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Pareto Optimal Allocations of Non-Convex Economies in Locally Convex Spaces

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Pareto Optimal Allocations of Non-Convex Economies in Locally Convex Spaces[†]

by

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Abstract. We extend the second fundamental theorem of welfare economics to economies with non-convex production sets, public goods and with an ordered locally convex space of commodities. Our work applies the hypertangent cone to this problem and makes essential use of Rockafellar's extension of the Clarke tangent cone to locally convex spaces.

Key Words: Pareto optimal allocations, hypertangent cones, epi-Lipschitzian, non-convex sets, public goods.

AMS(MOS) Subject Classifications (1979): Primary 90A14, 49B34. Secondary 90C48.

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I. Introduction

The second fundamental theorem of welfare economics, as formulated by Arrow [1] and Debreu [5], states that every Pareto optimal allocation of a convex economy with a finite number of commodities can be sustained as a competitive equilibrium with an appropriate redistribution of income. In subsequent work, Debreu [6] extended this theorem to a general setting of a linear topological space of commodities and formalized the notion of a price system as a non-zero element of the topological dual of such a space. The only additional requirement for the infinite-dimensional set-up was the assumption that the aggregate production set has an interior point.

More than twenty years after the initial work of Arrow-Debreu, Guesnerie [8] extended the second fundamental theorem to non-convex economies; in particular to economies with certain kinds of non-convex technologies (production sets). Guesnerie showed that corresponding to every Pareto optimal allocation of such an economy, there exists a system of prices which are marginal cost prices for the non-convex production sets and such that, with redistribution, competitive behavior at these prices by the convex sector of the economy sustains the given Pareto optimal allocation. Guesnerie formalized the notion of marginal cost prices as the Dubovickii-Miljurin normal cone to a given non-convex set. In a recent paper [10], Khan-Vohra have generalized Guesnerie's results to economies with arbitrarily non-convex production sets through the use of the Clarke-Rockafeller normal cone. It is worth noting, however, that both [8] and [10] remain in the setting of an Euclidean space of commodities.

At this stage, a natural question arises as to whether the results of Guesnerie-Khan-Vohra can be placed in a general setting of an infinite dimensional commodity space much in the same spirit as Debreu's extension of the Arrow-Debreu theory. We report such an extension here. Our generalization is based on Rockafellar's [14, 15] extension to locally convex spaces of the Clarke-Rockafellar theory of non-smooth optimization (see [4]) and we make essential use of his concept of sets which are epi-Lipschitzian at a particular point.

It is worth emphasizing that the full power of Rockafellar's theory is needed for the results that we present in the sequel. Even though economists use specific Banach spaces as the underlying commodity space, typically they formulate the price system as a non-zero element of the predual of such a Banach space; in addition to [6], see [2], [11] or [12]. As such, they are in a locally convex set-up and without the metric necessary for Clarke's [3] definition of his tangent cone.

Two final introductory remarks. So far, public goods have been studied only in a finite dimensional setting and a secondary contribution of our paper is its treatment of public goods in the full generality of a locally convex space. How fruitful such an extension proves to be in terms of the economics must remain an open question. Secondly, it is worth pointing out that our results have no bearing for exchange economies whose agents have preferences defined on consumption sets with empty topological interiors. As of this writing, the problems arising in this context have not been fully resolved even in the convex setting, see Mas-Colell [13].

The plan of the paper is as follows. Section 2 presents the mathematical preliminaries, Section 3 the economic model and principal results and Section 4 the proofs.

2. Mathematical Preliminaries

Let E be an ordered linear topological space with a locally convex Hausdorff topology τ and with the ordering $\geq$ induced by the closed convex cone K. Let $E_+ = \{x \varepsilon E | x \geq 0\}$.

Let E* be the topological dual of E and for any $f \varepsilon E^*$, $x \varepsilon E$, let the evaluation be denoted by $\langle f,x \rangle$.

For any positive integer k, let E^k denote the k-fold Cartesian product of E and endowed with the product topology.

For any set $C \subseteq E$, Cl C denotes the closure of C in E and for any set $B \subseteq E$, C/B denotes set-theoretic subtraction. For any $z \varepsilon E$, $\mathfrak{D}(z)$ denotes the collection of all neighborhoods of z.

We can now develop the mathematical concepts that we shall be using. The following definition is taken from [15, p. 262-263].

__Definition 2.1__. For any $C \subseteq E$, and any $x \varepsilon C$, the __tangent cone__ to C at x, T(C,x), consists of $y \varepsilon E$ if and only if for all $Y \varepsilon \mathfrak{D}(y)$, there exist $X \varepsilon \mathfrak{D}(x)$ and $\lambda > 0$ such that $(x' + \mu Y) \cap C \neq \phi$ for all x' in $C \cap X$ and for all $\mu \varepsilon (0,\lambda)$.

__Definition 2.2__. For any $C \subseteq E$, the __polar cone__ of C, C^+, is given by $\{p \varepsilon E^* | \langle p,x \rangle \leq 0\}$.

__Definition 2.3__. For any $C \subseteq E$, and any $x \varepsilon C$, the __normal cone__ to C at x, N(C,x), is defined as $T(C,x)^+$.

Definition 2.4. For any $C \subseteq E$, and any $x \in C$, the hypertangent cone to C at x, H(C,x), is defined as $\{y \in E \mid$ there exist $X \in \mathfrak{X}(x)$, $Y \in \mathcal{B}(y)$, $\lambda > 0$ such that $(x' + \mu y') \in C$ for all $x' \in C \cap X$, $y' \in Y$ and $\mu \in (0,\lambda)\}$.

Note that our definition of H(C,x) is identical to that of Clarke [4, p. 57] and is slightly different from that of Rockafellar [15, p. 267].

Definition 2.5. A set $C \subseteq E$ is said to be epi-Lipschitzian at x if there exists $y \in E$ such that $y \in H(C,x)$.

Proposition 2.1 (Rockafellar). For any set $C \subseteq E$ and any $x \in C$, H(C,x) is an open, convex cone with vertex 0. If C is epi-Lipschitzian at x, $T(C,x) = \text{Cl } H(C,x)$. If, in addition, x is a boundary point of C, then the set $C' = (E/C) \cup \{x\}$ is likewise epi-Lipschitzian at x, and $T(C,x) = -T(C',x)$.

Proof: Follows directly from Corollary 2 of Rockafellar [15, p. 268] given that our definition of H(C,x) coincides with that of the set K defined in the last paragraph of p. 269 in [15]. $\square$

From Rockafellar's Proposition we can deduce certain useful properties of epi-Lipschitzian sets. These are well known and we provide proofs for completeness.

Lemma 2.1. For any $C \subseteq E$ and $x \in C$,

(a) $H(C,x)^+ = N(C,x)$ if $H(C,x) \neq \phi$.

(b) If C is epi-Lipschitzian at x and x is a boundary point of C, then $N(C,x) \neq \{0\}$.

(c) If E_+ has a non-empty interior and $C + E_+ \subseteq C$ or $C - E_+ \subseteq C$, then C is epi-Lipschitzian at any $x \in C$.

<u>Proof of Lemma 2.1</u>

(a) By Proposition 2.1, $H(C,x) \subseteq T(C,x)$ and therefore $N(C,x)^+ \subseteq H(C,x)$. To show the reverse inequality, suppose $p \in H(C,x)^+$ and there exists $z \in T(C,x)$ such that $\langle p,z \rangle > 0$. Since the set $A = \{v \in E \mid \langle p,v \rangle > 0\}$ is open, and $T(C,x) = \text{Cl } H(C,x)$, there exists $\bar{z}$ such that $\langle p,\bar{z} \rangle > 0$ and $\bar{z} \in H(C,x)^+$, a contradiction to the fact that $p \in H(C,x)^+$.

(b) If $N(C,x) = \{0\}$, then $T(C,x) = E$ and 0 is an interior point of $T(C,x)$ and since $H(C,x) \neq \phi$, $0 \in H(C,x)$, i.e., there exists $z \in \mathcal{B}(0)$, $X \in \mathcal{B}(x)$ and $\lambda > 0$ such that $x' + \mu z' \in C$ for $x' \in X \cap C$, $z' \in Z$ and $\mu \in (0,\lambda)$. But this implies that $x + \mu z' \in C$ for all $z' \in Z$, i.e., x is an interior point of C, a contradiction.

(c) Since E_+ has an interior point there exists a z and $Z \in \mathcal{B}(z)$ such that $Z \subseteq E_+$. If $C + E_+ \subseteq C$, for any $x \in C$, $x + \mu z' \in C$ for $z' \in Z$, $\mu \in (0,1)$. This means that for any $x \in C$, there exist $Z \in \mathcal{B}(z)$, $X \in \mathcal{B}(x)$ and $\lambda = 1$ such that $x' + \mu z' \in C$ for $x' \in X \cap C$, $z' \in Z$ and $\mu \in (0,1)$. The other case is identical. □

<u>Lemma 2.2</u>. Let $x = (x_1,x_2) \in C^1 \times C^2$ where C^1 and C^2 are subsets of E. Then, $H(C^1 \times C^2,x) = H(C^1,x^1) \times H(C^2,x^2)$ and $N(C^1 \times C^2,x) = N(C^1,x^1) \times N(C^2,x^2)$.

<u>Proof</u>. Follows from the definitions as in [4, Corollary 2.4.5]. □

Lemma 2.3. Let E^{*k} be the k-fold Cartesian product of E*. Then E^{*k} can be identified with the dual of E^k such that for any $x = (x^i)\varepsilon E^k$, $x^* = (x^{*i})\varepsilon E^{*k}$, the canonical bilinear form is given by

$$\langle x^*, x\rangle = \sum_{i=1}^{k} \langle x^{*i}, x^i\rangle.$$

Proof. See [9, p. 266].

3. The Model and Results

An economy consists of a finite number of consumers and a finite number of firms. We shall index consumers by t, t = 1,...,T, and shall assume that each has a consumption set $X^t \subseteq E$ and a reflexive preference relation $\succsim_t$. Let the "better-than-set" for t at x^t be given by $P^t(x^t) = \{y\varepsilon X^t | y \succ_t x^t\}$ and the "no-worse-than-set" by $\overline{P}^t(x^t) = \{y\varepsilon X^t | y \succsim_t x^t\}$. Firms are indexed by j, j = 1,...,F, and each has a production set $Y^j \subseteq E$. The aggregate endowment is denoted by $w\varepsilon E_+$. An economy is thus denoted by $\mathcal{E} = ((X^t, \succsim_t)_1^T, (Y^j)_1^F, w)$ and we shall need the following concepts for it.

Definition 3.1. $((x^{*t}),(y^{*j}))$ is an allocation of $\mathcal{E}$ if for all $t = 1,\dots,T$, $x^{*t}\varepsilon X^t$, for all $j = 1,\dots,F$, $y^{*j}\varepsilon Y^j$ and $\sum_t x^{*t} - \sum_j y^{*j} \leq w$.

Definition 3.2. An allocation $((x^{*t}),(y^{*j}))$ is Pareto optimal if there does not exist any other allocation $((x^t),(y^j))$ such that $x^t\varepsilon\overline{P}^t(x^{*t})$ for all t and $x^t\varepsilon P^t(x^{*t})$ for at least one t.

For our first result, we shall need the following assumptions.

A.1 For all t and all $x^t\varepsilon X^t$, $\overline{P}^t(x^t) = \mathrm{Cl}\ P^t(x^t)$ and $P^t(x^t) = \mathrm{Int}\ \overline{P}^t(x^t)$.

Theorem 3.1. If $((x^{*t}),(y^{*j}))$ is a Pareto optimal allocation, (A1) is satisfied and $\overline{P}^t(x^{*t})$ and Y^j are respectively epi-Lipschitzian at $((x^{*t}),(y^{*j}))$, then there exists $p^* \varepsilon E^*$, $p^* \neq 0$ such that

(a) $-p^* \varepsilon N(\overline{P}^t(x^{*t}),x^{*t})$ for all t;

(b) $p^* \varepsilon N(Y^j,y^{*j})$ for all j.

Remark 3.1. While A.1 is a standard assumption in the economics literature, Theorem 3.1 would not be of much economic interest if the epi-Lipschitzian hypothesis for $\overline{P}^t(x^{*t})$ and Y^j is not generally satisfied. It is, therefore, important that "desirability" and "free disposal," which are assumptions frequently used in economics, are sufficient conditions for these sets to be epi-Lipschitzian, provided E_+ has a non-empty interior. Formally, "desirability," is the assumption that for all $x^t \varepsilon X^t$, $\overline{P}^t(x^t) + E_+ \subseteq \overline{P}^t(x^t)$ and can be loosely stated as "more is better." Likewise, "free disposal" is the assumption that

$$Y_j - E_+ \subseteq Y_j,$$

and simply means that if a production plan is technologically feasible, any other production plan which does not provide more of any output and uses less of any output is also technologically feasible. In either case, Lemma 2.1(c) guarantees that the sets are epi-Lipschitzian.

Corollary 3.1. If for any t, $\overline{P}^t(x^{*t})$ is convex, then (a) in Theorem 3.1 implies that there does not exist $x^t \varepsilon P^t(x^{*t})$ such that $\langle p^*,x^t\rangle < \langle p^*,x^{*t}\rangle$. If for any j, Y^j is convex then (b) in Theorem 3.1 implies that there does not exist $\overline{y} \varepsilon Y^j$ such that $\langle p^*,y^j\rangle > \langle p^*,y^{*j}\rangle$.

We invite the reader to compare our Corollary 3.1 with Theorem 2 in [6] and our Theorem 3.1 to Theorem 1 in [10].

Our next result extends Theorem 3.1 to economies with public goods. Recall that a public good is a commodity whose consumption is identical across individuals and such that each individual's consumption is equal to aggregate supply, see [7] and the references therein to the papers of Samuelson. Let E_π refer to the commodity space for private goods and E_g to that for public goods. We shall assume that both E_π and E_g are real, ordered, locally convex Hausdorff spaces. Let $E = E_\pi \times E_g$ where E is endowed with the product topology and the induced ordering.

An economy with public goods $\mathcal{E}^G = ((X^t, \succsim_t)_1^T, (Y^j)_1^F, w)$ is such that for all t, $X^t = (X_\pi^t, X_g^t)$ where $X_\pi^t \subseteq E_\pi$, $X_g^t \subseteq E_g$ are its projections onto the space of private and public goods respectively. We assume that $X_g^t = X_g$ for all t; that $Y^j \subseteq E$ for all j and that $w \varepsilon E$, $w = (w_\pi, 0)$, $w_\pi \varepsilon E_{\pi+}$. Let x_π^t and x_g^t refer to the consumption of the private and public goods respectively. $((x^{*t}),(y^{*j}))$ is an allocation for $\mathcal{E}^G$ if for all j, $y^{*j} \varepsilon Y^j$, $x^{*t} \varepsilon X^t$, $x_g^{*t} = x_g^*$ for all t, and $(\sum_t x_\pi^{*t}, x_g^*) - \sum_j y^{*j} \leq w_\pi$. The definition of a Pareto optimal allocation for $\mathcal{E}^G$ is then identical to the one given in Definition 3.2. We shall also need the following assumption on the desirability of public goods.

(A.2) For all t, $(\overline{x}_\pi^t, \overline{x}_g^t) \varepsilon P^t(x^t)$, $\hat{x}_g \geq \overline{x}_g^t$ implies $(\overline{x}_\pi^t, \hat{x}_g^t) \varepsilon P^t(x^t)$.

We can now present our second result.

Theorem 3.2. If $((x_\pi^{*t}, x_g^{*t}), (y^{*j}))$ is a Pareto optimal allocation, (A1)-(A2) are satisfied, and $\overline{P}^t(x^{*t})$ and Y^j are respectively epi-Lipschitzian at $(x^{*t}), (y^{*j})$, then there exist $p_\pi^* \in E_\pi^*$, $p_g^* \in E_g^*$, $(p_\pi^*, p_g^*) \neq 0$, $p_g^{*t} \in E_g^*$ such that

(a) $\sum_t p_g^{*t} = p_g^*$,

(b) $-(p_\pi^*, p_g^{*t}) \in N(P^t(x^{*t}), x^{*t})$ for all t,

(c) $p^* \in N(Y^j, y^{*j})$ for all j.

Theorem 3.2 is a direct generalization of Theorem 2 in [10] which in turn generalized Foley's Theorem 1 in [7] to the non-convex setting.

4. Proofs

In this section we present proofs of the results stated in Section 3. We begin by proving Theorem 3.2. The proof of Theorem 3.1 will then follow as an easy consequence.

Let $k = T+F$ and for $v \in E^k$, let the t^{th} projection of v, t=1, ..., T, be denoted by x^t and the j^{th} projection of v, j = T+1, ..., F, be denoted by y^j. We can thus write $v = ((x^t), (y^j))$.

We can now present a

Proof of Theorem 3.2. Let $v^k = ((x^t), (y^j)) \in E^k$ where $k = T+F$. Define the following sets in E^k.

$$V(x^*) = \Pi_t \overline{P}^t(x^{*t}) \times \Pi_j Y^j$$

$$W_\pi = \{v \in E^k \mid \sum_t x_\pi^t \leq \sum_j y_\pi^j + w_\pi\}$$

$W_g^t = \{v \varepsilon E^k | x_g^t \leq \Sigma y_g^j\}$ $\qquad t = 1, \ldots, T$

$W = W_\pi \bigcap_t W_g^t$

Since W is non-empty and $v^* \varepsilon W$, $T(W,v^*)$ is non-empty. Given the definition of $V(x^*)$, Lemma 2.2 implies that

(1) $\qquad H(V(x^*),v^*) = \Pi_t H(\overline{P}^t(x^{*t}),x^{*t}) \times \Pi_j H(Y^j,y^{*j}).$

Since the individual hypertangent cones are, by hypothesis, non-empty, $H(V(x^*),v^*) \neq \{\phi\}$.

Next, we prove that $H(V(x^*),v^*) \cap T(W,v^*) = \{\phi\}$. If not, there exists $v \varepsilon ((x^t), (y^j))$ such that $v \varepsilon H(V(x^*),v^*) \cap T(W,v^*)$. Since $v \varepsilon H(V(x^*),v^*)$, we appeal to (1) to assert that for all t, $x^t \varepsilon H(\overline{P}^t(x^{*t}),x^{*t})$ and for all j, $y^j \varepsilon H(Y^j,y^{*j})$. This implies that there exist $U_t \varepsilon \mathcal{B}(0)$, $\lambda_t > 0$ for all t and $U_j \varepsilon \mathcal{B}(0)$, $\lambda_j > 0$ for all j such that

(2) $\qquad (x^{*t} + \mu(x^t + U_t)) \subseteq \overline{P}^t(x_t^*)$ for all $\mu \varepsilon (0, \lambda_t)$, for all t,

(3) $\qquad (y^{*j} + \mu(y^j + U_j)) \subseteq Y^j$ for all $\mu \varepsilon (0, \lambda_j)$, for all j.

Now let $U = \Pi_t U_t \times \Pi_j U_j$. Certainly $U \varepsilon \mathcal{B}(0)$. Since $v \varepsilon T(W,v^*)$, we appeal to Definition 2.1 to assert the existence of $\lambda_w > 0$ such that for all $\mu \varepsilon (0, \lambda_w)$,

(4) $\qquad (v^* + \mu(v+U)) \cap W \neq \phi.$

Now let $\lambda = \text{Min}[(\lambda_t), (\lambda_j), \lambda_w]$. Certainly $\lambda > 0$. Pick $\overline{\mu} \varepsilon (0,\lambda)$, $\overline{\mu} \neq 0$. Then there exists $\alpha = ((\alpha^t), (\alpha^j)) \varepsilon U$ such that

(5) $(v^* + \bar{\mu}(v+\alpha)) \in W$.

Since $(x^{*t} + \mu(x^t+U_t))$ is an open set for all $\mu > 0$, we use A.1 and (2) to assert that

(6) $(x^{*t} + \bar{\mu}(x^t+\alpha^t)) \in P^t(x^*_t)$ for all t.

From (3), we obtain

(7) $(y^{*j} + \bar{\mu}(y^j+\alpha^j)) \in Y^j$ for all j.

Moreover, (5) can be rewritten as

(8) $\sum_t (x_\pi^{*t} + \bar{\mu}(x_\pi^t+\alpha_\pi^t)) \leq \sum_j (y_\pi^{*j} + \bar{\mu}(y_\pi^j+\alpha_\pi^j) + w_\pi$

(8') $(x_g^{*t} + \bar{\mu}(x_g^t+\alpha_g^t)) \leq \sum_j (y_g^{*j} + \bar{\mu}(y_g^j+\alpha_g^j))$ for all t.

Given A.2, (6) and (8') imply

(9) $((x^{*t} + \bar{\mu}(x_\pi^t+\alpha_\pi^t), \sum_j (y_g^{*j}+\bar{\mu}(y_g^j+\alpha_g^j))) \in P^t(x^{*t})$ for all t.

But (7), (8) and (9) contradict the Pareto optimality of v*.

Since H(V(x*),v*) and T(W,v*) are non-empty, convex and disjoint and H(V(x*),v*) is open (by Proposition 2.1), by the Hahn-Banach theorem, there exists $\rho \neq 0$, $\rho \in N(W,v^*)$ and $-\rho \in N(V(x^*),v^*)$.

We shall now use the properties of T(W,v*) to characterize the normal cone to W at v*, N(W,v*). Since W is a convex set, by [15, Theorem 1], T(W,v*) is the same as its closed tangent cone in the sense of convex analysis, i.e., the set $C(W,v^*) = \{C\ell \bigcup_{\lambda>0} \lambda^{-1}(W-\{v^*\})\}$.

For any $z \varepsilon E_\pi$ and any i, i=1, ..., T, define $m^i(z) \varepsilon E^k$ to be the vector of zeros in all coordinates except for (z,0) in the i-th and last coordinates.

Clearly $(m^i(z) + v^*)$ and $(-m^i(z) + v^*)$ are elements of W. Hence $m^i(z)$ and $-m^i(z)$ belong to $C(W,v^*) = T(W,v^*)$. By Lemma 2.1(a), $\langle \rho, v\rangle \leq 0$ for all $v \varepsilon T(W,v^*)$. This implies $\langle \rho, m^i(z)\rangle = 0$ for any $z \varepsilon E_\pi$ and i = 1,...,T. By Lemma 2.3, there exist $\rho_{x^t} \varepsilon E^*$ and $\rho_{y^j} \varepsilon E^*$ such that for any $v = ((x^t),(y^j)) \varepsilon E^k$, $\langle \rho, v\rangle = \sum_t \langle \rho_{x^t}, x^t\rangle + \sum_j \langle \rho_{y^j}, y^j\rangle$. Thus, for any $z \varepsilon E_\pi$ and i, k = 1,...,T, we have $\langle \rho, m^i(z)\rangle = \langle \rho_{x^i_\pi}, z\rangle + \langle \rho_{y^F_\pi}, z\rangle = 0 = \langle \rho, m^k(z)\rangle = \langle \rho_{x^k_\pi}, z\rangle + \langle \rho_{y^F_\pi}, z\rangle$. This implies that for any t, t = 1,...,T, $\rho_{x^t_\pi} = p^*_\pi$. Similarly, by defining $\overline{m}^i(z)$ to be the vector of zero's except for (z,0) in the first and the (T+i)-th coordinate, we can show that for any j, j = 1,...,F, $\rho_{y^j_\pi} = p^*_{y_\pi}$. Moreover, since for any $z \varepsilon E_\pi$, $\langle p^*_\pi, z\rangle + \langle p^*_{y_\pi}, z\rangle = 0$, $-p^*_{y_\pi} = p^*_\pi$.

For any $z_g \varepsilon E_g$, define $m^j_g(z)$ to be the vector consisting of $(0,z_g)$ in every coordinate 1, ..., T and T+j and zero everywhere else. Let $\rho_{x^t_g} = p^{*t}_g$. By the same argument as before, it can be shown that for all j, j=1, ..., F, $\rho_{y^j_g} = -p^*_g = -\sum_t p^{*t}_g$. Let $p^* = (p^*_\pi, p^*_g)$. Thus,

(10) $$\rho = ((p^*_\pi, p^{*1}_g), \ldots, (p^*_\pi, p^{*T}_g), -p^*, \ldots, -p^*).$$

Since $-\rho \in N(V(x^*),v^*)$, we can use (10), (1) and Lemma 2.2 to assert that

$$-(p_\pi^*, p_g^{*t}) \in N(\overline{P}^t(x^{*t}), x^{*t}) \text{ for all } t,$$

$$p^* \in N(Y^j, y^{*j}) \text{ for all } j.$$

Next, we show that $p_g^{*t} \geq 0$ for all t. If not, there exists t and $\overline{x} \in E_{g+}$ such that $\langle p_g^{*t}, \overline{x} \rangle < 0$. Since $H(\overline{P}^t(x^{*t}), x^{*t}) \neq \phi$, pick $z \in H(\overline{P}^t(x^{*t}), x^{*t})$. This implies that there exist $Z \in \mathcal{B}(z)$, $X \in \mathcal{J}(x^{*t})$ and $\lambda > 0$ such that for all $\mu \in (0,\lambda)$

$$(X \cap \overline{P}^t(x^{*t})) + \mu Z \subseteq \overline{P}^t(x^{*t}).$$

This implies, given A.1, that $(x'+\mu z') \in P^t(x^{*t})$ for any $x' \in X \quad \overline{P}^t(x^{*t})$ and any $z' \in Z$. This implies, given A.2, that for any positive integer n, $(x'+\mu z'+n\mu(0,\overline{x})) \in P^t(x^{*t})$. But then we have shown that

$$(X \quad \overline{P}^t(x^{*t})) + \mu(Z+n(0,\overline{x})) \subseteq \overline{P}^t(x^{*t}).$$

Hence $(z+n(0,\overline{x})) \in H(\overline{P}^t(x^{*t}), x^{*t})$ for all positive integers n. This implies

$$-\langle (p_\pi^*, p_g^{*t}), z \rangle - n \langle p_g^{*t}, \overline{x} \rangle \leq 0.$$

Sinze $z \in H(\overline{P}^t(x^{*t}), x^{*t})$,

$$-\langle (p_\pi^*, p_g^{*t}), z \rangle \leq 0,$$

and we obtain an absurdity.

We can now assert that $p^* = (p_\pi^*, p_g^*) \neq 0$. If not, then $p_g^{*t} = 0$ for all t and hence from (10), $\rho = 0$, a contradiction.

The proof is now complete. □

Remark 4.1. The version of the Hahn Banach theorem that we use in the proof of Theorem 3.2 does not require E to be locally convex, only that it be a topological vector space.

Proof of Theorem 3.1

Let $V(x^t)$ be defined as in the proof of Theorem 3.2 and let

$$\overline{W} = \{v \in E^k \mid \sum_t x^t \leq \sum_j y^j + w\}.$$

The proof now follows from that of Theorem 3.2 if we use $\overline{W}$ instead of W. □

Proof of Corollary 3.1. The corollary follows easily from the result that for a convex set $A \subseteq E$, $z \in A$, $T(A,z)$ is identical to the closed tangent cone to A at z in the sense of convex analysis (see Theorem 1 in Rockafellar [15]). □

Remark 4.2. Remark 4.1 also applies to the proof of Theorem 3.1.

REFERENCES

[1] Arrow, K. J., "An Extension of the Basic Theorem of Classical Welfare Economics," Proceedings of the Second Berkeley Symposium (1951), University of California Press.

[2] Bewley, T. F., "Existence of Equilibria in Economies with Infinitely Many Commodities," Journal of Economic Theory, 4 (1972), 514-540.

[3] Clarke, F. H., "Generalized Gradients and Applications," Transactions of the American Math. Society, 205 (1975), 247-262.

[4] Clarke, F. H., Optimization and Nonsmooth Analysis, John Wiley (1983).

[5] Debreu, G., "The Coefficient of Resource Utilization," Econometrica 19 (1951), 273-292.

[6] Debreu, G., "Valuation Equilibrium and Pareto Optimum," Proceedings of the National Academy of Sciences, 40 (1954), 588-92.

[7] Foley, D. K., "Lindahl's Solution and the Core of an Economy with Public Goods," Econometrica, 38 (1970), 66-72.

[8] Guesnerie, R., "Pareto Optimality in Non-Convex Economies," Econometrica, 43 (1975), 1-29.

[9] Horváth, J., Topological Vector Spaces and Distributions, Addison-Wesley (1966).

[10] Khan, M. Ali and R. Vohra, "An Extension of the Second Welfare Theorem to Economies with Non-Convexities and Public Goods," Quarterly Journal of Economics, (102) (1987), 223-241.

[11] Majumdar, M., "Some General Theorems on Efficiency Prices with an Infinite-Dimensional Commodity Space," Journal of Economic Theory, 5 (1972), 1-13.

[12] Mas-Colell, A., "A Model of Equilibrium with Differentiated Commodities," Journal of Mathematical Economics, 2 (1975), 263-295.

[13] Mas-Colell, A., "Notes on Pareto Optima in a Linear Space," Harvard University preprint, 1984.

[14] Rockafellar, R. T., "Directionally Lipschitzian Functions and Subdifferential Calculus," Proceedings of the London Mathematical Society, 39 (1979), 331-355.

[15] Rockafellar, R. T., "Generalized Directional Derivatives and Subgradients of Nonconvex Functions," Canadian Journal of Mathematics, XXXII (1980), 257-280.