .908 | .972 | .990 | .970 | .974 | .994 | .982 |
| | 1.0 | 1.0 | .992 | .960 | .982 | .994 | .994 | .992 | .992 |
| | 1.6 | 0.0 | .618 | .692 | .698 | .684 | .636 | .720 | .732 |
| | 1.6 | 0.5 | .672 | .672 | .688 | .690 | .634 | .740 | .730 |
| | 1.6 | 1.0 | .810 | .660 | .682 | .786 | .776 | .788 | .812 |
| | 1.9 | 0.0 | .298 | .268 | .266 | .290 | .230 | .310 | .312 |
| | 1.9 | 0.5 | .284 | .272 | .266 | .290 | .250 | .314 | .308 |
| | 1.9 | 1.0 | .286 | .256 | .260 | .292 | .292 | .306 | .306 |
| n = 50 | 2.0 | 0.0 | .074 | .094 | .074 | .086 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .930 | .988 | .992 | .990 | .986 | .994 | .992 |
| | 1.0 | 0.5 | .938 | .988 | .994 | .994 | .992 | .996 | .996 |
| | 1.0 | 1.0 | 1.000 | .986 | .998 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1.6 | 0.0 | .712 | .806 | .808 | .802 | .692 | .828 | .850 |
| | 1.6 | 0.5 | .740 | .798 | .804 | .810 | .714 | .830 | .836 |
| | 1.6 | 1.0 | .856 | .716 | .742 | .834 | .818 | .840 | .860 |
| | 1.9 | 0.0 | .304 | .316 | .306 | .322 | .224 | .346 | .354 |
| | 1.9 | 0.5 | .330 | .328 | .304 | .328 | .256 | .354 | .374 |
| | 1.9 | 1.0 | .386 | .322 | .316 | .368 | .308 | .372 | .406 |
| n = 100 | 2.0 | 0.0 | .068 | .108 | .114 | .112 | | .100 | .100 |
| | 1.0 | 0.0 | .946 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 0.5 | .980 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 1.0 | 1.000 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.6 | 0.0 | .828 | .954 | .950 | .938 | | .952 | .970 |
| | 1.6 | 0.5 | .870 | .948 | .952 | .950 | | .968 | .966 |
| | 1.6 | 1.0 | .992 | .938 | .952 | .990 | | .992 | .992 |
| | 1.9 | 0.0 | .416 | .462 | .452 | .470 | | .498 | .514 |
| | 1.9 | 0.5 | .420 | .460 | .438 | .468 | | .510 | .500 |
| | 1.9 | 1.0 | .528 | .444 | .426 | .498 | | .520 | .556 |
| n = 200 | 2.0 | 0.0 | .102 | .106 | .108 | .110 | | .100 | .100 |
| | 1.0 | 0.0 | .984 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 0.5 | .980 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 1.0 | 1.000 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.6 | 0.0 | .892 | .998 | .998 | .996 | | .996 | .996 |
| | 1.6 | 0.5 | .932 | .996 | .998 | .994 | | .998 | .998 |
| | 1.6 | 1.0 | 1.000 | .994 | .996 | 1.000 | | 1.000 | 1.000 |
| | 1.9 | 0.0 | .522 | .652 | .632 | .668 | | .696 | .696 |
| | 1.9 | 0.5 | .620 | .664 | .624 | .672 | | .714 | .710 |
| | 1.9 | 1.0 | .786 | .694 | .682 | .774 | | .774 | .800 |

above but we compute them using the ordinary least squares (OLS) residuals rather than the true disturbances u_j . The result of Pierce and Kopecky (1979) indicates that this modification is valid asymptotically. We denote the modified statistics by $\sqrt{\tilde{b}_1}$, \tilde{b}_2 etc. and consider the linear regression model

$$y_j = \sum_{i=1}^4 \beta_i x_{ji} + u_j \quad j = 1, 2, \dots, n.$$

Simulation results studying the relative powers of tests for normality of $u = (u_1, u_2, \dots, u_n)'$, computed using the OLS residuals $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n)'$, depend on the design matrix X since $\tilde{u} = Mu$ where M is an $n \times n$ matrix defined by $M = I - X(X'X)^{-1}X'$. Therefore, we consider two sets of regressors. The first set is defined as in White and MacDonald (1980, p. 20), i.e., we set $x_{j1} = 1$ ($j = 1, 2, \dots, n$), and generate X_2 , X_3 and X_4 from a uniform (U) distribution. The last three regressions are transformed to have mean zero and variance 25. The specific values of the means and variances of these regressors have no effect on the simulation results. For the second set, we set $x_{j1} = 1$ ($j = 1, 2, \dots, n$), and generate X_2 from a $N(10, 25)$, X_3 from a $U[7.5, 12.5]$ and X_4 from a χ^2_{10} .

The results corresponding to these two data sets are given in Tables II and III. To calculate the powers we utilize the same significance points used in the calculation of powers in Table I, except for \tilde{W} , \tilde{W}' and \tilde{LM}_N , for which we use empirical 10 percent critical points. For instance, the significance point for \tilde{W} is $\tilde{W}(50)$.

The first point to note from these tables is that in almost all cases the modified tests give lower powers than the corresponding tests using

TABLE II

Estimated powers of the tests against stable alternatives
using the OLS residuals

(Regressors: $X_1 = 1, X_2, X_3, X_4 \sim U$)

| | α | δ | $\sqrt{b_1}$ | \tilde{b}_2 | \tilde{D}^* | \tilde{R} | \tilde{W} | \tilde{W}' | \tilde{LM}_N |
|---------|----------|----------|--------------|---------------|---------------|-------------|-------------|--------------|----------------|
| n = 20 | 2.0 | 0.0 | .080 | .090 | .112 | .092 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .734 | .778 | .792 | .794 | .782 | .822 | .816 |
| | 1.0 | 0.5 | .760 | .754 | .772 | .788 | .768 | .814 | .830 |
| | 1.0 | 1.0 | .876 | .756 | .786 | .860 | .866 | .874 | .884 |
| | 1.6 | 0.0 | .402 | .348 | .362 | .368 | .370 | .414 | .434 |
| | 1.6 | 0.5 | .408 | .366 | .352 | .398 | .396 | .446 | .448 |
| | 1.6 | 1.0 | .486 | .366 | .384 | .450 | .456 | .474 | .508 |
| | 1.9 | 0.0 | .198 | .186 | .190 | .184 | .188 | .208 | .200 |
| | 1.9 | 0.5 | .200 | .188 | .184 | .184 | .180 | .192 | .208 |
| | 1.9 | 1.0 | .174 | .192 | .176 | .192 | .184 | .172 | .194 |
| n = 35 | 2.0 | 0.0 | .096 | .098 | .102 | .082 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .854 | .966 | .976 | .964 | .948 | .974 | .968 |
| | 1.0 | 0.5 | .904 | .966 | .976 | .970 | .960 | .982 | .980 |
| | 1.0 | 1.0 | .988 | .940 | .964 | .984 | .992 | .992 | .988 |
| | 1.6 | 0.0 | .612 | .632 | .646 | .630 | .582 | .672 | .696 |
| | 1.6 | 0.5 | .634 | .614 | .638 | .654 | .600 | .684 | .686 |
| | 1.6 | 1.0 | .758 | .594 | .614 | .722 | .696 | .742 | .754 |
| | 1.9 | 0.0 | .258 | .236 | .248 | .252 | .238 | .276 | .282 |
| | 1.9 | 0.5 | .266 | .258 | .244 | .270 | .248 | .286 | .296 |
| | 1.9 | 1.0 | .268 | .226 | .230 | .266 | .258 | .280 | .276 |
| n = 50 | 2.0 | 0.0 | .086 | .090 | .074 | .066 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .906 | .982 | .990 | .982 | .978 | .984 | .984 |
| | 1.0 | 0.5 | .932 | .986 | .992 | .994 | .988 | .996 | .994 |
| | 1.0 | 1.0 | 1.000 | .980 | .992 | .998 | .998 | .998 | .998 |
| | 1.6 | 0.0 | .688 | .772 | .786 | .780 | .682 | .794 | .808 |
| | 1.6 | 0.5 | .744 | .766 | .782 | .802 | .708 | .800 | .814 |
| | 1.6 | 1.0 | .842 | .696 | .708 | .810 | .774 | .824 | .838 |
| | 1.9 | 0.0 | .290 | .310 | .294 | .302 | .232 | .308 | .324 |
| | 1.9 | 0.5 | .322 | .330 | .314 | .326 | .236 | .328 | .352 |
| | 1.9 | 1.0 | .348 | .312 | .290 | .338 | .302 | .366 | .388 |
| n = 100 | 2.0 | 0.0 | .066 | .098 | .098 | .090 | | .100 | .100 |
| | 1.0 | 0.0 | .936 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 0.5 | .978 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 1.0 | 1.000 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.6 | 0.0 | .818 | .948 | .940 | .934 | | .946 | .960 |
| | 1.6 | 0.5 | .866 | .936 | .942 | .940 | | .958 | .958 |
| | 1.6 | 1.0 | .986 | .926 | .934 | .986 | | .986 | .986 |
| | 1.9 | 0.0 | .410 | .452 | .430 | .458 | | .484 | .496 |
| | 1.9 | 0.5 | .432 | .444 | .432 | .470 | | .486 | .500 |
| | 1.9 | 1.0 | .520 | .436 | .418 | .496 | | .528 | .548 |
| n = 200 | 2.0 | 0.0 | .092 | .122 | .114 | .100 | | .100 | .100 |
| | 1.0 | 0.0 | .980 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 0.5 | .982 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 1.0 | 1.000 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.6 | 0.0 | .888 | .998 | .998 | .998 | | .996 | .996 |
| | 1.6 | 0.5 | .924 | .996 | .996 | .994 | | .996 | .996 |
| | 1.6 | 1.0 | 1.000 | .994 | .994 | 1.000 | | 1.000 | 1.000 |
| | 1.9 | 0.0 | .512 | .652 | .638 | .660 | | .674 | .678 |
| | 1.9 | 0.5 | .620 | .650 | .630 | .664 | | .694 | .694 |
| | 1.9 | 1.0 | .772 | .686 | .682 | .770 | | .778 | .800 |

TABLE III

Estimated powers of the tests against stable alternatives
using the OLS residuals

(Regressors: $X_1 = 1$, $X_2 \sim N$, $X_3 \sim U$, $X_4 \sim X_{10}^2$)

| | α | δ | \sqrt{b}_1 | \tilde{b}_2 | \tilde{D}^* | \tilde{R} | \tilde{W} | \tilde{W}' | \tilde{LM}_N |
|---------|----------|----------|--------------|---------------|---------------|-------------|-------------|--------------|----------------|
| n = 20 | 2.0 | 0.0 | .080 | .104 | .106 | .088 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .726 | .766 | .780 | .782 | .764 | .802 | .822 |
| | 1.0 | 0.5 | .766 | .740 | .778 | .790 | .770 | .812 | .822 |
| | 1.0 | 1.0 | .864 | .700 | .746 | .826 | .854 | .862 | .868 |
| | 1.6 | 0.0 | .396 | .404 | .418 | .416 | .390 | .444 | .452 |
| | 1.6 | 0.5 | .432 | .396 | .404 | .426 | .414 | .446 | .484 |
| | 1.6 | 1.0 | .450 | .338 | .370 | .414 | .442 | .456 | .480 |
| | 1.9 | 0.0 | .202 | .162 | .170 | .184 | .176 | .170 | .206 |
| | 1.9 | 0.5 | .202 | .166 | .158 | .186 | .178 | .182 | .206 |
| | 1.9 | 1.0 | .196 | .176 | .176 | .194 | .226 | .242 | .234 |
| n = 35 | 2.0 | 0.0 | .068 | .092 | .084 | .088 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .870 | .958 | .974 | .960 | .952 | .972 | .964 |
| | 1.0 | 0.5 | .900 | .960 | .976 | .966 | .966 | .982 | .970 |
| | 1.0 | 1.0 | .988 | .932 | .966 | .984 | .986 | .990 | .986 |
| | 1.6 | 0.0 | .602 | .648 | .646 | .658 | .584 | .676 | .694 |
| | 1.6 | 0.5 | .646 | .622 | .648 | .650 | .626 | .680 | .694 |
| | 1.6 | 1.0 | .776 | .602 | .630 | .734 | .720 | .750 | .776 |
| | 1.9 | 0.0 | .296 | .260 | .256 | .262 | .258 | .310 | .306 |
| | 1.9 | 0.5 | .294 | .268 | .258 | .290 | .266 | .310 | .316 |
| | 1.9 | 1.0 | .284 | .236 | .244 | .264 | .290 | .304 | .312 |
| n = 50 | 2.0 | 0.0 | .062 | .074 | .080 | .066 | .100 | .100 | .100 |
| | 1.0 | 0.0 | .910 | .982 | .990 | .982 | .976 | .986 | .986 |
| | 1.0 | 0.5 | .930 | .988 | .992 | .992 | .990 | .996 | .996 |
| | 1.0 | 1.0 | 1.000 | .978 | .996 | .998 | .998 | .998 | 1.000 |
| | 1.6 | 0.0 | .684 | .758 | .780 | .766 | .678 | .798 | .810 |
| | 1.6 | 0.5 | .714 | .760 | .770 | .780 | .688 | .796 | .810 |
| | 1.6 | 1.0 | .842 | .686 | .722 | .804 | .792 | .840 | .848 |
| | 1.9 | 0.0 | .280 | .280 | .270 | .294 | .230 | .332 | .326 |
| | 1.9 | 0.5 | .332 | .300 | .298 | .318 | .244 | .338 | .366 |
| | 1.9 | 1.0 | .340 | .294 | .294 | .330 | .294 | .368 | .370 |
| n = 100 | 2.0 | 0.0 | .070 | .096 | .090 | .078 | | .100 | .100 |
| | 1.0 | 0.0 | .940 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 0.5 | .976 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 1.0 | 1.000 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.6 | 0.0 | .810 | .948 | .944 | .928 | | .948 | .962 |
| | 1.6 | 0.5 | .870 | .942 | .944 | .948 | | .966 | .964 |
| | 1.6 | 1.0 | .986 | .936 | .938 | .986 | | .986 | .988 |
| | 1.9 | 0.0 | .416 | .464 | .436 | .466 | | .478 | .510 |
| | 1.9 | 0.5 | .428 | .458 | .432 | .472 | | .498 | .512 |
| | 1.9 | 1.0 | .516 | .438 | .408 | .498 | | .514 | .552 |
| n = 200 | 2.0 | 0.0 | .068 | .112 | .130 | .124 | | .100 | .100 |
| | 1.0 | 0.0 | .986 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 0.5 | .976 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.0 | 1.0 | 1.000 | 1.000 | 1.000 | 1.000 | | 1.000 | 1.000 |
| | 1.6 | 0.0 | .882 | .998 | .998 | .998 | | .998 | .998 |
| | 1.6 | 0.5 | .926 | .996 | .996 | .994 | | .998 | .998 |
| | 1.6 | 1.0 | 1.000 | .996 | .996 | 1.000 | | 1.000 | 1.000 |
| | 1.9 | 0.0 | .520 | .648 | .632 | .662 | | .694 | .688 |
| | 1.9 | 0.5 | .622 | .664 | .638 | .670 | | .708 | .710 |
| | 1.9 | 1.0 | .774 | .678 | .666 | .766 | | .774 | .790 |

the original disturbances and that these power differences diminish as n increases. This is easily seen by comparing the powers in Table I with those in Tables II and III. This illustrates the observed tendency of the OLS residuals to be "more normal" than the true disturbances. Gnanadesikan (1977, p. 265) calls this tendency the "supernormality" of the residuals. Bassett and Koenker (1982) provide a simple explanation of this behavior. Each residual can be decomposed into true disturbance and a term, which behaves asymptotically as normal irrespective of the true nature of the disturbance, pushing the distribution of the OLS residuals towards normality. We did not make any attempt to use Theil's (1965) best linear unbiased scalar (BLUS) residuals since the Monte Carlo results of Huang and Bolch (1974) show the superiority of OLS over BLUS residuals for testing the normality of the regression disturbances. We also find that the ranking of the tests in these tables is approximately the same as in Table I. \tilde{LM}_N has the highest power in around 75 percent of the cases. After \tilde{LM}_N , the next best test is \tilde{W}' .

Earlier we noted that LM_N is asymptotically distributed as a χ^2_2 . However, for small samples, the actual critical values can be very different from the χ^2_2 critical points. So the use of asymptotic critical values would lead to misleading inference in small samples. In Table IV we report the ratios of empirical mean values LM_N under H_0 and the corresponding χ^2_2 mean which is equal to 2, for some selected values of n . For this table observations under normality are generated using the IMSL subroutine GGNML. The empirical mean values are based on 10,000 replications. Therefore, the maximum standard error of each entry will be $\sqrt{1/10000} = .01$ and we can expect the true ratios to be

TABLE IV

Ratio of empirical and asymptotic mean values of
the LM test statistic

| Sample size | Ratio | Sample size | Ratio |
|-------------|--------|-------------|--------|
| 10 | .46420 | 90 | .90105 |
| 20 | .65435 | 100 | .91255 |
| 30 | .73690 | 120 | .91560 |
| 40 | .77990 | 140 | .93845 |
| 50 | .83260 | 160 | .94680 |
| 60 | .84245 | 180 | .97440 |
| 70 | .86580 | 200 | .98145 |
| 80 | .89150 | 400 | .98620 |

within $\pm .02$ of the reported values. From the table critical values for any given significance level and sample size can be obtained by multiplying the given ratio with the χ^2_2 critical value. Let $n = 30$ and the significance level is .01. The χ^2_2 critical value is 9.21, and hence it would be more appropriate to use $.73690 \times 9.21 \approx 6.79$ as the critical value for the sample size 30. For intermediate values of n , the ratios can be obtained by interpolation.

4. CONCLUSION

In this paper we have investigated the performances of some of the available tests for normality against stable alternatives by simulation. It has been observed that the test suggested in Bera and Jarque (1981) and Shapiro and Francia (1972) W' test are most effective in detecting non-normality of both observations and regression disturbances. It seems no computationally simple test can be developed using the LM principle by directly considering the density function of the stable family. Possibly, a simple test can be constructed from its characteristic function. Further work in this direction will be fruitful.

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APPENDIX

Let $\ell_i(\theta)$ denote the log-density function for the i -th observation where θ is the parameter vector. If we have n independent observations, the log-likelihood function is $\ell(\theta) = \sum_{i=1}^n \ell_i(\theta)$. Let $\tilde{\theta}$ denote the MLE under the null hypothesis H_0 . Then the Rao's score or the LM test statistic to test H_0 is given by

$$LM = d'(\tilde{\theta})I^{-1}(\tilde{\theta})d(\tilde{\theta}) \quad (A.1)$$

where $d(\theta) = \partial \ell(\theta) / \partial \theta$, $I(\theta) = -E[\partial^2 \ell(\theta) / \partial \theta \partial \theta']$, and " \sim " denotes the expressions are evaluated at $\theta = \tilde{\theta}$. We assume that the PDF of u is a member of the Pearson family, i.e.,

$$f(u_j) = \frac{\exp\left[\int \frac{c_1^{-u_j}}{c_0 - c_1 u_j + c_2 u_j^2} du_j\right]}{\int_{-\infty}^{\infty} \exp\left[\int \frac{c_1^{-u}}{c_0 - c_1 u + c_2 u^2} du\right] du}$$

$$-\infty < u_j < \infty$$

$$j = 1, 2, \dots, n.$$

When $c_1 = c_2 = 0$, $f(u_j)$ reduces to a normal density function with mean zero and variance c_0 . Therefore, we test $H_0: \theta = (c_0, c_1, c_2)' = (c_0, 0, 0)'$. Let us denote

$$h(\theta; u) = \int \frac{c_1^{-u}}{c_0 - c_1 u + c_2 u^2} du.$$

Then

$$\ell(\theta) = -n \cdot \ln\left[\int_{-\infty}^{\infty} \exp\{h(\theta; u)\} du\right] + \sum_{j=1}^n h(\theta; u_j).$$

Using this expression of $\ell(\theta)$ we can show that

$$d'(\tilde{\theta}) = n[0, -\frac{\mu_3}{3\mu_2^2} + \frac{\mu_1}{\mu_2}, \frac{\mu_4}{4\mu_2^2} - \frac{3}{4}]$$

and

$$I(\tilde{\theta}) = n \begin{bmatrix} \frac{1}{2\mu_2^2} & 0 & \frac{3}{2\mu_2} \\ 0 & \frac{2}{3\mu_2} & 0 \\ \frac{3}{2\mu_2} & 0 & 6 \end{bmatrix}$$

where $\mu_i = \sum_{j=1}^n u_j^i/n$, $i = 1, 2, 3, 4$ and $\tilde{\theta} = (\mu_2, 0, 0)'$. Substituting $d(\tilde{\theta})$ and $I(\tilde{\theta})$ in (A.1), we obtain

$$LM = n \left[\frac{\mu_3^2}{6\mu_2} + \frac{1}{24} \left(\frac{\mu_4}{\mu_2} - 3 \right)^2 \right] + n \left[\frac{3\mu_1^2}{2\mu_2} - \frac{\mu_3\mu_1}{\mu_2} \right].$$

If observations are measured from the sample mean, $\mu_1 = 0$, and then LM reduces to LM_N given in equation (3) by defining $b_1 = \mu_3^2/\mu_2^3$ and $b_2 = \mu_4/\mu_2^2$.

REFERENCES

- Bartels, R. (1977), On the use of limit theorem arguments in economic statistics, Amer. Statistn. 31, 85-87.
- Bassett, G. W. and R. Koenker (1982), An empirical quantile function for linear models with iid errors, J. Amer. Statist. Assoc. 77, 407-415.
- Bera, A. K. (1982), A new test for normality, Econ. Letts. 9, 263-268.
- Bera, A. K. and C. M. Jarque (1981), An efficient large-sample test for normality of observations and regression residuals, Working Papers in Economics and Econometrics, 40, The Australian National University.
- Bera, A. K. and S. John (1983), Tests for multivariate normality with Pearson alternatives, Commun. Statis. Theor. Meth. A12(1), 103-117.
- Carlson, J. A. (1975), Are price expectations normally distributed?, J. Amer. Statist. Assoc. 70, 749-754.
- Chambers, J. M., C. L. Mallows and B. W. Stuck (1976), A method for simulating stable random variables, J. Amer. Statist. Assoc. 71, 340-344.
- D'Agostino, R. B. (1971), An omnibus test for normality for moderate and large size samples, Biometrika 58, 341-348.
- D'Agostino, R. B. and E. S. Pearson (1973), Tests for departure from normality. Empirical results for the distributions of b_2 and $\sqrt{b_1}$, Biometrika 60, 613-622.
- DuMouchel, W. (1971), Stable Distributions in Statistical Inference, Ph.D. Thesis, Yale University.
- DuMouchel, W. (1981), Stable distributions and infinite variance models in statistics, Technical Report No. 16, Department of Mathematics, Massachusetts Institute of Technology.
- Epps, T. W. and L. B. Pulley (1982), A test for normality based on the empirical characteristic function, mimeo., Department of Economics, University of Virginia.
- Fama, E. F. and R. Roll (1971), Parameter estimates for symmetric stable distributions, J. Amer. Statist. Assoc. 66, 331-338.
- Gnanadesikan, R. (1977), Methods for Statistical Data Analysis of Multivariate Observations (John Wiley and Sons, New York).

- Gnedenko, B. V. and A. N. Kolmogorov (1954), Limit Distributions for Sums of Independent Random Variables, Translated from the Russian by K. L. Chung (Addison-Wesley, Cambridge).
- Huang, C. J. and B. W. Bolch (1974), On the testing of regression disturbances for normality, J. Amer. Statist. Assoc. 69, 330-335.
- Koenker, R. and G. W. Bassett (1978), Regression quantiles, Econometrica 46, 33-50.
- Mandelbrot, B. (1963), The variation of certain speculative prices, J. Business 36, 394-419.
- Mardia, K. V. (1980), Tests of univariate and multivariate normality, in: P. R. Krishnaiah, ed., Handbook of Statistics, Volume 1 (North-Holland, Amsterdam), pp. 279-320.
- Pearson, E. S. and H. O. Hartley (1962), Biometrika Tables for Statisticians, Volume 1 (Cambridge University Press, Cambridge).
- Pearson, E. S., R. B. D'Agostino and K. O. Bowman (1977), Tests for departure from normality: Comparison of powers, Biometrika 64, 231-246.
- Pierce, D. A. and K. J. Kopecky (1979), Testing goodness of fit for the distribution of errors in regression models, Biometrika 66, 1-5.
- Poirier, D., M. D. Tello and S. E. Zin (1984), A diagnostic test for normality within the power exponential family, Working Paper Series, Number 8412, Department of Economics and Institute for Policy Analysis, University of Toronto.
- Rao, C. R. (1948), Large sample tests of statistical hypotheses concerning several parameters with applications to problems of estimation, Proc. Camb. Phil. Soc. 44, 50-57.
- Saniga, E. M. and H. Hayya (1977), Simple goodness-of-fit test for symmetric stable distributions, J. Financial and Quantitative Analysis 12, 276-289.
- Saniga, E. M. and J. A. Miles (1979), Power of some standard goodness-of-fit tests of normality against asymmetric stable alternatives, J. Amer. Statist. Assoc. 74, 861-865.
- Saniga, E. M., R. Pfaffenberger and J. Hayya (1975), Estimation and goodness-of-fit tests for symmetric stable distributions, Proc. Amer. Statist. Assoc., Bus. & Eco. Stat. Sec., 530-534.
- Shapiro, S. S. and R. S. Francia (1972), An approximate analysis of variance test for normality, J. Amer. Statist. Assoc. 67, 215-216.

- Shapiro, S. S. and M. B. Wilk (1965), An analysis of variance test for normality (complete samples), Biometrika 52, 591-611.
- Smith, K. V. (1975), A simulation analysis of the power of several tests for detecting heavy-tailed distributions, J. Amer. Statist. Assoc. 70, 662-665.
- Theil, H. (1965), The analysis of disturbances in regression analysis, J. Amer. Statist. Assoc. 60, 1067-1079.
- White, H. and G. M. MacDonald (1980), Some large-sample tests for non-normality in the linear regression model, J. Amer. Statist. Assoc. 75, 16-28.

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