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On Symmetric Cournot-Nash Equilibrium Distributions in a Finite-Action, Atomless Game

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On Symmetric Cournot-Nash Equilibrium
Distributions in a Finite-Action,
Atomless Game†

by

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Abstract. We show that in a finite-action, atomless game, every Cournot-Nash equilibrium distribution can be "symmetrized." This yields an elementary proof of a result of Mas-Colell.

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I. Introduction

In [4], Mas-Colell showed the existence of a Cournot-Nash equilibrium distribution (CNED) as a consequence of the Fan-Glicksberg theorem. Mas-Colell also showed the existence of a symmetric CNED in finite-action, atomless games as a consequence of the Kakutani fixed point theorem and results in the theory of integration of correspondences. These results consist, in particular, of Lyapunov's theorem on the range of a vector measure, Aumann's measurable selection theorem, as well as his theorem on the upper hemicontinuity of the integral of a correspondence with upper-hemicontinuous values; on all of this [1] is a standard reference.

In this note, we show that in a finite-action, atomless game every CNED can be "symmetrized" to yield a symmetric CNED. This allows us to deduce Mas-Colell's result on the existence of a symmetric CNED from his first result on the existence of a CNED. The proof of our result is elementary in the sense that it uses only Lyapunov's theorem on the convexity of the range of a scalar measure.

Section 2 recalls the model and presents the results. Section 3 gives the basic idea of the proof and Section 4 is devoted to the formalities of the proof. Section 5 concludes with a remark.

2. The Model and Results

We recall for the reader's convenience the basic definitions from [4]. Let A be a compact, metric space of actions, \mathcal{M} the set of Borel probability measures on A endowed with the weak $*$ topology and \mathcal{U}_A is the space of continuous from $A \times \mathcal{M}$ into \mathbb{R} and endowed with the supremum-norm topology. A game is a Borel probability measure on \mathcal{U}_A .

A Borel probability measure τ is said to be a Cournot-Nash equilibrium distribution (CNED) of the game μ if the marginal of τ on \mathcal{U}_A , τ_u , is μ and $\tau(B_\tau) = 1$ where $B_\tau = \{(a, u) \in A \times \mathcal{U}_A : u(a, \tau_A) \geq u(a, \tau_A)\}$ for all $a \in A\}$ and τ_A denotes the marginal of τ on A . τ is said to be a symmetric Cournot-Nash equilibrium distribution if τ is a CNED and there exists a measurable function $h: \mathcal{U}_A \rightarrow A$ such that $\tau(\text{graph } h) = 1$. We shall say that every CNED τ can be symmetrized if there exists a symmetric CNED τ^s such that $B_{\tau^s} = B_\tau$.

We can now state

Theorem. Every Cournot-Nash equilibrium distribution of a game μ with action set A can be symmetrized if μ is atomless and A is finite.

This yields as a corollary

Corollary (Mas-Colell): A symmetric Cournot-Nash equilibrium distribution exists for a game μ with action set A whenever μ is atomless and A is finite.

The Corollary is an easy consequence of our theorem and Theorem 1 of [4].

3. Heuristics of the Proof

We illustrate the basic idea of the proof of our theorem by considering an action set with two elements. The reader may wish to keep Figure 1 in mind as we go through the argument.

Let τ be the CNED of a game μ with action set $\{a_1, a_2\}$. Let the set B_τ of all pay-offs and corresponding pay-off maximizing actions be

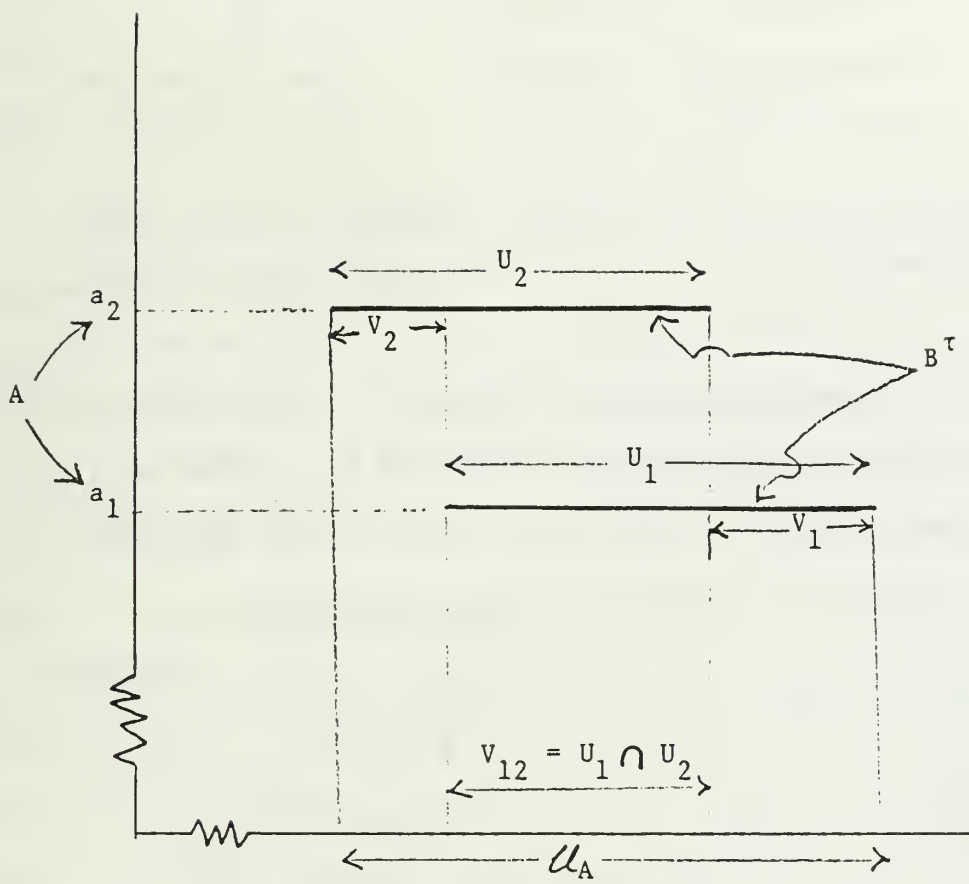


Figure 1

denoted by the set $(a_1 \times U_1) \cup (a_2 \times U_2)$. Unlike Figure 1, U_1 and U_2 need not necessarily be connected sets. Suppose, again unlike Figure 1, that $U_1 \cap U_2 = \phi$. Since $U_1 \cup U_2 = \mathcal{U}_A$, τ can be shown to be symmetric CNED simply by letting $h(u) = a_i$ for all $u \in U_i$, for all $i = 1, 2$. Certainly h is measurable and $\tau(\text{graph } h) = 1$. Thus, in the case $U_1 \cap U_2 = \phi$, there is nothing to prove.

Suppose $U_1 \cap U_2 \neq \phi$. The basic idea in this case is to "disjointify" U_1 and U_2 , i.e., to construct measurable subsets $U_i^* \subset U_i$ for all $i = 1, 2$, such that $U_1^* \cap U_2^* = \phi$. Since μ is atomless, this can be done in a number of ways but the important consideration is to do this in such a way that the marginal of τ on A , τ_A , does not change. Since B_τ depends only on τ_A , this ensures that B_τ does not change. We now briefly spell out the mechanics of such a procedure.

Let $V_i = U_i - U_j$, $i = 1, 2$, $j \neq i$, and $V_{12} = U_1 \cap U_2$. Find measurable subsets V_{12}^1, V_{12}^2 of V_{12} such that $V_{12}^1 \cap V_{12}^2 = \phi$, $V_{12}^1 \cup V_{12}^2 = V_{12}$ and $\mu(V_{12}^1) = \tau(a_1 \times V_{12})$, $i = 1, 2$. Since $\sum_{i=1}^2 \tau(a_i \times V_{12}^i) = \tau(\bigcup_{i=1}^2 (a_i \times V_{12}^i)) = \tau(\{a_1, a_2\} \times V_{12}) = \tau_A(V_{12}) = \mu_A(V_{12})$, Lyapunov's theorem on the range of an atomless scalar measure guarantees that V_{12}^1 and V_{12}^2 can be found. Now let $U_i^* = V_i \cup V_{12}^i$, $i = 1, 2$. These are the sets that work by letting $h: \mathcal{U}_A \rightarrow A$ be a function such that $h(u) = a_i$ for all $u \in U_i^*$, for all $i = 1, 2$. Now let $\tau^S(B) = \mu\{u \in \mathcal{U}_A: (h(u), u) \in B\}$ for any measurable subset B of $A \times \mathcal{U}_A$. τ^S is the symmetric CNED. The only point which needs to be checked is that $\tau_A^S = \tau_A$. But $\tau_A(\{a_i\}) = \tau(a_i \times \mathcal{U}_A) = \tau(a_i \times U_i) = \tau(a_i \times V_i) + \tau(a_i \times V_{12}^i) = \mu(V_i) + \mu(V_{12}^i) = \mu(V_i \cup V_{12}^i) = \mu(U_i^*) = \mu\{u \in \mathcal{U}_A: (h(u), u) \in (a_i \times \mathcal{U}_A)\} = \tau_A^S(\{a_i\})$.

4. Proof of the Theorem

We begin with an elementary lemma.

Lemma 1. Let A_i ($i = 1, \dots, k$) and B be arbitrary sets. Then

$$\bigcup_{i=1}^k (A_i \times B) = ((\bigcup_{i=1}^k A_i) \times B).$$

Proof: Straightforward. ||

Our next lemma is a simple consequence of Lyapunov's theorem on the range of a scalar measure.

Lemma 2. Let (S, \mathcal{S}, μ) be an atomless measure space. If $V \in \mathcal{S}$,

$$\mu(V) = \sum_{i=1}^n \lambda_i \text{ with } \lambda_i \geq 0 \text{ for all } i, \text{ there exist for all } i = 1, \dots, n,$$

$$V^i \in \mathcal{S} \text{ such that } V^i \cap V^j = \emptyset \text{ (} i \neq j \text{), } \bigcup_{i=1}^n V^i = V \text{ and } \mu(V^i) = \lambda_i.$$

Proof: We shall prove the lemma by induction. The lemma is trivially true for $n = 1$. Assume it to be true for $n = k$ and let $V \in \mathcal{S}$ with

$$\mu(V) = \sum_{i=1}^{k+1} \lambda_i, \lambda_i \geq 0 \text{ for all } i = 1, \dots, k+1.$$
If $\lambda_i = 0$ for any i , we are reduced to the case of $n = k$ and the proof is completed by letting $V_i = \emptyset$ for that i . Thus, suppose $\lambda_i > 0$ for all i . Let

$$\lambda(1) = \sum_{i=1}^k \lambda_i / \sum_{i=1}^{k+1} \lambda_i \text{ and } \lambda(2) = 1 - \lambda(1).$$
By Lyapunov's theorem [1, p. 45], we can find $V^{k+1} \in \mathcal{S}$ such that $\mu(V^{k+1}) = \lambda_{k+1}$. Since $(V - V^{k+1}) \in \mathcal{S}$, and $\mu(V - V^{k+1}) = \sum_{i=1}^k \lambda_i$, we use the induction hypothesis to complete the proof. ||

Before we present the proof of Theorem, we develop some notation. Let I denote the set $\{1, 2, \dots, n\}$ and $P(I)$ the set of subsets of I , including the empty set. For any $\pi \in P(I)$, let π^c denote the complement of π in I . Let $P^m(I) = \{\pi \in P(I) : m \in \pi\}$. We shall use the

convention that a union over the empty set is the empty set. We also use the same notation for a point and a set consisting solely of that point.

Proof of Theorem

Let τ be the Cournot-Nash equilibrium distribution of the game μ .

Let $U_i = \text{proj}_{\mathcal{U}_A}(B_\tau \cap (a_i \times \mathcal{U}_A))$ for all $i \in I$

$$(1) \quad \bigcup_{i \in I} U_i = \mathcal{U}_A$$

Certainly $U_i \subset \mathcal{U}_A$ for $i \in I$. On the other hand, let $u \in \mathcal{U}_A$. Certainly there exists $k \in I$ such that $u(a_k, \tau) \geq u(a_i, \tau)$. Then $(a_k, u) \in B_\tau$ and hence $u \in U_k$.

$$(2) \quad B_\tau = \bigcup_{i \in I} (a_i \times U_i)$$

Certainly $(a_i \times U_i) \subset B_\tau$ for all $i \in I$. Now any element x of B_τ can be written as (a_i, u) for some $i \in I$ and some $u \in \mathcal{U}_A$. Hence $u \in U_i$ and $x \in (a_i \times U_i)$.

$$(3) \quad \tau(a_i \times U_i) = \tau(a_i \times \mathcal{U}_A)$$

Since $(a_i \times U_i) \subset (a_i \times \mathcal{U}_A)$, certainly $\tau(a_i \times U_i) \leq \tau(a_i \times \mathcal{U}_A)$. Suppose there exists $i \in I$ such that strict inequality holds for that i .

Then $1 = \tau(B_\tau) = \tau(\bigcup_{i \in I} (a_i \times \mathcal{U}_A)) = \tau(A \times \mathcal{U}_A)$, a contradiction to the fact that τ is a probability measure.

For any $\pi \in P(I)$, let $V_\pi = (\bigcap_{i \in \pi} U_i) - (\bigcup_{i \in \pi^c} U_i)$.

$$(4) (a) \bigcup_{\pi \in P(I)} V_{\pi} = \mathcal{U}_A, \quad (b) V_{\pi} \cap V_{\sigma} = \phi(\pi, \sigma \in P(I), \pi \neq \sigma), \quad (c) \bigcup_{\pi \in P^1(I)} V_{\pi} = U_i.$$

For (a), pick $u \in \mathcal{U}_A$. Let $\sigma = \{i \in I : u \in U_i\}$. By (1), $\sigma \neq \phi$. Then $u \in V_{\sigma}$. On the other hand, $u \in \bigcup_{\pi \in P(I)} V_{\pi}$ implies that there exists $\sigma \in P(I)$, $\sigma \neq \phi$ such that $u \in V_{\sigma}$. Hence $u \in U_i$ for all $i \in \sigma$ and hence, by (1), $u \in \mathcal{U}_A$. For (b), suppose there exists π, σ in $P(I)$ such that $\pi \neq \sigma$ and $V_{\pi} \cap V_{\sigma} \neq \phi$. Since V_{π} and V_{σ} are nonempty, π and σ are nonempty. Then there exists $i \in \pi$, $i \notin \sigma$. Now $u \in V_{\pi} \cap V_{\sigma}$ implies $u \in U_i$. Since $i \in \sigma^c$, $u \notin V_{\sigma}$ which is a contradiction. For (c), pick $u \in \bigcup_{\pi \in P^1(I)} V_{\pi}$. Then there exists $\pi \in P^1(I)$ such that $u \in V_{\pi}$. Since $i \in \pi$, $u \in U_i$. On the other hand, for any $u \in U_i$, let $\sigma = \{j \in I, u \in U_j\}$ and $\pi = \{i\} \cup \sigma$. Certainly $u \in V_{\pi}$ and $\pi \in P^1(I)$.

$$(5) \text{ For any } \pi \in P(I), \exists \text{ measurable } V_{\pi}^i (i \in I), V_{\pi}^i \cap V_{\pi}^j = \phi(i \neq j), \bigcup_{i \in \pi} V_{\pi}^i = V_{\pi}$$

$$\text{and } \mu(V_{\pi}^i) = \tau(a_i \times V_{\pi})$$

Observe that $\mu(V_{\pi}) = \tau_{\mathcal{U}_A}(V_{\pi}) = \tau(A \times V_{\pi}) = \tau((\bigcup_{i \in I} a_i) \times V_{\pi})$ which, by Lemma 1, equals $\tau((\bigcup_{i \in I} a_i \times V_{\pi})) = \sum_{i \in I} \tau(a_i \times V_{\pi})$. We can now apply Lemma 2 to complete the proof of (5).

$$\text{Now let } U_i^* = \bigcup_{\pi \in P^1(I)} V_{\pi}^i.$$

$$(6) \text{ For all } i \in I, (a) U_i^* \subset U_i, (b) U_i^* \cap U_j^* = \phi(i \neq j), (c) \bigcup_{i \in I} U_i^* = \mathcal{U}_A$$

To see (a), pick $u \in U_i^*$. Then there exists $\pi \in P^1(I)$ such that $u \in V_{\pi}^i$. This implies $u \in V_{\pi}$. Since $i \in \pi$, $u \in U_i$. (b) follows from the fact that for $i \neq j$, $V_{\pi}^i \cap V_{\pi}^j = \phi$ on the one hand, and from

$V_\pi \cap V_\sigma = \emptyset$ for $\pi \neq \sigma$ on the other. For (c), note that

$$\bigcup_{i \in I} U_i^* = \bigcup_{i \in I} \bigcup_{\pi \in P^i(I)} V_\pi^i = \bigcup_{i \in I} \bigcup_{\pi \in P(I)} V_\pi^i = \bigcup_{\pi \in P(I)} \bigcup_{i \in I} V_\pi^i = \bigcup_{\pi \in P(I)} V_\pi =$$

\mathcal{U}_A , the last step from (4a).

$$(7) \quad \mu(U_i^*) = \tau(a_i \times U_i) \text{ for all } i \in I.$$

The left hand side equals $\mu(\bigcup_{\pi \in P^i(I)} (V_\pi^i))$. Since $V_\pi^i \subset V_\pi$ by (5), and

$V_\pi \cap V_\sigma = \emptyset$ for $\pi \neq \sigma$ by (4c), this equals $\sum_{\pi \in P^i(I)} \mu(V_\pi^i)$. By (5), this

equals $\sum_{\pi \in P^i(I)} \tau(a_i \times V_\pi)$ which equals $\tau(\bigcup_{\pi \in P^i(I)} (a_i \times V_\pi))$. By Lemma 1,

this can be written as $\tau(a_i \times \bigcup_{\pi \in P^i(I)} V_\pi)$ and hence by (4b) as $\tau(a_i \times U_i)$.

We are now ready to construct our symmetric Cournot-Nash equilibrium distribution. Let $h: \mathcal{U}_A \rightarrow A$ be such that $h(u) = a_i$ for all $u \in U_i^*$, for all $i \in I$. Since V_π^i are measurable, U_i^* are measurable. Moreover, from (vi), h is a well-defined function. Now let τ^S be a measure on $A \times \mathcal{U}_A$ such that for any measurable B , $\tau^S(B) = \mu\{u \in \mathcal{U}_A: (h(u), u) \in B\}$. Given measurability of h and the identity map, τ^S is well-defined. Also

$$\tau^S(\text{graph } h) = \mu\{u \in \mathcal{U}_A: (h(u), u) \in (\text{graph } h)\} = \mu\{u \in \mathcal{U}_A\} = 1.$$

All that remains to be shown is that τ^S is a Cournot-Nash equilibrium distribution. Towards this end, we first show that $\tau^S_{\mathcal{U}_A} = \mu$. Pick any measurable subset W of \mathcal{U}_A . Then $\tau^S_{\mathcal{U}_A}(W) = \tau^S(A \times W) = \mu\{u \in \mathcal{U}_A: (h(u), u) \in A \times W\} = \mu\{u \in (\mathcal{U}_A \cap W)\} = \mu(W)$.

Next, we show $\tau^S_A = \tau_A$. Pick any measurable subset of A . If this set is empty, there is nothing to be shown. Hence, let this set be

$$\bigcup_{i \in \pi} a_i \text{ for some } \pi \in P(I). \text{ Now } \tau_A^S(\bigcup_{i \in \pi} a_i) = \tau^S(\bigcup_{i \in \pi} a_i \times \mathcal{U}_A) =$$

$$\mu\{u \in \mathcal{U}_A : (h(u), u) \in ((\bigcup_{i \in \pi} a_i) \times \mathcal{U}_A)\} = \mu\{u \in \mathcal{U}_A : h(u) = a_i, \\ i \in \pi\} = \mu(\bigcup_{i \in \pi} h^{-1}(a_i)) = \sum_{i \in \pi} \mu(U_i^*). \text{ Now}$$

$$\sum_{i \in \pi} (U_i^*) = \sum_{i \in \pi} \tau(a_i \times U_i) \quad (\text{by (7)})$$

$$= \sum_{i \in \pi} \tau(a_i \times \mathcal{U}_A) \quad (\text{by (2)})$$

$$= \tau(\bigcup_{i \in \pi} (a_i \times \mathcal{U}_A))$$

$$= \tau((\bigcup_{i \in \pi} a_i) \times \mathcal{U}_A) \quad (\text{by Lemma 1})$$

$$= \tau_A(\bigcup_{i \in \pi} a_i).$$

We are done.

Since $\tau_A^S = \tau_A$ and since B_τ depends only on τ_A , $B_{\tau_s} = B_\tau$. Thus to show $\tau^S(B_{\tau_s}) = 1$. But by the definition of h , $\text{graph } h \subset B_\tau$. Since $\tau^S(\text{graph } h) = 1$, $\tau^S(B_{\tau_s}) = \tau^S(B_\tau) = 1$. The proof of the theorem is complete. ||

5. Concluding Remark

In [2, 3], the authors present an alternative formulation of Mas-Colell's result in games where pay-offs are represented by preference relations or by functions which are upper-semicontinuous in actions. We remark that the theorem proved here applies to that generalized set-up.

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