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December 1989

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May 1988

Revised: December 1989


We wish to thank Bhaskar Chakravorti, Tomoichi Shinotsuka and especially William Thomson, for their advice and comments. All errors remain our own.

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## Abstract

An  $n$ -person *Bargaining Problem* consists of a pair  $(S, d)$  where  $S \subset \mathbf{R}^n$  is a set of feasible utility vectors which the players may obtain through cooperation, and the point  $d \in S$ , called the *disagreement point*, is interpreted as the utility that players receive if they fail to reach an agreement. Given a class of bargaining problems,  $\Sigma^n$ , a *solution* is a map that associates with each problem  $(S, d)$  in  $\Sigma^n$  a unique point in  $S$ . In this paper, we relax the common assumption that  $S$  is convex and examine the implications for well known solution concepts. We argue that even with von Neumann–Morgenstern utilities, this restriction is substantive and limits the application of the theory. Without convexity, the solution introduced by Nash(1950) is no longer well defined. We propose a new solution called the Nash Extension. This solution coincides with the Nash solution when  $S$  is convex and is the unique solution satisfying weak Pareto optimality, symmetry, scale invariance, continuity, and a new axiom, ethical monotonicity. We explore the relationship between ethical monotonicity and Nash's independence of irrelevant alternatives. The solution introduced by Kalai and Smorodinsky(1975) remains well defined on our domain and the characterization of the solution provided in their paper can be obtained without the assumption of convexity of  $S$ . Similarly, the Egalitarian solution is well defined on our domain and the characterization provided by Kalai(1977) does not require the convexity of  $S$ .



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## 1. Introduction

An *n*-person bargaining problem consists of a pair  $(S, d)$  where  $S$  is a non-empty subset of  $\mathbf{R}^n$ , and  $d \in S$ . The set  $S$  is interpreted as the set of utility allocations that are attainable through joint action on the part of all  $n$  agents. If the agents fail to reach an agreement, then the problem is settled at the point  $d$ , which is called the *disagreement point*. A bargaining solution  $F$ , defined on a class of problems  $\Sigma^n$ , is a map that associates with each problem  $(S, d) \in \Sigma^n$  a unique point in  $S$ . In the axiomatic approach to bargaining we start by specifying a list of properties (Pareto optimality, for example) that we would like a solution to have. If it can be shown that there is a unique solution that satisfies a given list of axioms, then the solution is said to be *characterized* this list.

It is common to restrict the domain to problems with convex feasible sets. However, bargaining problems can arise from a variety of political, social and economic situations. The requirement that  $S$  be convex seems to remove many important cases from consideration. For example, the image in utility space of a finite set of resource allocations will be a finite set of points, not a convex set. Or consider the bargaining problem associated with an economy in which strong externalities are present. It is quite likely that the feasible set of such a problem will be non-convex.

This restriction of domain is often justified by assuming that agents' preferences can be represented by von Neumann–Morgenstern utility functions. Feasible sets may then be convexified by running lotteries over the original elements. It is claimed that since agents care only about the expected utility of settlements, these random allocations just as good as the non-random ones. Restricting attention to bargaining problems with convex feasible sets, therefore, has no economic consequences other than those associated with the von Neumann–Morgenstern axioms on agents' preferences.

We disagree with this conclusion. We argue that the use of lotteries prevents us from applying axiomatic bargaining theory to a large class of interesting problems.

For example, one common way of using bargaining theory in a cooperative context is as a method of prescribing settlements to social distribution problems that are “fair” or

“ethical”. Under this interpretation, the axioms that characterize the solution employed are equivalent to a description of the ethical values of the society. We run into difficulties, however, when a solution recommends that a problem be settled at an utility allocation attainable only through a lottery. This is because agents do not walk away from the bargaining table with lotteries in their pockets, they walk away with *outcomes* of lotteries. We must therefore choose whether the fairness test is to be applied to expected utility allocations or to actual utility allocations. This might be viewed as a choice between fairness of opportunity and fairness of result.<sup>1</sup> Allowing problems to be settled at lotteries is the same as deciding that outcomes don’t matter. In many situations, this may be inappropriate and restrictive.

Take the case of an “egalitarian” society in which two identical free agents toss a fair coin to decide who is to be slave to the other as an illustration. Each may prefer the gamble to the status quo of both agents being free. So before the coin is tossed, the gamble represents an individually rational, Pareto optimal, and in particular, equal division of the surplus. However, the agent who ends up being the slave is substantially worse off than the agent who becomes the slave holder. One would be hard pressed to argue that this is an egalitarian society after the lottery is held. One would be even harder pressed to convince the slave to accept his fate as “fair”, and as the inevitable result of his egalitarian beliefs.<sup>2</sup>

The axiomatic approach has also been applied to non-cooperative bargaining games. Labor disputes, for example, may be interpreted as unanimity games in which all agents must agree to a division of the jointly produced surplus or gain nothing. Such games may have many Nash equilibria and axiomatic methods give us a sensible way of selecting a single outcome.

One interesting way to use bargaining theory in this situation is to imagine a mediator attempting to settle a labor dispute. Mediation is predicated on the belief that if agents are persuaded that a particular division is “fair”, then they will voluntarily agree to coordinate their actions and accept the allocation as a settlement to their problem. Thus, the axioms

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<sup>1</sup> We thank Charles Kahn for this suggestion.

<sup>2</sup> A similar point is argued by Myerson(1981)

summarize what the mediator believes the agents are likely to accept as fair.

Alternatively, we could view the theory as being purely predictive in nature. The axioms are then interpreted as statements about the behavior of rational agents given the institutional framework. For example, we might believe that rational agents will never settle at a point that is not Pareto optimal or individually rational.

Now consider both of these non-cooperative interpretations in an institutional setting that does not include the possibility of signing binding contracts. Then in particular, agents can not credibly commit to abide by the outcomes of lotteries. Take the slave game described above. Recall that after the lottery is held, one player ends up being worse off than he is at the disagreement point. So regardless of whether individual rationality is imposed because it is “fair”, as in the mediation model, or because it is “rational”, as in the behavioral model, the agent who loses the lottery will surely renege on his agreement to be a slave if there does not exist a binding contract to hold him. Agents will not voluntarily accept a settlement that is unfair or irrational, respectively, under these two interpretations. Therefore, the use of lotteries in the non-cooperative context requires that binding contracts be part of the institutional structure. <sup>3</sup>

In this paper we investigate the behavior of certain solution concepts on a domain that admits non-convex problems. Our primary contribution is a generalization of the Nash solution. The Nash solution is not well defined on the domain of non-convex bargaining problems. Maximizing the “welfare function” proposed by Nash, or indeed any “social welfare function”, over a non-convex set will not necessarily yield a unique point.<sup>4</sup> The Nash solution, therefore, does not give a clear recommendation of how to settle some problems. To overcome this difficulty we suggest and characterize a new solution, called the Nash Extension. This new solution is well defined on the domain of problems that admit freely disposable utility, and coincides with the Nash solution on the domain of convex problems. The axioms employed in the characterization are the same as those employed

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<sup>3</sup> Binmore(1988) forcefully presents a similar argument in the non-cooperative framework, concluding: “Thus convexity necessarily has to appear as a *substantive* assumption rather than a near tautology.” p51.

<sup>4</sup> It is possible, however, to characterize the Nash solution on the domain on non-convex bargaining problems where the Nash solution happens to be well defined. See Foster and Vohra (1988)



by Nash, except that Nash's Independence of Irrelevant Alternatives, or Contraction Independence as it will be called in this paper, is replaced by a new axiom, called Ethical Monotonicity.

We motivate this axiom in the following way. It is undesirable to settle problems at allocations attainable only through lotteries. Nevertheless, if agents have von Neumann-Morgenstern preferences, then lotteries must be included in the totality bargaining opportunities. So if a solution is to be sensitive to all these opportunities then we should somehow take into account the convex hull of the original  $S$ . We take the settlement recommended by a solution for the convex hull of a bargaining problem as a bench mark allocation. In a sense, this *ethical point* summarizes all the attainable allocations since it is the ethically desirable point of the true feasible set (including the inadmissible lotteries) under the chosen solution concept. Now suppose that we have a problem  $S$  and a smaller problem  $S'$  that is contained in  $S$ . Then we say that the two problems are ethically similar if the ethical point of the larger one is an element of the convex hull of the smaller, and the two problems have the same disagreement point. Ethical monotonicity says that if two problems are ethically similar, then no agent should benefit from this decrease in opportunities from  $S$  to  $S'$ . Similarly, no agent should be hurt by an expansion of opportunities if the expanded problem is ethical comparable to the original problem.

A secondary contribution of this paper is to show that convexity is not required for the characterizations of the Egalitarian and Kalai-Smorodinsky solutions provided by Kalai (1977) and Kalai and Smorodinsky (1975), if utility is assumed to be freely disposable. The paper concludes by suggesting how the methods used can be extended to other solution concepts and to other domains.

## 2. Definitions and Axioms

We start with some definitions and formal statements of the axioms used in the characterizations. Given a point  $d \in \mathbf{R}^n$ , and a set  $S \subset \mathbf{R}^n$ , we say  $S$  is *d-comprehensive* if  $d \leq x \leq y$  and  $y \in S$  implies  $x \in S$ .<sup>5</sup>

The *comprehensive hull* of a set  $S \subset \mathbf{R}^n$ , with respect to a point  $d \in \mathbf{R}^n$  is the smallest d-comprehensive set containing  $S$ :

$$\text{comp}(S; d) \equiv \{x \in \mathbf{R}^n \mid x \in S \text{ or } \exists y \in S \text{ such that } d \leq x \leq y\}. \quad (1)$$

The *convex hull* of a set  $S \subset \mathbf{R}^n$  is the smallest convex set containing  $S$ :

$$\text{con}(S) \equiv \left\{ x \in \mathbf{R}^n \mid x = \sum_{i=1}^{n+1} \alpha_i y_i \text{ for some } \alpha_i \geq 0, \sum_{i=1}^{n+1} \alpha_i = 1 \text{ and } y_i \in S \forall i \right\}. \quad (2)$$

The *convex and comprehensive hull* of a set  $S \subset \mathbf{R}^n$ , with respect to a point  $d \in \mathbf{R}^n$  is the smallest convex, d-comprehensive set containing  $S$ :

$$\text{concomp}(S; d) \equiv \text{con}(\text{comp}(S; d)). \quad (3)$$

Let  $\mathcal{C}$  denote the space of compact subsets of  $\mathbf{R}^n$ . The Hausdorff distance  $\rho : \mathcal{C} \times \mathcal{C} \rightarrow \mathbf{R}$  is defined by,

$$\rho(S, S') \equiv \left\{ \max \left[ \max_{x \in S'} \min_{y \in S} \|x - y\| ; \max_{x \in S} \min_{y \in S'} \|x - y\| \right] \right\} \quad (4)$$

where  $\|\bullet\|$  is the Euclidean norm. An *closed  $\epsilon$ -ball around  $x$*  is defined as:

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<sup>5</sup> The vector inequalities are represented by  $\geq, >, \text{ and } \gg$ .



$$B_\epsilon(x) \equiv \{z \in \mathbf{R}^n \mid \|x - z\| \leq \epsilon\} \quad (5)$$

Let  $\text{int}(S)$  denote the *interior* of  $S$ , and  $\partial(S)$  the boundary of  $S$ . Define the *weak Pareto frontier* of  $S$  as:

$$WP(S) \equiv \{x \in S \mid y \gg x \text{ implies } y \notin S\}. \quad (6)$$

Define the *strong Pareto frontier* of  $S$  as:

$$P(S) \equiv \{x \in S \mid y \geq x \text{ implies } y \notin S\}. \quad (7)$$

The domain of bargaining problems considered in this paper is  $\Sigma_c^n$ . This is defined as the class of pairs  $(S, d)$  where  $S \subset \mathbf{R}^n$  and  $d \in \mathbf{R}^n$  such that:

- A1)  $S$  is compact.
- A2)  $S$  is  $d$ -comprehensive.
- A3) There exists  $x \in S$  and  $x \gg d$ .

This differs from the usual domain, which we denote  $\Sigma_{con}^n$ , in that we do not assume that the set of feasible utility allocations is convex. A *bargaining solution*,  $F$ , is a function from  $\Sigma_c^n$  to  $\mathbf{R}^n$  such that for each  $(S, d) \in \Sigma_c^n$ ,  $F(S, d) \in S$ .

A list of axioms that will be used to characterize the solutions discussed in this paper follows. Readers familiar with axiomatic bargaining theory may wish to skip to the definition of Ethical Monotonicity. All bargaining problems mentioned below are assumed to be elements of  $\Sigma_c^n$ .

*Weak Pareto-Optimality* (W.P.O.):  $F(S, d) \in WP(S)$ .

*Contraction Independence* (C.IND)<sup>6</sup>: If  $S' \subset S$ ,  $d' = d$ , and  $F(S, d) \in S'$ , then  $F(S', d') = F(S, d)$ .

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<sup>6</sup> C.IND was introduced by Nash(1950) with the label "Independence of Irrelevant Alternatives".

A *permutation operator*,  $\pi$ , is a bijection from  $\{1, 2, \dots, n\}$  to  $\{1, 2, \dots, n\}$ .  $\Pi^n$  is the class of all such operators. Let  $\pi(x) = (x^{\pi(1)}, x^{\pi(2)}, \dots, x^{\pi(n)})$ .<sup>7</sup> and  $\pi(S) = \{y \in \mathbf{R}^n \mid y = \pi(x), x \in S\}$ .

*Symmetry* (SYM): If for all permutation operators  $\pi \in \Pi^n$ ,  $\pi(S) = S$  and  $\pi(d) = d$ , then  $F^i(S, d) = F^j(S, d) \forall i, j$ .

An *affine transformation on  $\mathbf{R}^n$*  is a map,  $\lambda : \mathbf{R}^n \rightarrow \mathbf{R}^n$ , where  $\lambda(x) = a + bx$  for some  $a \in \mathbf{R}^n, b \in \mathbf{R}_{++}^n$ .  $\Lambda^n$  is the class of all such transformations. Let  $\lambda(S) = \{y \in \mathbf{R}^n \mid y = \lambda(x), x \in S\}$ .

*Scale Invariance* (S.INV):  $\forall \lambda \in \Lambda^n, F(\lambda(S), \lambda(d)) = \lambda(F(S, d))$ .

*Translation Invariance* (T.INV):  $\forall x \in \mathbf{R}^n, F(S + \{x\}, d + x) = F(S, d) + x$ .

*Continuity* (CONT): For all sequences  $\{(S^\nu, d)\}_{\nu=1}^\infty$ , if  $\rho(S, S^\nu) \rightarrow 0$ , then  $F(S^\nu, d) \rightarrow F(S, d)$ .

*Strong Monotonicity* (S.MON): If  $S \subset S'$  and  $d = d'$ , then  $F(S', d') \geq F(S, d)$ .

The *Ideal Point* of a problem  $(S, d)$  is defined as:

$$a(S, d) \equiv (\max_{\substack{x \in S \\ x \geq d}} x^1, \max_{\substack{x \in S \\ x \geq d}} x^2, \dots, \max_{\substack{x \in S \\ x \geq d}} x^n). \quad (8)$$

*Restricted Monotonicity* (R.MON): If  $S \subset S'$ ,  $d = d'$ , and  $a(S, d) = a(S', d')$ , then  $F(S', d') \geq F(S, d)$ .

The *Ethical Point with respect to  $F$  of a problem  $(S, d)$* , is defined as:

$$e^F(S, d) \equiv F(\text{con}(S), d). \quad (9)$$

Now *Ethical Monotonicity* is formally defined:

*Ethical Monotonicity* (E.MON): If  $S' \subset S$ ,  $d' = d$ , and  $e^F(S, d) \in \text{con}(S')$ , then  $F(S, d) \geq F(S', d')$ .

Although this new axiom is a monotonicity requirement, it turns out that Ethical Monotonicity with Pareto Optimality implies Contraction Independence on the convex

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<sup>7</sup> Superscripts stand for the components of a vector

domain. On the domain of comprehensive problems, there is no logical relation between the two axioms.

### 3. The Solutions

In his 1950 paper, Nash considers the domain  $\Sigma_{con}^n$  of convex problems. He proposes the following solution:

$$N(S, d) \equiv \left\{ \operatorname{argmax}_{\substack{x \in S \\ x \succeq d}} \prod_{i=1}^n (x_i - d_i) \right\}, \quad (10)$$

and demonstrates that it is the only solution that satisfies W.P.O, SYM, S.INV, and C.IND. On the domain  $\Sigma_c^n$ , however, the mapping  $N$  fails to be a solution as defined the introduction. This is because  $N$  will not be single-valued on  $\Sigma_c^n$  in general. A natural suggestion would be to define a new solution by taking a selection from the set of maximizers of the Nash product. Unfortunately, it is impossible to do this in a way that satisfies the properties enjoyed by  $N$  on the convex domain. Obviously, no such selection satisfies SYM. Additionally, it can be shown that *any* such selection must also fail to satisfy the axioms CONT and C.IND.

To remedy this deficiency we propose a new solution, the *Nash Extension*, which is constructed as follows. First define the mapping  $L : \Sigma_c^n \rightarrow \mathbf{R}^n$  as:

$$L(S, d) \equiv \operatorname{con}\left(N(\operatorname{con}(S), d), d\right). \quad (11)$$

$L(S, d)$  is the line segment connecting the disagreement point  $d$ , to the

Nash solution of the problem composed of the convex hull of  $S$ , and  $d$ . Now we define the solution  $NE$ :

$$NE(S, d) \equiv \left\{ \max x \mid x \in L(S, d) \cap S \right\}, \quad (12)$$

where  $\max$  indicates the maximal element with respect to the partial order on  $\mathbf{R}^n$ . The construction of  $NE$  is illustrated figure 1. The point  $NE(S, d)$  is the intersection of the

weak Pareto frontier of  $S$  and the line segment connecting the disagreement point and Ethical Point under the Nash solution the problem  $(S, d)$ . Obviously,  $NE$  coincides with  $N$  on the domain of convex problems.

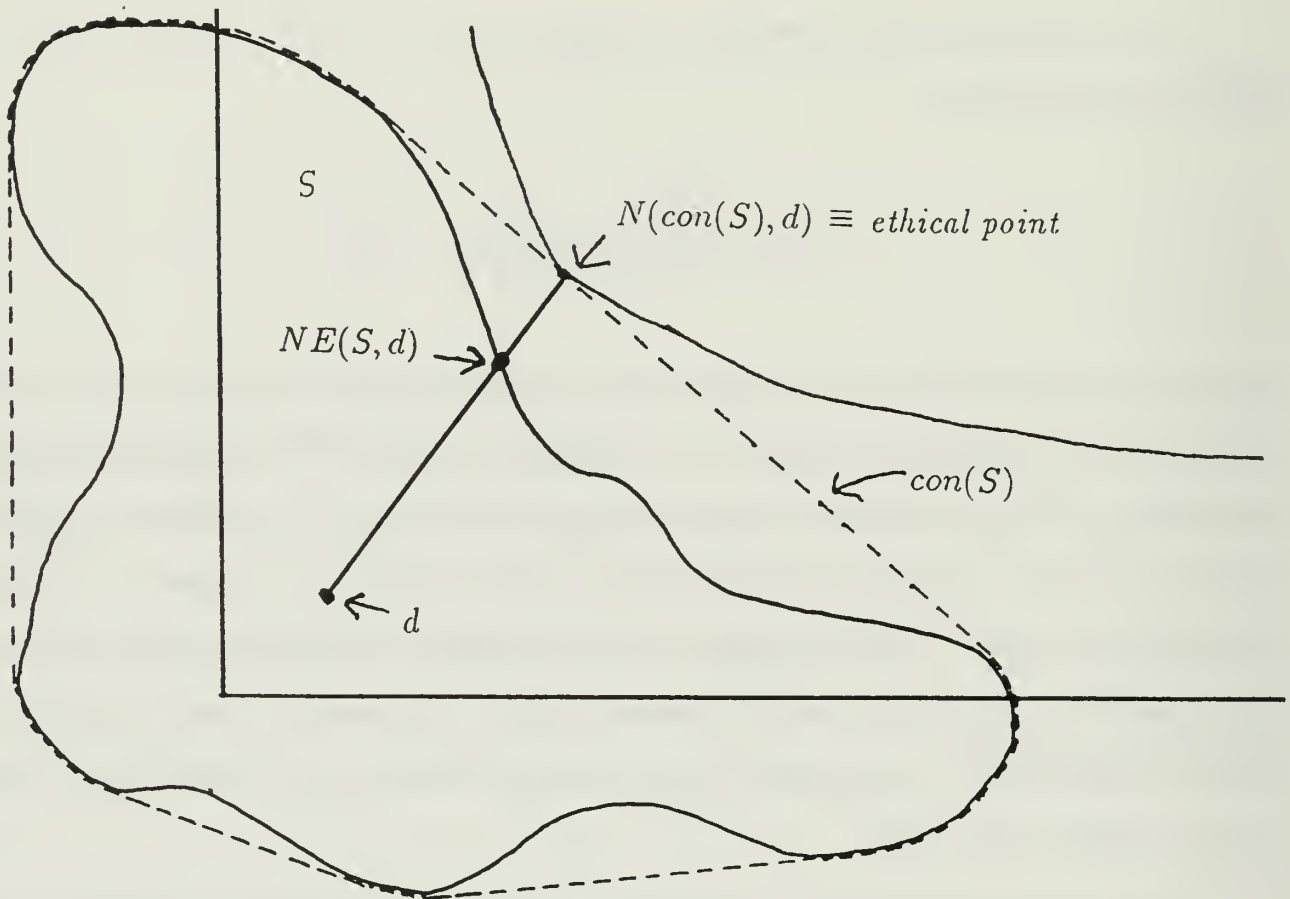


Figure 1: The Nash extension solution

Nash wrote that the minimal standard that any solution to a bargaining problem ought to meet is that it be single valued and continuous. We pause to show that the Nash Extension meets both of these requirements. To see that the  $NE$  solution is single valued, notice that  $L$  is a non-empty, compact-valued correspondence. Then since  $L$  is also a line segment, its maximal element exists and is unique. Thus,  $NE$  is non-empty and single-valued on  $\Sigma_c^n$ . We now prove that it is continuous.

Lemma 1.  $NE$  is continuous on  $\Sigma_c^n$ .

Proof/

Let  $S'' \rightarrow S$ . We begin by showing that  $con$  is a  $\rho$ -continuous correspondence. To see this, suppose that for any given  $\epsilon > 0$ ,  $\rho(S, S') \leq \epsilon$ . Then for any  $y = \sum_{i=1}^{n+1} \alpha_i x_i \in con(S)$  there is a  $y' = \sum_{i=1}^{n+1} \alpha_i x'_i \in con(S')$  such that  $y \in B_\epsilon(y')$ . By reversing the argument, we



also find that for any  $y' \in \text{con}(S')$  there is a  $y \in \text{con}(S)$  such that  $y' \in B_\epsilon(y)$ . Thus if  $\rho(S, S') \leq \epsilon$  then  $\rho(\text{con}(S), \text{con}(S')) \leq \epsilon$ , and so  $\text{con}$  is  $\rho$ -continuous.

Therefore, since  $\text{con}$  is continuous, and  $N$  is continuous on  $\sum_{\text{con}}^n$ , the composition map  $e^N$ , where  $e^N(S, d) \equiv N(\text{con}(S), d)$ , is continuous by Hildenbrand(1974) proposition B.7. We conclude that if  $S^\nu \rightarrow S$ , then  $e^N(S^\nu, d) \rightarrow e^N(S, d)$ .

By Definition,  $NE(S, d) \in L(S, d)$ . So  $NE(S, d) = (1 - \lambda^*)d + \lambda^*e^N(S, d)$  for some  $\lambda^* \in [0, 1]$ . Also, for each  $S^\nu$ ,  $NE(S^\nu, d) = (1 - \lambda^\nu)d + \lambda^\nu e^N(S^\nu, d)$  for some  $\lambda^\nu \in [0, 1]$ . Notice that the sequence  $\{\lambda^\nu\}$  is drawn from the compact set  $[0, 1]$ . Thus, given any sequence of sets  $\{S^\nu\}$  converging to  $S$ , if it can be shown for every convergent subsequence  $\{\lambda^{\nu_k}\}$  of  $\{\lambda^\nu\}$  that  $\lambda^{\nu_k} \rightarrow \lambda^*$  then the lemma is proven. Suppose not. Then there are two cases:

1. Suppose first that for some subsequence  $\{S^{\nu_k}\}$ ,  $\lambda^{\nu_k} \rightarrow \hat{\lambda}$  and  $\hat{\lambda} > \lambda^*$ . Then the definition of  $NE$  implies  $(1 - \hat{\lambda})d + \hat{\lambda}e^N(S, d) \equiv \hat{x} \notin S$ . Thus the sequence  $\{NE(S^{\nu_k})\}$  converges to a point not  $S$ , contradicting the hypothesis  $S^{\nu_k} \rightarrow S$ .
2. Now suppose that for some subsequence  $\{S^{\nu_k}\}$  that  $\lambda^{\nu_k} \rightarrow \hat{\lambda}$  and  $\hat{\lambda} < \lambda^*$ . Then  $(1 - \hat{\lambda})d + \hat{\lambda}e^N(S, d) \equiv \hat{x} \ll NE(S, d)$ . Additionally, the existence of a point  $S$  that strictly dominates  $d$  implies that  $d \ll x$ . Hence by the d-comprehensiveness of  $S$ ,  $\hat{x} \in \text{int}(S)$ . Thus there exists  $\epsilon > 0$  and  $\nu_1$  such that for  $\nu > \nu_1$ ,  $B_\epsilon(\hat{x}) \subset S^{\nu_k}$  and  $NE(S, d) \notin B_\epsilon(\hat{x})$ . Since  $e^N(S^{\nu_k}, d) \rightarrow e^N(S, d)$ , there exists  $\nu_2$  such that  $\nu > \nu_2$  implies  $L(S^{\nu_k}, d) \cap B_\epsilon(\hat{x}) \neq \emptyset$ . Now, for each  $\nu_k$  let  $y^{\nu_k} = \max\{L(S^{\nu_k}, d) \cap B_\epsilon(\hat{x})\}$ . Let  $\nu' \equiv \max\{\nu_1, \nu_2\}$ . Clearly for  $\nu_k > \nu'$ ,  $y^{\nu_k}$  exists and  $y^{\nu_k} \in \partial B_\epsilon(\hat{x}) \cap S^{\nu_k}$ . However, by hypothesis  $NE(S^{\nu_k}, d) \rightarrow \hat{x}$ , so there exists  $\nu''$  such that  $\nu_k > \nu''$  implies  $NE(S^{\nu_k}, d) \in \text{int}(B_\epsilon(\hat{x}))$ . Then for  $\nu_k > \max\{\nu', \nu''\}$ , we have  $y^{\nu_k} \in S^{\nu_k} \cap L(S^{\nu_k}, d)$  and  $y^{\nu_k} \gg NE(S^{\nu_k}, d)$ , contradicting the definition of  $NE$ .

Hence for every subsequence of  $S^\nu$ , we have that  $\hat{\lambda} = \lambda^*$ . Therefore,  $\hat{x} = NE(S, d)$ .

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Our main result is a characterization of the new solution  $NE$ .

**Theorem 1.** *A solution on  $\Sigma_c^n$  satisfies W.P.O, S.V, SYM, E.MON, and CONT if and only if it is the Nash extension.*

Proof/

(a) First it is shown that the  $NE$  solution satisfies the axioms.

W.P.O: Let  $x = NE(S, d)$ . Assume there exists  $y \in S$  such that  $y \gg x$ . Then since  $S$  is  $d$ -comprehensive there exists  $z \in L(S, d) \cap S$  such that  $z \gg x$ . However, this contradicts the hypothesis  $x = NE(S, d)$ .

S.INV: Let  $(S, d) \in \Sigma_c^n$  and  $\lambda \in \Lambda^n$  be any affine transformation. Since  $con(\lambda(S)) = \lambda(con(S))$ , and  $N$  satisfies S.INV on  $\Sigma_{con}^n$ , we conclude that  $N(\lambda(con(S)), \lambda(d)) = \lambda(N(con(S), d))$ . Thus  $L(\lambda(S), \lambda(d)) = \lambda(L(S, d))$ . Therefore,  $\max\{L(\lambda(S), \lambda(d)) \cap \lambda(S)\} = \max\{\lambda(L(S, d) \cap S)\} = \lambda(NE(S, d))$ , as required.

SYM: Let  $(S, d)$  be a symmetric problem. Then  $(con(S), d)$  is also a symmetric problem. Since  $N$  satisfies SYM on  $\Sigma_{con}^n$ ,  $N(con(S), d)$  is a point of equal coordinates. But so is  $d$ , and so all elements  $L(S, d)$  are points of equal coordinates. Consequently,  $NE(S, d) \in L(S, d)$  is symmetric.

E.MON: Let  $(S, d), (S', d')$  be such that;  $S \subset S'$ ,  $d = d'$  and  $e^{NE(S', d')} \in con(S)$ . Then  $N(con(S'), d') = NE(con(S'), d') = e^{NE(S', d')}$ , and therefore  $N(con(S'), d') \in con(S)$ . Since  $con(S) \subset con(S')$ , and  $N$  satisfies C.IND on  $\Sigma_{con}^n$ ,  $N(con(S), d) = N(con(S'), d')$ . Furthermore since  $d = d'$  by hypothesis,  $L(S, d) = L(S', d')$ . Therefore  $S \subset S'$  implies  $NE(S, d) \leq NE(S', d')$ , as required.

CONT: See lemma 1.

(b) Conversely let  $F$  be a solution on  $\Sigma_c^n$  satisfying the five axioms, and consider any problem  $(S, d)$ . By S.INV, we can set  $d = 0$  and  $N(con(S), d) = (1, 1, \dots, 1) \equiv e$ . Then  $NE(S, d) = (\alpha, \dots, \alpha) \equiv x$  for some  $\alpha > 0$ . We distinguish two cases:

Case 1)  $S \subset \mathbf{R}_+^n$ .

Let the sets  $T$  and  $V$  be defined as follows,

$$T \equiv concomp[(n, 0, \dots, 0), (0, n, \dots, 0), (0, \dots, n); d] \quad (13)$$

$$V \equiv T \setminus \{x + \mathbf{R}_{++}^n\}. \quad (14)$$

Since  $e = N(\text{con}(S), d)$ , the hyperplane defined by  $\sum_{i=1}^n x_i = n$  supports  $\text{con}(S)$  at  $e$ . Hence  $S \subset T$ . Also, since  $F$  satisfies W.P.O, and  $S$  is comprehensive,  $z \in \{x + \mathbf{R}_{++}^n\}$  implies that  $z \notin S$ . Thus  $S \subseteq V$ .

Now, since  $(V, 0)$  is a symmetric problem, and  $x$  is the only symmetric point in  $WP(V)$ , by W.P.O. and SYM,  $F(V, d) = x$ . Also, since  $e$  is the only symmetric point in  $WP(T)$ , by W.P.O. and SYM,  $F(T, d) = e$ . But  $\text{con}(V) = T$ , and so  $e^F(V, d) = e$ . Therefore, since  $S \subseteq V$  and  $e^F(V, d) = e \in \text{con}(S)$ , by E.MON,  $F(S, d) \leq F(V, d) = x$ . There are two possibilities.

- i)  $x \in P(S)$ . Then by W.P.O,  $F(S, d) = x = NE(S, d)$  and the proof is complete.
- ii)  $x \notin P(S)$ . Then consider the sequence of problems  $\{(V^\nu; 0)\}$  and  $\{(S^\nu; 0)\}$  defined by:

$$V^\nu \equiv \left\{ V \cup_{\text{comp}} \left[ \frac{1}{\nu} e + \left(1 - \frac{1}{\nu}\right) x; 0 \right] \right\} \quad (15)$$

$$S^\nu \equiv \left\{ S \cup_{\text{comp}} \left[ \frac{1}{\nu} e + \left(1 - \frac{1}{\nu}\right) x; 0 \right] \right\}. \quad (16)$$

Since  $V^\nu$  is symmetric and  $d = 0$ , by W.P.O. and SYM,  $F(V^\nu, 0) = (\alpha + \frac{1}{\nu}, \dots, \alpha + \frac{1}{\nu}) \equiv x^\nu$ . Since  $S^\nu \subseteq V^\nu$ ,  $e^F(V^\nu, 0) = e$  and  $e \in \text{con}(S^\nu)$ , by E.MON and W.P.O. we have  $F(S^\nu, d) = F(V^\nu, d) = x^\nu$ . But since  $S^\nu \rightarrow S$ , by CONT  $F(S^\nu, d) \rightarrow F(S, d)$ . Thus since  $x^\nu \rightarrow x$ , we conclude that  $F(S, d) = x = NE(S, d)$ .

Case 2)  $S \not\subseteq \mathbf{R}_+^n$ .

Let  $V' \equiv \{V \cup_{\pi \in \Pi^n} \pi(S)\}$ . Since  $V'$  is symmetric, and  $x \in WP(V')$ , we can replace  $(V, 0)$  above with  $(V', 0)$  and replicate argument given for Case 1. •

Kalai(1977) examines the properties of an alternative to the Nash solution called the egalitarian solution. Although idea of dividing surpluses equally is not new, Kalai was the first to present an axiomatic characterization of this solution. Formally, let us define the *egalitarian solution*,  $E$ , as:

$$E(S, d) \equiv \{\max[x \in S \mid x_i - d_i = x_j - d_j \forall i, j \in (1, \dots, n)]\}. \quad (17)$$

The axioms we use here are the same as those employed by Kalai to characterize the solution  $E$  on the domain of convex bargaining problems. Further properties of the egalitarian solution are discussed Kalai(1977) and Thomson (1986).

**Theorem 2.** *A solution  $F$  on  $\Sigma_c^n$  satisfies SYM, T.INV, W.P.O, and S.MON if and only if it is the egalitarian solution.*

Proof/

The proof that  $E$  satisfies the four axioms is elementary and is omitted. Conversely let  $F$  be a solution satisfying the four axioms. Given any  $(S, d) \in \Sigma_c^n$ , we can assume by T.INV that the problem has been normalized such that  $d = 0$ . Thus  $E(S, d) = (\alpha, \dots, \alpha) \equiv x$  for some  $\alpha > 0$ . Now let  $T$  be defined by:

$$T \equiv \text{comp}(x; 0), \quad (18)$$

and consider the problem  $(T, 0)$ . Since  $T$  is symmetric,  $d = 0$ , and  $x$  is the only symmetric element of  $WP(T)$ , by W.P.O. and SYM,  $F(T, d) = x$ . Also, since  $S$  is comprehensive  $T \subseteq S$ . Hence, by S.MON,  $F(S, d) \geq x$ .

By assumption,  $S$  is compact. Thus, there exists  $\beta \in \mathbf{R}$  such that  $x \in S$  implies  $(-\beta, -\beta, \dots, -\beta) \leq (x^1, x^2, \dots, x^n) \leq (\beta, \beta, \dots, \beta)$ . Let  $Z$  be the symmetric closed hypercube defined by:

$$Z \equiv \{y \in \mathbf{R}^n \mid |y| \leq (\beta, \beta, \dots, \beta)\}. \quad (19)$$

Also define  $T'$  as:

$$T' \equiv Z \setminus \{x + \mathbf{R}_{++}^n\}. \quad (20)$$



Consider the problem  $(T'; 0)$ . Since  $T'$  is symmetric,  $d = 0$  and  $x$  is the only symmetric element of  $WP(T')$ , by W.P.O. and SYM,  $F(T', d) = x$ . But since  $S \subseteq T'$ , by S.MON,  $F(S, d) \leq x$ . Thus,  $F(S, d) = x = E(S, d)$ . •

The next solution considered is the Kalai-Smorodinsky solution, K:

$$(S, d) \equiv \max [x \in S \mid x \in \text{con}(a(S, d), d)]. \quad (21)$$

The axioms used are those employed by Kalai and Smorodinsky(1975) to characterize  $K$  on the convex domain with two agents, except that only weak Pareto optimality is used. The generalization to more agents is not immediate since  $K$  does not satisfy even Weak Pareto Optimality on  $\Sigma_{con}^n$  for  $n > 2$ . No such problem exists on the comprehensive domain. For further discussion see Kalai and Smorodinsky(1975) and Thomson(1986).

**Theorem 3.** *A solution  $F$  on  $\Sigma_c^n$  satisfies SYM, S.INV, W.P.O, and R.MON if and only if it is the Kalai-Smorodinsky solution.*

Proof/

The proof that  $K$  satisfies the axioms is elementary and is omitted. Conversely let  $F$  be a solution satisfying the four axioms. Given any  $(S, d) \in \Sigma_c^n$ , assume by S.INV that the problem has been normalized such that  $d = 0$  and  $a(S, d) = (\beta, \dots, \beta) \equiv y$ . Then  $K(S, d) = (\alpha, \dots, \alpha) \equiv x$  for some  $\alpha > 0$ . Let  $T$  be defined as:

$$T \equiv \text{comp}(y; 0) \setminus \{x + \mathbf{R}_{++}^n\} \quad (22)$$

and consider the problem  $(T, 0)$ . We distinguish two cases:

Case i)  $S \subset \mathbf{R}_{++}^n$ . Since  $S$  is comprehensive and  $x \in WP(S)$ , we have  $S \subseteq T$ . Also, since  $T$  is symmetric,  $d = 0$ , and  $x$  is the only symmetric element  $WP(T)$ , by W.P.O. and SYM,  $F(T, 0) = x$ . However, since  $S \subset T$ , and  $a(S, 0) = a(T, 0) = y$ , by R.MON  $F(S, 0) \leq F(T, 0) = x$

Now let  $T'$  be defined by,



$$T' \equiv comp[(\beta, 0, \dots, 0), (0, \beta, \dots, 0), 0, \dots, \beta), x; 0]. \quad (23)$$

Consider the problem  $(T', 0)$ . Since  $T$  is symmetric,  $d = 0$ , and  $x$  is the only symmetric element in  $WP(T')$ , then by W.P.O. and SYM,  $F(T', 0) = x$ . Also, since  $T' \subset S$  and  $a(S, d) = a(T', 0) = y$ , by R.MON,  $F(S, d) \geq F(T', d) = x$ . Thus  $F(S, d) = x = K(S, d)$ .

Case ii)  $S \not\subset \mathbf{R}_+^n$ . Let  $V$  be defined as follows,

$$V \equiv T \cup \left\{ \bigcup_{\pi \in \Pi} \pi(S) \right\}. \quad (24)$$

Note that  $V$  is symmetric and  $S \subset V$ . If we replace

$(T, 0)$  the previous argument with  $(V, 0)$  the proof goes through as before. •

## 4. Conclusion

Since Nash's pioneering treatment of the bargaining problem, most authors have maintained the assumption of convexity of the feasible set. In this paper we have dispensed with convexity. We proposed a new solution, the Nash Extension, and demonstrated that it retains several of the desirable features of the Nash solution on the domain of comprehensive problems, while coinciding with the Nash solution when the problem is convex.

Our main result is a characterization of the new solution, employing a new axiom, Ethical Monotonicity. Additionally we demonstrate that axiomatic characterizations of several well known solutions can be extended to the domain of problems that are merely comprehensive. This does not seem to be a very strong restriction on the domain since it is implied by an assumption of freely disposable utility.

This work suggests that the assumption of a convex feasible set is not essential for any Monotone Path Solution. Since any Monotone Path Solution is well-defined on the domain of comprehensive problems any characterization found on the domain of convex problems should be easy to adapt. This class of solutions is discussed and axiomatized Thomson (1986), pp52-57. A second class of solutions that are well defined on the domain of convex problems is the class of strictly concave social welfare functions. The Nash solution is the most widely known of these. The class of solutions represented by an additively separable social welfare function has recently been axiomatized by Lensberg(1988). It would be of interest to see if our method of constructing the Nash Extension could be employed to define a new solution or solution class on the domain of comprehensive problems, which coincides with the selection of the given social welfare function or class of functions, when the problem is convex. A characterization could then be attempted using E.MON or some similar axiom.

We close by remarking that it may also be of interest to study more general domains. For example, suppose agents cannot necessarily dispose of utility freely, but they can "agree some of the time and disagree some of the time," then we have a domain of problems which the feasible sets are star-shaped with respect to the disagreement point.

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