

A NUMERICAL STUDY OF 3D BLACK HOLE SPACETIMES

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A NUMERICAL STUDY OF 3D BLACK HOLE SPACETIMES

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We present results from numerical evolutions of vacuum black hole spacetimes in 3D Cartesian coordinates. We first studied the Schwarzschild spacetime, comparing extensively with 1D studies. We show that although accurate 3D evolutions are possible, there are a number of difficulties in evolving 3D black holes, for which we suggest approaches to overcome. With current limits on computer memory sizes, we show that with certain slicing conditions, the black hole can be evolved to about $t = 50M$, where M is the black hole mass. We also present the first 3D evolutions of colliding black holes, with evolutions of the axisymmetric Misner two-black hole initial data sets. Here we demonstrate that the techniques we developed for the Schwarzschild case carry over to other spacetimes. We also demonstrate the feasibility of extracting gravitational wave signals during 3D evolutions. We present new, fully 3D distorted black hole initial data sets, extending Bernstein's axisymmetric data sets in a straightforward way. Metric functions and waveforms from evolutions of these data sets in the axisymmetric limit are shown to agree with 2D calculations. For evolutions of the non-axisymmetric sets, which are the first truly 3D black hole data sets to be evolved, all non-trivial wave modes through $\ell = 4$ are presented. Finally, we point the way towards studying the evolution of these data sets with perturbation theory.

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Contents

1	Introduction.....	1
1.1	Organization	2
1.2	Notation and Conventions	3
2	Modeling Black Holes Numerically	4
2.1	The 3+1 (ADM) Formalism	4
2.2	Solving the Initial Data Problem	6
2.3	Evolving the Initial Data	7
2.3.1	Gauge conditions	7
2.3.2	Boundary conditions	8
2.4	The ADM Mass	10
2.5	Extracting Gravitational Radiation	11
2.6	The Apparent Horizon	12
2.7	Computational Issues in 3D	13
3	The Schwarzschild Black Hole	14
3.1	Schwarzschild Initial Data	15
3.2	The Numerical Grid	17
3.3	Results: Geodesic Slicing	21
3.4	Results: Maximal Slicing	29
3.5	Results: Algebraic Slicing	36

3.6	Summary	38
4	Two Black Hole Collision: Misner Initial Data	46
4.1	Misner Initial Data	48
4.2	The Numerical Grid	50
4.3	Evolution of Data Sets with $\mu < 1.8$	52
4.3.1	Slicing Conditions	52
4.3.2	Radiation Extraction	58
4.3.3	Horizons	60
4.4	Evolution of Data Sets with $\mu > 1.8$	61
4.4.1	Metric Functions	63
4.4.2	Radiation Extraction	63
4.4.3	Horizons	63
4.4.4	Newman-Penrose Quantities	68
4.5	Summary	69
5	The 3D Black Hole Plus Brill Wave Spacetime: Initial Data	72
5.1	Background	72
5.2	Extension to 3D	74
5.3	Limits on the Brill Wave Amplitude	75
5.4	Masses	77
5.5	Apparent Horizons	88
5.6	Summary	98
6	The 3D Black Hole Plus Brill Wave Spacetime: Nonlinear Evolu- tions	99
6.1	Slicing Condition	100
6.2	Evolution of Axisymmetric Initial Data Sets	100

6.2.1	Metric Functions	101
6.2.2	Radiation Extraction	106
6.2.3	Horizons	109
6.3	Evolution of 3D Initial Data Sets	111
6.3.1	Metric Functions	114
6.3.2	Radiation Extraction	117
6.4	Summary	125
7	Linear Evolution of Perturbations	126
7.1	The Theory of Schwarzschild Perturbations	127
7.2	Axisymmetric Tests	129
7.3	Summary	131
8	Conclusions and Future Directions	133
 Appendices		
A	Analysis of Numerical Methods	137
A.1	Finite Differences and Convergence	137
A.2	Initial Data	138
A.3	Evolutions	141
B	Numerical Location of Apparent Horizons in 3D	144
B.1	The Method	145
B.2	Code Tests	148
C	Gauge Invariant Radiation Extraction in 3D	161
C.1	Constructing the Zerilli Function	161
C.2	Implementation in 3D	163

Bibliography	165
Vita.....	175

Chapter 1

Introduction

Einstein's theory of general relativity predicts that certain astrophysical phenomena will produce gravitational radiation. Although there exists indirect evidence for the existence of these waves [78], they have never been measured directly. Within the next decade, a number of gravitational wave detectors should go on-line, allowing the observation of gravitational radiation for the first time [5].

One class of events which is expected to produce detectable radiation is the coalescence of two black holes. In order to distinguish such an event from others, and to interpret the data from such an event, it will be necessary to predict the parameters of the waveform for a collision between two black holes as a function of the impact parameter, and the masses and angular momenta of the black holes [49].

In order to predict these parameters, it will be necessary to solve Einstein's equations for the relevant physical situations. Einstein's equations are a set of coupled, nonlinear, partial differential equations. There is no analytic solution for the problem that we want to solve. Perturbation theory is not generally applicable, due to the strong gravitational fields involved. The problem must be solved numerically.

One finds that solving Einstein's equations numerically for even a spherical, stationary black hole is a challenging problem in itself. The general problem is currently intractable. Therefore, a series of small steps has been made over the last decade to work towards the final goal. Much work has been done evolving single, axisymmetric,

distorted black holes [21, 28, 29]. Work has also been done evolving the axisymmetric collision of two black holes initially at rest [14]. In 3D, a simple method has been developed for deriving initial data for multiple black holes with arbitrary linear and angular momenta [26], and evolutions of these data are under investigation [33].

The above evolved spacetimes are axisymmetric, and can be solved with codes that only have two spatial dimensions. The general problem that we want to solve will require a code that can solve problems in three spatial dimensions. In this thesis we describe results obtained by evolving vacuum black hole spacetimes with Einstein's equations of general relativity using a 3D numerical code. We first discuss the evolution of axisymmetric spacetimes, which we studied in order to test the code and develop 3D analysis tools. We then direct our attention to the development of truly 3D initial data sets and their evolution. These data sets represent single, distorted black holes, which we study as a model of the late time behavior of the collision of two black holes.

The work presented in this thesis has been carried out as part of a 5-year, NSF funded, national Grand Challenge collaboration involving the University of Illinois and seven other institutions. The goal of the collaboration is to solve numerically the general 3D black hole collision, with the aim of providing the gravitational wave detection community with a catalog of waveforms for various impact parameters, masses and angular momenta of the black holes. One important step towards this goal is the development of a 3D numerical code which can accurately evolve a single black hole for a long time, as this is the end state of any black hole collision. The work described here represents significant progress towards such a code.

1.1 Organization

This thesis is organized as follows: in Chapter 2 we describe the theoretical framework for the numerical analysis of black hole spacetimes. In Chapters 3 and 4 we

describe the evolutions of two axisymmetric initial data sets with the full 3D code. In Chapters 5, 6, and 7 we describe the generation of truly 3D initial data and its evolution. In Chapter 8 we summarize the work and point towards future directions.

The appendices deal with topics which did not fit naturally into the body of the thesis. Appendix A contains a numerical analysis of the various time-stepping schemes that were used. Appendix B discusses the problem of locating apparent horizons on 3D numerical slices. Finally, Appendix C describes the numerical extraction of gravitational radiation in 3D.

1.2 Notation and Conventions

Throughout this thesis, we will use the unit convention that the gravitational constant G and the speed of light c are unity. Greek indices are taken to run from 0 to 3, and Latin indices from 1 to 3. The metric signature used is $-+++$.

Partial derivatives with respect to a coordinate x^i will be indicated with either the symbol ∂_i , or simply with a comma (*e.g.*, $f_{,i}$). Covariant differentiation with respect to a coordinate x^α on a 4-dimensional spacetime will be indicated with the symbol ∇_α , and covariant differentiation with respect to a coordinate x^i on 3-dimensional slices will be indicated with the symbol D_i . Lie differentiation with respect to a vector field n^α will be indicated with the symbol $\mathcal{L}_n$.

We will be discussing results from numerical computations on 3D Cartesian grids. All results will be for computations performed on grids with the same number of grid points in each direction, and with the same resolution in each direction. Thus, if a value is given for nx , the number of grid points in the x direction, or Δx , the spacing between grid points in the x direction, it is to be assumed that $nx = ny = nz$ and $\Delta x = \Delta y = \Delta z$.

Chapter 2

Modeling Black Holes Numerically

Einstein's equations treat the time and space variables on an equal footing. In order to solve these equations numerically, it is convenient to cast the equations in the form of a Cauchy problem. Once that is done, one must find an appropriate initial data set. Then, to evolve the initial data set, one must specify spatial boundary conditions and a gauge condition, as described below.

2.1 The 3+1 (ADM) Formalism

In the ADM splitting of the Einstein equations, one considers the spacetime to consist of a foliation of three dimensional spatial hypersurfaces, each the level surface of some time coordinate (see, *e.g.*, [85]). The invariant distance between two infinitesimally separated events can then be written as

$$ds^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j. \quad (2.1)$$

The so-called lapse function α determines the proper time $d\tau = \alpha dt$ measured by an observer falling normal to the hypersurface, or slice. The shift vector β^i determines the coordinate distance a constant coordinate point moves away from the normal vector to the slice as one advances from one slice to the next. Finally, γ_{ij} is the metric induced on the given 3D hypersurface.

Einstein's equations can be written in the deceptively simple form $G_{\mu\nu} = 8\pi T_{\mu\nu}$,

where the Einstein tensor $G_{\mu\nu}$ is a nonlinear differential operator on the metric, and $T_{\mu\nu}$ is the stress energy tensor. In vacuum spacetimes $T_{\mu\nu}$ vanishes, and the Einstein equations reduce to setting each component of the 4-dimensional Ricci tensor, ${}^{(4)}R_{\mu\nu}$, to zero.

It is convenient to introduce a tensor known as the extrinsic curvature, K_{ij} , which describes the curvature of the 3D slice in the 4D Lorentzian space in which it is embedded. This quantity is defined via the Lie derivative of the 3-metric with respect to the future-pointing unit normal vector to the 3-surface, *i.e.*,

$$\begin{aligned} K_{ij} &= -\frac{1}{2}\mathcal{L}_n\gamma_{ij} \\ &= -\frac{1}{2\alpha}(\partial_t\gamma_{ij} - D_i\beta_j - D_j\beta_i). \end{aligned} \quad (2.2)$$

In order to express the Einstein equations in terms of the 3+1 variables, one decomposes the Ricci tensor into its timelike and spacelike components. Setting the timelike components ${}^{(4)}R_{0\alpha}$ to zero leads to the Hamiltonian and momentum constraints for the 3-metric and the extrinsic curvature:

$$R + (tr K)^2 - K^{ij}K_{ij} = 0 \quad (2.3)$$

$$D_j(K^{ij} - \gamma^{ij}tr K) = 0. \quad (2.4)$$

Setting the spacelike components ${}^{(4)}R_{ij}$ to zero, and using the definition of the extrinsic curvature, results in twelve evolution equations:

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \quad (2.5)$$

$$\begin{aligned} \partial_t K_{ij} = & -D_i D_j \alpha + \alpha[R_{ij} + (tr K)K_{ij} - 2K_{ik}K^k_j] \\ & + \beta^k D_k K_{ij} + K_{ik}D_j \beta^k + K_{jk}D_i \beta^k. \end{aligned} \quad (2.6)$$

In these equations, R_{ij} is the Ricci tensor, R the scalar curvature, and D_i the covariant derivative associated with the metric of the spacelike slices, γ_{ij} .

Analytically, the evolution equations are guaranteed to preserve the constraint equations. Numerically, however, they do not. Therefore, it is important to monitor the constraint equations during the evolution, in order to gauge the size of numerical errors.

2.2 Solving the Initial Data Problem

One needs to specify values for the 3-metric γ_{ij} and the extrinsic curvature K_{ij} on some initial time slice before one can use equations (2.5) and (2.6) to evolve them. The initial data are not arbitrary: they must satisfy the constraint equations (2.3) and (2.4). To solve the constraints, it is common to use York's conformal decomposition method [85]. This method starts by factoring out a function ψ , known as the conformal factor, from the 3-metric and extrinsic curvature tensors in the following way:

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij} \tag{2.7}$$

$$K_{ij} = \psi^{-2} \hat{K}_{ij}. \tag{2.8}$$

If we use this decomposition, and restrict ourselves to initial slices with vanishing $trK = \gamma^{ij}K_{ij}$, the constraint equations take the following form:

$$\hat{\Delta}\psi = \frac{1}{8}\psi\hat{R} - \frac{1}{8}\psi^{-7}\hat{K}^{ij}\hat{K}_{ij} \tag{2.9}$$

$$\partial_i \hat{K}^{ij} + \hat{\Gamma}_{ik}^i \hat{K}^{kj} + \hat{\Gamma}_{ik}^j \hat{K}^{ik} = 0, \tag{2.10}$$

where $\hat{\Delta}$ is the Laplacian, $\hat{R}$ the scalar curvature, and $\hat{\Gamma}_{ik}^j$ the Christoffel symbol associated with the conformal 3-metric. Note that the conformal factor does not appear in the momentum constraint equations. This allows us to use the following procedure to derive initial data: first, one specifies the conformal 3-metric $\hat{\gamma}_{ij}$ freely. Then, the momentum constraint equations are solved for the conformal extrinsic curvature. Finally, one solves the Hamiltonian constraint for the conformal factor.

This is the procedure we will use to create the 3D black hole initial data sets described in Chapter 5.

2.3 Evolving the Initial Data

Once one has specified the initial data, there are two main concerns to address before the evolution can proceed. One needs to specify both the gauge and the boundary conditions that will be used.

2.3.1 Gauge conditions

The Einstein equations do not specify evolution equations for the lapse function α or the shift vector β^i . These quantities are gauge quantities and can be chosen at will. Thus, the first thing one needs to do before evolving the initial data is to decide on a gauge condition. We discuss here a few choices for a lapse function, also known as the slicing condition, because it determines which spatial slices will be used in the evolution. Results presented in this thesis have all been obtained with a vanishing shift vector, so we will not discuss other choices here.

The simplest slicing condition is geodesic slicing, which amounts to setting $\alpha = 1$ and $\beta^i = 0$. In geodesic slicing, grid points correspond to freely falling observers. This slicing condition provides a good test of a numerical black hole code, since for a Schwarzschild black hole, an observer initially on the horizon will hit the singularity in a time of πM (see, *e.g.*, [62]). Thus, a computer code should “crash” at this time.

Although geodesic slicing is simple, it does not allow one to cover enough of the spacetime because it allows slices to hit singularities very early in the evolution. One would like a “singularity avoiding slicing”. Such a slicing results in a lapse function which is very small in regions close to a singularity, effectively stopping evolution there. One such slicing is maximal slicing. Maximal slicing is so-called because it produces the lapse that maximizes the volume of the spatial hypersurface. Because

volume elements are small near a singularity, these slices avoid singularities. It can be shown that this property is equivalent to setting the trace of the extrinsic curvature equal to zero. Thus, to obtain the maximal slicing equation for the lapse, one takes the trace of the evolution equation for the extrinsic curvature (2.6) and sets $tr K$ equal to zero. This yields the following elliptic equation for the lapse:

$$D^i D_i \alpha = R\alpha. \quad (2.11)$$

Elliptic equations are time-consuming to solve numerically, especially in 3D, so we have experimented with algebraic singularity-avoiding conditions on the lapse. These slicings are based on the fact that $\det(\gamma_{ij})$ will shrink in the vicinity of a curvature singularity. The algebraic slicings we tried all had the general form $\alpha = \alpha_0 f(\hat{\gamma})$, where $\hat{\gamma} = \det(\hat{\gamma}_{ij})$. One that worked well was setting $f(\hat{\gamma})$ to $1 + \ln \hat{\gamma}$, as will be discussed in Section 3.5.

2.3.2 Boundary conditions

When one is evolving a black hole spacetime numerically, one needs to specify boundary conditions at two places: the inner boundary, generally chosen at the black hole’s “throat”, and the outer boundary, far away from the black hole.

The throat of a black hole is the inner-most surface which is locally an areal minimum. For a Schwarzschild black hole, the throat is the initial event horizon. The usual way to apply a boundary condition on the throat is to require the geometry, and thus the metric, to be identical under a certain mapping from the outside of the throat to the inside. This mapping is known as an isometry condition.

The isometry condition takes the form of a map J which identifies two asymptotically flat sheets through the throat [43]:

$$\gamma_{ij}(\vec{x}) = \pm J_i^k J_j^\ell \gamma_{k\ell}(J(\vec{x})) \quad (2.12)$$

with

$$J(\vec{x}) = a^2 \frac{\vec{x} - \vec{c}}{|\vec{x} - \vec{c}|^2} + \vec{c} \quad (2.13)$$

and $J_i^j = \frac{\partial J^j}{\partial x^i}$, where a is the radius of the throat centered at $\vec{c}$. The mapping (2.12) is applicable to both the metric and the extrinsic curvature tensor components. The evolution equations themselves also respect this symmetry if the lapse and shift variables obey the isometry as well. Generally, the radial component of the shift must vanish on the throat, and the lapse and extrinsic curvature components must have the same sign (positive or negative). We note that for certain gauge conditions, it will not be necessary to apply an isometry condition, as discussed in Section 3.4 below.

Appropriate conditions for the outer boundary have yet to be derived for 3D. In 1D and 2D codes, the outer boundary is simply placed far enough away that the spacetime is nearly flat there, and static or flat boundary conditions can be specified for the evolved functions. However, due to the constraints placed on us by limited computer memory, this is not currently possible in 3D. The results in this work have all been computed with the evolved functions kept static at the outer boundary.

In order to avoid unnecessary computations, when possible, we take advantage of the symmetry of a situation to evolve only one octant of the Cartesian grid. When this is done, boundary conditions must also be supplied for the planes $x = 0$, $y = 0$, and $z = 0$. The setting of boundary conditions on these planes is straightforward as they are determined by the symmetries of the problem. For axisymmetric, equatorial-plane symmetric, non-rotating spacetimes, these boundary conditions are

$$\begin{aligned} 0 &= \gamma_{xy}|_{x=0} = \gamma_{xy}|_{y=0} = \gamma_{xz}|_{z=0} \\ &= \gamma_{xz}|_{x=0} = \gamma_{yz}|_{y=0} = \gamma_{yz}|_{z=0} \end{aligned} \quad (2.14)$$

and

$$0 = \partial_x \alpha|_{x=0} = \partial_y \alpha|_{y=0} = \partial_z \alpha|_{z=0}$$

$$\begin{aligned}
&= \partial_x \gamma_{xx}|_{x=0} = \partial_y \gamma_{xx}|_{y=0} = \partial_z \gamma_{xx}|_{z=0} \\
&= \partial_x \gamma_{yy}|_{x=0} = \partial_y \gamma_{yy}|_{y=0} = \partial_z \gamma_{yy}|_{z=0} \\
&= \partial_x \gamma_{zz}|_{x=0} = \partial_y \gamma_{zz}|_{y=0} = \partial_z \gamma_{zz}|_{z=0} \\
&= \partial_x \gamma_{yz}|_{x=0} = \partial_y \gamma_{xz}|_{y=0} = \partial_z \gamma_{xy}|_{z=0}.
\end{aligned} \tag{2.15}$$

The extrinsic curvature components obey identical conditions as the corresponding metric components.

2.4 The ADM Mass

We would like to be able to define some sort of energy for our spacetimes. There are two reasons for this. First, we would like to be able to characterize our initial data sets. Second, we would like to use such an energy as a test of our evolutions.

Although one cannot define a local energy density of the gravitational field in general relativity, one can define the energy of an isolated source as measured by a distant observer. Arnowitt, Deser, and Misner computed such an energy for their 3+1 formalism [18]. Ó Murchadha and York modified their expression to use the variables in Yorks conformal decomposition method [64]. For conformal metrics which fall off fast enough with radius (as ours will), their expression for the ADM mass is

$$M_{ADM} = -\frac{1}{2\pi} \oint_{\infty} D_i \psi dS^i. \tag{2.16}$$

Because our numerical domains will have finite extent, we will be performing this integral over a sphere of large constant radius R , resulting in the following expression:

$$M_{ADM} = -\frac{R^2}{2\pi} \int_0^{2\pi} d\phi \int_0^\pi \frac{\partial \psi}{\partial r} \sin \theta d\theta. \tag{2.17}$$

When we state the ADM mass for a spacetime, we have confirmed that the error involved in computing the integral at a finite radius is not significant by making sure that the integral as a function of R is approaching a constant value where we are performing the integration.

2.5 Extracting Gravitational Radiation

Once one has the metric and extrinsic curvature for all time, one would like to extract some physical, coordinate independent quantities. In particular, one would like to measure the amount of gravitational radiation in a spacetime. One measure of this radiation is the Zerilli function, ψ , which measures the deviation of the spacetime from sphericity in a gauge-invariant way. The Zerilli function can be found by writing the 4D metric as the sum of a spherically symmetric part and a non-spherical perturbation: $g_{\alpha\beta} = \overset{\circ}{g}_{\alpha\beta} + h_{\alpha\beta}$. One then specifies $\overset{\circ}{g}_{\alpha\beta}$ and a form for the perturbation $h_{\alpha\beta}$, which is normally expanded in spherical tensor harmonics. To compute the elements of $h_{\alpha\beta}$, one integrates the evolved metric components against appropriate spherical harmonics, as described in Appendix C. The resulting functions can then be combined in a gauge-invariant way to produce the Zerilli function, as described in [1] and Appendix C.

Each ℓm -mode of the Zerilli function satisfies the Zerilli equation

$$\frac{\partial^2 \psi^{(\ell m)}}{\partial t^2} - \frac{\partial^2 \psi^{(\ell m)}}{\partial r^{*2}} + V_Z^{(\ell)}(r) \psi^{(\ell m)} = 0, \quad (2.18)$$

where the gravitational scattering potential $V_Z^{(\ell)}$ is given by

$$V_Z^{(\ell)}(r) = \left(1 - \frac{2M}{r}\right) \times \left\{ \frac{1}{\Lambda^2} \left[\frac{72M^3}{r^5} - \frac{12M}{r^3} (\ell - 1)(\ell + 2) \left(1 - \frac{3M}{r}\right) \right] + \frac{\ell(\ell - 1)(\ell + 2)(\ell + 1)}{r^2 \Lambda} \right\}, \quad (2.19)$$

with

$$\Lambda = \ell(\ell + 1) - 2 + \frac{6M}{r}. \quad (2.20)$$

Here, r is the standard Schwarzschild radial coordinate, and r^* is the so-called tortoise coordinate, given by $r^* = r + 2M \ln(r/2M - 1)$. Note that the Zerilli equation has the form of a second-order wave equation.

We note here that the so-called quasinormal modes of a Schwarzschild black hole are defined as those solutions to the Zerilli equation which are purely ingoing at

$r^* = -\infty$ and purely outgoing at $r^* = +\infty$. This will be discussed in more detail in Chapter 7.

2.6 The Apparent Horizon

Other measures of interest in a black hole simulation are the location of the apparent horizon, and quantities related to it. The apparent horizon is the outer most of what is known as a marginally trapped surface on a spatial slice. A marginally trapped surface has vanishing volume expansion (Θ) of its outgoing null normal vectors. In terms of the variables on a given 3D spacelike hypersurface, this condition can be written [86]

$$\Theta = D_i s^i + K_{ij} s^i s^j - \text{tr} K = 0, \quad (2.21)$$

where s^i is the spacelike unit normal vector to the surface. As opposed to an event horizon, an apparent horizon can be used during the numerical construction of the spacetime, since it is defined locally in time.

Apparent horizons are useful for studying the dynamics of black hole spacetimes [7]. For example, the wavelength and damping time of the quasinormal modes of a black hole can be read off of a plot of the ratio of polar to equatorial circumference of the apparent horizon as a function of time, at least in axisymmetry. Apparent horizons can also provide an inner boundary in numerical evolutions of black hole spacetimes [13, 71, 80], although such a boundary condition will not be discussed in this work.

The numerical location of apparent horizons in 3D is a subject of ongoing research. In Appendix B we discuss this in more detail.

2.7 Computational Issues in 3D

In 1D and 2D black hole evolution codes, the coordinate system has almost always been specialized to the particular problem. Some work has been done in 3D to evolve a spherically symmetric, fully relativistic star with spherical coordinates, but those attempts were unsuccessful, due to the coordinate singularity on the z -axis [76]. Our 3D codes use Cartesian coordinates. Although this coordinate system has no coordinate singularities, it does introduce problems of its own. The main problem is that the boundary of the computational domain is not a natural boundary on which to apply boundary conditions: the outer boundary is not spherical, and the throat is located inside the computational domain.

Another complication in 3D is that the size of computer memory places a restriction on the number of grid points which may be evolved. This is not much of a problem in 1D and 2D codes, but in 3D it is a serious problem. It is not currently possible to use the desired resolution while simultaneously putting the outer boundary where we would like it. We can alleviate this problem somewhat by evolving only one octant of the spacetime when symmetries allow. However, this trick does not get us far and we will not be able to use it for general problems. This problem as it relates to the Schwarzschild black hole is discussed in detail in Section 3.2.

The large number of grid points needed in 3D also results in a large number of computations that need to be done on each time slice. To obtain results in a reasonable amount of time, we have found it necessary to use large, parallel computers. The use of parallel computers introduces problems, however: if an algorithm is not chosen carefully, the code might run even slower than it would have on a single-processor machine.

Chapter 3

The Schwarzschild Black Hole

We have developed a 3D code which solves the Einstein equations in Cartesian coordinates using the ADM 3+1 decomposition of the Einstein equations described in Section 2.1. The numerical methods used in this code are described in Appendix A. The first black hole spacetimes which have been evolved have had axial symmetry. These problems have been solved numerically with 1D and 2D codes, and thus serve as good test problems for our 3D code. Results from the first of these problems, the spherically symmetric Schwarzschild black hole, are described in this Chapter, and in our paper [12], from which this Chapter was developed. Results from the second of these problems, the Misner two black hole collision, are described in the next Chapter.

It is important for us to be able to evolve the Schwarzschild black hole. It represents the end state of any black hole system without angular momentum. Thus, we must be able to evolve it accurately before 3D coalescence can be studied. Also, we expect that techniques developed here will be useful in more complex spacetimes, since it contains the main feature present in all black hole spacetimes: a singularity.

3.1 Schwarzschild Initial Data

In 1916 Schwarzschild solved the Einstein equations for a spherically symmetric vacuum spacetime [70]. In the standard Schwarzschild coordinates, this solution is

$$ds^2 = - \left(1 - \frac{2M}{r_s}\right) dt_s^2 + \left(1 - \frac{2M}{r_s}\right)^{-1} dr_s^2 + r_s^2 d\theta^2 + r_s^2 \sin^2 \theta d\phi^2. \quad (3.1)$$

This solution is static in the coordinates given. However, the gauge freedom inherent in the Einstein equations allows us to slice spacetime however we please. In particular, we can choose a lapse function such that the 3-metric and extrinsic curvature depend on the resulting time coordinate. Such a lapse function would allow one to numerically evolve this simplest time-dependent spacetime that contains a black hole.

A convenient way to understand this coordinate freedom for the Schwarzschild spacetime is to consider the Schwarzschild solution in the coordinates used by Kruskal and others. In these coordinates, the coordinate singularity at $r_s = 2M$ is removed, and it is seen that the Schwarzschild solution in Schwarzschild coordinates actually only covers half of the manifold on which it is a solution. In Figure 3.1, we show where certain constant- r_s and t_s surfaces lie in this coordinate system. The Kruskal coordinates u and v are related to the Schwarzschild coordinates by

$$\left(\frac{r_s}{2M} - 1\right) e^{r_s/2M} = u^2 - v^2 \quad (3.2)$$

$$t_s = \begin{cases} 4M \tanh^{-1}(v/u) & \text{in I and III} \\ 4M \tanh^{-1}(u/v) & \text{in II and IV} \end{cases} \quad (3.3)$$

Surfaces of constant time t_s are straight lines which intersect the origin on this diagram, with $t_s = 0$ corresponding to the u axis. Surfaces of constant r_s are hyperbolae, with the exception of $r_s = 2M$, which is represented by two lines. We note that in this coordinate system, the metric is perfectly regular at $r_s = 2M$. It is of course singular at $r_s = 0$, which is a true curvature singularity. We will refer back to this diagram as we discuss the various slicing conditions.

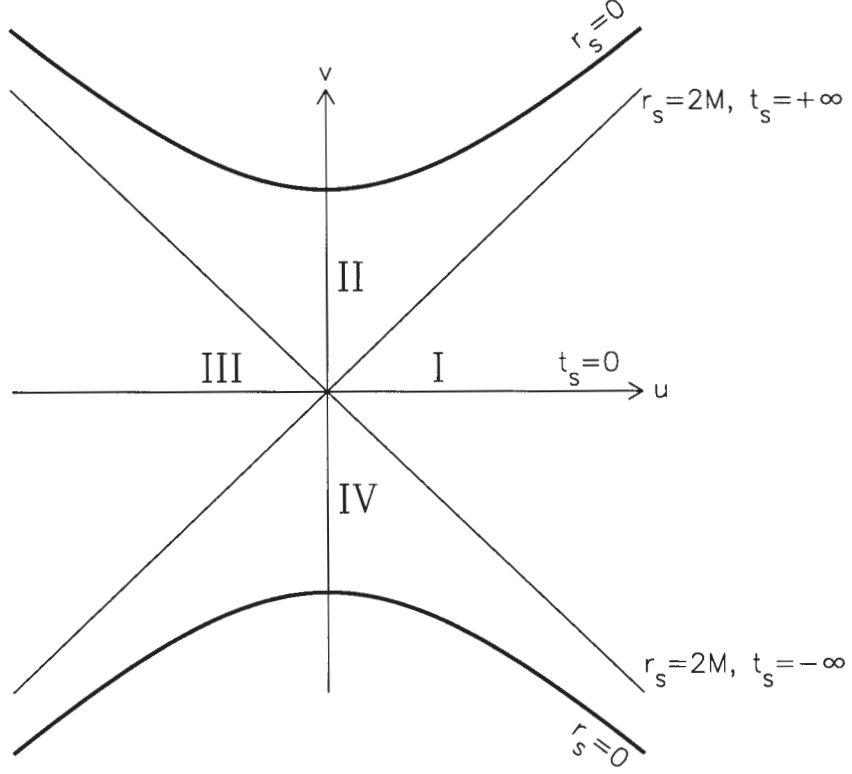


Figure 3.1: We show the Kruskal diagram for the Schwarzschild spacetime.

Primarily due to the coordinate singularity at $r_s = 2M$, previous work evolving the Schwarzschild spacetime numerically used a different radial coordinate, resulting in the following metric:

$$ds^2 = - \left(\frac{2M - r}{2M + r} \right)^2 dt^2 + \left(1 + \frac{M}{2r} \right)^4 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \quad (3.4)$$

Here, r is the isotropic radius, related to the standard Schwarzschild radial coordinate r_s by $r_s = (1 + \frac{M}{2r})^2 r$. Another advantage to this coordinate system is that it is invariant under the isometry transformation $r \rightarrow \frac{M^2}{4r}$, which will allow us to apply the isometry boundary condition described in Section 2.3.2 at the throat, with $a = \frac{M}{2}$ and $\vec{c} = (0, 0, 0)$.

There has been much study of the Schwarzschild black hole with 1D computer codes. These codes can be run with high resolution, providing very accurate results that can be compared with 3D results.

To obtain 3D initial data, we first converted the spatial part of the Schwarzschild metric (3.4) to Cartesian coordinates:

$$d\ell^2 = \psi^4 (dx^2 + dy^2 + dz^2), \quad (3.5)$$

where $\psi = 1 + \frac{M}{2r}$. This 3-metric, along with $K_{ij} = 0$, is used as the initial data for the evolution.

3.2 The Numerical Grid

Before going into the details of the calculations, it is instructive to compare the current Cartesian grid with the logarithmic η grid used in other numerical work on black holes [21, 22]. In that system, a radial coordinate η is defined by

$$r = \frac{M}{2} e^\eta, \quad (3.6)$$

where r is the Schwarzschild isotropic radius and M is the Schwarzschild mass of the black hole. This coordinate has the advantage of providing fine resolution near the throat of the black hole and also near the peak that develops in the radial metric function, while also allowing the outer boundary to be placed far from the hole. Typical high resolution calculations in 1D and 2D based on this coordinate use $\Delta\eta = 0.03$, with the outer boundary placed at $\eta = 6$, or $r \approx 200M$. Low resolution calculations are performed with $\Delta\eta = 0.06$. Disadvantages of this coordinate are that: (i) The throat region remains extremely well resolved, even after the horizon has moved significantly away from the hole. Therefore much computational effort is wasted well inside the horizon where the lapse is typically near zero and the region is causally disconnected from the outside. (ii) The grid becomes very coarse outside

the horizon in the radiation zone, because equal spacing in the η coordinate leads to larger and larger spacing in the more physical r coordinate. Under these conditions waves may be reflected back toward the black hole as they are scattered off of the coarse grid at larger radii, as discussed in references [1, 21].

One can estimate what kind of resolutions will be necessary to solve the spherical black hole problem in 3D by doing runs with a spherically symmetric (1D) code in η coordinates. It is instructive to study a high resolution 1D black hole run ($\Delta\eta = 0.03$) with maximal slicing to $t = 50M$, as shown in Figure 3.2. At that time, the peak in the radial metric function is located at approximately $\eta = 2$. This corresponds to an isotropic radius r of approximately $3.7M$. The effective resolution at the peak in isotropic radial coordinates is thus $\Delta r \approx 0.1M$. Given that this peak is the sharpest feature in the domain, one might expect that it would be possible to obtain reasonable results with a 3D code with $\Delta x = \Delta y = \Delta z \approx 0.1M$. We also have a 1D code which uses isotropic radial coordinates, which are more closely related to our 3D Cartesian coordinates. Experiments with this code suggest that a resolution of $\Delta r \approx 0.05M$ is actually needed to obtain reasonably accurate results to a time of $50M$. This difference is due to the fact that although the equally spaced η coordinates cover the throat region extremely well, where the peak begins to develop at early times, the equally spaced r coordinates do not, and therefore higher resolution is required. There are also important geometric factors to consider in 3D, such as the length of the diagonal of a Cartesian cube being $\sqrt{3}\Delta x$, which make the resolution requirements more stringent yet. We show the size of the black hole throat on a grid of typical resolution ($\Delta x = 0.15M$) in Figure 3.3. In this Figure the throat is located at $r = 0.5M$, so we see that the throat itself is not extremely well resolved. (Recall that r denotes the isotropic, not Schwarzschild radius.)

As mentioned above, the 1D code with η coordinates was run with an $\eta_{max} = 6$, which corresponds to $r_{max} \approx 200M$. With equally spaced Cartesian zones such a

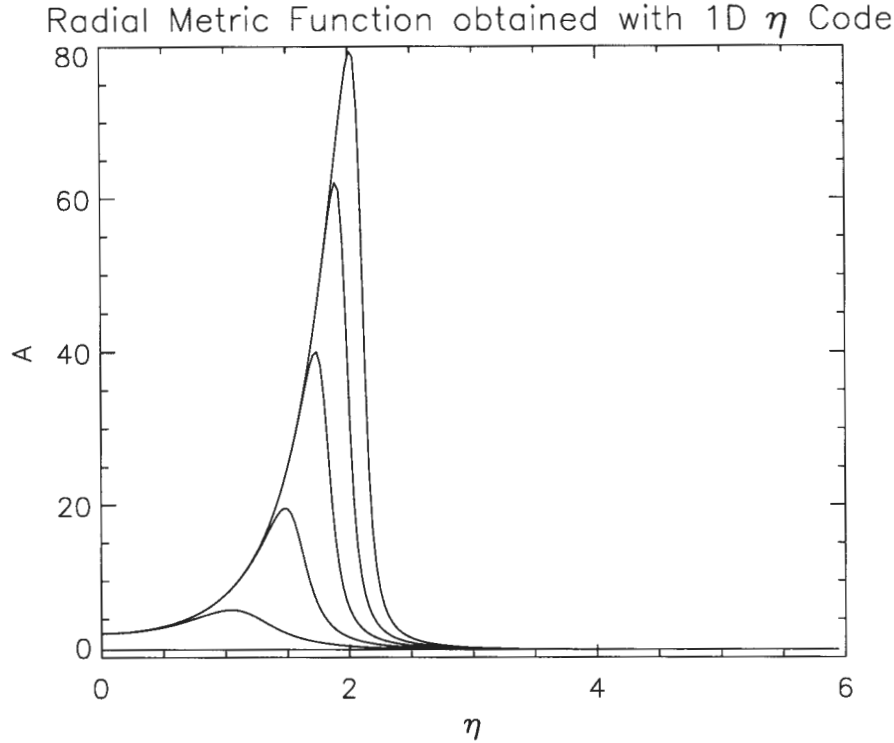


Figure 3.2: We show the radial metric function $A = \gamma_{rr}/\psi^4$ for a spherical black hole evolved with a 1D code with maximal slicing. Time slices are shown at intervals of $t = 10M$ until the final time of $t = 50M$. The spacetime shown here was evolved with the logarithmic coordinate η described in the text, with a resolution of $\Delta\eta = 0.3$.

luxury is impractical with present computer memories and speeds: the resolution $\Delta r = 0.05M$ recommended above would require 4000^3 zones. Experiments with 1D codes show that if one is interested in following the metric and extrinsic curvature components only to a time of $50M$, the outer boundary should be placed at a radius of greater than about $30M$. When the outer boundary is too close, its influence will be felt by the interior solution if the boundary conditions are not properly formulated, producing error in the height of the peak. Common treatments of the outer boundary involve holding the metric functions fixed or extrapolating them to the outer zones [21], but these are adequate only when the boundary is placed quite far away. When the boundary is placed at $r_{max} = 30M$, by $t = 50M$ the error in the peak of γ_{rr} (reconstructed from the six Cartesian metric functions) due to boundary effects (using

Numerical Grid

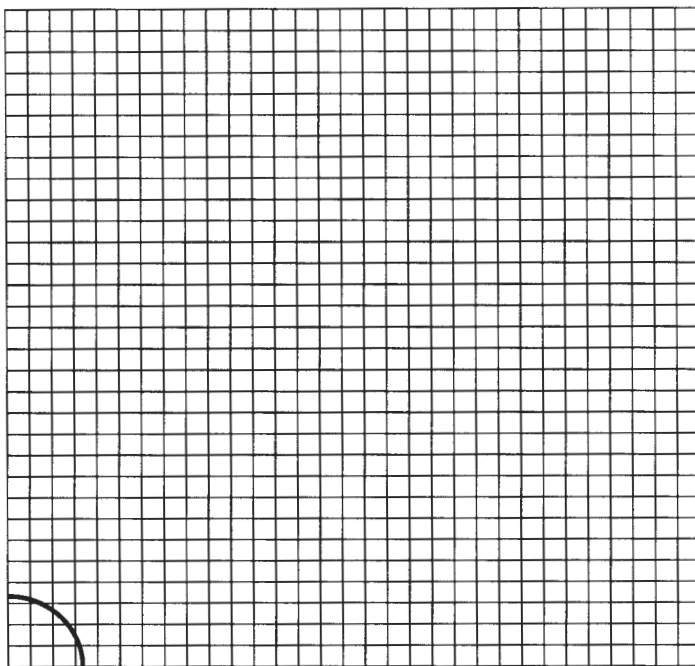


Figure 3.3: We show the Cartesian coordinate grid with the black hole throat superimposed as a dark solid line at the lower left corner. The throat is located at $r = 0.5M$, where r is the isotropic radius, and the resolution of the grid is $\Delta x = 0.15M$.

maximal slicing) can be about 10%, depending on the treatment of the outer boundary conditions. These effects clearly point to the need for appropriate outer boundary conditions for black hole spacetimes.

Although experiments with 1D codes allow us to estimate how much resolution will be needed, they are incomplete guides to the 3D problem. First of all, resolution issues will be different in Cartesian coordinates. In 1D and 2D (axisymmetric) codes, constant coordinate lines are essentially perpendicular to the gradients, so that symmetries are easily preserved. However, with a Cartesian grid the constant coordinate lines cross developing features at all angles, causing finite difference errors to be larger than a 1D treatment would show. Thus, one expects that even better resolution will

be needed than that implied by the 1D code tests. Also, when considering how far out to put the boundary of a Cartesian grid, one must take into consideration that the outer boundary does not have the shape which is characteristic to the problem (it is a cube, not a sphere). Thus, boundary conditions must be applied carefully there. However, in spite of these problems we believe that Cartesian coordinates are to be preferred over other specialized coordinate systems. The symmetries of this first 3D black hole problem are artificial. A general 3D spacetime with multiple black holes will not suggest a preferred coordinate system, as a single hole does, so a Cartesian mesh should be just as effective as a spherical or boundary fitted coordinate system (*e.g.* Čadež coordinates [15]), at least as far as the near zone is concerned. Furthermore, the singularity free nature of Cartesian coordinates is clearly desirable. All axisymmetric numerical relativity codes (written in either $\rho - z$ or $r - \theta$ type coordinates) that we are aware of have difficulties near the symmetry axes and origin (see, *e.g.*, discussion in Ref. [21]), and in 3D these problems are much more severe [76].

Present memory available on the AEI 32-node Origin 2000 (8 GBytes) allows calculations of up to about 200^3 zones with our present code¹. With the above discussion in mind, it is clear that in 3D black hole simulations, balancing the demands of high resolution with the need to place the outer boundary sufficiently far from the hole are quite difficult with a fixed, equally spaced grid.

3.3 Results: Geodesic Slicing

In this Section we discuss results obtained with the geodesic slicing condition described in Section 2.3.1. As mentioned in Section 2.3.1, with this slicing condition, grid points correspond to freely falling observers, and an observer initially on the horizon ($r_s = 2M$ or $r = M/2$) will hit the singularity in a time of πM . In the

¹Some results in later chapters were obtained with the 64-node and 128-node NCSA Origin 2000 systems under development in August 1997, allowing 300^3 simulations.

Kruskal diagram, geodesic slicing will produce slices which are parallel to the u -axis. If one specifies data everywhere at $t = 0$, slices will advance in the positive- v direction, with the point initially on the horizon hitting the singularity at $r = 0$. Thus, using this slicing leaves much of the spacetime uncomputed.

Despite this, evolving the black hole with geodesic slicing also allows one to perform serious tests of the evolution equations without the complication of the lapse computation. We test the evolution equations by evolving a black hole with geodesic slicing with both our 3D code and a well-tested, spherical, 1D code, and compare metric functions and certain derived quantities which will be discussed below. When one runs the codes with high spatial resolution, *e.g.* $\Delta x = \Delta y = \Delta z = 0.05M$, the peaks in $\hat{\gamma}_{rr}$ and $\hat{\gamma}_{\theta\theta}$ line up almost exactly, as shown in Figures 3.4 and 3.5. In the 3D case, these functions have been constructed from the Cartesian metric function that are actually evolved. In Figure 3.6 we show a 2D slice through the plane $z = 0$ of the Cartesian metric function $\hat{\gamma}_{xx} = \gamma_{xx}/\psi^4$ evolved to a time $t = 3M$. (We factor out the dependence of the conformal factor ψ to show the dynamical evolution more clearly.) Along the x -direction, this function behaves like $\hat{\gamma}_{rr} = \gamma_{rr}/\psi^4$, but in other directions it does not. It is only through the combination of all Cartesian metric functions that the spherical behavior of γ_{rr} is seen.

As mentioned above, it is not necessary in principle to apply the isometry condition across the throat. We have shown, however, that for accurate evolution it is necessary for the geodesic slicing case. When the isometry is not applied, the peak in the radial metric function grows faster than it should, causing the code to crash earlier than it would otherwise. This should not be surprising, because the isometry condition maps the entire spacetime region exterior to the throat to the domain inside the throat. As we are covering this interior region with a very small number of zones, it is impossible to evolve the interior accurately in this manner. By mapping the well resolved exterior numerical solution to the poorly resolved interior, we are able to

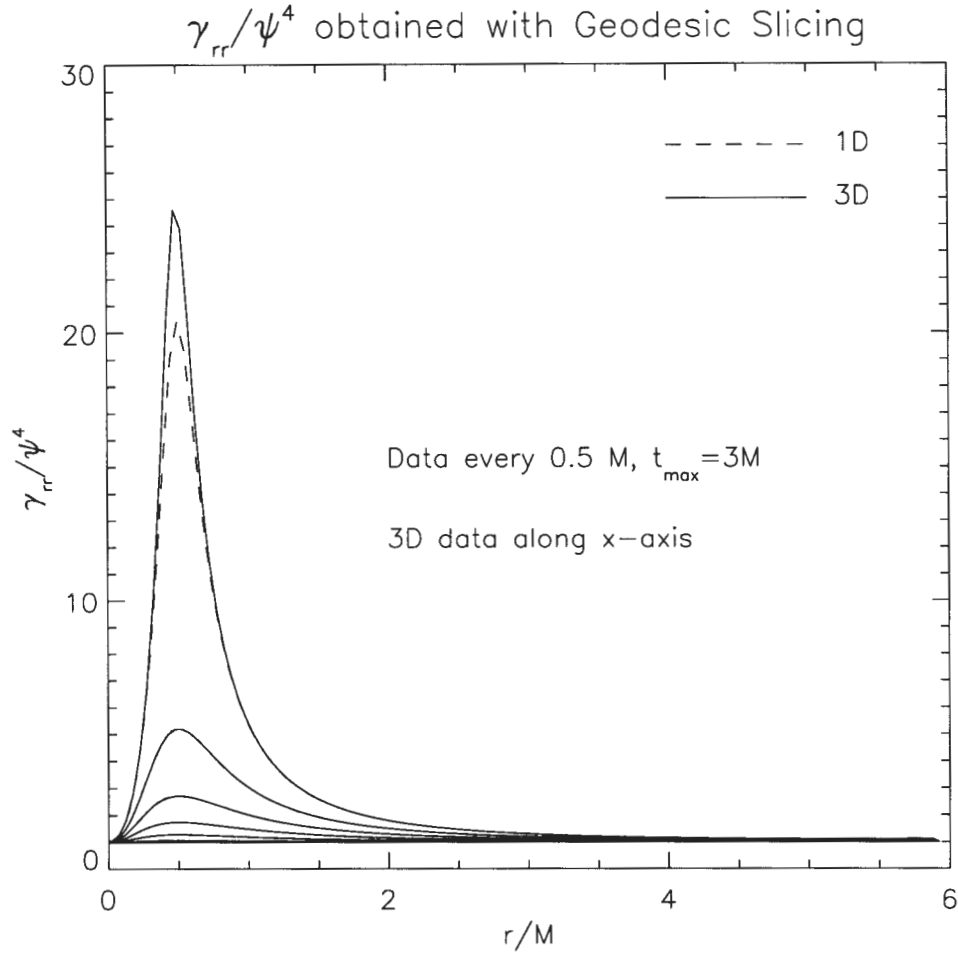


Figure 3.4: We show a comparison of γ_{rr}/ψ^4 obtained with 1D and 3D codes using geodesic slicing. The 1D data were obtained with 120 radial zones and $\Delta r = 0.05M$. The 3D data were obtained with 120^3 Cartesian grid zones with $\Delta x = \Delta y = \Delta z = 0.05M$. In 1D we evolve γ_{rr} , while in 3D we reconstruct γ_{rr} from the 3D Cartesian metric functions.

achieve very high effective resolution inside the throat.

The 3D results discussed thus far were obtained with the black hole throat located in a corner of the 3D grid, thereby permitting simulations that cover only one-eighth of the total spacetime volume. Symmetry conditions were applied at the faces of the cube that match up to other interior regions of the spacetime, as discussed in Section 2.3.2. To show that this technique does not affect our results, we show results obtained by evolving the full spacetime domain with the black hole throat located at the center of the 3D grid. In Figure 3.7, we show a plot of the metric function

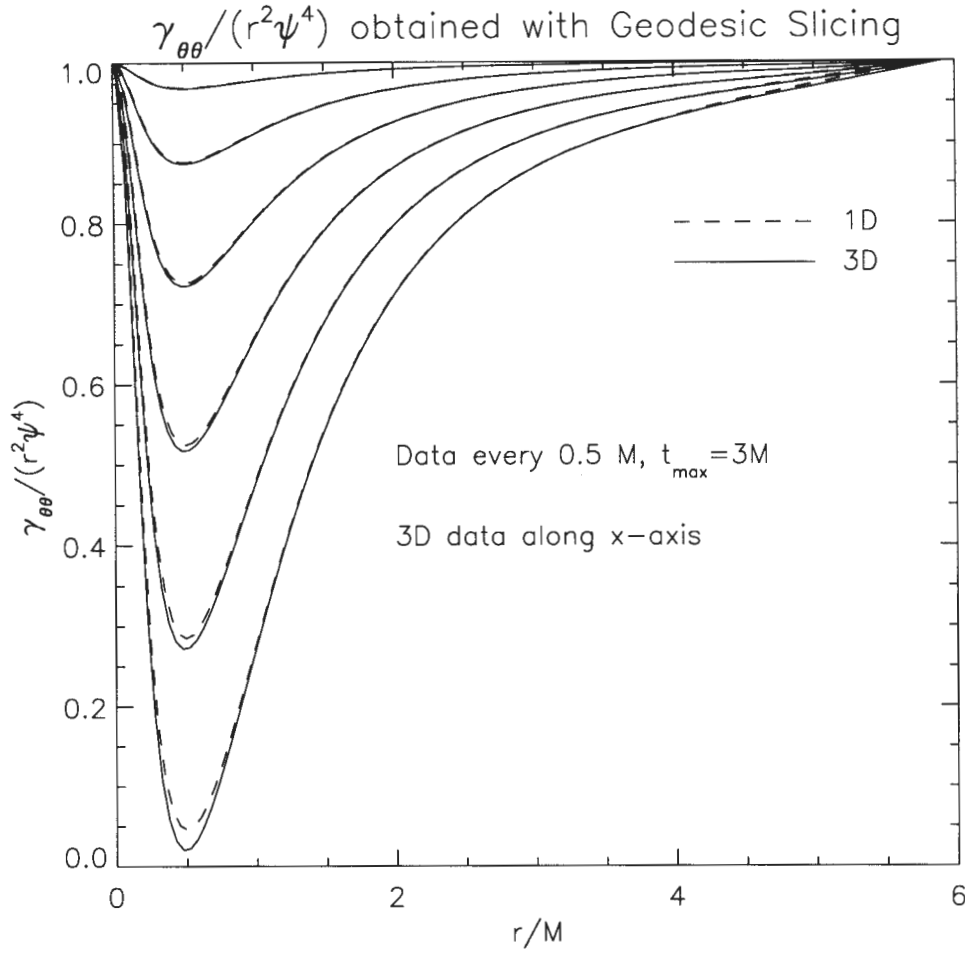


Figure 3.5: As in Figure 3.4, except that we plot $\gamma_{\theta\theta}/(r^2\psi^4)$.

$\hat{\gamma}_{xx}$ along the x -axis obtained with both a full and an octant grid evolution. For positive- x , data from both runs are present, but they are indistinguishable in this plot.

We also took advantage of the simplicity of geodesic slicing to perform convergence tests on the evolution equations. Convergence tests done on the code with second order spatial derivatives show that our code is indeed second order. A number of metric functions have been tested with similar results.

In addition to comparing directly the behavior of metric functions obtained with our 3D Cartesian and 1D spherical codes, we have also compared derived quantities that are sensitive measures of the physics being computed. For example, the location

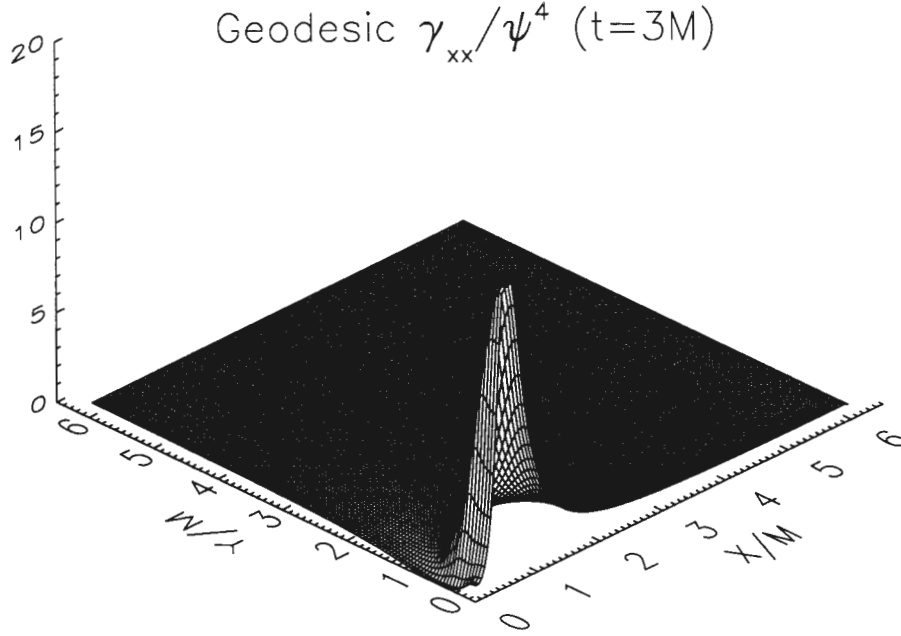


Figure 3.6: The metric function $\hat{\gamma}_{xx} = \gamma_{xx}/\psi^4$ obtained with geodesic slicing is shown at time $t = 3M$. The data are from the $z = 0$ plane, and were obtained using 120^3 grid zones with a resolution of $\Delta x = 0.05M$.

and mass of the apparent horizon of a black hole evolved numerically are particularly sensitive to errors in the calculations [16, 29, 71]. Although we had not yet connected a full 3D apparent horizon solver to our code at the time these results were derived, we could take advantage of the fact that we expected our data to be spherical. (A 3D apparent horizon finder is under development, as described in Appendix B and [11].) Therefore we transformed the Cartesian metric and extrinsic curvature components into their spherical coordinate counterparts, interpolated them onto a radial line which intersects the origin, and located the apparent horizon along that line using a 1D apparent horizon finder. Using this method, we were able to locate the position

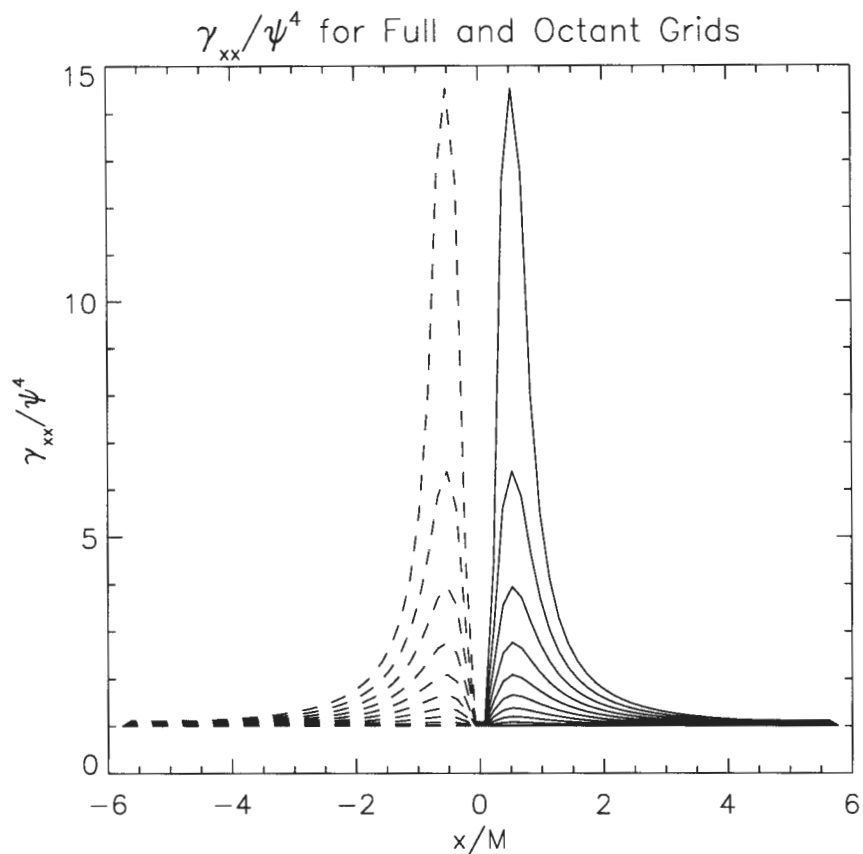


Figure 3.7: We show the metric function $\hat{\gamma}_{xx}$ along the x -axis obtained with full and octant grid resolutions. For positive- x , data from both evolutions are present, but they are indistinguishable in this plot.

of the apparent horizon and compute its mass M_{AH} using the area relation

$$M_{AH} = \sqrt{\frac{A}{16\pi}}, \quad (3.7)$$

where A is the surface area of the horizon. For a Schwarzschild spacetime, M_{AH} should be equal to the ADM mass of the spacetime throughout the evolution, but due to numerical error this is difficult to achieve at late times, even in 1D and 2D codes [16, 29, 71]. With our 3D code we have computed these quantities along four different lines (the x -, y -, and z - axes and the diagonal). Using the location of the horizon found along these lines, we can compute an effective mass at each point by taking the metric functions found there and computing the area, assuming spherical symmetry. Both the mass M_{AH} and the location of the horizon agree well both along

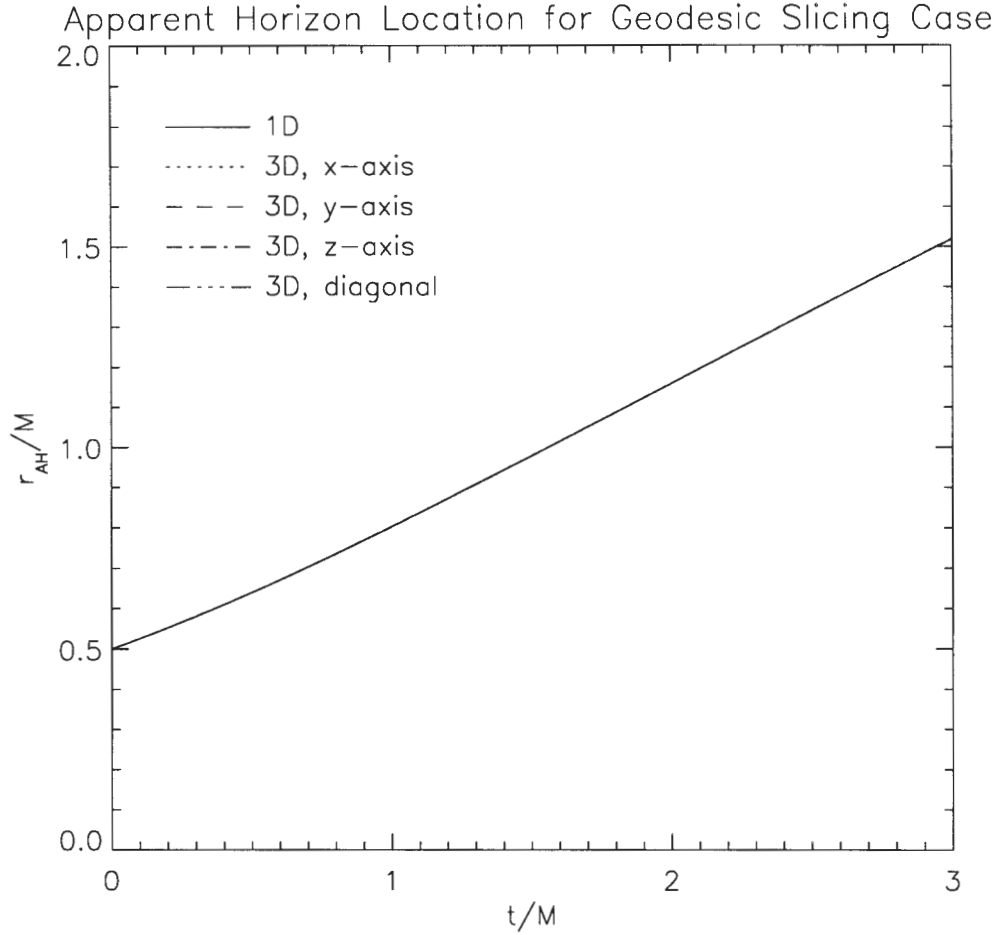


Figure 3.8: We plot the position of the apparent horizon for 1D and 3D runs with geodesic slicing. The 1D data were obtained using 128 grid points and resolution $\Delta r = 0.0375M$. The 3D data were obtained using 128^3 grid points and resolution $\Delta x = 0.0375M$ using a procedure described in the text. All five lines are plotted, but they are virtually indistinguishable in this Figure.

different lines in the 3D domain, and with the values obtained with a 1D code. The position of the apparent horizon is shown in Figure 3.8. All lines are shown, but they are indistinguishable in this plot. In Figure 3.9 we plot the apparent horizon mass computed along all four lines in 3D, as well as the 1D result. The masses are all within 0.07% of each other and the ADM mass at $t = 3M$.

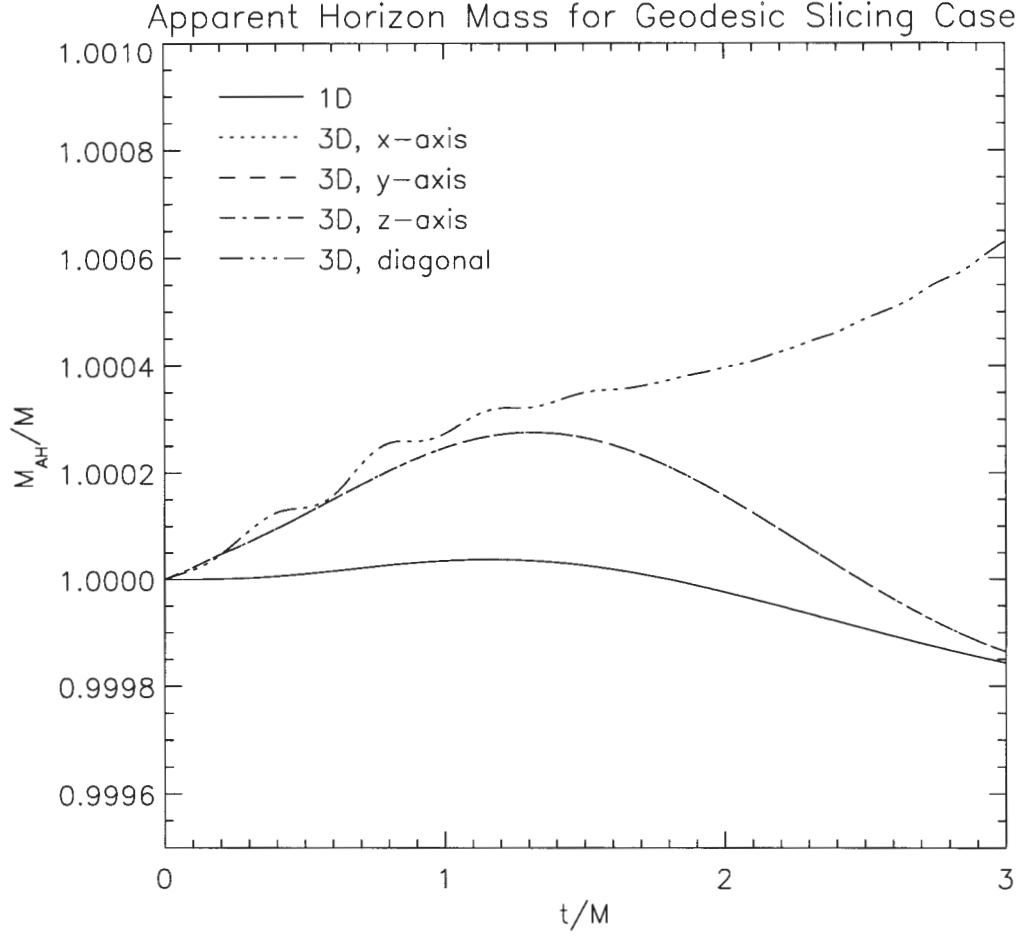


Figure 3.9: We show the mass of the apparent horizon for 1D and 3D runs with geodesic slicing. The 1D data were obtained using 128 radial points and resolution $\Delta r = 0.0375M$. The 3D data were obtained using 128^3 grid points and resolution $\Delta x = 0.0375M$. All results agree with the analytic result to better than 0.07% for the entire evolution.

3.4 Results: Maximal Slicing

We now turn to results obtained using maximal slicing. Earlier work on numerical black hole evolution with maximal slicing in 1D (see, *e.g.* [22]) and 2D (see, *e.g.*, [9, 29]) has taken advantage of the isometry to use the throat as an inner boundary on both the evolution and the solution to the maximal slicing equation. In those cases, the isometry was a simple differential condition across the throat in the spherical coordinate system used. In this 3D case the Cartesian boundary conditions on the lapse are cumbersome to apply in the elliptic solver, so we have chosen to evolve the entire domain inside and outside the throat. As we will show below, the lapse collapses quickly both outside *and* inside the throat, halting the evolution there. Although in this case the evolution inside the throat is no longer isometric to that outside, it is of little consequence.

What do maximal slices look like on the Kruskal diagram? As with geodesic slicing, one specifies initial data everywhere at $t = 0$. Again, the slices advance forward in the positive- v direction. However, with maximal slicing, time is slowed for points which would have reached the singularity at $r = 0$. In fact, no point goes beyond the limit surface of $r_s = 3M/2$ [48].

The first case we consider is a simulation evolving a single octant of the full 3D spacetime. This is a run with a resolution of 130^3 equally spaced zones, with $\Delta x = 0.1M$. The outer boundary was located at $x = 12.95M$. The value of the lapse at the outer boundary was held at the Schwarzschild value, which in isotropic coordinates takes the form

$$\alpha = \frac{2r - M}{2r + M}. \quad (3.8)$$

This treatment at the outer edges of the cube was crucial in maintaining a stable evolution there. If the lapse is taken to be, say, $\alpha = 1$ in the outer region, serious edge effects and non-spherical behavior develop quickly.

First we show the lapse function α at a late time of $t = 28M$. In Figure 3.10 a 2D slice of the lapse through the plane $z = 0$ is shown. It has collapsed throughout the throat region and also in a region outside it, and then climbs steeply towards its outer Schwarzschild value in spherical step function fashion.

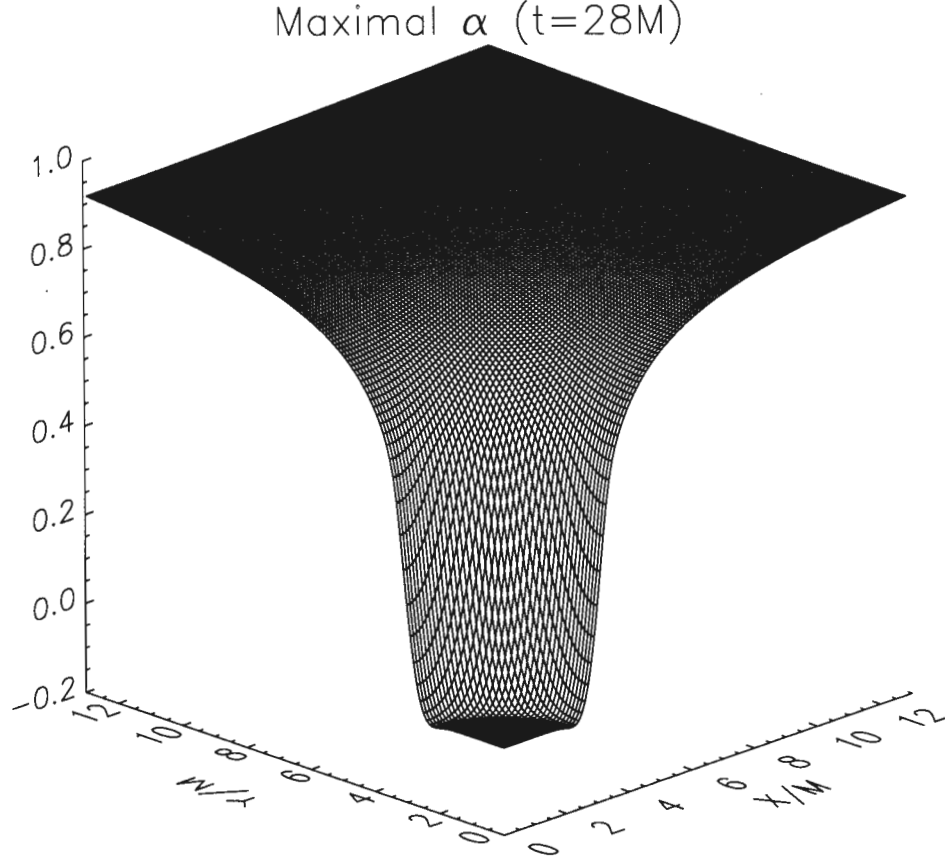


Figure 3.10: A 2D slice through the plane $z = 0$ is shown for the lapse function α at time $t = 28M$ for maximal slicing. The resolution is $\Delta x = 0.1M$, and the total number of zones is 130^3 . The singularity avoiding properties of the maximal lapse create a steep 3D well in α surrounding the throat.

In Figure 3.11 we show a 2D slice (at the plane $z = 0$) of the conformal metric function $\hat{\gamma}_{xx} = \gamma_{xx}/\psi^4$ at the same time $t = 28M$. By this time serious gradients and shearing are developing in the metric functions due to the grid stretching effects that result from maximal slicing. Along the x -direction, the metric function $\hat{\gamma}_{xx}$

behaves much like a radial metric function $\hat{\gamma}_{rr}$. Along the y -direction, however, the function $\hat{\gamma}_{xx}$ is essentially flat, and along the diagonal there is a very sharp transition region. This is quite typical of the effects of singularity avoiding lapse conditions. In Figure 3.12 we show a 2D slice (through the plane $x = 0$) of the metric function $\hat{\gamma}_{rr} = \gamma_{rr}/\psi^4$ reconstructed from the six Cartesian metric functions at time $t = 28M$ for the same simulation. The familiar spherical peak is developing around the black hole, as in 1D and 2D calculations.

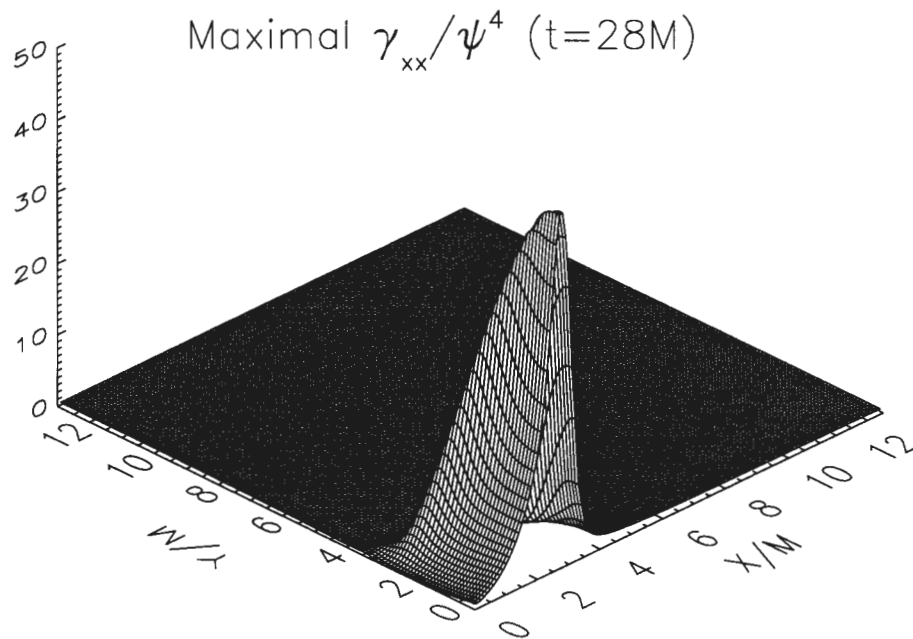


Figure 3.11: A 2D slice through the plane $z = 0$ for the metric function $\hat{\gamma}_{xx}$ is shown at time $t = 28M$ for maximal slicing. The resolution is $\Delta x = 0.1M$ and the total number of zones is 130^3 . The sharp peak developing in this 3D calculation cannot be resolved, causing difficulty with the calculation at late times.

As in the previous Section on geodesic slicing, we have tracked and analyzed the

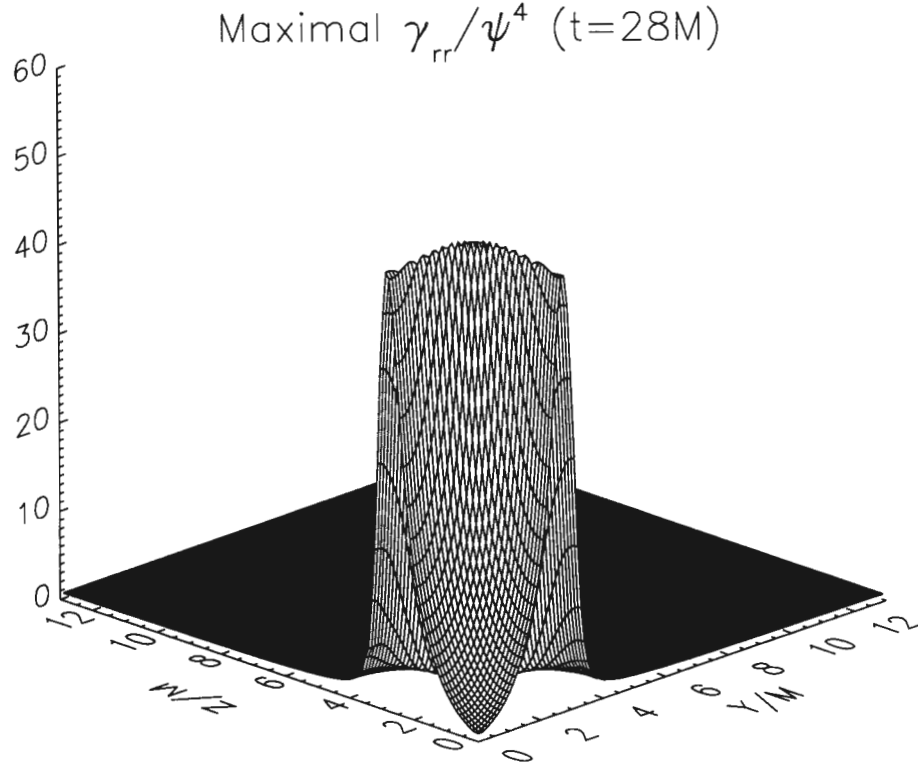


Figure 3.12: A 2D slice through the plane $x = 0$ is shown for the metric function $\hat{\gamma}_{rr} = \gamma_{rr}/\psi^4$ at time $t = 28M$ for maximal slicing. The resolution is $\Delta x = 0.1M$, and the total number of zones is 130^3 . The sharp spherical peak develops in this 3D calculation just as in the 1D and 2D calculations.

apparent horizon in these simulations, and compared the results to those obtained using a spherical, 1D code. In Figure 3.13 we show the apparent horizon location from our 3D code as determined by considering radial lines along the x -, y -, and z - axes and the diagonal of the cube. The independent results obtained by evolving a maximally sliced spherical black hole with a 1D code are also shown. The agreement is quite good, within one 3D grid zone at late times. We do not expect perfect agreement in the location of the horizon in coordinate space, since the slicing and boundary conditions are slightly different. In the 1D case, the evolution was performed with

an isometry condition across the throat. Consequently, the maximal slicing condition was implemented with a symmetric boundary condition there. The 3D calculation was performed *without* an isometry at the throat. Furthermore, the maximal slicing condition becomes a simple ordinary differential equation in 1D, whereas in 3D it was solved as a full 3D elliptic equation with boundary conditions applied on the faces of a cube. For these reasons the slicing and geometric meaning of the coordinates will differ somewhat from the 1D case.

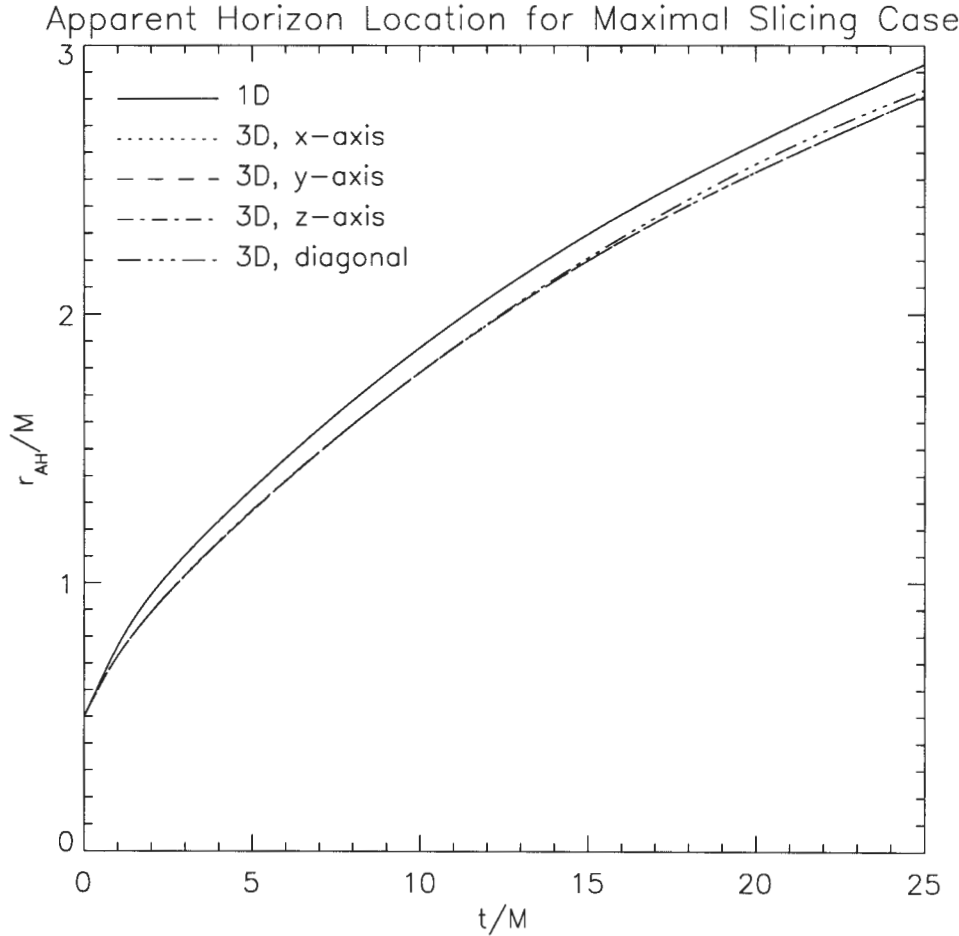


Figure 3.13: We plot the position of the apparent horizon for 1D and 3D runs with maximal slicing. The 1D data were obtained using 130 grid zones with a resolution $\Delta r = 0.1M$. The 3D data were obtained using 130^3 grid zones with a resolution of $\Delta x = 0.1M$.

A more geometrically meaningful measurement is of the *mass* of the apparent

horizon as defined by equation (3.7), which we compute in each of four directions, as described in the geodesic slicing case above. Results are plotted in Figure 3.14, and compared with results obtained for the 1D simulation. As this is a Schwarzschild spacetime, M_{AH} should be the Schwarzschild mass and constant in time. From the Figure one can see that the error is slightly larger along the diagonal than in the other cases after about $t = 10 - 15M$. This can be understood by considering that the effective resolution in the radial direction is less along the diagonal due to geometric effects. The measurements made along the three axes agree with each other, as expected, and agree reasonably well with measurements made along the diagonal line. Also, the measurements made along all four lines agree well with the 1D result, although at the end of the calculation, when the peaks in metric functions are growing dramatically (see, *e.g.*, Figure 3.12), the 1D result is slightly better.

These results are typical of what is achieved at “medium” resolution. With this resolution, instabilities develop in the region of strong metric peaks and the code crashes by about $t = 30M$. At the highest resolution achievable with our present code and the NCSA CM-5 (about 180^3 zones with maximal slicing), we could reach about $t = 35M$. As the code is second order accurate, the results are better at higher resolution. However, with this highest resolution achievable, we are still short of the best 2D codes, which can run a maximally sliced black hole to approximately $t = 100M$.

To demonstrate that our code can also evolve a black hole in full 3D, without symmetries used at the boundaries, we show results for a maximally sliced black hole placed at the center of the computational grid. Because this calculation requires eight times the memory and computer time, it cannot be run at the same resolution as the calculation discussed above. In Figure 3.15 we show a 2D slice through the $z = 0$ plane of the metric function $\hat{\gamma}_{rr}$ on the full grid at a time $t = 15M$. The spherical nature of this function is evident. In Figure 3.16 we show a comparison of a 1D

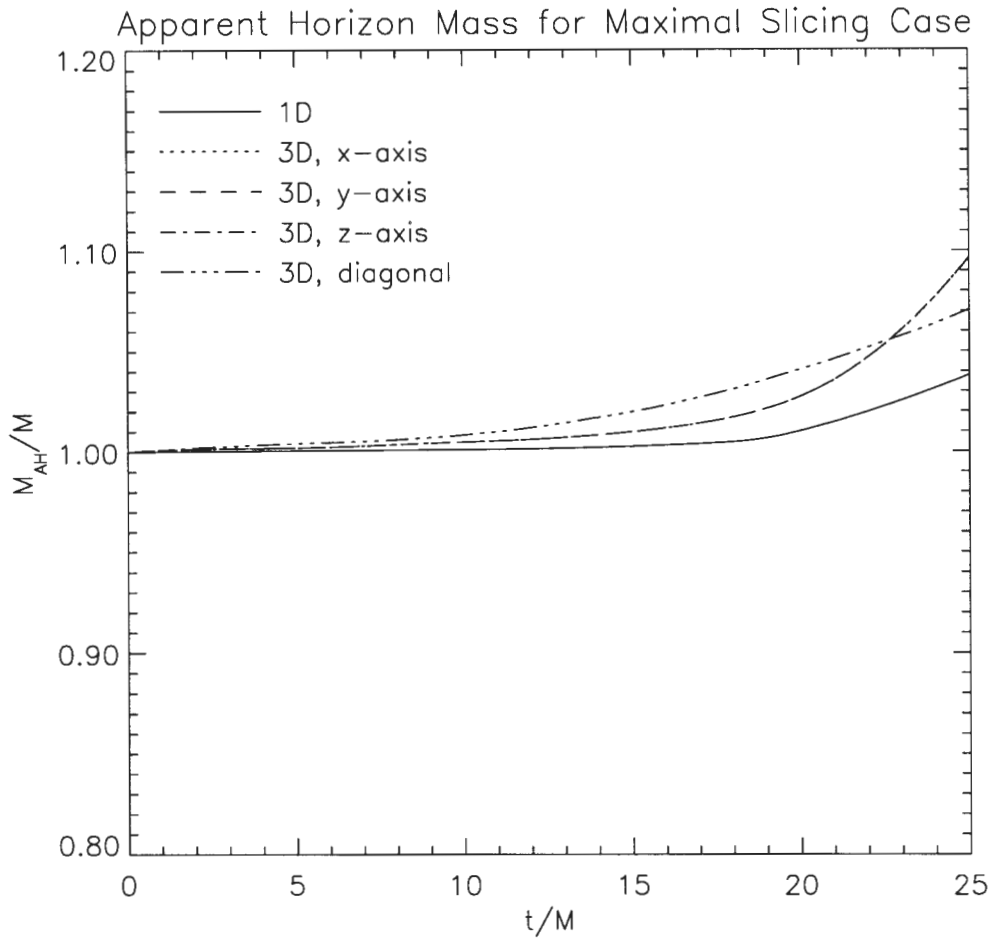


Figure 3.14: As in Figure 3.13, except that we plot the mass of the apparent horizon. The 1D results are reproduced to within about 5% by the end of the calculation.

cross section of this slice with the same function obtained by evolving a single octant of the spacetime at the same resolution with the same outer boundary location. In the plot, it is clear that the functions agree extremely well. It turns out, however, that for the maximal slicing case the results are not *exactly* the same because of the iterative solver used to compute the lapse function. However, the resulting difference is negligible.

It is clear that maximal slicing presents a number of problems for 3D black hole evolution. Not only is it very time (and memory) consuming to solve a 3D elliptic equation on every time slice, but it is also difficult to resolve on a 3D Cartesian grid

the kinds of very sharp, spherical step function features that develop in the solution. Tolerances on the solver must be set quite tightly, especially at late times, in order to get a good solution. Towards the end of the calculation, just before instabilities in the hyperbolic solution set in, the solution for α can actually become slightly negative at the “base” of the step function, and the derivatives of α can become unsmooth, both due to difficulties associated with solving for this function on a 3D Cartesian grid. These problems can be delayed by increasing tolerances or resolution, but at some point they will develop in all cases studied.

3.5 Results: Algebraic Slicing

As shown above, maximal slicing is effective for avoiding a black hole singularity, but in Cartesian coordinates it can develop problems at late times, and it is very time consuming. Algebraic slicings are cost effective alternatives to maximal slicing that have not been explored extensively in numerical relativity. In this Section we discuss results obtained for the condition

$$\alpha = f(x^a)(1 + \ln(\hat{\gamma})), \quad (3.9)$$

where $f(x^a)$ is an arbitrary function of the spatial coordinates.

As the function $f(x^a)$ can be chosen at will, this slicing condition still has considerable freedom. The grid stretching problems associated with black hole evolutions are severe enough in 3D to halt the evolution due to the unbounded growth in metric functions, as shown above for maximal slicing. Therefore, in choosing $f(x^a)$ one would like to delay the grid stretching effects while allowing true dynamics to evolve. A good choice for this is a lapse condition that vanishes on the black hole throat. In this case, we choose the function $f(x^a)$ to have the value of the Schwarzschild lapse given in eqn. 3.8. In Figure 3.17 we show the Schwarzschild lapse in the $z = 0$ plane. In principle this choice of slicing condition makes the time coordinate t the

Killing time, so that if the system were evolved analytically the metric would be truly static. However, this requires perfect cancellation between second derivatives of the lapse and Ricci tensor components in the extrinsic curvature evolution equations (see equation (2.6)), which will not occur due to discretization error. Therefore, the black hole will evolve in time, although the initial grid stretching effects will be slowed dramatically due to the lapse profile. At later times the lapse collapses and develops a profile similar to the maximal slicing discussed above. Such “antisymmetric” (in the appropriate sense across the throat) lapse conditions have been used extensively in other calculations, such as the collision of two black holes [14, 16] and rotating black holes [28, 29].

The algebraic slicings do have one notable drawback relative to maximal slicing in that they tend to be a bit too “local”. If some feature develops at a particular point in the 3-metric, the algebraic lapse responds instantly and locally. This is then fed back into the evolution equations through first and second spatial derivatives of the lapse, which can exaggerate undesirable features in the solution. The solution to an elliptic equation, on the other hand, tends to smooth over any local inhomogeneities. In order to decrease the locality of the algebraic lapse conditions, we have taken advantage of the evolution equation for the determinant of the 3-metric and used it to actually evolve the lapse. Then we may add a diffusion term to the evolution equation for the lapse. The diffusion tends to smooth out any local, higher frequency features that develop, leading to a more stable evolution. This approach leads to the following scheme for the lapse:

$$\dot{\alpha} = f(x^a)g'(\gamma)\dot{\gamma} + \epsilon\nabla^2\alpha, \quad (3.10)$$

where $\dot{\gamma}$ is given by

$$\dot{\gamma} = \gamma(-2\alpha tr K + 2D_a\beta^a). \quad (3.11)$$

In Figures 3.18a,b we show the lapse at times $t = 33M$ (partially collapsed) and

$t = 48M$ respectively for a simulation run with 128^3 zones with $\Delta x = 0.06M$. As the lapse is antisymmetric across the throat, it approaches the value $\alpha = -1$ at the origin, and its gradient becomes undefined there. For this reason it is essential to perform the evolution with the isometry condition imposed so that the throat interior is obtained by mapping the exterior solution rather than through evolution equations. By time $t = 30M$ the lapse has begun to collapse around the throat and by $t = 48M$ it has collapsed dramatically with a profile very similar to that seen with maximal slicing.

In Figure 3.19 the metric function $\hat{\gamma}_{rr} = \gamma_{rr}/\psi^4$ is shown, also at $t = 48M$. As before, this function has been constructed from the Cartesian metric coefficients, and shows the familiar peak surrounding the black hole throat. The prominent peak structure *inside* the throat results from the mapping of the exterior to the interior region. As in the previous cases, we have also compared the evolution with the black hole placed in the center of the grid to the evolution with the hole in the corner, with the same results.

Because of the growth of the large gradients in the metric functions, the evolution becomes unstable shortly after this time, causing the code to crash by $t = 50M$ at this resolution. Higher resolution can stall the development of this instability somewhat, but at some point it develops for all computational parameters tested to date (*e.g.*, artificial viscosity parameters, resolution, outer boundary, slicing choice, *etc.*).

3.6 Summary

In studying the evolution of the Schwarzschild black hole in 3D, Cartesian coordinates, we have learned much about the issues that arise, and some partial solutions.

The boundary conditions on the spacetime can be treated in a number of ways. We have demonstrated that the isometry that is commonly applied at the black hole throat can be applied effectively in a 3D black hole evolution in Cartesian coordinates, but it is not necessary in all cases, such as when maximal slicing is being used.

The boundary conditions at the outer edge of the grid are more delicate, however. In previous calculations carried out in 1D and 2D, the outer boundary has been placed far enough away that the metric could be held static, but in 3D this does not work well. We have tested several conditions that use extrapolation, but in most cases, extrapolation boundary conditions give worse results than keeping the outer boundary static.

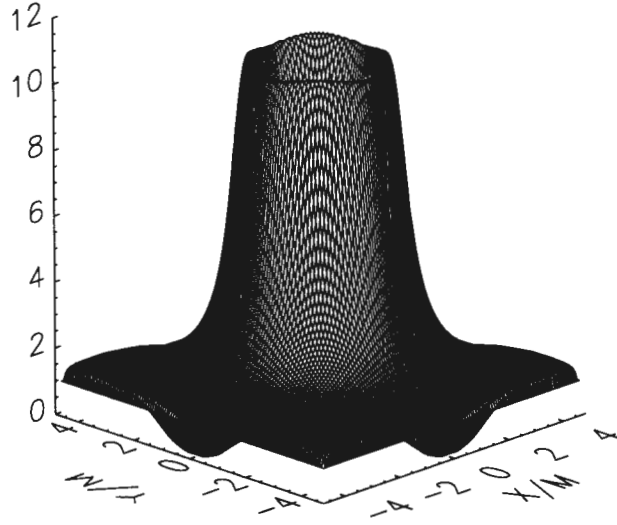
Many slicing conditions have been tested. For geodesic slicing, we demonstrated that the 3D code reproduces the results from a 1D code with a high degree of accuracy, both by comparing metric functions and by looking at derived quantities such as apparent horizon locations and masses. A number of singularity avoiding slicings have been developed for 3D, including maximal and a class of algebraic slicings. Maximal slicing works well, as in 1D and 2D simulations, until the gradients in the metric become very pronounced. At that time very tight tolerances on the elliptic solver are required to improve the accuracy and smoothness of the lapse solution, but problems develop at late times in all cases studied. We have shown that the algebraic slicings are quite promising and economical, and in some cases we have been able to evolve beyond $t = 50M$. We regard the algebraic slicings as a major step forward for 3D black hole evolutions, but in all slicings studied the evolution cannot be carried beyond a certain point due to extreme grid stretching effects. These difficulties have been long recognized in 1D and 2D studies, but they are more severe in 3D. By adding more resolution or viscosity terms one can delay the growth of instabilities, but a more fundamental approach to the problem is needed. Different formulations of the equations, such as those discussed in [25], allow for the use of other numerical techniques that may be able to handle the peaks that show up in the metric functions better.

In order to solve these problems, apparent horizon boundary conditions are under development by a number of researchers [13, 69, 71]. It has been demonstrated that

a shift vector designed to prevent coordinates from falling into the hole, combined with cutting away the singular region inside, can work quite well in a 3D Cartesian black hole simulation [12].

Because we made no assumptions about the symmetries of the black hole that we evolved, we assume that much of what we have learned will be applicable to the evolution of more general black hole spacetimes. In the next Chapter we apply what we have learned to the evolution of another black hole initial data set, the Misner two black hole initial data.

(a) Maximal γ_{rr}/ψ^4 ($t=15M$)



(b) Maximal γ_{rr}/ψ^4 ($t=15M$)

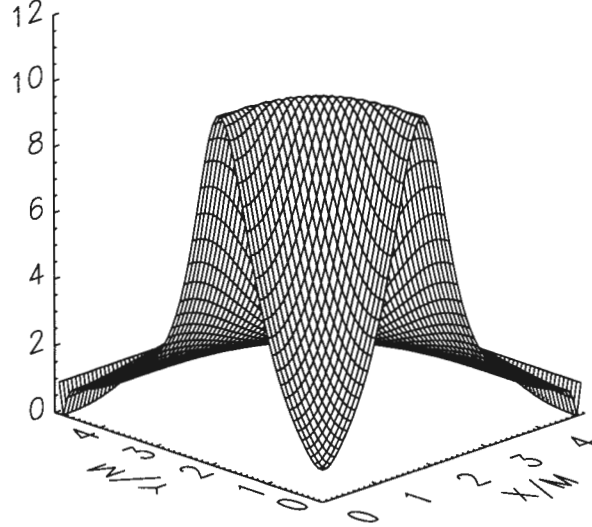


Figure 3.15: We show a comparison of the metric function $\hat{\gamma}_{rr}$ computed for a black hole placed at the center of the numerical grid with one placed in the “corner”. In both cases, the resolution was $\Delta x = 0.075M$ and maximal slicing was used. In (a), the number of grid points used was 126^3 . However, in (b), the number of grid points used was 64^3 , with symmetry boundary conditions across the coordinate planes, giving the same results as in (a). Although it is not obvious from these surface plots that we are obtaining the same results in both cases, we can see in the line plot 3.16 the the functions line up perfectly.

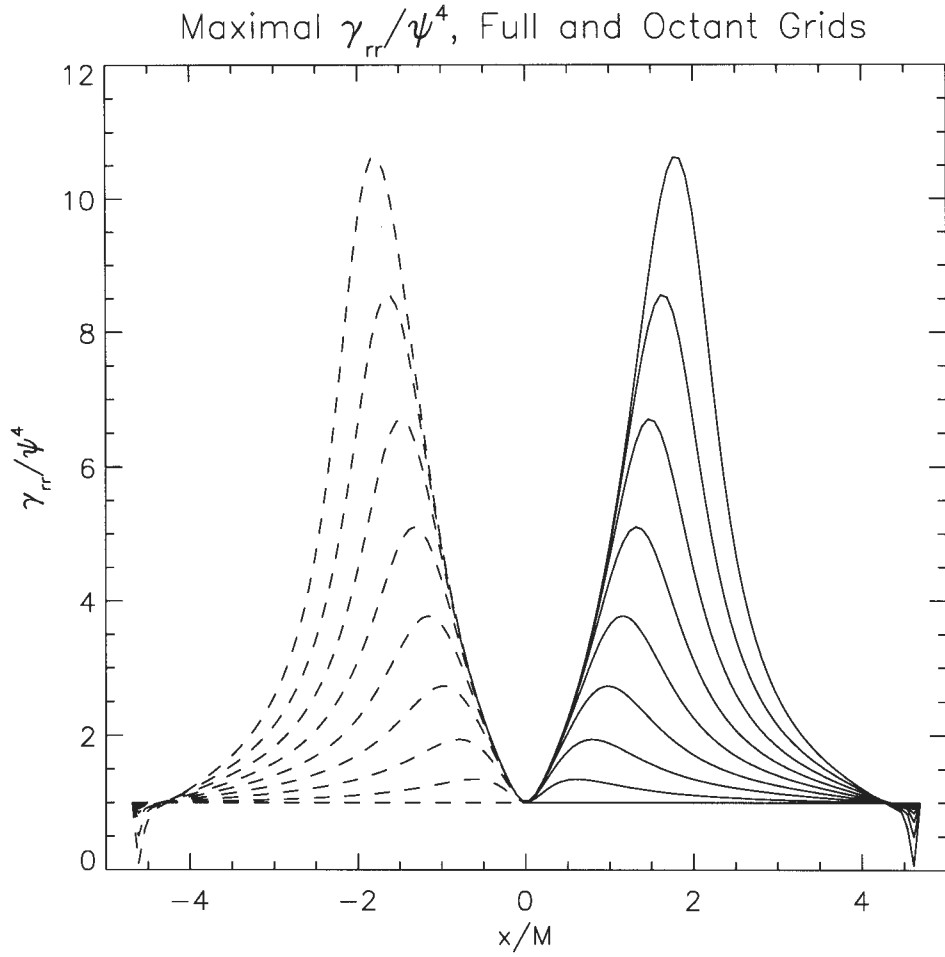


Figure 3.16: A plot of the cross sectional line of $\hat{\gamma}_{rr}$ for full and octant grid cases for maximal slicing. Data are taken from the x axis. Note that for $x > 0$, both cases are present, but the data for the two cases are indistinguishable.

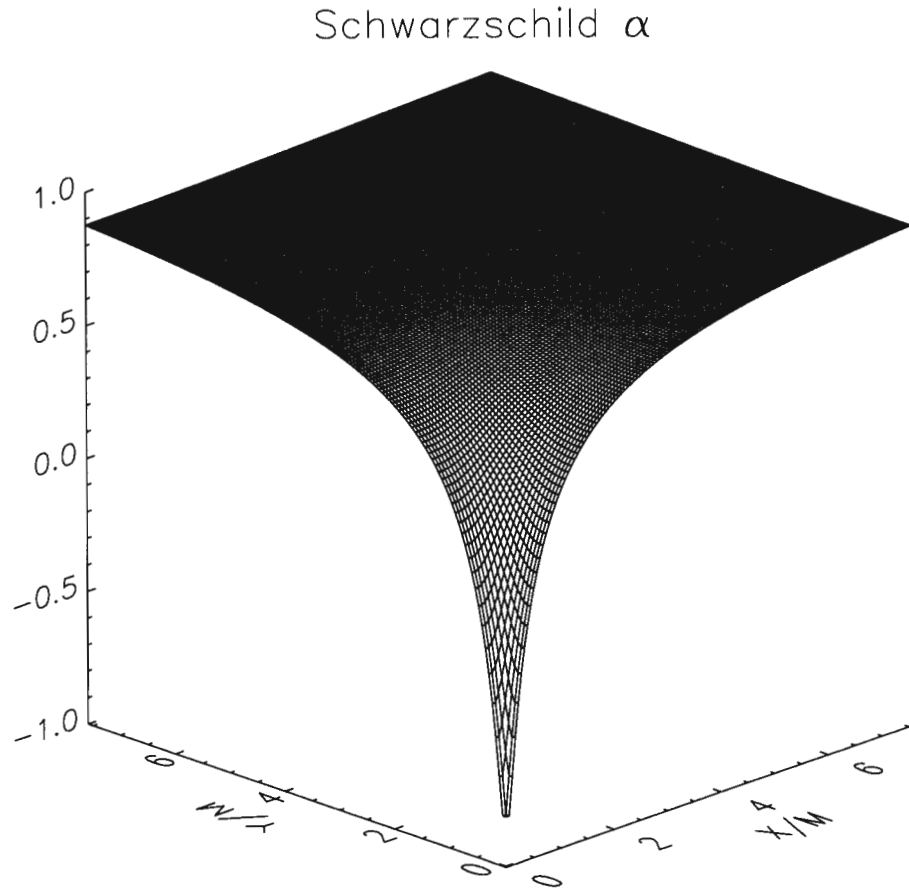


Figure 3.17: We show the Schwarzschild lapse in isotropic coordinates in the $z = 0$ plane. This lapse was used as the initial lapse for the algebraic slicing runs.

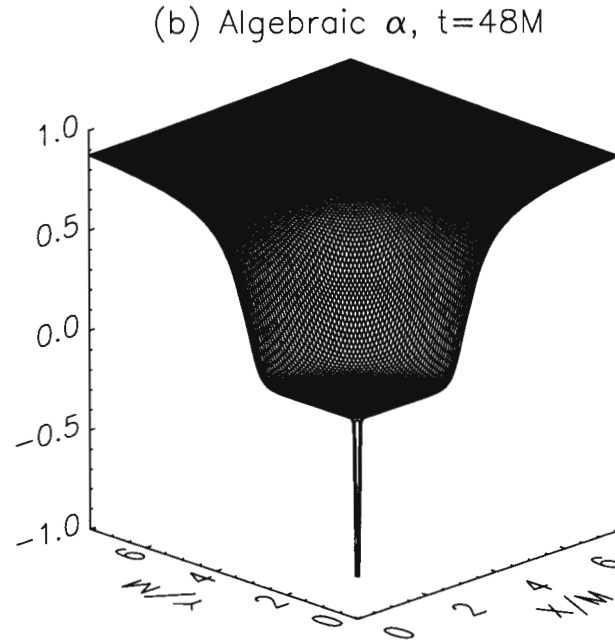
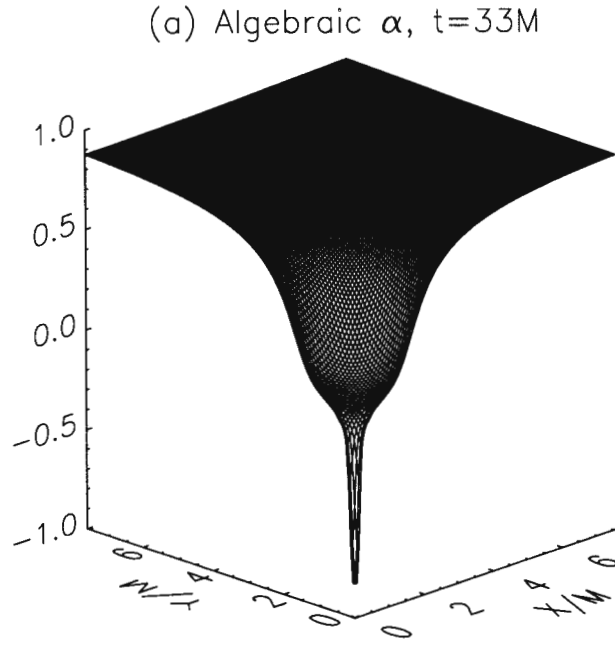


Figure 3.18: We show the evolution of the lapse for the “1+ln” algebraic slicing case discussed in the text. Initially, the lapse in Schwarzschild, as shown in (a). After a while, the lapse begins to collapse (b) as in the maximal case. Eventually, the lapse is completely collapsed (c) and the troubles due to steep gradients that also occur in maximal slicing cause the code to crash shortly after this time.

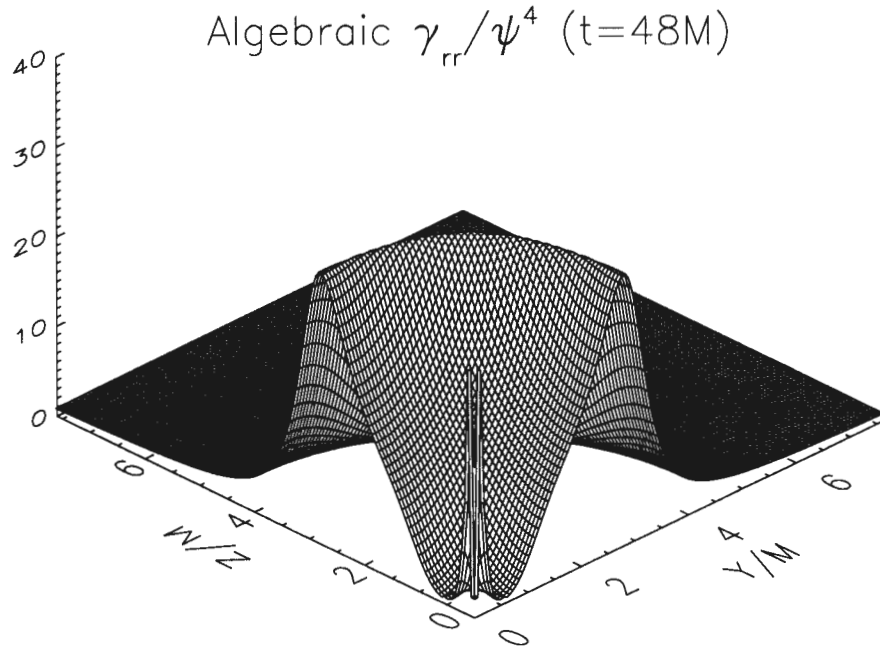


Figure 3.19: We show the function γ_{rr}/ψ^4 obtained with algebraic slicing. The data are taken from the $x = 0$ plane. 128^3 grid points were used, with a resolution of $\Delta x = 0.06M$. The peak inside the throat is a result of the isometry condition, which maps data outside the throat to points inside.

Chapter 4

Two Black Hole Collision: Misner Initial Data

The second black hole initial data set that we chose to evolve was the Misner two-black-hole initial data set. This data set represents two equal-mass, non-rotating black holes momentarily at rest. Although there are other initial data sets which correspond to the same physical situation, the Misner data have the advantage of obeying an isometry condition across the throats of the black holes.

The study of this classic initial data set and its evolution has a long history, dating back 35 years. In 1960, Misner first proposed his analytic, time symmetric, conformally flat initial data set for two equal mass black holes [61]. In 1963, Brill and Lindquist extended his work, deriving time symmetric, conformally flat, analytic initial data for N black holes with arbitrary masses [32]. The first numerical evolutions of the Misner initial data were attempted in 1964 by Hahn and Lindquist [54]. They were able to demonstrate the feasibility of this approach to finding solutions of the dynamical Einstein equations, but were hampered by a number of things beyond their control. In particular, the notion of black holes and horizons had not yet been properly formulated. Also, the bispherical coordinates that Misner used to find his initial data, which they also used for their evolutions, did not work for numerical calculations. In the late 1960's, DeWitt started a research effort on the numerical computation of two black hole collisions. This effort led to theses by Čadež [35],

Smarr [72], and Eppley [46], in addition to follow-up papers [73, 74, 75]. They were able to show that the quasinormal modes of the final black hole are excited, and that the total energy released in gravitational radiation during the collision is typically less than 0.1% of the mass of the final black hole.

Although DeWitt, Čadež, Smarr, and Eppley made much progress, the computer power at the time did not allow for highly resolved evolutions, and they did not have at their disposal waveform extraction techniques for determining gauge-invariant waveforms. For these reasons, they quoted results as uncertain as a factor of two. In the early 1990's, their work was revisited by Anninos, *et. al.* [14, 15, 16]. With the higher resolution available with 1990's computer technology, techniques to better handle coordinate singularities, and advanced wave extraction techniques [2], they were able to produce much more accurate evolutions and waveforms.

The evolution of the Misner initial data has also been studied with perturbation theory in two regimes. First, it was studied in the “close-limit” regime in which the holes are close enough together that they are surrounded by a common apparent horizon, and are thus one black hole [17, 67]. In this case the spacetime can be treated as a perturbation of Schwarzschild. Second, it was studied in the limit that the holes are far enough apart that the “particle-membrane method” is valid [17]. In this case, one of the black holes can be treated as a point particle falling into the other using perturbation theory, but with corrections due to the internal dynamics of the black holes using the membrane paradigm. The results from these studies were compared to the numerical results of Anninos, *et. al.*, with strikingly good agreement. Further development of the “close-limit” approximation to second order perturbation theory was even more spectacular, although beyond a certain limit, it also breaks down [51]. As a result of these studies, an enhanced physical understanding of the system was developed.

We will now present results from the first 3D evolutions of the Misner initial data.

The fact that this data set has been studied so extensively in axisymmetry and with perturbation theory makes it an ideal candidate for a test problem in our code. Also, unlike the Schwarzschild black hole, it will allow us to develop and test 3D tools to extract measures of gravitational radiation. Also, it marks the first attempt to study evolutions of multiple black holes in 3D, where horizons evolve and can change topology. This important first step is the simplest in a progression of simulations leading ultimately to the 3D coalescence of two black holes.

4.1 Misner Initial Data

Misner's 3-metric for two black holes momentarily at rest takes the conformally-flat form [61]

$$d\ell^2 = \psi^4 (dx^2 + dy^2 + dz^2), \quad (4.1)$$

with the conformal factor ψ given by

$$\psi = 1 + \sum_{n=1}^{\infty} \frac{1}{\sinh(n\mu)} \left(\frac{1}{+r_n} + \frac{1}{-r_n} \right), \quad (4.2)$$

where

$$\pm r_n = \sqrt{x^2 + y^2 + (z \pm \coth(n\mu))^2}. \quad (4.3)$$

In Figure 4.1 we show a diagram of where the throats of the black holes are located in the $x = 0$ plane for this choice of conformal factor and $\mu = 1$. The black holes are on the z -axis, with their centers at $z = \pm \coth \mu$ and with throat radii $a = 1/\sinh \mu$. As μ is increased, the centers of the holes approach each other in coordinate space, and their throat radii decrease.

The parameter μ determines the ADM mass M of the spacetime and the proper distance L between the throats of the two holes along the z -axis:

$$M = 4 \sum_{n=1}^{\infty} \frac{1}{\sinh(n\mu)} \quad (4.4)$$

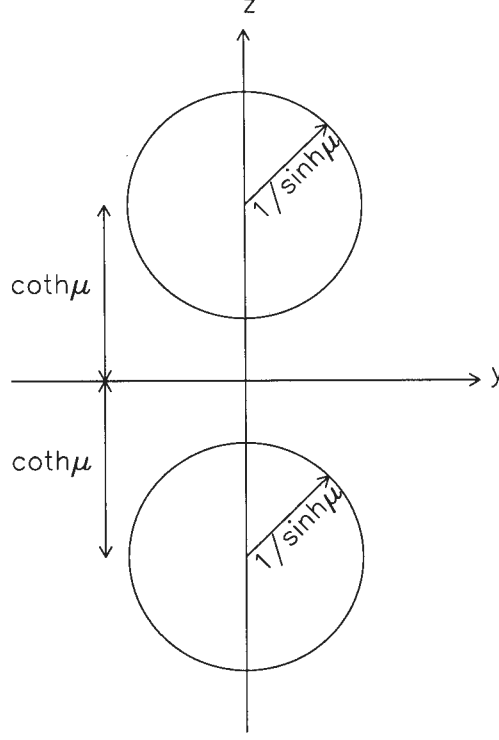


Figure 4.1: We show the location of the throats of the black holes in the $x = 0$ plane for the Misner initial data. The particular value of μ used here was $\mu = 1$. Increasing μ will move the hole centers closer together and decrease the hole radii in coordinate space.

$$L = 2 \left(1 + 2\mu \sum_{n=1}^{\infty} \frac{n}{\sinh(n\mu)} \right) \quad (4.5)$$

The Misner metric satisfies the isometry condition (2.12) with the throat radii $a = 1/\sinh \mu$ and throat centers $\vec{c} = (0, 0, \pm \coth \mu)$. Thus, if the gauge condition is chosen appropriately, we can use an isometry condition to update the evolved functions at points within the throats during the evolution. The extrinsic curvature K_{ij} is initially zero, as in the Schwarzschild case.

In 2D studies, it was found that for $\mu < 1.36$, the throats are surrounded by a common apparent horizon and are actually a single black hole [75]. Further research

which investigated the event horizon location found that there is actually only a single black hole for values of μ up to approximately 1.8, corresponding to a proper separation of about $L = 3.5M$ [8]. In this work we will focus on the evolution of two Misner data sets which have been studied extensively in axisymmetry: $\mu = 1.2$, corresponding to a single black hole, and $\mu = 2.2$, corresponding initially to two black holes.

Like Schwarzschild, this data set has symmetries which allow us to evolve only one octant of the spacelike slices, which we do. Note that this results in only one black hole in our computational domain. Also, only a finite number of terms in the infinite sums that appear when considering this data set can be included. We have found that setting $n_{max} = 20$ is adequate, and have done so in all results presented here.

4.2 The Numerical Grid

Before discussing the results, it is useful to compare our numerical grid to that used in previous 2D work. The grid used in 2D was that developed by Čadež [35]. In Figure 4.2 we show where these coordinates lie on the Cartesian grid for the $\mu = 2.2$ case. Here we show only the very inner-most region of the computational domain. The grid in the 2D code actually extends to a maximum radius of approximately $240M$. This grid approaches the standard spherical-polar coordinate system at large radius, and is fitted to the throats of the black holes at small radius. As in the 1D studies discussed in Chapter 3, the radial coordinate is logarithmic, providing higher resolution near the throat and less resolution in the asymptotic region.

The main drawback of the Čadež coordinate system from a numerical point of view is that there is a coordinate singularity at the origin, in addition to the standard one on the axis. Our Cartesian grid has the advantage of not having any coordinate singularities. However, as discussed in the 1D case, we are not able to go out nearly

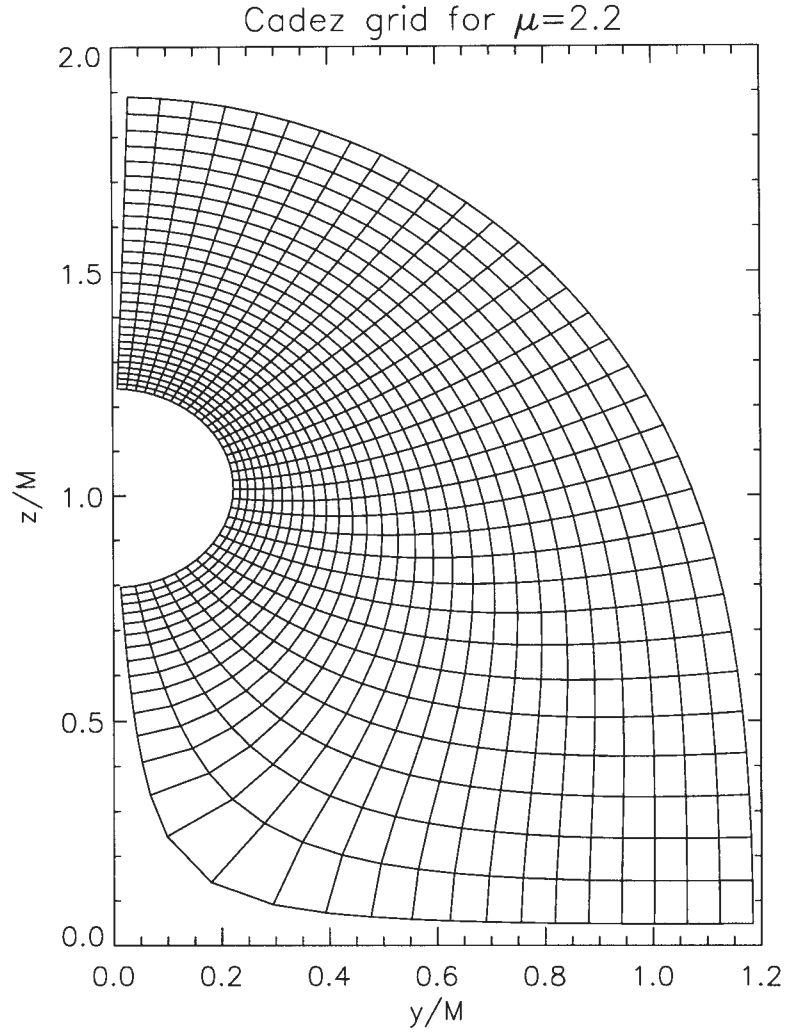


Figure 4.2: We show the inner part of the $\mu = 2.2$ Čadež grid used in the NCSA 2D Misner evolution code. The actual grid used extends past $r = 200M$ in this case, where it approaches a standard spherical-polar coordinate system.

as far in radius. For a very good $\mu = 2.2$ run, on the AEI Origin 2000, we can run jobs of 200^3 grid points with a resolution of $\Delta x = 0.06M$, putting the outer boundary along an axis at approximately $r = 12M$.

4.3 Evolution of Data Sets with $\mu < 1.8$

As mentioned above, in Misner initial data sets with $\mu < 1.8$ a common event horizon surrounds the two throats, resulting in what is actually a single black hole. In this Section we investigate data sets of this type by studying evolutions of the $\mu = 1.2$ initial data set. In this case, the two throats are also surrounded by a common apparent horizon.

4.3.1 Slicing Conditions

As in the Schwarzschild case, we worked extensively with two singularity-avoiding slicings: maximal slicing, and the “ $1 + \ln$ ” algebraic slicing. In this Section, we discuss results obtained with these two slicings.

The first slicing condition with which we evolved this data set was maximal slicing. As in the Schwarzschild case, we chose to evolve the entire domain, without the application of an isometry condition. Also as in the Schwarzschild case, we held the lapse on the outer boundary to a time-independent function which reflected the symmetry of the situation. For the Schwarzschild black hole, this function was the Schwarzschild lapse. For the Misner initial data set, this function was the Čadež lapse, given by [35]:

$$\alpha\psi = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sinh(n\mu)} \left(\frac{1}{+r_n} + \frac{1}{-r_n} \right), \quad (4.6)$$

where ψ is the conformal factor, and $\pm r_n$ are as given in equation (4.3). The initial lapse was obtained by solving the maximal slicing equation, given the Čadež lapse as an initial guess, and holding the outer boundary fixed. This method results in a lapse

which is nearly unity everywhere, but which has symmetry on the outer boundary appropriate to the problem.

As an illustration we consider the evolution of a $\mu = 1.2$ initial data set. This run was computed with 96^3 grid points with a resolution of $\Delta x = 0.0911M$ (M for this case is 3.69). In Figure 4.3, we show the lapse at different times along the z -axis. The lapse can be seen to collapse first inside the throat, then in the region surrounding it. In Figure 4.4 we show a plot of the metric function $\hat{\gamma}_{rr}$ at time $t = 20M$ in the $x = 0$ plane. As is characteristic with singularity avoiding slicings, this metric function is developing a sharp peak which will soon cause the code to crash.

We also evolved these data sets with the “1+ln” algebraic slicing condition described in Section 3.5. As in the Schwarzschild case, we choose an initial lapse which is antisymmetric across the throats of the black holes, and which obeys an isometry condition. This lapse is the Čadež lapse (4.6). Because this lapse has undefined gradients within the throats of the black holes as the Schwarzschild lapse does, it is necessary to update the evolved functions at points within the throat with the isometry condition.

As an illustration, we consider another evolution of the $\mu = 1.2$ initial data set. This run was done with 100^3 grid points using a resolution of $\Delta x = 0.0911M$. In Figure 4.5 we show the initial lapse in the $x = 0$ plane for this data set. In Figure 4.6 we show the lapse function in the $x = 0$ plane at two subsequent times: $t = 10M$ and $t = 20M$. We see that the lapse collapses just outside the throat.

It seems that our conclusions about slicing conditions for the evolution of the Misner initial data are similar to those that we drew for the Schwarzschild case. Both maximal slicing and the “1+ln” algebraic slicing exhibit singularity avoiding properties, allowing evolution for a few 10’s of M given appropriate initial lapse functions. However, with current limits on computer memory, the growth in the metric functions eventually becomes unresolvable and the code crashes. We need

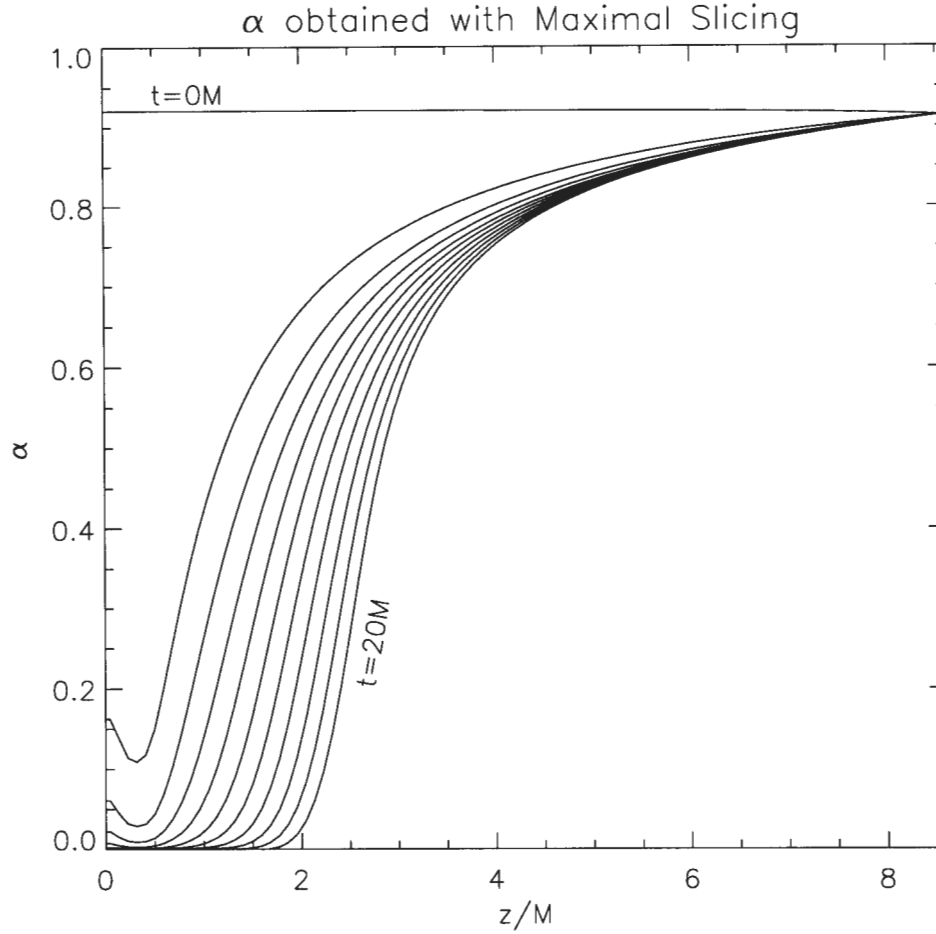


Figure 4.3: The lapse function α along the z -axis is shown as a function of time for the evolution of the $\mu = 1.2$ Misner initial data set evolved with maximal slicing. The data were obtained with 96^3 grid zones with a resolution of $\Delta x = 0.0911M$. The lapse function is plotted every $2M$ in time up to a time of $20M$.

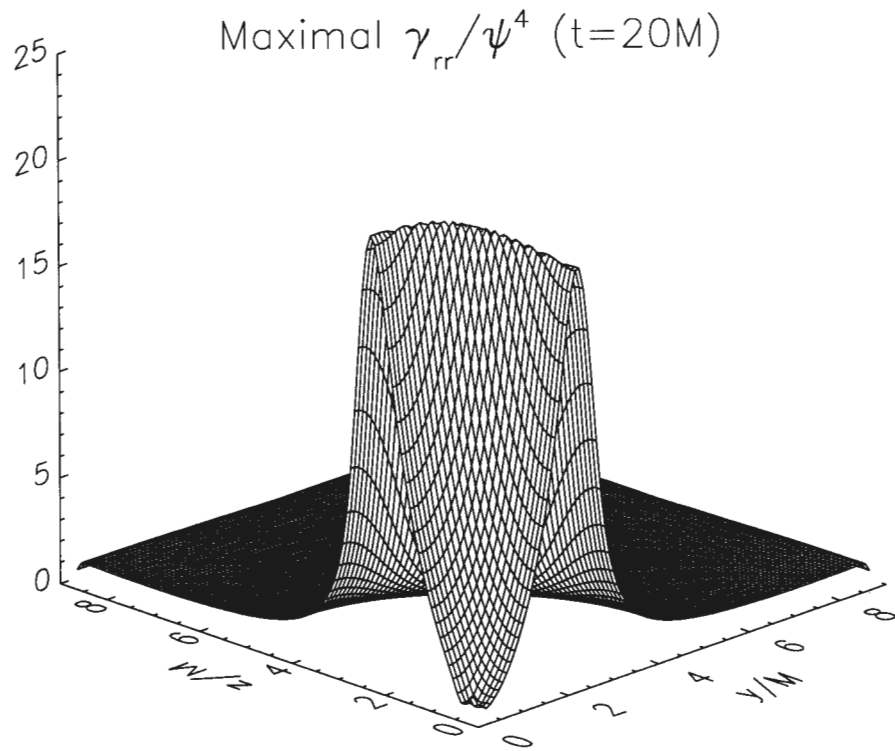


Figure 4.4: As in Figure 4.3, except that we plot the conformal metric function $\hat{\gamma}_{rr}$ in the $x = 0$ plane at $t = 20M$. $\hat{\gamma}_{rr}$ is constructed from the Cartesian metric functions actually evolved.

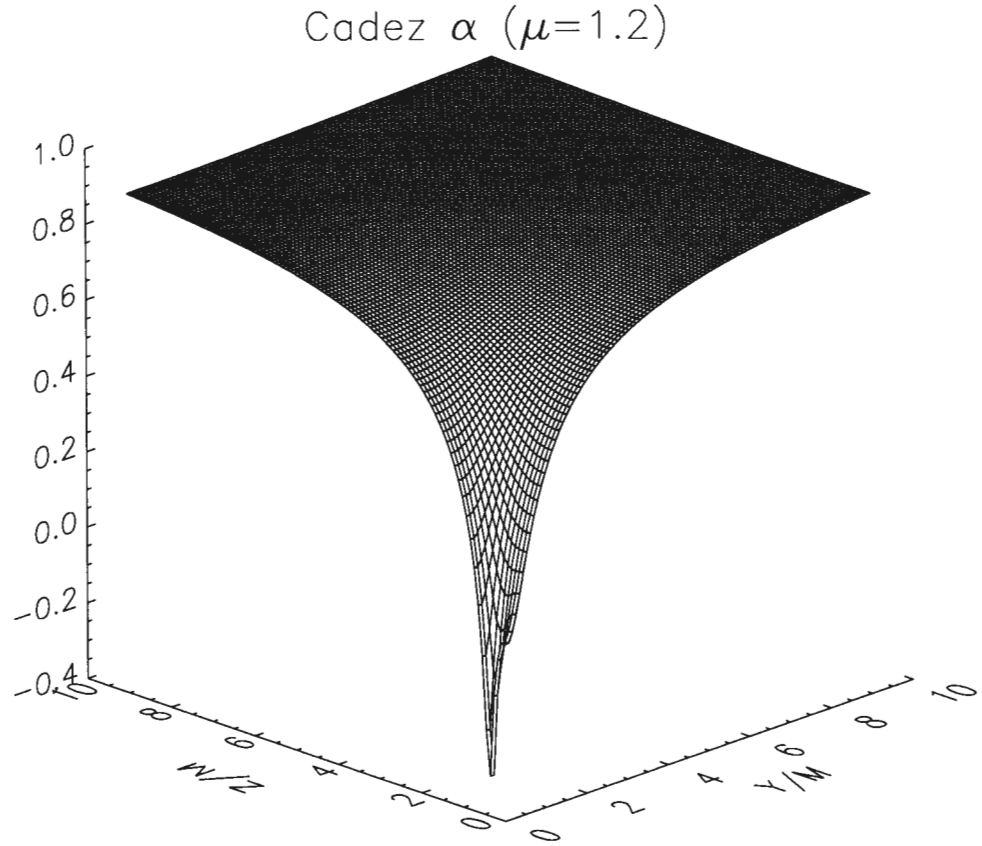


Figure 4.5: We show the Čadež lapse in one quadrant of the $x = 0$ plane for $\mu = 1.2$. This lapse vanishes at the throats of the two black holes in the Misner initial data set, and is used as the initial lapse for the algebraic slicing runs.

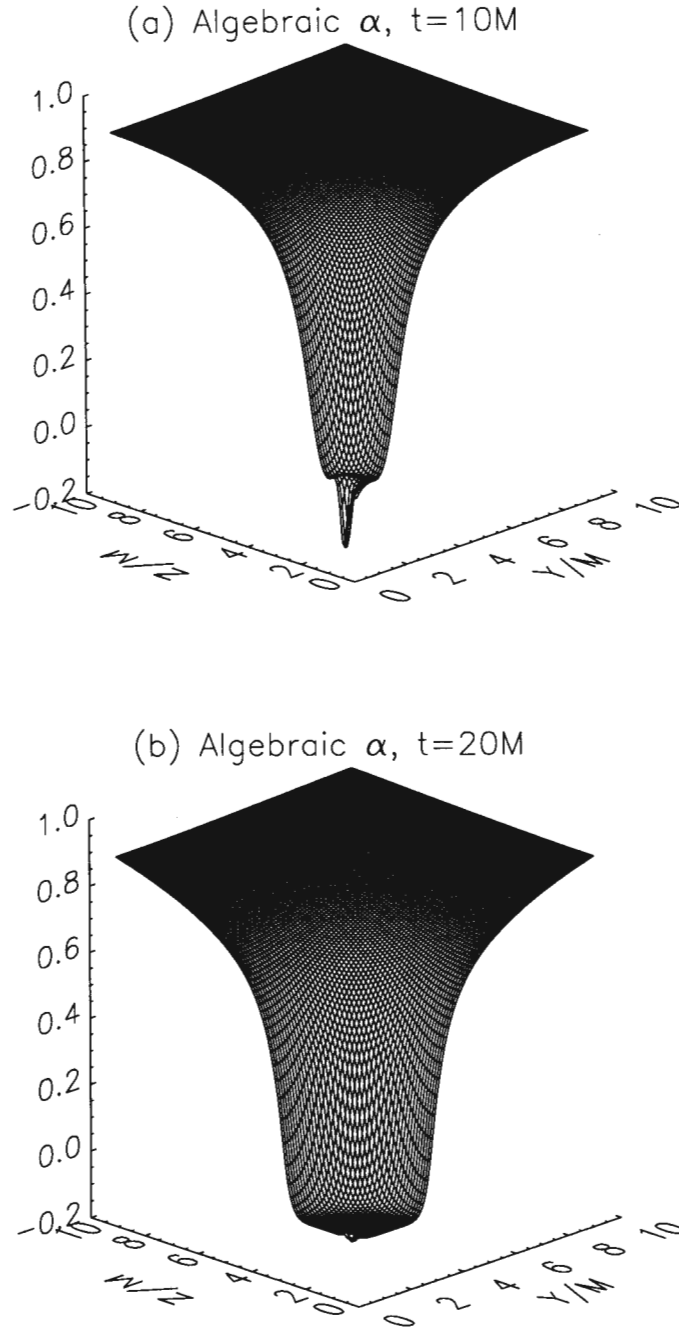


Figure 4.6: We show the lapse function α at times (a) $t = 10M$ and (b) $t = 20M$ for the evolution of the $\mu = 1.2$ Misner initial data set evolved with algebraic slicing. These data were computed with 100^3 grid points. The resolution used in this run was $\Delta x = 0.0911M$.

more sophisticated methods to solve these problems in 3D.

4.3.2 Radiation Extraction

During the evolution of the Misner initial data sets, as the black holes fall together, gravitational radiation is emitted. We have written a routine to compute a measure of this radiation, the Zerilli function, described in Section 2.5. We use the gauge invariant Zerilli function extraction method described in [1] and in Appendix C.

As mentioned above, a problem with 3D simulations is that when using standard methods, with current limits on computer memory and speed, it is not possible to place the outer boundary far enough away from the black hole. This has deleterious effects on the wave form. Of course, if the outer boundary is not far enough away, the whole simulation becomes unacceptably inaccurate. However, before this happens, the wave extraction can become inaccurate if the outer boundary is too close to the radius at which the radiation is extracted. In Figure 4.7 we show the $\ell = 2$ Zerilli function extracted at a radius of $r = 5.42M$ for three different locations of the outer boundary. We see that the waveform found at late times when the outer boundary was closest is quite different from that found when the outer boundary was further away. However, a point is reached at which the outer boundary is far enough away at the given maximum time. At this time, it seems that the accuracy of the waveform is limited by resolution.

Eventually (or for small μ , initially), the evolved spacetime should represent a perturbed Schwarzschild black hole. Thus, far from the black holes, we should be able to measure the quasinormal mode frequencies in the radiation. In Figure 4.8 we show the $\ell = 2$ Zerilli function extracted during the evolution of the $\mu = 1.2$ initial data set using 200^3 grid points and $\Delta x = 0.0813M$. Superimposed on this plot is a fit of the two lowest $\ell = 2$ quasinormal modes. We see reasonably good agreement, indicating that the frequency of the extracted wave is correct. The fit is only done

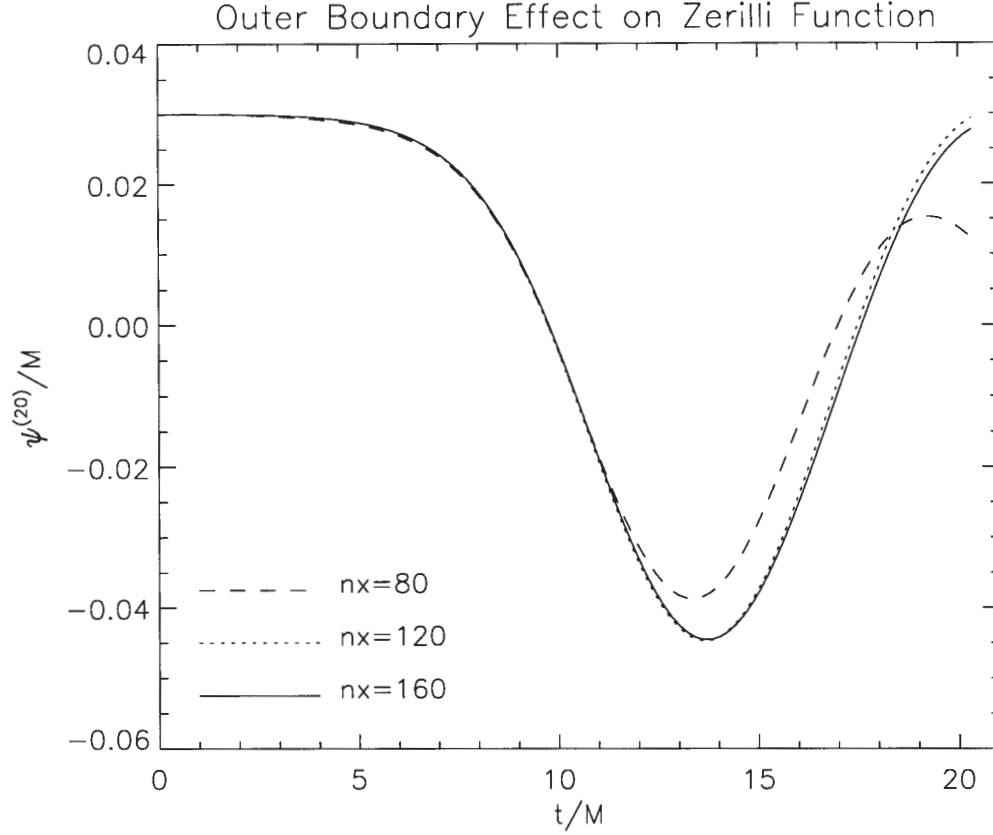


Figure 4.7: We show the $\ell = 2$ Zerilli function extracted from $\mu = 1.2$ evolutions for three different locations of the outer boundary. In all three cases, the resolution used was $\Delta x = 0.0813M$, and the wave was extracted at a radius of $r = 5.42M$. The three grid sizes used were 80^3 , 120^3 , and 160^3 , resulting in distances to the outer boundary along a coordinate axis of $6.38M$, $9.63M$, and $12.9M$. We see that for the time that this evolution was carried out, the outer boundary can be put far enough away that it no longer significantly affects the waveform at this radius.

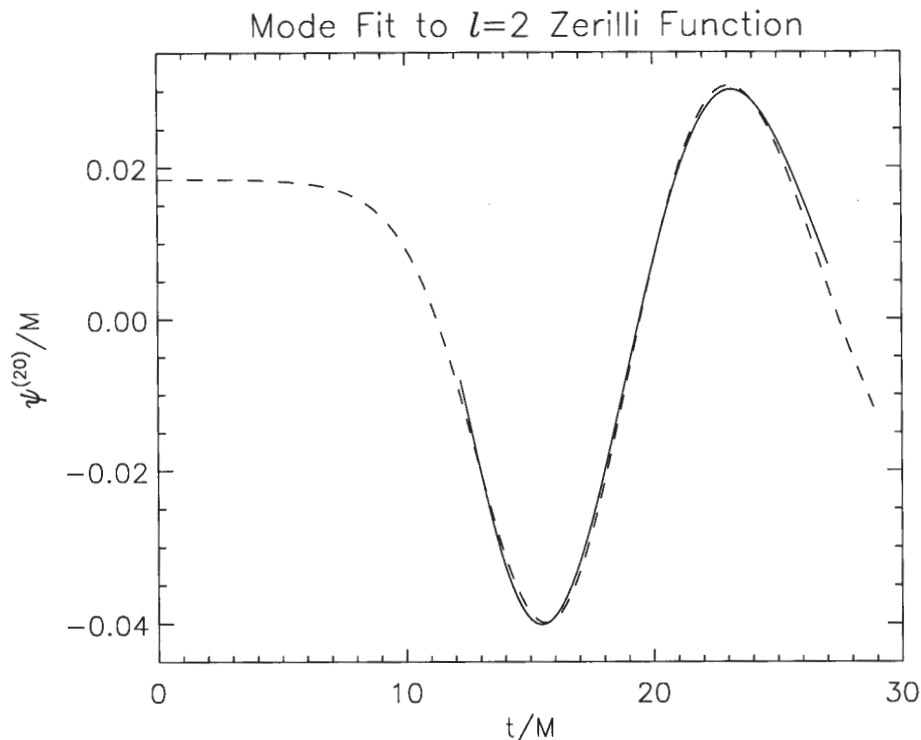


Figure 4.8: We show the quasinormal mode fit to the $\ell = 2$ Zerilli function extracted at a radius of $r = 7.05M$ from the evolution of the $\mu = 1.2$ initial data. The evolution was run with 200^3 grid points and a resolution of $\Delta x = 0.0813M$.

on part of the waveform. We do not do the fit at early times, because the waveform there is very dependent on the details of the dynamics, and the spacetime is not simply distorted Schwarzschild. We do not do the fit at late times, because by then the evolution is becoming inaccurate.

4.3.3 Horizons

We computed the location of the apparent horizon (AH) during the evolution of the $\mu = 1.5$ initial data set, using the 3D AH finder described in Appendix B. Here, $\mu = 1.5$ is studied instead of $\mu = 1.2$ because in this data set, the AH initially consists of the two distinct trapped surfaces at the throats, allowing us to follow the topology change of the AH. From 2D studies, we know that a common AH encircling both throats forms at time $t \approx 1.6M$.

The calculation was run on a 64^3 grid with $\Delta x = 0.0418M$ for a time of $t = 10M$ using the “1+ln” slicing condition. The AH finder was run with only axisymmetric terms in the expansion, up to and including order $L = 4$. Figure 4.9 plots the surfaces found at each time the finder was called ($t = 0, 2.5, 5, 7.5$, and $10M$). The surfaces increase in radius as time increases. Unfortunately, there is no analytic result to compare against the computed location of the AH for $t > 0$. Instead, we overlay the 3D computed AH surfaces with the results from our 2D axisymmetric code evolving the same initial data. The surfaces basically agree with one another. Nevertheless, we note that in the Figure the two sets of surfaces are not exactly the same in coordinate space, as they should *not* be, since we are using different lapse and shift functions in the two cases. Part of the difference may also be attributed to inaccuracies in the 3D evolution; as born out in computing the mass of the AH at different times: at $t = 2.5M$, after a common surface forms to surround both throats, we find $M_{AH} = 2.28$, compared to the 2D result of 2.36. However, by $t = 10M$, the difference in the mass of the computed surface increases to roughly 18% as the evolution becomes less accurate. These errors are attributable to numerical inaccuracies in the evolution and not the AH finder routine; in particular, they are unrelated to the multipole order used, since the surface becomes more spherical in time.

4.4 Evolution of Data Sets with $\mu > 1.8$

Next we consider the evolution of initial data sets which really represent two black holes initially. As an illustration we use the $\mu = 2.2$ initial data set. For this data set, the throats are centered at $z = \pm 1.02$, and have a radius of $a = 0.224$. The ADM mass of the spacetime is $M = 1.01$. For the evolutions in this Section, we will use the “1+ln” algebraic slicing, with the Čadež initial lapse.

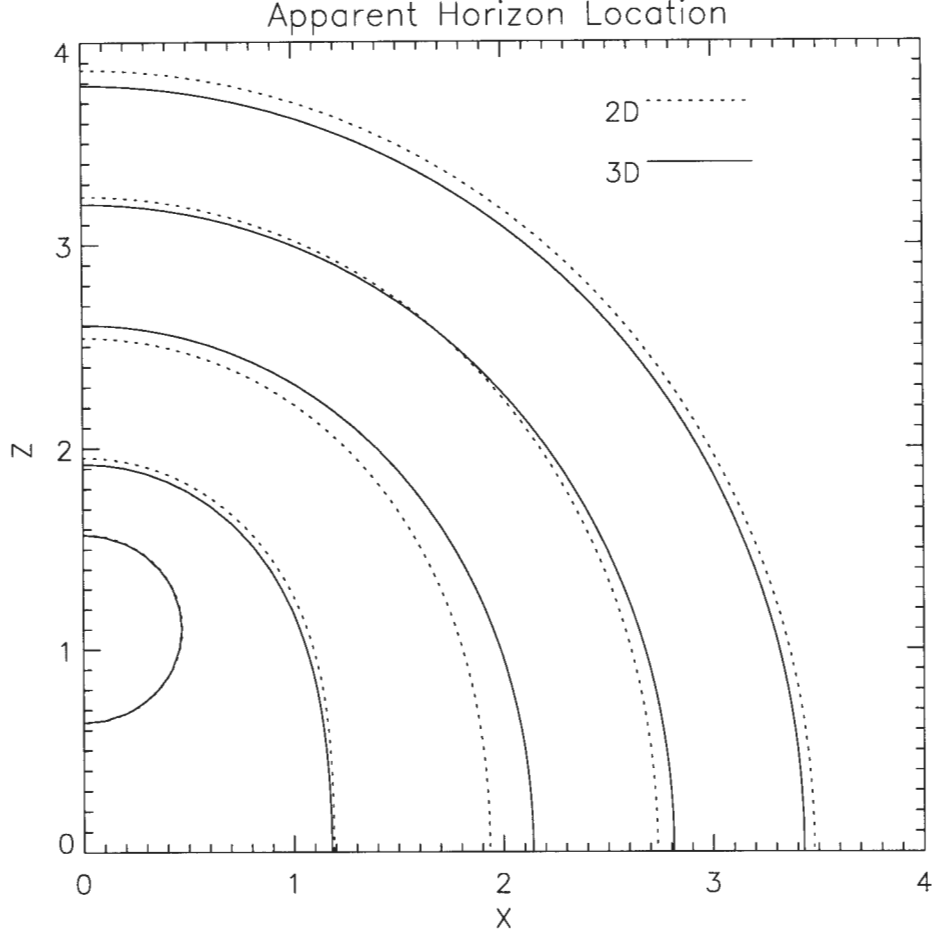


Figure 4.9: Apparent horizon locations in the 3D evolution of the $\mu = 1.5$ Misner two black hole data. The surfaces are plotted at times $t = 0, 2.5, 5, 7.5$ and $10M$, where M is the ADM mass of the spacetime, and the surfaces increase in radius with time. The evolution is performed using a 64^3 grid with resolution $\Delta x = 0.0418M$ and a multipole expansion of order $L = 4$. Also shown are the corresponding horizons found in the 2D axisymmetric evolutions. The surfaces from the two calculations agree nicely, although we note that an exact correspondence is not expected due to different shift and lapse functions.

4.4.1 Metric Functions

In this Section we show data from an evolution done with 170^3 grid points and a resolution of $\Delta x = 0.0596M$. This run crashed at a time of $32.4M$. In Figure 4.10 we show the lapse along the z -axis every $2.98M$ in time, to a maximum time of $t = 29.8M$. As usual, the lapse collapses inside and around the throat. The last plot shows some rockiness, indicating that the code is about to crash. In Figure 4.11 we show the metric function $\hat{\gamma}_{rr}$ in one quadrant of the $x = 0$ plane at a time $t = 29.8M$. The peak in this function is becoming quite sharp relative to the grid resolution.

4.4.2 Radiation Extraction

We extracted the $\ell = 2$ Zerilli function at radii $r = 4.95M, 5.94M, 6.93M, 7.92M$, and $8.91M$ during the evolution discussed in the previous Section. In Figure 4.12 we show the Zerilli function at these different radii. The Zerilli function extracted at the smallest radius looks strange at late times, because there the detector is actually on part of the peak of the metric functions.

4.4.3 Horizons

The location of the event horizon (EH) in numerical relativity is a hard problem. This is because the EH location depends on the entire spacetime, and at any given time, one only has a finite part of this spacetime. We computed the EH location during the evolution of the $\mu = 2.2$ Misner initial data. The method used to compute the EH location is that described in [8], which we briefly review here.

The EH is by definition a null surface. If we express this surface by $f(t, x^i) = 0$, the 4-normal to the surface is given by the 4-gradient of f , $\partial_\mu f$. The magnitude of the 4-normal to a null surface vanishes, *i.e.*,

$$g^{\mu\nu} \partial_\mu f \partial_\nu f = 0, \tag{4.7}$$

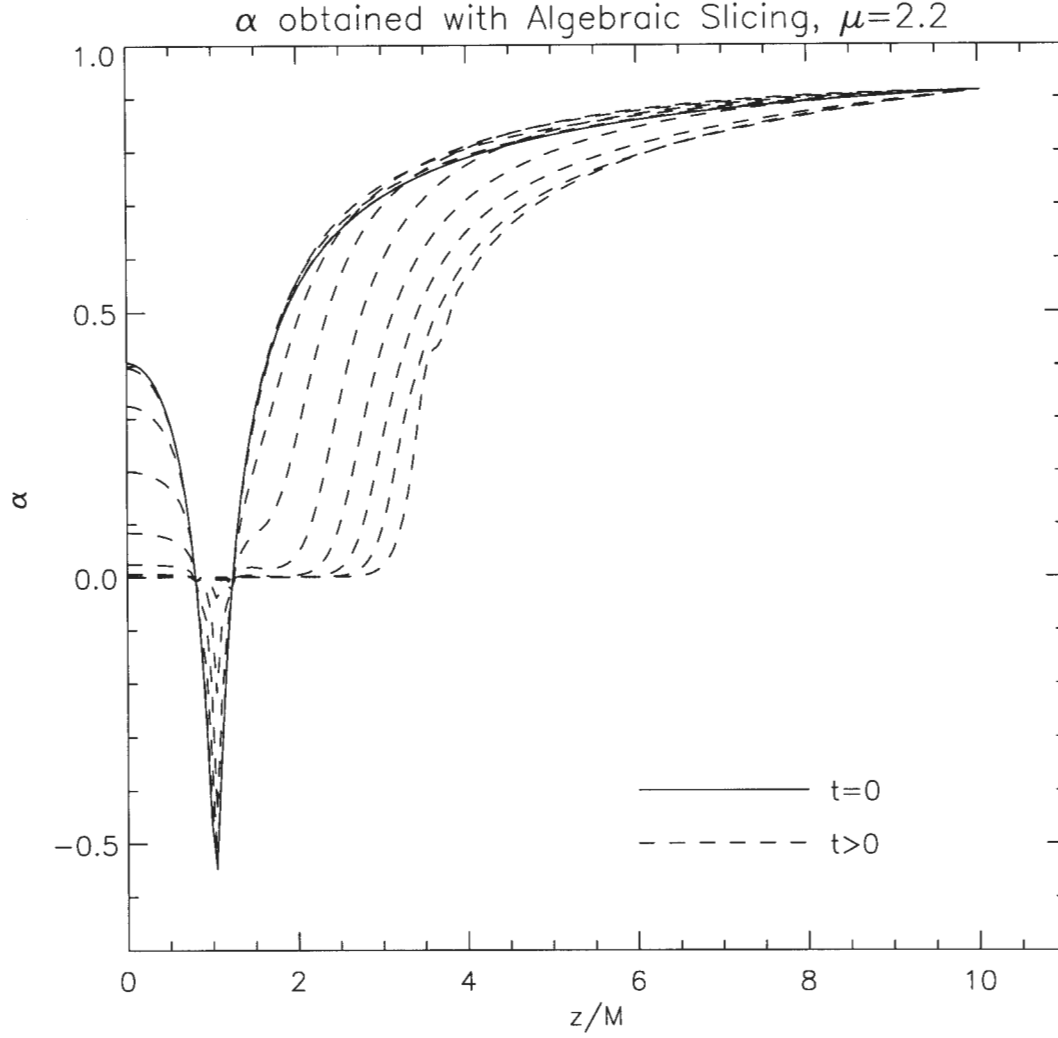


Figure 4.10: We show the lapse function α along the z -axis for the evolution of the $\mu = 2.2$ Misner initial data set. Data are shown every $t = 2.98M$, to a maximum time of $t = 29.8M$. The evolution was done with 170^3 grid zones and a resolution of $\Delta x = 0.0596M$.

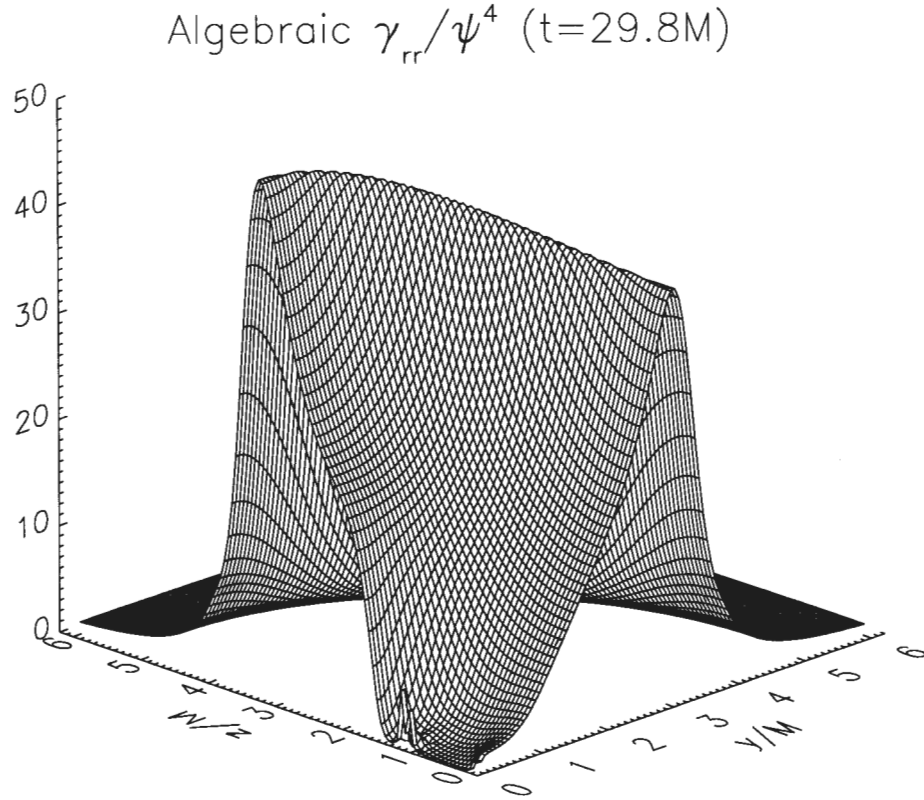


Figure 4.11: We show the metric function $\hat{\gamma}_{rr}$ in one quadrant of the $x = 0$ plane for the evolution of the $\mu = 2.2$ initial data set. The evolution was performed with 170^3 grid zones and a resolution of $\Delta x = 0.0596M$. Data from only the inner 100^2 points in this plane are shown, to bring out detail.

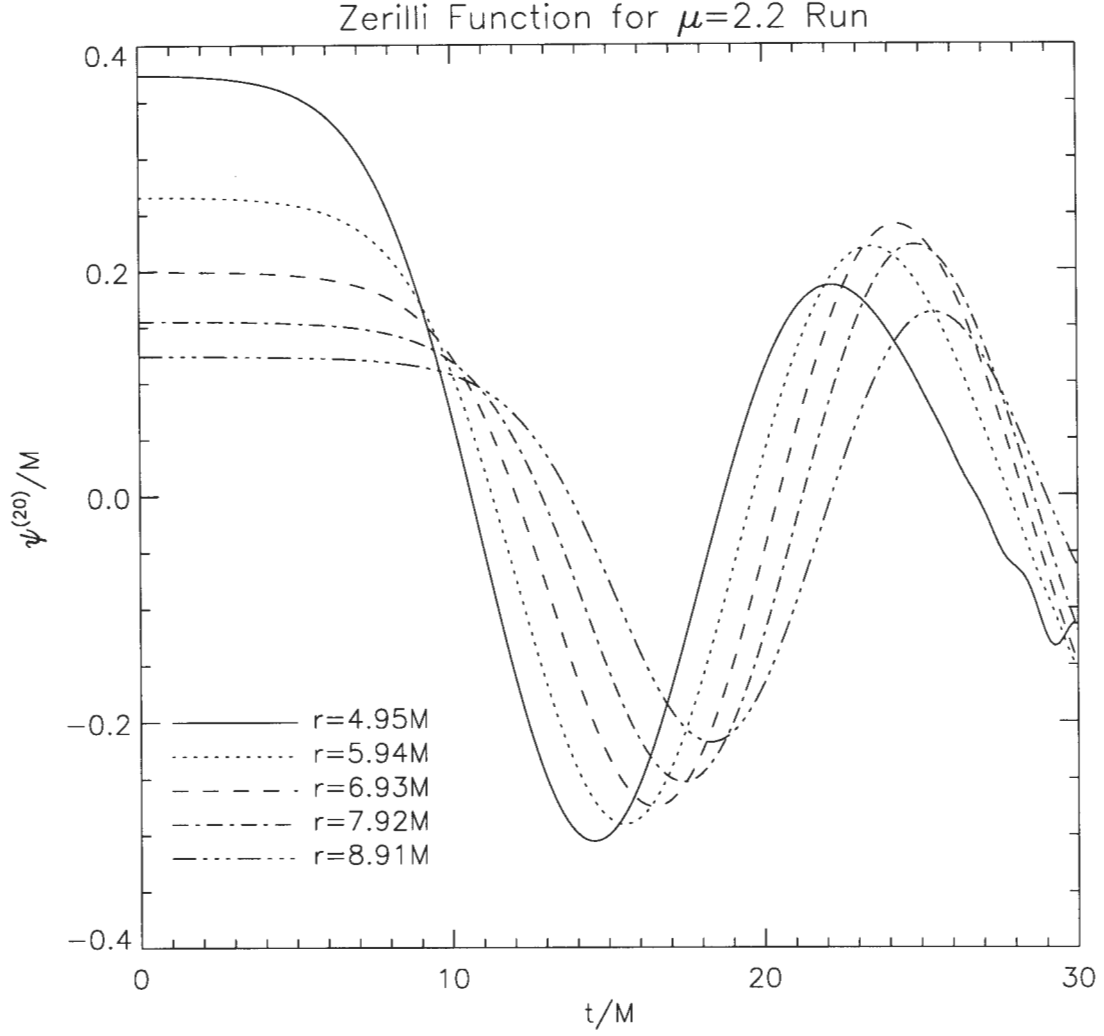


Figure 4.12: We show the $\ell = 2$ Zerilli function extracted during the evolution of the $\mu = 2.2$ Misner initial data set. The different lines shown are extractions at different radii. The evolution was run with 170^3 grid zones and a resolution of $\Delta x = 0.0596M$.

which can also be written

$$\partial_t f = \frac{-g^{ti}\partial_i f + \sqrt{(g^{ti}\partial_i f)^2 - g^{tt}g^{ij}\partial_i f \partial_j f}}{g^{tt}}, \quad (4.8)$$

where the positive sign has been chosen to pick out outgoing null surfaces. The EH is the boundary between those surfaces which escape to infinity and those which fall into the singularity. In principle, one can choose a number of null surfaces to integrate in time as the evolution progresses. Those surfaces which eventually escape to infinity were initially outside the EH, and those surfaces which do not escape were initially inside. It turns out, however, that this procedure is unstable, in that small errors away from the true EH surface grow exponentially in time. Thus, in practice, one runs the numerical evolution, saving all needed data relatively frequently. Then, one uses a guess for the EH location at late time, and evolves this surface backwards in time, using the saved data. This way, small errors in the location of the EH decrease exponentially going backwards in time. In principle, the guess for the EH location at late time could be chosen to be the apparent horizon, since at late times the apparent horizon and the EH will be coincident. However, in practice, it has been shown that many different initial surfaces will rapidly converge to the same surface as the integration proceeds, so that it is not necessary to have an accurate guess.

At the time this calculation was done, there were 2D EH finders, but not 3D. Thus, to find the EH in this simulation, we used the fact that the spacetime is actually axisymmetric, and interpolated data from the $y = 0$ plane onto the appropriate Čadež grid, and used the 2D finder, written by Paul Walker.

We compared the mass of the EH and the coalescence time of the holes with what was obtained from the 2D evolution code, with good agreement to a time of about $10M$. In Figure 4.14, we show the coordinate location of the EH at four different times during the 3D evolution.

4.4.4 Newman-Penrose Quantities

The Newman-Penrose quantities ψ_0 , ψ_1 , ψ_2 , ψ_3 , and ψ_4 are a set of five complex numbers which are scalars given an orthogonal set of four null vectors. These ten numbers contain all the information of the ten independent components of the Weyl tensor. Szekeres showed that the quantities ψ_0 and ψ_4 correspond to transverse waves in a spacetime [77]. We thus use them as a measure of radiation in our spacetimes. Steve Brandt has written a routine to compute these quantities in 3D in the 3+1 formalism, using the scheme described in [53].

As an illustration, in Figure 4.13 we show the quantity ψ_4 computed at time $t = 39M$ for the evolution of the $\mu = 2.2$ initial data set with a 2D code. The tetrad used in this case was that described in [16]. The quantity ψ_4 is indicated by the color map, while the height map measures the lapse function. Note the banana-shaped lobes in ψ_4 . These lobes can be observed to travel outwards as time progresses.

In Figure 4.14 we show the same quantity in the $x = 0$ and $y = 0$ planes computed during the evolution of the same data set by a 3D code. Data are shown at four different times during the evolution, which ran to just past $t = 30M$. Qualitatively, we have good agreement between the 2D and 3D results. However, the tetrads chosen for these computations are not exactly the same, so it is difficult to make a quantitative comparison.

Also shown in Figure 4.14 is the coordinate location of the event horizon, computed as described above in Section 4.4.3. The event horizon location is represented by the surface visible in the first three frames. The surface visible in the last frame is an isosurface of ψ_4 , which is obscuring the event horizon.

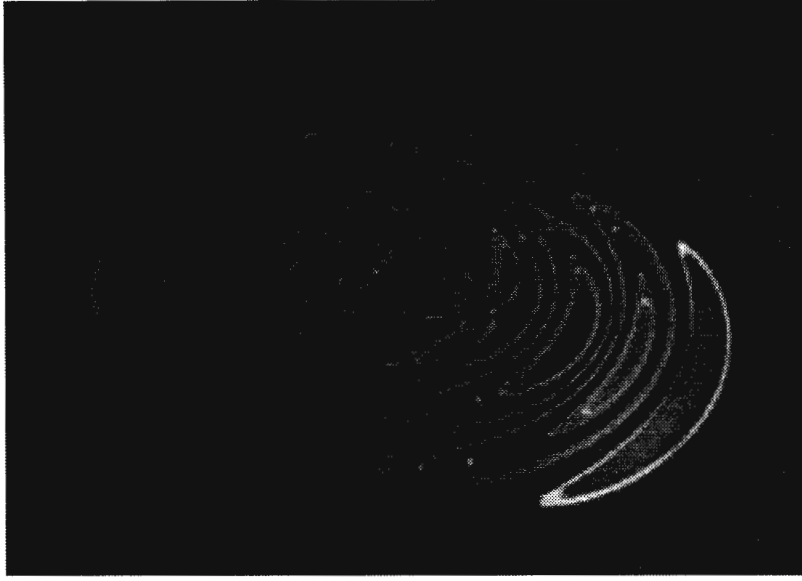


Figure 4.13: We show the Newman-Penrose quantity ψ_4 computed with a 2D code evolving the $\mu = 2.2$ initial data set. The value of ψ_4 is indicated by the color map, while the height map is a measure of the lapse function.

4.5 Summary

In evolving the Misner initial data sets, we have learned that much of what worked well in the Schwarzschild evolutions also works well here. Maximal slicing and the “1+ln” algebraic slicing performed analogously, with an analogous initial lapse. In the algebraic slicing case, we applied the isometry condition to find the values of functions inside the throat, which continued to be an appropriate procedure. Also, the use of a static outer boundary, while still not ideal, proved adequate for the methods being used.

Unlike Schwarzschild, the evolution of the Misner initial data produces gravitational radiation. We have developed two tools to measure this radiation: Zerilli function extraction and calculation of the Newman-Penrose quantities. These tools produced results which were qualitatively similar to results derived with a 2D code, although direct comparisons were not made due to the drastically different coordinate systems.

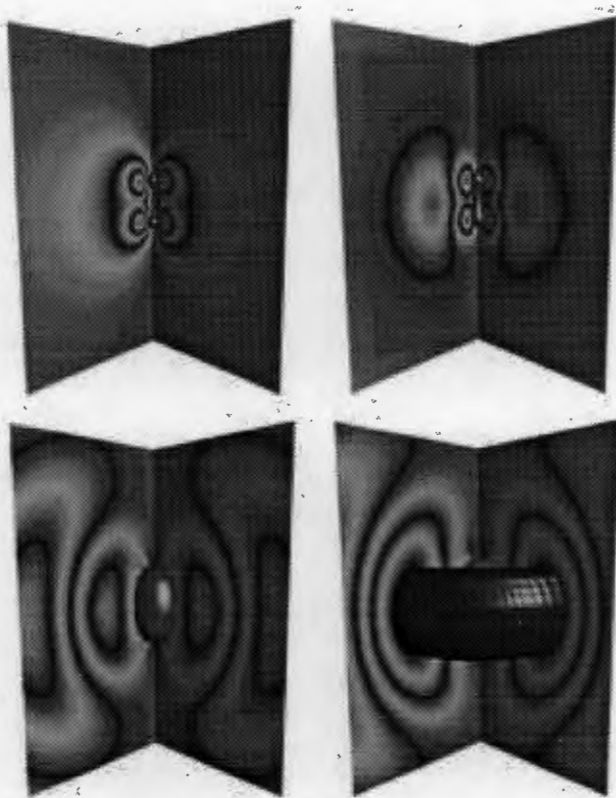


Figure 4.14: We show the Newman-Penrose quantity ψ_4 at four different times computed with a 3D code evolving the $\mu = 2.2$ initial data set. ψ_4 is indicated by the color map. The 3D surfaces visible in the first three frames indicate the coordinate location of the event horizon. The surface visible in the final frame is an isosurface of ψ_4 , which obscures the event horizon.

The accuracy of the metric functions and radiation indicators, and the time a simulation can be run, are limited by computational resources, which limit the resolution and outer boundary location that can be used. Higher resolution simulations are in progress at this time, and will be reported in [10].

In the next Chapters, we return to single-black hole data sets, but we relax the requirement of axisymmetry. Here, we will continue to apply the slicing and boundary conditions that have worked so far, and we will extend the radiation indicators to extract the full set of gravitational wave modes.

Chapter 5

The 3D Black Hole Plus Brill Wave Spacetime: Initial Data

We now turn our attention to the evolution of initial data sets which are truly 3D. To develop such data sets, we will extend the thesis work of David Bernstein, who studied non-rotating, distorted, single black holes in axisymmetry. In this Chapter, we will discuss the form of the initial data that we will evolve. In Chapter 6 we will consider the evolution of these data sets with the full 3D nonlinear code. In Chapter 7 we will consider the evolution of these data sets in linear theory, in the regime in which they can be considered perturbations of a Schwarzschild black hole.

5.1 Background

In his thesis work, Bernstein considered the evolution of axisymmetric black hole initial data sets which could be roughly considered to consist of a Schwarzschild black hole spacetime distorted by a toroidal, time-symmetric gravitational wave [20]. The form of the initial 3-metric was that used by Brill [31]. In spherical-polar coordinates, this 3-metric takes the form

$$d\ell^2 = \psi^4 \left(e^{2q} (dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta d\phi^2 \right), \quad (5.1)$$

where the Brill wave function q and the conformal factor ψ are functions of r and θ only. When q vanishes, we are left with a conformally flat 3-metric. It can be shown

that the only conformally flat metric with a single throat is the spherically symmetric solution, *i.e.*, the Schwarzschild solution. Thus, when $q = 0$, ψ is the Schwarzschild conformal factor in isotropic coordinates, as given in Section 3.1.

Brill used his form of the metric to prove that gravitational waves could contain energy, and were not just coordinate effects. He found that in order for the slice to have a well-defined mass, q has to obey the following boundary conditions:

$$q(r, 0) = 0 \quad (5.2)$$

$$q \sim O(r^{-2}) \text{ as } r \rightarrow \infty. \quad (5.3)$$

Bernstein chose to use a logarithmic radial coordinate η , related to the above coordinate r by $\eta = \ln(2r/m)$, where m is a scale parameter, equal to the mass of the Schwarzschild black hole obtained when $q = 0$. This coordinate has the advantage that the proper distance between equally-spaced points increases with increasing radius, so that fewer computations are done in the outer regions of the domain where functions vary less rapidly. The 3-metric in this coordinate system takes the form¹

$$d\ell^2 = \tilde{\psi}^4 \left(e^{2q} (d\eta^2 + d\theta^2) + \sin^2 \theta d\phi^2 \right), \quad (5.4)$$

and the Schwarzschild solution is

$$\tilde{\psi} = \sqrt{2M} \cosh\left(\frac{\eta}{2}\right). \quad (5.5)$$

The Hamiltonian constraint in this coordinate system looks like

$$\frac{\partial^2 \tilde{\psi}}{\partial \eta^2} + \frac{\partial^2 \tilde{\psi}}{\partial \theta^2} + \cot \theta \frac{\partial \tilde{\psi}}{\partial \theta} = -\frac{1}{4} \tilde{\psi} \left(\frac{\partial^2 q}{\partial \eta^2} + \frac{\partial^2 q}{\partial \theta^2} - 1 \right). \quad (5.6)$$

As can be seen from the transformation equation between η and r , the throat in the η coordinates is located at $\eta = 0$. The isometry condition across the throat in this coordinate system takes the particularly simple form $\gamma_{ij}(\eta) = \pm \gamma_{ij}(-\eta)$.

¹We have changed our notation for the conformal factor from ψ to $\tilde{\psi}$ because in the η coordinate system, a factor of $r^{1/2}$ has been absorbed into the conformal factor, and it is thus not the same quantity in the two coordinate systems. We will be solving for $\tilde{\psi}$ and evolving with ψ .

The Brill wave function q was chosen to have the following form, which obeys the boundary conditions (5.2) and (5.3):

$$q(\eta, \theta) = a \sin^n \theta \left(e^{-\left(\frac{\eta+b}{w}\right)^2} + e^{-\left(\frac{\eta-b}{w}\right)^2} \right). \quad (5.7)$$

Regularity along the axis requires that the exponent n must be even. In his work, Bernstein mostly concentrated on initial data sets in which $n = 2$, although $n = 4$ was also considered. Thus, an initial data set was characterized by the parameters (a, b, w) , where, roughly speaking, a was the amplitude of the Brill wave, b was its radial location, and w its width.

Note that the requirement that n must be even results in data that have equatorial plane symmetry, so that only θ in the range $0 \leq \theta \leq \pi/2$ (or equivalently, $\pi/2 \leq \theta \leq \pi$) need to be considered.

5.2 Extension to 3D

We chose to extend Bernstein's work to 3D by multiplying the Brill wave function q by a factor which has azimuthal dependence. The particular form of this factor was $1 + c \cos^2 \phi$, giving the q -function

$$q(\eta, \theta, \phi) = a \sin^n \theta \left(e^{-\left(\frac{\eta+b}{w}\right)^2} + e^{-\left(\frac{\eta-b}{w}\right)^2} \right) (1 + c \cos^2 \phi). \quad (5.8)$$

Note that the axisymmetric case is recovered by setting $c = 0$. Now the Hamiltonian constraint takes on the more complicated form

$$\begin{aligned} & \frac{\partial^2 \tilde{\psi}}{\partial \eta^2} + \frac{\partial^2 \tilde{\psi}}{\partial \theta^2} + \cot \theta \frac{\partial \tilde{\psi}}{\partial \theta} + \csc^2 \theta e^{2q} \frac{\partial^2 \tilde{\psi}}{\partial \phi^2} + 2 \csc^2 \theta e^{2q} \frac{\partial q}{\partial \phi} \frac{\partial \tilde{\psi}}{\partial \phi} = \\ & -\frac{1}{4} \tilde{\psi} \left(\frac{\partial^2 q}{\partial \eta^2} + \frac{\partial^2 q}{\partial \theta^2} + 2e^{2q} \csc^2 \theta \frac{\partial^2 q}{\partial \phi^2} + 3e^{2q} \csc^2 \theta \left(\frac{\partial q}{\partial \phi} \right)^2 - 1 \right). \end{aligned} \quad (5.9)$$

Note that with this q , we still have equatorial plane symmetry. We also have sufficient symmetry in the azimuthal angle ϕ that we only need to consider ϕ within

the range $0 \leq \phi \leq \pi/2$. This allows us only to perform computations in the first octant, as we have done with previous data sets.

In order to limit our parameter space, in this thesis we limit ourselves to initial data sets in which the exponent n is 2. Thus, our initial data sets are characterized by the parameters (a, b, w, c) . This is the notation we will use to refer to these data sets in the rest of this work.

The method by which we will obtain the conformal factor for our 3D, Cartesian evolution code is the following: First, we will solve for the conformal factor $\tilde{\psi}$ on the spherical (η, θ, ϕ) grid with a 3D elliptic solver, as described in Appendix A. We will also compute the derivatives of $\tilde{\psi}$ on this grid, where it will be in general better resolved than on the Cartesian grid. We will then interpolate these data onto the 3D Cartesian grid. The form of the initial 3-metric in Cartesian coordinates is

$$\gamma_{ij} = \psi^4 \begin{pmatrix} \frac{y^2+x^2e^{2q}}{x^2+y^2} & \frac{xy(e^{2q}-1)}{x^2+y^2} & 0 \\ \frac{xy(e^{2q}-1)}{x^2+y^2} & \frac{x^2+y^2e^{2q}}{x^2+y^2} & 0 \\ 0 & 0 & e^{2q} \end{pmatrix}, \quad (5.10)$$

where $\psi = \tilde{\psi}r^{-1/2}$.

5.3 Limits on the Brill Wave Amplitude

Cantor and Brill showed that for a certain class of spacetimes, there are limits on the Brill wave amplitude beyond which the conformal metric is no longer consistent with asymptotic flatness [40]. Brandt extended their analysis to find limits on the Brill wave amplitude for his axisymmetric, rotating spacetimes [30]. In this Section, we use Brandt's approach to find limits on the Brill wave amplitude for our 3D initial data sets.

One would like to find an inequality which involves the conformal metric, but which does not involve the conformal factor, and which assumes asymptotic flatness. The violation of such an inequality would indicate a conformal metric which is inconsistent with asymptotic flatness.

We work with the 3-metric in isotropic radial coordinates, given by equation (5.1).

Consider the volume integral

$$\int \hat{D}_i f \hat{D}^i f dV, \quad (5.11)$$

where f is for now an arbitrary function, and the integration is over the three-dimensional volume extending from the throat to the sphere at infinity. Let $f = u\psi$.

One can show that in terms of u , the integrand can be written

$$\hat{D}_i f \hat{D}^i f = \hat{D}_i (u^2 \psi \hat{D}^i \psi) - u^2 \psi \hat{D}_i \hat{D}^i \psi + \psi^2 \hat{D}_i u \hat{D}^i u. \quad (5.12)$$

If we now replace u with $f\psi^{-1}$ in the first two terms, and apply Gauss' law, the integral (5.11) becomes

$$\int \hat{D}_i f \hat{D}^i f dV = \oint f^2 \psi^{-1} \hat{D}^i \psi dS_i - \int f^2 \psi^{-1} \hat{D}_i \hat{D}^i \psi dV + \int \psi^2 \hat{D}_i u \hat{D}^i u dV. \quad (5.13)$$

We would like to eliminate ψ from this equation. Let us start by looking at the surface integral. It has two components: the throat and the sphere at infinity. Let us first consider the throat component. If we use the isometry condition $\partial \tilde{\psi} / \partial \eta = 0$, we find that the contribution of the throat to the surface integral is

$$\frac{m}{4} \oint_{\eta=0} f_{\eta=0}^2 d\Omega, \quad (5.14)$$

where $d\Omega = \sin \theta d\theta d\phi$. Now consider the contribution to the surface integral of the sphere at infinity. If we restrict ourselves to f 's which have vanishing angular dependence at infinity, and we assume that $\psi \rightarrow 1 + O(1/r)$ there, the integral becomes

$$\oint_{\eta=\infty} f^2 \psi^{-1} \hat{D}^i \psi dS_i = f_{\eta=\infty}^2 \oint_{\eta=\infty} \hat{D}^i \psi dS_i. \quad (5.15)$$

Now, with the above asymptotic behavior for the conformal factor, this integral is finite, and is in fact proportional to the ADM mass as given in equation (2.17), *i.e.*,

$$\oint_{\eta=\infty} f^2 \psi^{-1} \hat{D}^i \psi dS_i = -2\pi f_{\eta=\infty}^2 M_{ADM}. \quad (5.16)$$

We can eliminate the conformal factor from the first volume integral in equation (5.13) simply by using the Hamiltonian constraint in conformal form (2.9), giving

$$\int f^2 \psi^{-1} \hat{D}^i \hat{D}_i \psi dV = \int \frac{1}{8} f^2 \hat{R} dV. \quad (5.17)$$

We cannot eliminate the conformal factor from the second volume integral in equation (5.13), but we note that this integral is manifestly positive. If we require that $\hat{D}_i u$ fall off at least as fast as $r^{-3/2}$, it will be finite. We can then drop this term to obtain the following inequality:

$$\int \hat{D}^i f \hat{D}_i f dV > \frac{m}{4} \oint_{\eta=0} f_{\eta=0}^2 d\Omega - 2\pi f_{\eta=\infty}^2 M_{ADM} - \int \frac{1}{8} f^2 \hat{R} dV, \quad (5.18)$$

or

$$2\pi f_{\eta=\infty}^2 M_{ADM} + \int \left(\hat{D}^i f \hat{D}_i f + \frac{1}{8} f^2 \hat{R} \right) dV > \oint_{\eta=0} f_{\eta=0}^2 d\Omega. \quad (5.19)$$

For a given Brill wave amplitude, if one can find an f for which this inequality is violated, that Brill wave amplitude is inconsistent with asymptotic flatness.

Brandt found two functional forms for f which gave good limits on the Brill wave amplitude for the axisymmetric initial data sets. The extension of this analysis to the fully 3D data sets will be discussed in [27].

5.4 Masses

In this Section, we investigate how the ADM mass described in Section 2.4 behaves as a function of the Brill wave parameters. We will consider varying each of the parameters a , b , and w together with the azimuthal parameter c in order to determine how the added azimuthal dependence affects the axisymmetric results observed by Bernstein. This Section amounts to a detailed survey of the 4-dimensional parameter space of this family of distorted black hole initial data. Although this Section is a bit dry, it provides an important guide to the physical characteristics of these data sets, which will be important in choosing interesting data for full 3D evolution. For

reference, in the η -coordinate system, the equation for the ADM mass (2.17) takes the form

$$M_{ADM} = -\frac{1}{2\pi} \sqrt{\frac{m}{2}} e^{\eta_{max}} \int_0^{2\pi} d\phi \int_0^\pi \left(\frac{\partial \tilde{\psi}}{\partial \eta} - \frac{\tilde{\psi}}{2} \right) \Big|_{\eta_{max}} \sin \theta d\theta. \quad (5.20)$$

We consider the family of data sets for which the Brill wave is located on the throat and has unit width, *i.e.*, the data sets $(a, b, w, c) = (a, 0, 1, c)$. First we use the ADM mass to determine the range of valid Brill wave amplitudes a for a given azimuthal parameter c . There are two effects which could make an amplitude “invalid”. The first is a physical consideration: it could be inconsistent with conformal flatness, as discussed in Section 5.3 above. In this case, the solver generally finds a solution to the Hamiltonian constraint, but we are able to discard it because the ADM mass (and parts of the conformal factor) become negative. The second effect is purely computational: a chosen amplitude could produce a slice which is so distorted that our numerical grid does not adequately resolve it. In order to determine if we are adequately resolving the conformal factor, we measured the ADM mass for the same set of parameters for the conformal factor computed on grids of two different resolutions: our standard grid, and a grid with half the resolution (*i.e.*, twice the grid spacing). If the ADM masses computed on these two grids differed from each other by more than 1% relative to the standard grid result, that set of parameters was considered to give too distorted a spacetime to compute adequately.

In Figure 5.1 we show the range of valid Brill wave amplitudes as a function of c . Let us first consider the maximum valid amplitudes. We see that there are two maxima. It is not surprising that the maximum valid amplitude for $c = 0$ is greater than that for $c > 0$, since in that case, the azimuthal factor is everywhere greater than unity, resulting in an effective amplitude increase. For $c < 0$ it seems that there are two competing factors. Note that the azimuthal factor can be written

$$1 + c \cos^2 \phi = 1 + \frac{c}{2} + \frac{c}{2} \cos(2\phi). \quad (5.21)$$

The factor can be considered to have two parts: a constant part, and an oscillatory part. Clearly, increasing the constant part should decrease the maximum valid a . If one considers the oscillatory part within our range of ϕ , $0 \leq \phi \leq \pi/2$, it becomes clear that increasing the absolute value of the amplitude of the oscillatory part should also decrease the maximum valid a . However, for c within the range $-2 \leq c \leq 0$, we see that increasing the absolute value of c decreases the constant term while increasing the magnitude of the oscillatory term. Apparently, for a certain range of c , this decrease in the constant term is enough to counteract the increase in the oscillatory term, resulting in the perhaps unexpected second maximum.

The minimum valid Brill wave amplitudes can be explained in a similar manner. Again, we expect the value at $c = 0$ to have a larger absolute value than at $c > 0$, due to the effective increased amplitude. For this case, however, there is no extremum at $c = 0$. Apparently, for negative Brill wave amplitudes, the relative effect of the constant and oscillatory terms in the azimuthal factor are different.

Now we consider the values of the ADM masses within the valid range of Brill wave amplitudes. In all results presented, we will scale our ADM mass by the scale parameter m , so that the spherical initial data set has an ADM mass of unity. In Figure 5.2 we show a plot of the ADM mass as a function of a and c . In Figure 5.3 we show line plots of the ADM mass as a function of a for four different values of c . The results are more or less what one might expect given Bernstein's results in axisymmetry, and the form of the azimuthal factor in the q -function. Bernstein found that in axisymmetry (Figure 5.3c), when one increases a , the ADM mass initially decreases, then increases with a . Although the initial increase seems counter-intuitive, he points out that if one considers, say, the data sets $(a, b, w) = (0, 0, 1)$ and $(0.1, 0, 1)$, with respective ADM masses m and $0.925m$, we are really considering completely unrelated hypersurfaces. Although one increases the amplitude of the Brill wave, it is possible to simultaneously decrease the mass of the underlying black hole, which

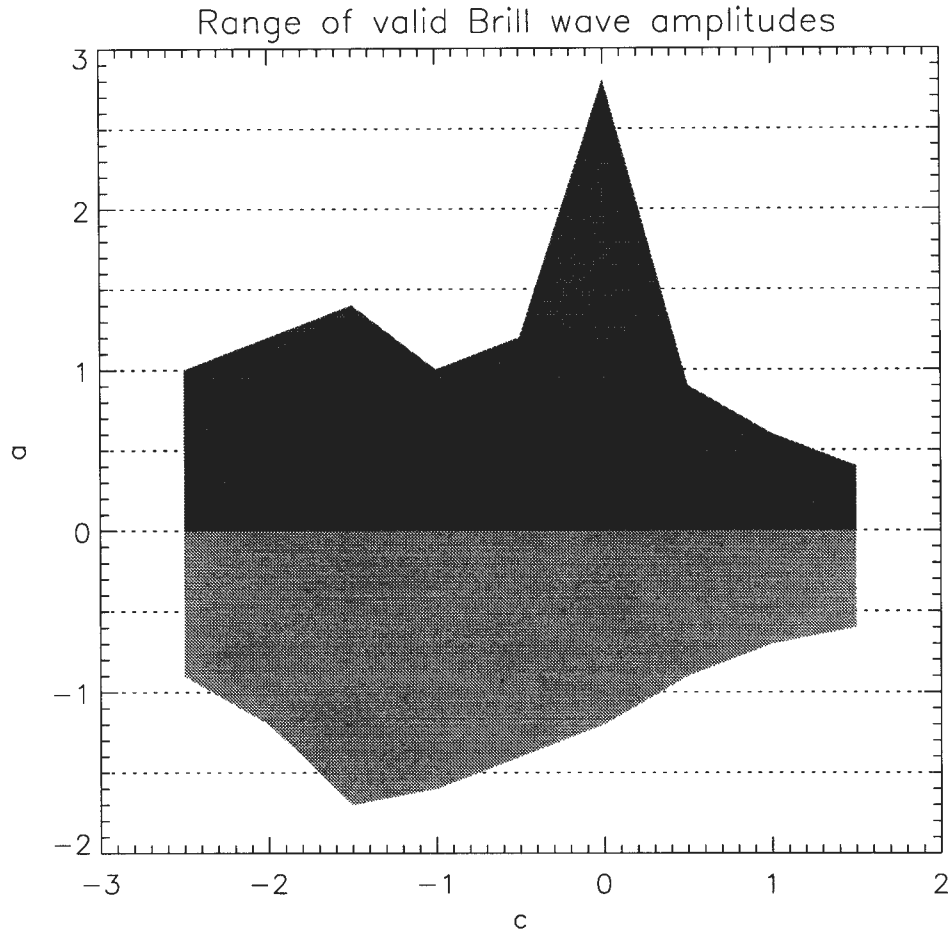


Figure 5.1: We show the range of valid Brill wave amplitudes a as a function of the azimuthal parameter c , determined as described in the text. The shaded regions correspond to acceptable amplitude parameters, based on physical and computational considerations. This plot shows us a section of the 3D distorted black hole parameter space we have chosen to analyze in this Chapter. The Brill wave parameters b and w were 0 and 1, respectively.

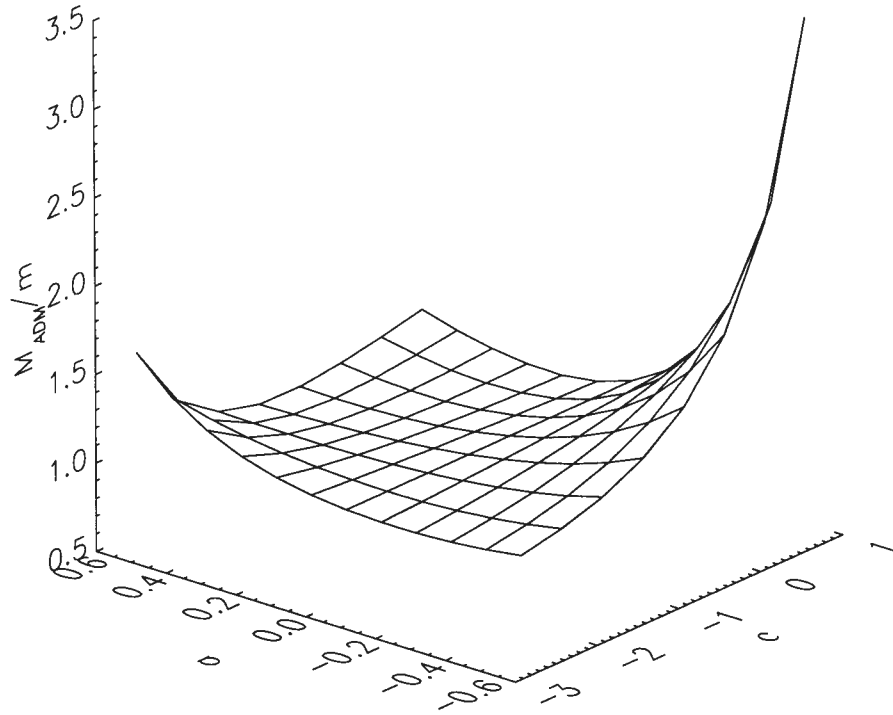


Figure 5.2: We show the ADM mass as a function of the Brill wave parameters a and c . The wave location and width parameters b and w were 0 and 1, respectively.

can give a net decrease in the mass. Bernstein also found that for negative Brill wave amplitudes, the ADM mass increases monotonically with decreasing a .

As expected, we reproduce Bernstein's results for $c = 0$. For positive c , *e.g.* in Figure 5.3d, we see results consistent with an effective amplitude increase. For $a < 0$, the mass increases monotonically with decreasing a , and has a larger value than the axisymmetric case at any given a . For $a > 0$, we see a decrease in the mass, then an increase with increasing a . As expected, the ADM mass for the $c = 1$ case achieves its minimum at a lower value of a than the $c = 0$ case, and is larger at $a = 0.6$ than the $c = 0$ case.

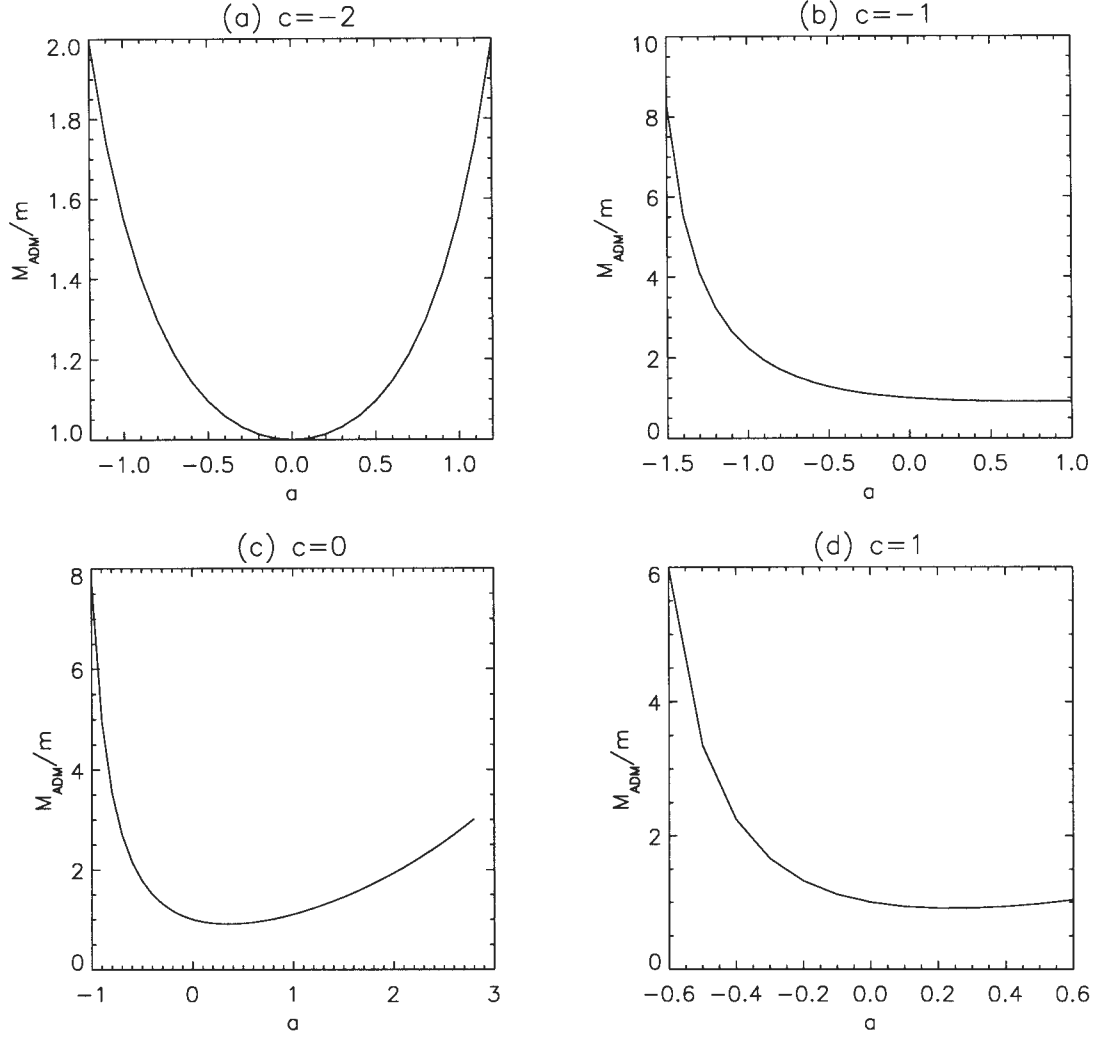


Figure 5.3: We show the ADM mass as a function of the Brill wave amplitude a for four different values of c . The data for $c = -2$ are for its whole range of validity. Data for the other values of c have been restricted in a to bring out the features in the other values. The parameters b and w were 0 and 1, respectively.

For $c = -1$ (Figure 5.3b), we see results consistent with an effective amplitude decrease. For $a < 0$, the mass increases with decreasing a , and has lower values than the corresponding a in axisymmetry. For $a > 0$, the minimum in the mass occurs at a higher value of a than the axisymmetric case, and it has a lower value at any given c . Although it is not clear from the plot, the mass at $a = 1$ is greater than that at $a = 0.9$.

As mentioned above, the case $c = -2$ is special because the constant term in the azimuthal factor vanishes. One expects that for this case, one will obtain the same spacetime if one changes the sign of the Brill wave amplitude, only phase shifted. Thus, the ADM mass should be symmetric about $a = 0$, which is what we see.

Now we consider the effect of varying the Brill wave location, determined by the parameter b . We hold the Brill wave amplitude fixed at $a = 0.5$ and the width fixed at $w = 1$. In Figure 5.4 we show the ADM mass as a function of b and c . In Figure 5.5 we show the ADM mass as a function of b for four values of c . We see that for each value of c considered, the mass initially decreases with increasing b , then increases. There are two counter-acting effects which cause this behavior, as pointed out by Bernstein. The first is that when the Brill wave is moved off the throat, it is split in two, due to the two exponentials in the Brill wave function (5.8). This causes an effective decrease in the Brill wave amplitude, and thus in the mass as the wave is moved off the throat. The second effect is that the Brill wave occupies more and more proper volume as it is moved off the throat. This effect is enhanced by the logarithmic radial coordinate. This second effect is what causes the mass increase with increasing b .

Now we consider the variation with c . For $c > 0$, we see that the mass has the same behavior as b is increased as in the axisymmetric case, but it is larger. This makes sense because for positive c , there is an effective increased amplitude. For $c < 0$ we see an initial decrease in the mass as c is decreased, then an increase.

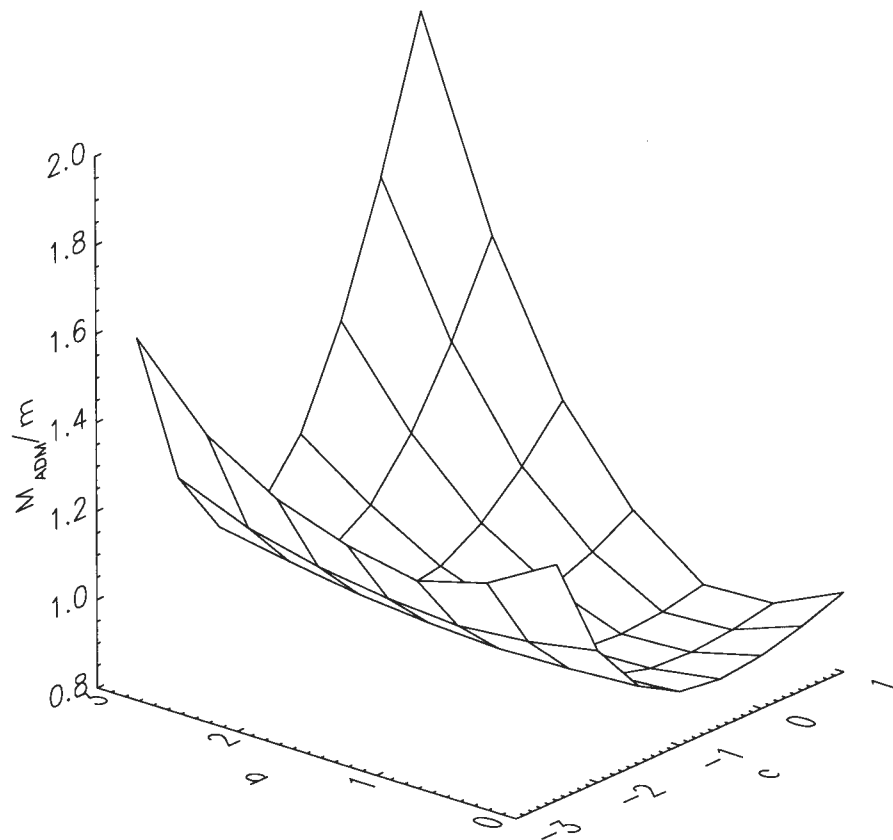


Figure 5.4: We show the ADM mass as a function of the Brill wave location parameter b and azimuthal parameter c . All data are for Brill amplitude $a = 0.5$ and width $w = 1$.

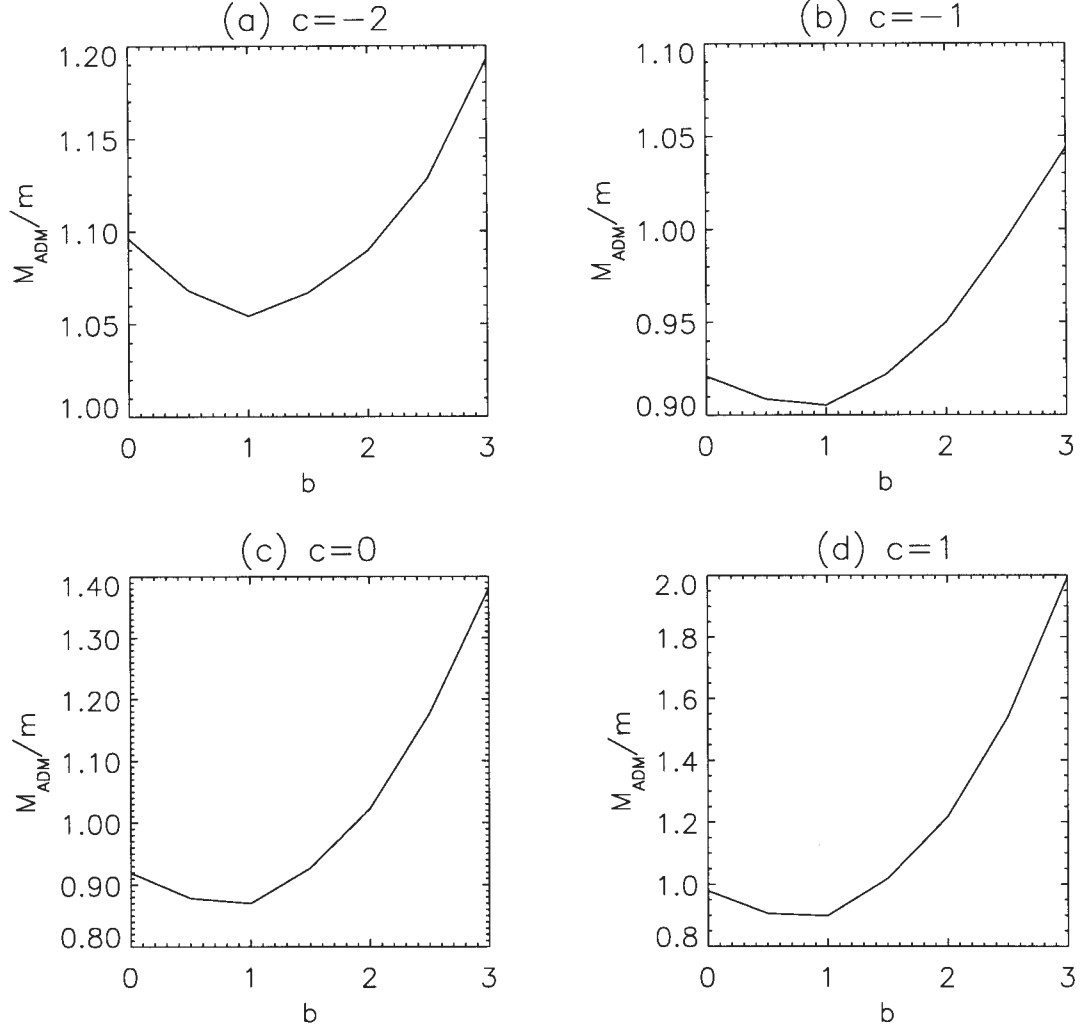


Figure 5.5: We show the ADM mass as a function of the Brill wave parameter b for different values of c . In all cases, $a = 0.5$ and $w = 1$. Each plot is on a different scale to bring out detail.

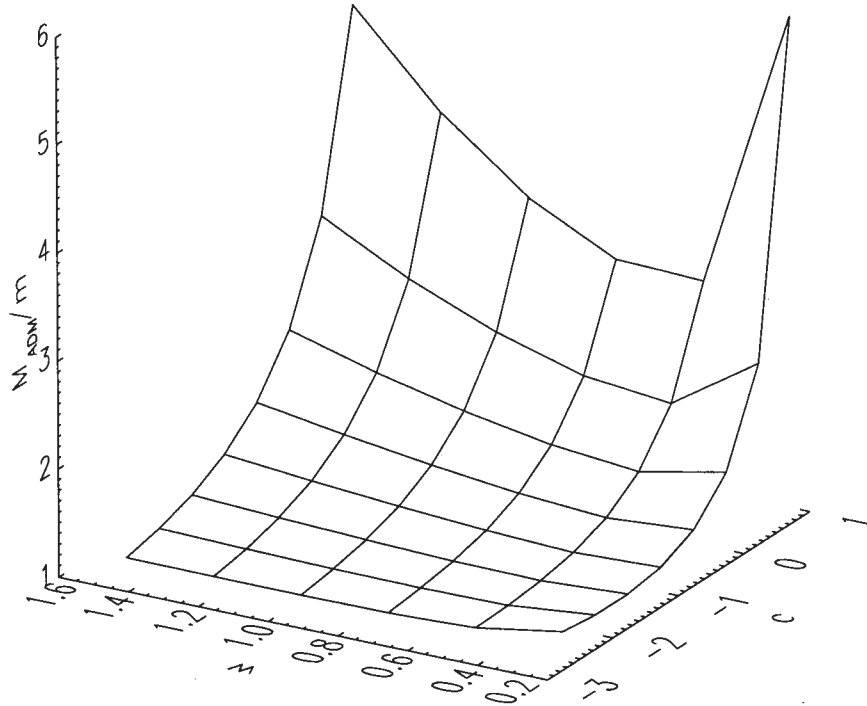


Figure 5.6: We show the ADM mass as a function of the Brill wave width and azimuthal parameters w and c . All data are for Brill wave amplitude $a = -0.5$ and location $b = 0$.

Now we consider the effect of varying the Brill wave width, by studying the parameter set $(a, b, w, c) = (-0.5, 0, w, c)$. In Figure 5.6 we show the ADM mass as a function of w and c . In Figure 5.7 we show the ADM mass as a function of w for four values of c . We see that for each value of c , increasing the width from unity causes an increase in the mass. This makes sense because we are increasing the volume of the wave. Conversely, decreasing the width from unity causes a decrease in the mass. However, decreasing the width past 0.5 or so causes an increase in the mass. Bernstein attributes this to the effect of high frequency components in the wave.

We now consider the effect of varying c . We see that for positive c , the mass is

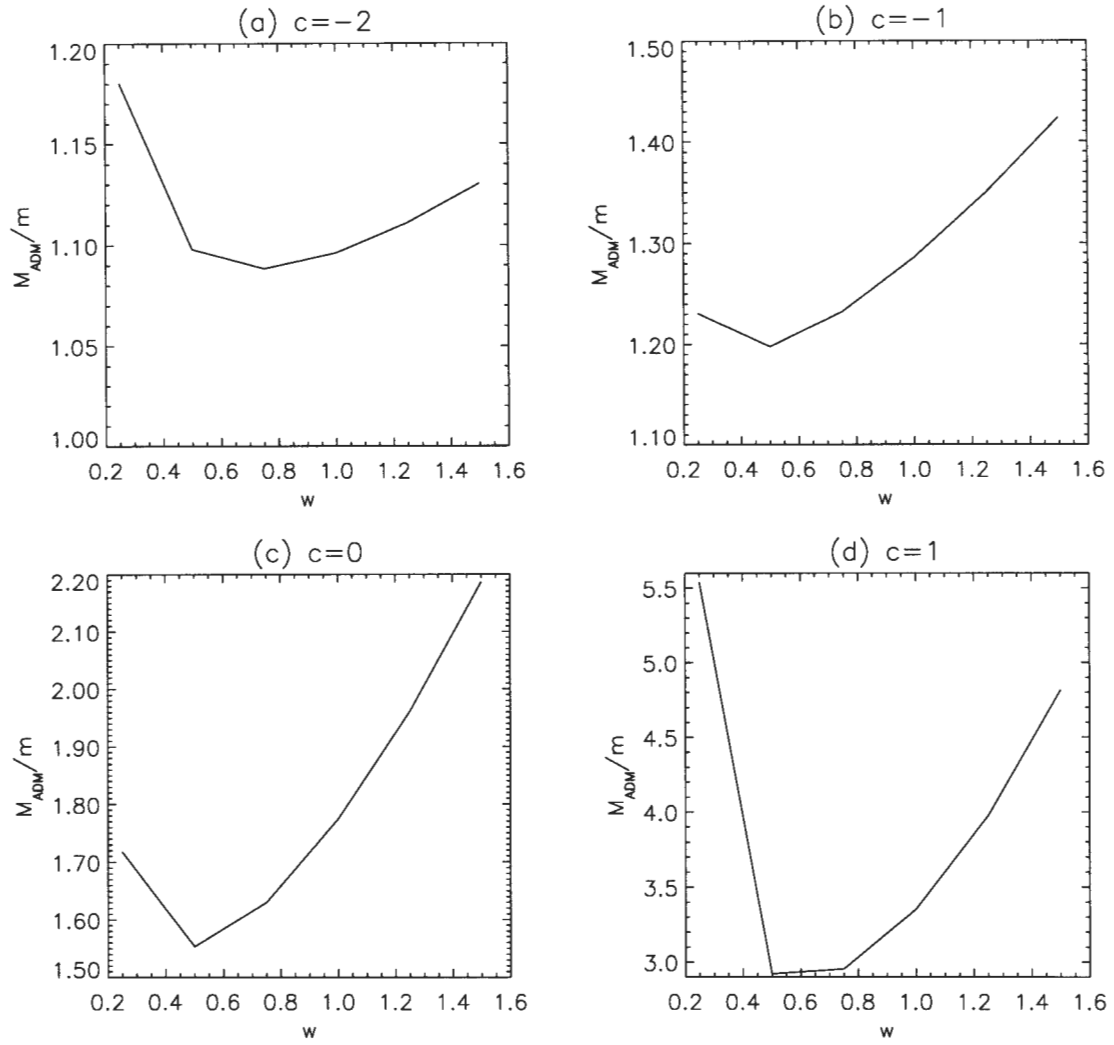


Figure 5.7: We show the ADM mass as a function of the Brill wave parameter w for different values of c . All data are for $a = -0.5$, $b = 0$. Each plot is on a different scale to bring out detail.

increased, consistent with our expectations. As we decrease c , the mass decreases.

We now summarize the effect of the azimuthal dependence on the ADM mass. We have seen that the chosen azimuthal dependence has two components: a constant component and an oscillatory component. The constant component is in some sense understood because by itself it corresponds directly to adjusting the Brill wave amplitude. The effect of the oscillatory part by itself can be seen in Figure 5.3a. Increasing the absolute value of c increases the ADM mass. For positive values of c , both the constant term and the oscillatory term are increased, and the effect of the ADM mass has the same character as increasing the Brill wave amplitude. For negative values of c , the situation is more complicated. For the range $-2 \leq c \leq 0$, the magnitude of the constant term decreases with decreasing c while the magnitude of the oscillatory term increases. The relative effect on the ADM mass within this range depends on the sign of the Brill wave amplitude. For neither sign is the effect monotonic in this range. For $c < -2$, this spacetime is becoming highly distorted, requiring significantly more resolution to study accurately.

5.5 Apparent Horizons

In this Section we discuss the location and properties of the apparent horizon (AH) in these initial data sets. Studying the apparent horizon will give us a detailed understanding of the shape and mass of the black hole, and therefore of the maximum amount of radiation we could expect the system to release during an evolution.

As discussed in Section 2.6, the apparent horizon is the outer-most marginally-trapped surface. For our data sets, it is also a minimal area surface. Gibbons showed that an isometry surface, such as the throats in our data sets, is an extremal area surface [50]. Thus, the throats in our data sets are marginally trapped surfaces. They are of course not always the outer-most one. In order to determine whether the AH is on or off the throat for a given set of parameters, we plotted the area of constant- η

surfaces as a function of η . Clearly, since the throat is a constant- η surface, if this area plot shows a minimum off the throat, the AH is off the throat. The converse is not necessarily true: if the throat is the only minimal area surface in the plot, there may still be an AH off the throat, since the AH need not be a constant- η surface. However, we expect that except in borderline cases, if the throat is the only minimum then it is the AH.

In Figure 5.8a, we show for the family of initial data sets $(a, b, w, c) = (a, 0, 1, c)$ which data sets considered have their apparent horizon on the throat, and which have it off the throat, as determined by the method described above. For the range of parameters a and c considered, only the set $a = -0.6, c = 1$ has its AH off the throat. This result is consistent with what Bernstein found in axisymmetry. He found that the AH is off the throat for Brill wave amplitudes outside of the range $-0.65 < a < 3.03$. Since our a 's are all well inside that range, we expect the AH to be on the throat unless the azimuthal factor is large enough that the effective amplitude increase results in another minimal surface forming. This is what we see.

In Figure 5.8b, we show for which sets of parameters the AH is on the throat for the family of initial data sets $(a, b, w, c) = (-0.5, 0, w, c)$. We see that the smaller the Brill wave width, the more likely a minimal surface is to form, which is consistent with Bernstein's results.

For the family of initial data sets $(a, b, w, c) = (0.5, b, 1, c)$, the AH is on the throat for all cases in the range studied ($0 \leq b \leq 3, -2.5 \leq c \leq 1$). This makes sense because, as pointed out by Bernstein, moving the Brill wave outward should decrease the chance of forming a new minimal surface.

When the apparent horizon is on the throat, it is of course a coordinate sphere. It is interesting, however, to study how distorted it is in physical space. Here we study this using two measures. The first measure is the ratio of the polar to equatorial circumference, C_P/C_E . The polar circumference, C_P , is the proper length of

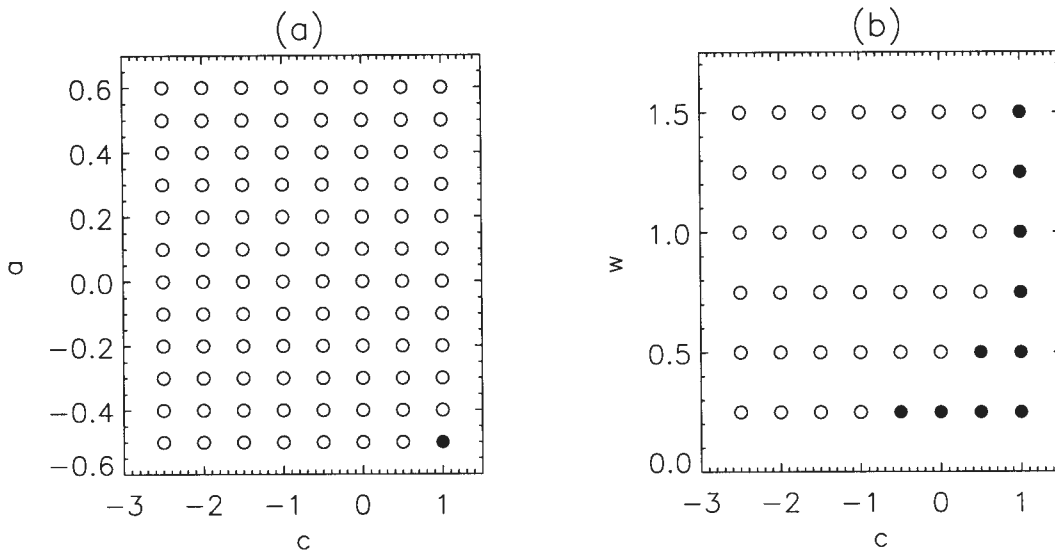


Figure 5.8: We show for which sets of parameters the apparent horizon is *not* on the throat, which we denote by a filled circle. For (a), the location and width parameters are $b = 0$ and $w = 1$. For (b), the amplitude and width parameters are $a = -0.5$ and $b = 0$.

the circumference which goes through the poles along a line of constant ϕ , and the equatorial circumference, C_E , is the proper length of the equator, defined as $\theta = \pi/2$. This ratio has been measured in axisymmetry as a straightforward way to measure the shape of the apparent horizon (see, *e.g.*, [30]). If $C_P/C_E = 1$, the horizon is spherical. If $C_P/C_E < 1$ it is oblate, and if $C_P/C_E > 1$ it is prolate. In 3D, this quantity is a function of the azimuthal angle ϕ , because C_P is.

The main drawback to using this quantity in 3D is that, although C_P and C_E are proper lengths, when measured in this way, they are not entirely coordinate independent. In axisymmetry, there are natural choices for ϕ and for the location of the equator. In 3D, one could transform the coordinate ϕ to something like $\phi' = \phi + A \sin^2 \theta$, which would be a perfectly valid thing to do, but which would give a different answer. We will of course use our coordinate ϕ . Also, in general 3D, the choice of what to call the equator is not obvious. For our problem, however, it is, due to our equatorial plane symmetry.

In Figure 5.9 we show the ratio C_P/C_E of the AH as a function of ϕ for different values of c for $a = 0.5$. Recall that for these parameters, the AH is on the throat. We see that for $c = -2$, this quantity is everywhere less than 1, indicating an oblate horizon. For the other c 's shown, this quantity is everywhere greater than 1, or prolate. In fact, we see that the horizon becomes more and more prolate with increasing c . We expect however, that it would not increase without bound. If we could look at more and more distorted initial data eventually the AH would pop off the throat, where it would be more spherical [30].

In Figure 5.10 we plot C_P/C_E of the AH as a function of ϕ for $b = 1.5$ and many values of c . We see that the horizon is nearly spherical: all values are near unity. This is because the distorting Brill wave is so far away that in the initial data it has little effect on the geometry of the throat.

The second quantity that we use to measure the curvature of the apparent horizon is its Gaussian curvature. The Gaussian curvature is the reciprocal of the product of the radii of curvature at a given point on the surface. Thus, a negative Gaussian curvature indicates a saddle structure. It can be shown that the Gaussian curvature is one-half the scalar curvature (*i.e.*, trace of the Ricci tensor) induced on the 2-surface, which is how we compute it. This is a completely geometric property intrinsic to the surface, independent of any coordinate choice, unlike our C_P/C_E measure.

In Figure 5.11 we show the Gaussian curvature of the AH for $c = -2$ and $a = \pm 0.5$. Recall that for $c = -2$, we should get the same initial slice, except phase shifted by $\phi = \pi/2$. This is consistent with what we see in these plots. We note that there are regions for which the Gaussian curvature is negative, indicating that the surface is not entirely convex.

Another quantity of interest which is related to the apparent horizon is the maximum radiation loss, or *MRL*. It comes about as follows. The second law of black hole thermodynamics states that the area of the event horizon of a black hole, which

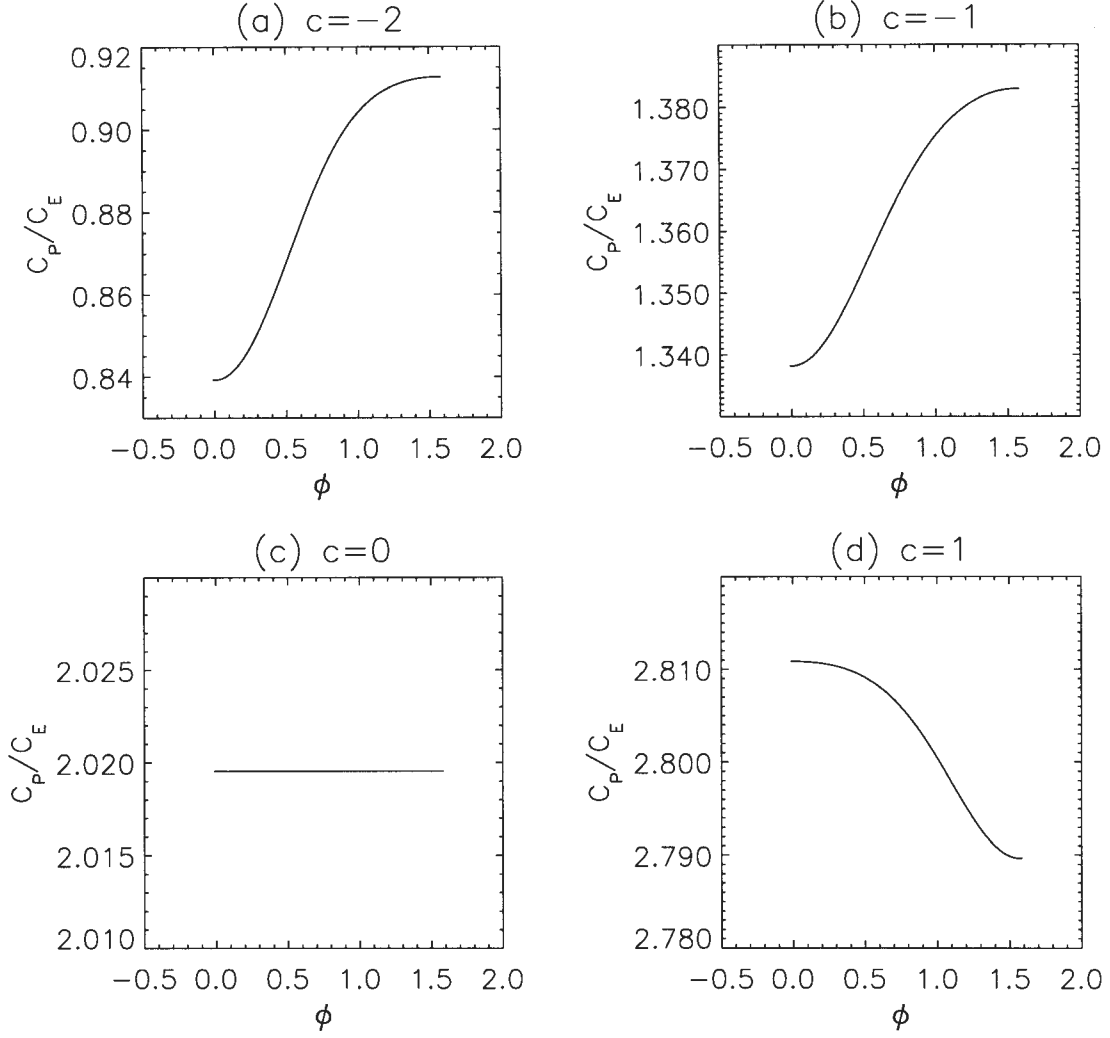


Figure 5.9: We show the ratio of polar to equatorial circumference of the apparent horizon as a function of ϕ for various values of c . For these data sets, $a = 0.5$, $b = 0$, and $w = 1$. The data are plotted on different scales to bring out detail.

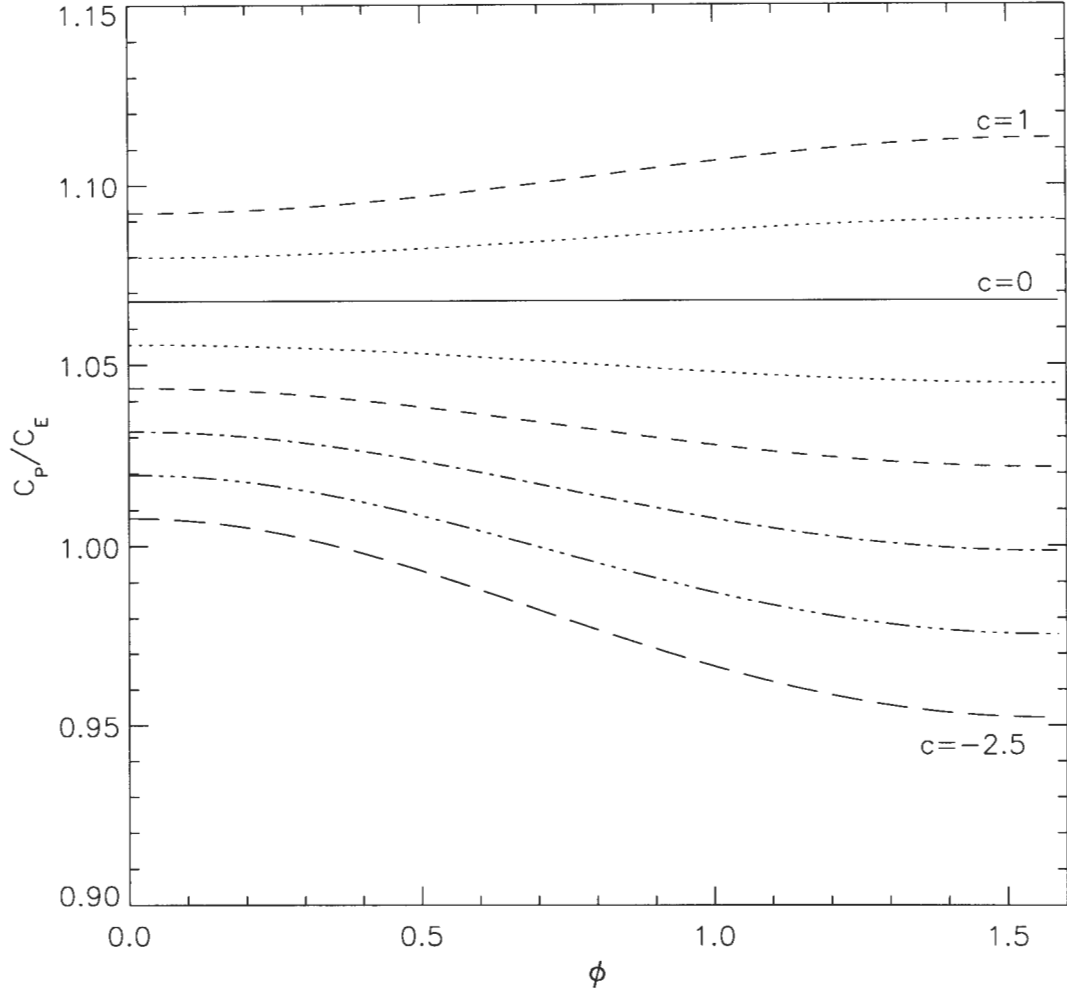


Figure 5.10: We show the ratio C_P/C_E of the apparent horizon for $(a, b, w) = (0.5, 1.5, 1)$ and several values of c . We see the horizon is nearly spherical, due to the Brill wave being far away.

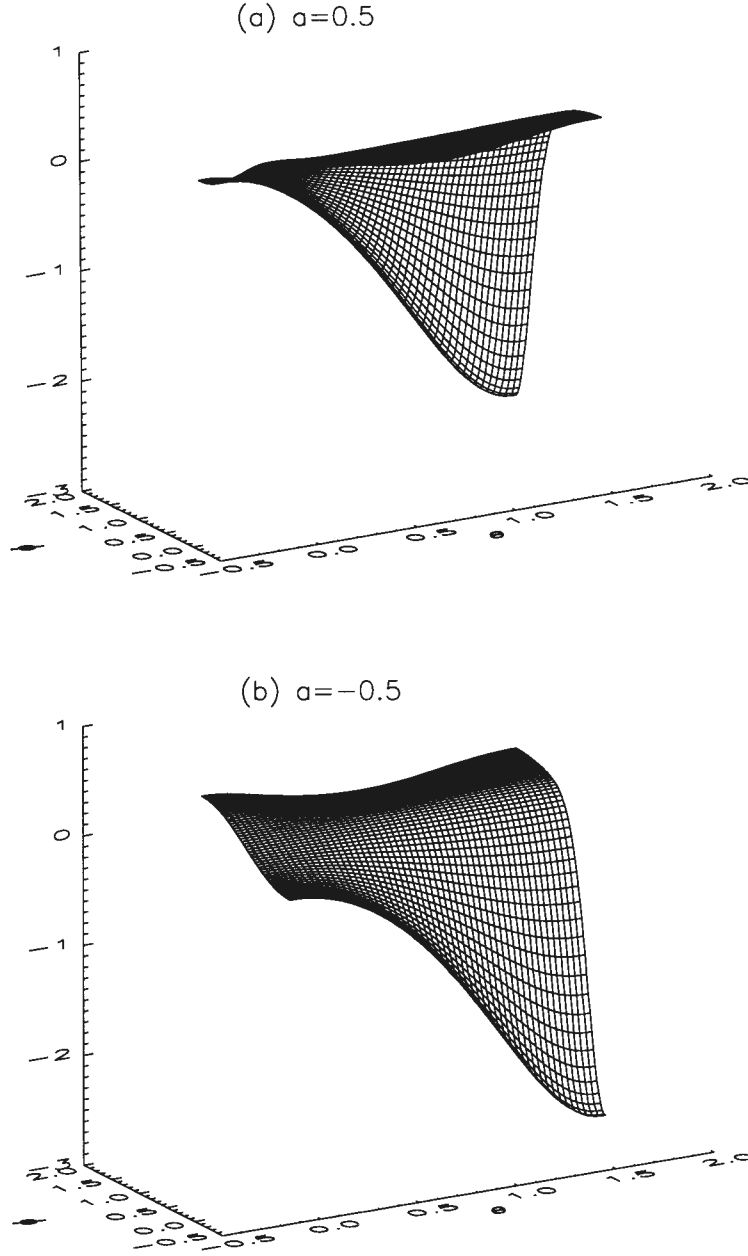


Figure 5.11: We show the Gaussian curvature of the apparent horizon as a function of the angular variables θ and ϕ for Brill wave amplitudes $a = \pm 0.5$. In both cases, $b = 0$, $w = 1$, and $c = -2$. As discussed in the text, when $c = -2$, changing the sign of a should only produce a phase shift in ϕ , as is seen here.

we will call A_{EH} , cannot decrease in time. One can define a quantity associated with the area of a black hole known as the irreducible mass, M_{EH} [42, 43]:

$$M_{EH} = \sqrt{\frac{A_{EH}}{16\pi}}. \quad (5.22)$$

By analogy to this quantity, one can define the mass of the apparent horizon given its area, A_{AH} :

$$M_{AH} = \sqrt{\frac{A_{AH}}{16\pi}}. \quad (5.23)$$

One defines the radiation efficiency RE as the difference between the ADM mass of a spacetime and its irreducible mass in its final state. *i.e.*,

$$RE = \frac{M_{ADM} - M_{EH,f}}{M_{ADM}}. \quad (5.24)$$

However, by the second law of black hole thermodynamics, the final irreducible mass must be greater than or equal to the initial irreducible mass. Thus,

$$RE \leq \frac{M_{ADM} - M_{EH,i}}{M_{ADM}}. \quad (5.25)$$

Because one needs to perform an evolution to find the location of the event horizon on any slice, we can not compute the above fraction, even though it is for quantities defined on our data set. However, for our initial data sets, we know that the apparent horizon is the outer-most minimal area surface. It can be shown that if an apparent horizon exists, it must lie inside or coincident with an event horizon [55]. Thus the area of the event horizon on our initial slice must be greater than or equal to the area of the apparent horizon. Thus we can define a quantity we call the maximum radiation loss, or MRL , by

$$MRL = \frac{M_{ADM} - M_{AH,i}}{M_{ADM}} \geq RE. \quad (5.26)$$

The fraction above is an upper bound on the radiation efficiency. Therefore, we know that during an actual evolution, the total energy radiated has to be less than this

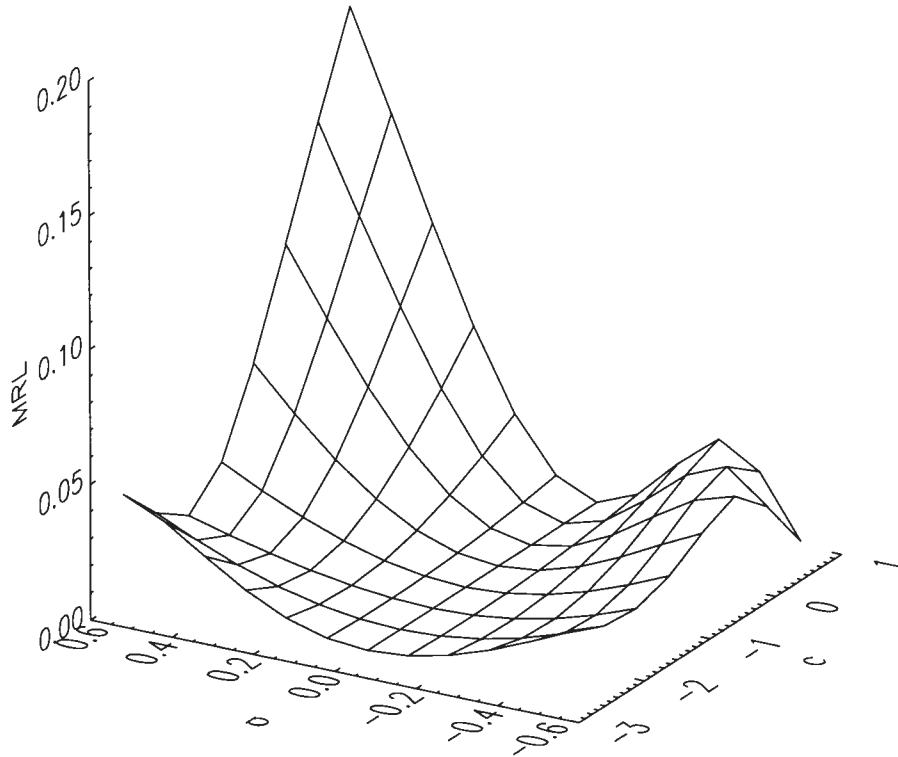


Figure 5.12: We show the MRL as a function of the Brill wave amplitude a and azimuthal parameter c . The location and width parameters b and w were 0 and 1, respectively.

amount. This will be a guide to choosing interesting initial data sets for evolution in the future.

In Figure 5.12 we show the MRL as a function of a and c for the family of initial data sets $(a, b, w, c) = (a, 0, 1, c)$. The sets within this family with the largest MRL 's are those with positive a and c , indicating that these might be interesting data sets to evolve.

In Figure 5.13 we plot the MRL as a function of b and c for this family of data sets. The MRL generally increases as b increases. This is expected because the further the wave is from the black hole, the easier it should be for it to escape to null

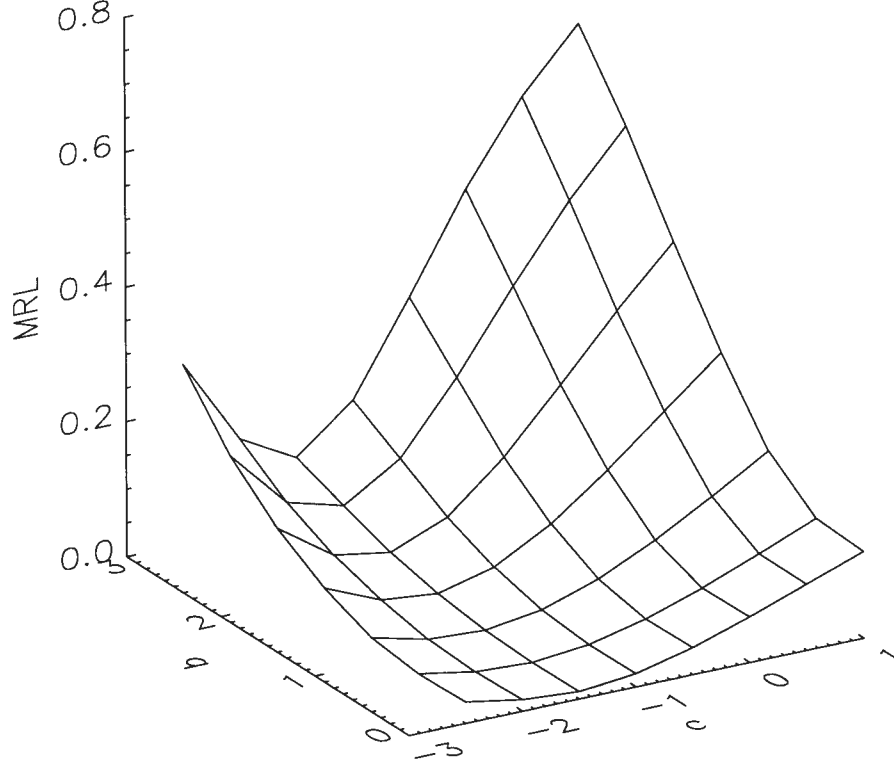


Figure 5.13: We show the MRL as a function of the width and azimuthal parameters b and c for Brill wave amplitude $a = 0.5$ and width $w = 1$.

infinity. However, for most c , there is also a minimum in the MRL at $b = 0.5$. This can be understood if one considers the Brill wave function (5.8). When the Brill wave is on the throat, the peaks of the exponentials add together. However, when the Brill wave is off the throat, there is an effective amplitude decrease. We saw above that the MRL generally increased with increasing amplitude for moderate a , so what we are seeing is consistent.

5.6 Summary

We have extended Bernstein’s axisymmetric, distorted black hole initial data sets to full 3D by giving them a particular azimuthal dependence. An extensive summary of these new data sets has revealed that their physical properties, such as their ADM mass and MRL, are consistent with what has been observed in axisymmetry. By studying the details of their geometries and physical properties, we have mapped out the initial data as a guide to future evolutions: which data sets correspond to highly distorted black holes, which are likely to radiate copious waves, and which are small perturbations. These results, and the extension to rotating black holes, are in preparation for publication in [27].

In the next Chapters, we will look at the evolution of these initial data sets. In the case of full 3D nonlinear evolutions, we will restrict ourselves to Brill waves which are on the throat of the black hole ($b = 0$), because of the necessary closeness of the outer boundary.

Chapter 6

The 3D Black Hole Plus Brill Wave Spacetime: Nonlinear Evolutions

We now turn our attention to the evolution of the 3D distorted black hole initial data sets discussed in Chapter 5. To our knowledge, these are the first fully nonlinear evolutions of black hole spacetimes which are not axisymmetric.

In order to obtain initial data for the 3-metric on the Cartesian grid, we interpolate the conformal factor from the spherical grid using linear interpolation, removing the extra factor of $r^{1/2}$. Once we have the conformal factor, we can specify the 3-metric, the form of which is given in equation (5.10). The extrinsic curvature of course vanishes, as these are time-symmetric initial data sets. In the Schwarzschild and Misner studies, we found the evolutions to be much more stable if we stored the (then analytic) derivatives of the conformal factor, and used the chain rule and these derivatives to compute the derivatives of the metric functions. Although our conformal factor is no longer analytic, because of the higher resolution of the spherical grid in the inner regions, computing derivatives on this grid and storing the interpolated values on the Cartesian grid should confer some advantage. Thus, we compute the first and second derivatives of the conformal factor $\tilde{\psi}$ with respect to the spherical coordinates (η, θ, ϕ) on the spherical grid. We interpolate these quantities onto the Cartesian grid using linear interpolation. Then, using simple coordinate transformations, we convert these quantities to derivatives of the conformal factor ψ with respect

to the Cartesian coordinates (x, y, z) .

6.1 Slicing Condition

For the evolutions of the Schwarzschild and Misner initial data sets, the slicing condition that worked the best was the evolution of an initially maximal, antisymmetric lapse with the “1+ln” algebraic slicing. With this slicing condition we were able to produce stable evolutions, and it is much more computationally efficient than maximal slicing. However, for the distorted black hole initial data sets there is no appropriate analytic antisymmetric initial lapse to use. We tried using the Schwarzschild lapse, but this resulted in evolutions which quickly became unstable. Thus, we decided to find an antisymmetric maximal lapse numerically. Because we cannot enforce a boundary condition inside the domain (such as at the throat in the Cartesian grid) with the current elliptic solver, we decided to find the antisymmetric maximal lapse on the spherical grid and interpolate that onto the Cartesian grid. We have shown that this procedure when applied to the Schwarzschild case produces the same results as when the analytic Schwarzschild lapse is used, although it is necessary to use at least quadratic interpolation. This is the slicing condition used in all runs in this Chapter, except where indicated otherwise. Note that as before, with this lapse we will need to use the isometry condition to update evolved functions within the throat.

6.2 Evolution of Axisymmetric Initial Data Sets

We first study the evolution of initial data sets which are axisymmetric, *i.e.*, the sets $(a, b, w, 0)$. Comparisons of these evolutions with those from 2D codes will help us check the 3D code.

6.2.1 Metric Functions

In this section we consider metric functions computed during evolutions of axisymmetric initial data sets. In most cases, we will not be able to compare these functions quantitatively with those produced by a 2D code, but we will be able to make qualitative comparisons. They will also be a reference for us to compare against when we perform evolutions of non-axisymmetric data sets.

The 2D code used in these comparisons is an extension of the 1D code described in Section 3.2. There, the radial coordinate was the logarithmic coordinate η , related to the standard isotropic radius by $\eta = \ln\left(\frac{2r}{m}\right)$, where m is a scale parameter. The 2D code introduces the standard polar angle θ . As all data sets evolved with this code had horizontal-plane symmetry, θ was restricted to lie between 0 and $\pi/2$. All 2D results presented in this Chapter and the next used 200 radial zones and a maximum radius of $\eta_{max} = 6$, giving a radial grid spacing of $\Delta\eta = 0.03$. The number of angular zones, chosen to make $\Delta\theta \approx \Delta\eta$, was approximately 50.

We start by considering evolutions of axisymmetric initial data sets with geodesic slicing ($\alpha = 1$). Geodesic slicing will of course not allow us to cover much of the spacetime, since grid points on the throat will rapidly fall into the singularity, as discussed in Section 2.3.1. However, it does allow us to compare evolved metric functions with those obtained from a 2D code without the complication of a lapse calculation.

In Figure 6.1 we show a plot of the conformal metric function $\hat{\gamma}_{rr}$ obtained from 2D and 3D codes along the z -axis. The initial data set evolved was $(0.5, 0, 1, 0)$. The 3D data were obtained using 70^3 grid points and a resolution of $\Delta x = 0.0544M$. For the 3D data, $\hat{\gamma}_{rr}$ was computed from the Cartesian metric functions actually evolved. In Figure 6.2 we show a comparison of $\hat{\gamma}_{\theta\theta}/r^2$ from the two codes for the same runs. In both plots we see that the metric functions line up reasonably well. The small differences can easily be explained by the better resolution that is present in the 2D

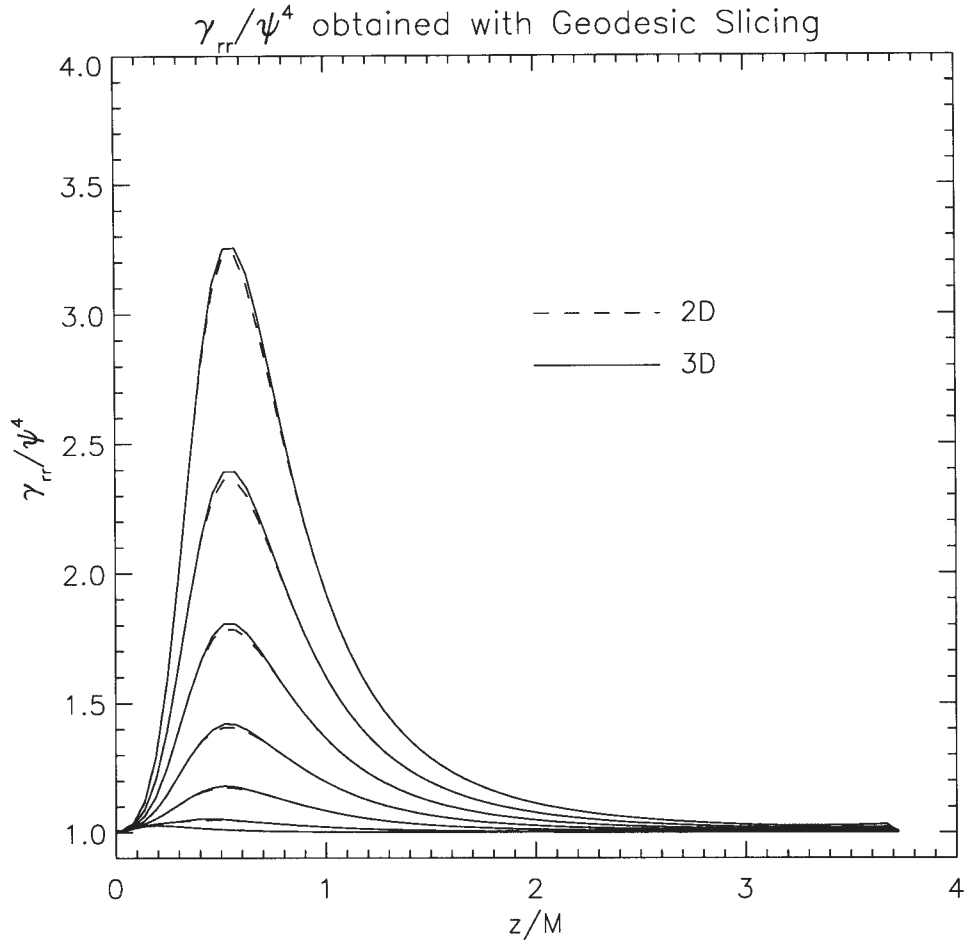


Figure 6.1: We show a comparison of γ_{rr}/ψ^4 obtained with 2D and 3D codes using geodesic slicing to evolve the axisymmetric data set $(a, b, w) = (0.5, 0, 1)$. The data are shown every $0.163M$, to a maximum time of $t = 0.979M$. The 3D data were obtained with 70^3 grid points with a resolution of $\Delta x = 0.0544M$.

code, as discussed in Section 3.2.

The data in Figures 6.1 and 6.2 are only shown to a time of $t = 0.979M$. The reason for this is not that the throat observers have already fallen into the singularity at that time. In fact, the 3D code with that resolution can be run past $t = 1.96M$ without crashing. The problem is that the 2D code uses the standard spherical polar coordinate system, which has a coordinate singularity on the axis. Although there are no grid points actually on the axis, the code becomes unstable for grid points near the axis, causing the 2D code to crash very early. This problem is usually solved by

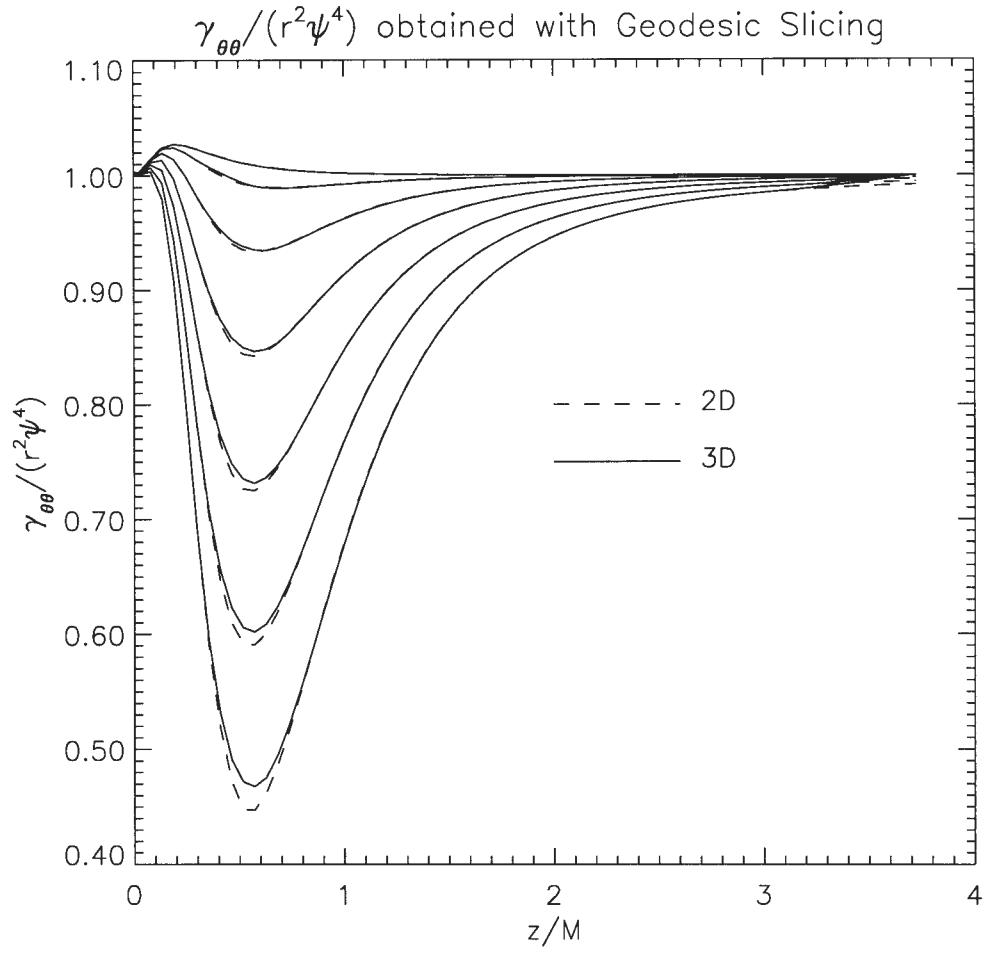


Figure 6.2: As in Figure 6.1, except that we plot $\gamma_{\theta\theta}/(r^2\psi^4)$.

using the gauge freedom of the Einstein equations to use the shift vector to maintain the metric function $\gamma_{r\theta} = 0$, thus keeping constant-coordinate lines perpendicular ($\gamma_{r\phi}$ and $\gamma_{\theta\phi}$ also vanish). However, the use of a non-zero shift results in a different 3-metric than in the vanishing shift case. Because this shift condition is not convenient to implement in the 3D Cartesian code, we were forced to compare with the zero-shift case, and thus for only a short time.

We next consider evolutions with the the “1+ln” slicing condition. The initial lapse is obtained as described above in Section 6.1. Here we only show that the metric functions obtained are qualitatively correct. We cannot compare directly with data from 2D simulations, again because of the different slicing and shift conditions used.

We first consider the evolution of the initial data set $(0.5, 0, 1, 0)$. This data set has an ADM mass of $M = 0.919m$, where m is the scale parameter discussed in Chapter 5. The run was done with 150^3 grid points with a resolution of $\Delta x = 0.0544M$. In Figure 6.3a, we show the metric function $\hat{\gamma}_{rr}$ in the $x = 0$ plane at the time $t = 27.2M$. We see the characteristic peak growing in this metric function, caused by the differential acceleration of grid points towards the hole. The shape of the metric function is similar to that obtained with the 2D code. As in the Schwarzschild case, the peak growing near the origin inside the throat is a result of the application of the isometry condition. In Figure 6.3b, we show the lapse function in the $x = 0$ plane at the same time. We see that, as in the Schwarzschild and Misner cases, the lapse which was initially antisymmetric at the throat, and evolved with “1+ln” slicing, has collapsed at and around the throat.

We next consider the evolution of the initial data set $(-0.5, 0, 1, 0)$. This data set has ADM mass $M = 1.77m$. The run was done with 150^3 grid points and a resolution of $\Delta x = 0.0282M$. In Figure 6.4a, we show the metric function $\hat{\gamma}_{rr}$ in the $x = 0$ plane at the time $t = 14.1M$. We again see the characteristic peak in this metric

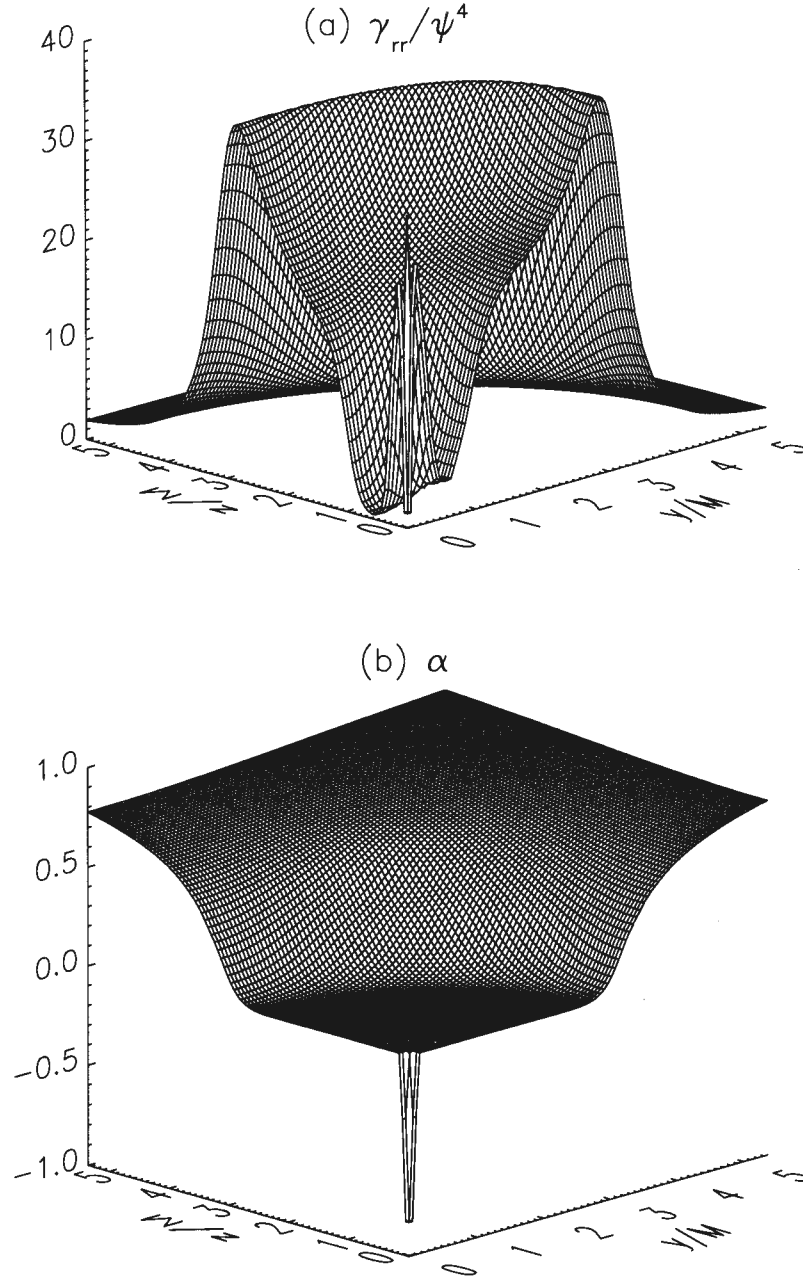


Figure 6.3: We show (a) the conformal metric function $\hat{\gamma}_{rr}$ and (b) the lapse function α in the $x = 0$ plane for the evolution of the axisymmetric initial data set $(0.5, 0, 1, 0)$. The data shown are at a time of $t = 27.2M$. The evolution was run on a 150^3 grid, although data from only the inner 100^3 points are shown. The resolution was $\Delta x = 0.0544M$.

function, although its shape is slightly different than in the positive amplitude case. In Figure 6.4b, we show the lapse function in the $x = 0$ plane at the same time. It again displays singularity avoiding properties.

6.2.2 Radiation Extraction

We extracted the $\ell = 2$ and $\ell = 4$ Zerilli functions during an evolution of the initial data set $(0.5, 0, 1, 0)$, using the extraction method described in Appendix C. These quantities are less sensitive to gauge conditions than individual metric functions (in fact, they are invariant to first order gauge transformations), and can be compared to 2D results meaningfully.

In Figure 6.5a we show the $\ell = 2$ Zerilli function extracted at a radius $r = 8.7M$ as a function of time. Superimposed on this plot is the same function computed during the evolution of the same initial data set with a 2D code. The agreement of the two plots is a strong affirmation of the 3D evolution code and extraction routine. It seems reasonable that the small phase difference that begins around $t = 15M$ could be due merely to differences in slicing. The 2D code uses maximal slicing, while the 3D code uses the “1+ln” slicing. These slicing differences would lead to differences in the physical location of the extraction coordinates, or differences in the proper time advanced. The difference could also be due to numerical inaccuracies in either of the evolution codes, but simulations run with 200^3 and 300^3 grid zones indicate that the waveform has already converged in the smaller grid case. In any case, the exact cause of this difference has not yet been determined, and is under investigation.

In Figure 6.5b we show the $\ell = 4$ Zerilli function extracted at the same radius, computed during evolutions with 2D and 3D codes. This waveform is more difficult to extract, because it has a higher frequency in both its angular and radial dependence, and it has a much lower amplitude. The agreement between the two plots is very good, although not as good as in the $\ell = 2$ case. If one looks closely, one can see that

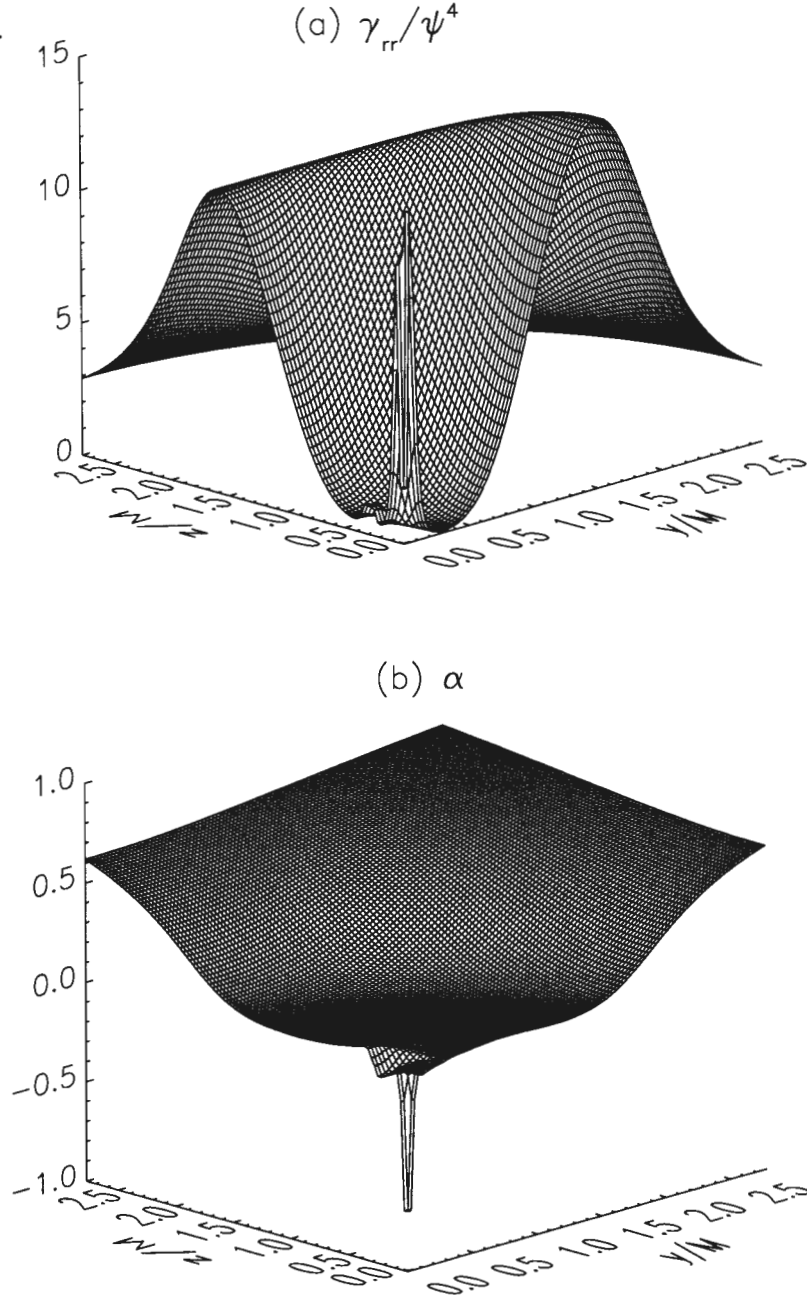


Figure 6.4: We show (a) the conformal metric function $\hat{\gamma}_{rr}$ and (b) the lapse function α in the $x = 0$ plane for the evolution of the axisymmetric initial data set $(-0.5, 0, 1, 0)$. The data shown are at a time of $t = 14.1M$. The evolution was run on a 150^3 grid, although data from only the inner 100^3 points are shown. The resolution was $\Delta x = 0.0282M$.

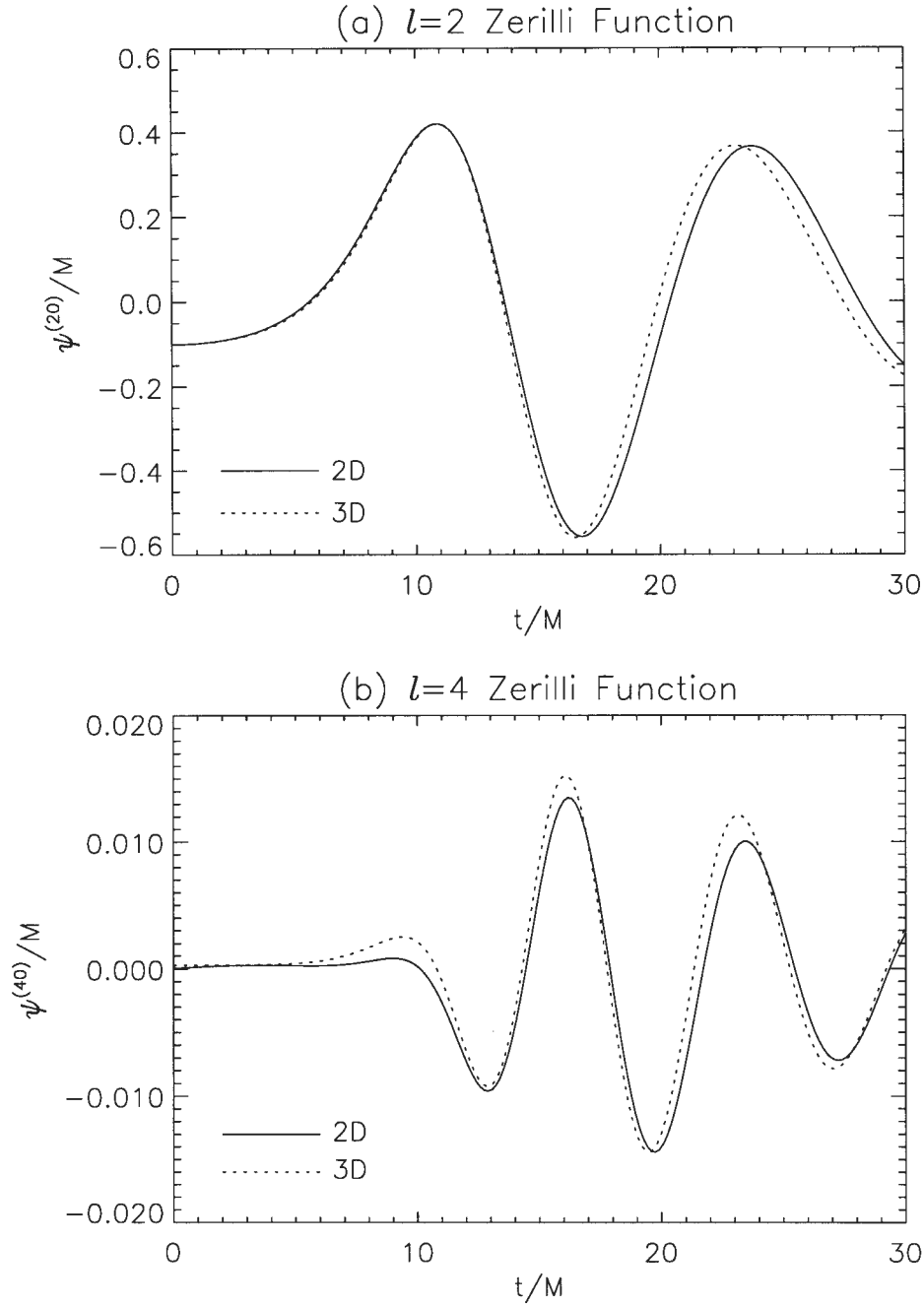


Figure 6.5: We show the (a) $\ell = 2$ and (b) $\ell = 4$ Zerilli functions as a function of time extracted during 2D and 3D evolutions of the data set $(a, b, w) = (0.5, 0, 1)$. The functions were extracted at a radius of $8.7M$. The 3D data were obtained using 300^3 grid points and a resolution of $\Delta x = 0.0816M$.

the functions do not exactly line up even at $t = 0$. Preliminary tests comparing the Zerilli function as a function of radius for initial data sets with $n = 4$ (in which the $\ell = 4$ Zerilli function is larger) suggest that at least part of this disagreement is due to interpolation errors being significant relative to the small $\ell = 4$ signal.

Since this data set is axisymmetric, the non-axisymmetric Zerilli functions of a perfect evolution of it should vanish. However, we expect that numerical errors, especially errors due to the distinctly non-axisymmetric outer boundary, will result in finite values for these Zerilli functions. Knowing the size of the error produced will help us gauge the accuracy of our results when we move to full 3D, where we expect real non-axisymmetric signals. In Figure 6.6, we show non-axisymmetric Zerilli functions as a function of time, extracted during an evolution of the same data set as above, at the same radius. We see that the $\ell = 2, m = 2$ and $\ell = 4, m = 2$ waveforms remain very small throughout the evolution. The $\ell = 4, m = 4$ waveform, while an order of magnitude smaller than the $\ell = 4, m = 0$ waveform, is still much larger than the other non-axisymmetric waveforms. Preliminary tests computing these quantities for Schwarzschild initial data, for which all Zerilli functions should vanish, indicate that the $\ell = 4, m = 4$ waveform is more sensitive to errors in the interpolation of metric functions onto the spherical extraction surface than the other Zerilli functions are. The reason for this sensitivity and how to correct for it are under investigation.

6.2.3 Horizons

We computed the location of the apparent horizon (AH) during an evolution of the initial data set $(a, b, w) = (-0.5, 0, 1)$. The calculation was performed with 66^3 grid points with a resolution of $\Delta x = 0.0564M$. For this run, maximal slicing was used. The AH was found using the 3D AH finder discussed in Appendix B. This calculation was originally done as a test of the AH finder.

Figure 6.7 shows the horizon shapes and locations in the 3D calculation every

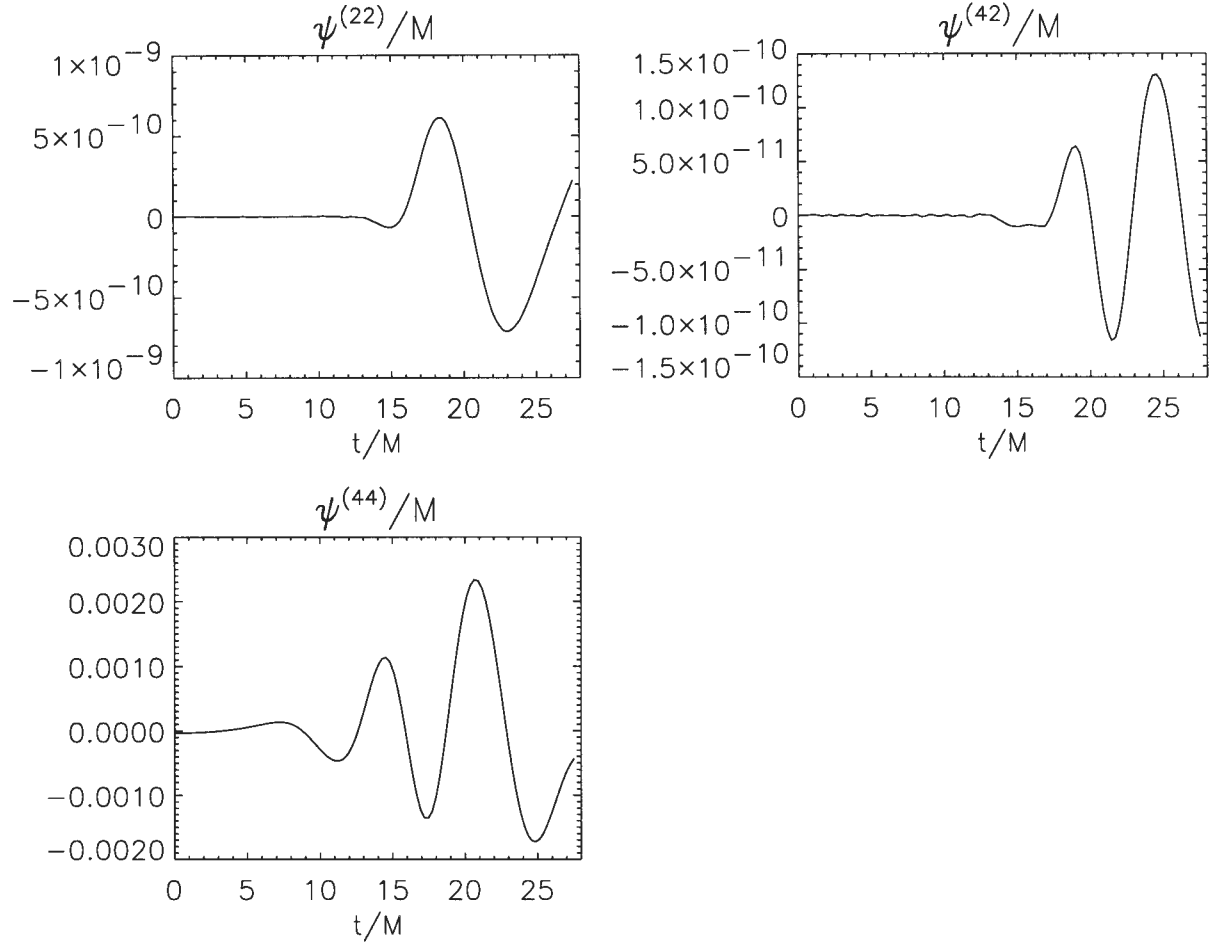


Figure 6.6: We show the non-axisymmetric Zerilli functions as a function of time extracted during an evolution of the axisymmetric data set $(a, b, w) = (0.5, 0, 1)$. The extraction was done at a radius of $r = 8.7M$. The calculation was performed with 200^3 grid points and a resolution of $\Delta x = 0.109M$.

$1.15M$ in time, starting at $t = 0$. The multipole order used to find the AH in this calculation is $L = 4$, with only axisymmetric terms considered to reduce the computational time. Also shown in Figure 6.7 are the corresponding surfaces found in our 2D axisymmetric code. We note that the slicing and shift conditions differ in the two cases, so we do not expect the surfaces to coincide precisely. In and around the x - y plane, the solutions match to within half a grid cell. Greater differences, however, are found along the z -axis, where the two surfaces are displaced by a maximum of roughly two grid cells. This is attributed in part to a bigger $\Delta\alpha$ (~ 0.1) near the z -axis, which results from the imposed asymmetry in the lapse function due to the nearness of the outer boundaries, where we enforce the spherical Schwarzschild lapse as a boundary condition in the maximal equation used in the 2D simulation.

A more geometrical quantity is the mass of the surface found. The horizon mass is defined as $M_{AH} = \sqrt{A_{AH}/16\pi}$, where A_{AH} is the area of the surface. Figure 6.8 plots the AH mass as a function of time for the 2D and the $L = 4$, 3D evolutions. In both cases, the mass increases at first, as the gravitational waves fall into the black hole, reaching $M_{AH} \sim 0.997M$ at $t \sim 6M$. The masses in the two cases differ by only 0.1% at $t \sim 6M$. For comparison we also plot M_{AH} for the surface found using a lower order $L = 2$ multipole expansion. At early times, when the horizon is most distorted, the $L = 2$ expansion is clearly not adequate to resolve the horizon shape, as evidenced by the AH mass which exceeds the ADM mass by about 1%. However, as the black hole settles down into a quasi-static state, the surface becomes more spherical and the $L = 2$ solution approaches both the 2D and the $L = 4$, 3D results, differing from the 2D result by about 0.35% at $t \sim 6M$.

6.3 Evolution of 3D Initial Data Sets

In Section 6.2 above, we showed by careful comparison to 2D results that our 3D code is able to accurately simulate distorted black holes, including the development

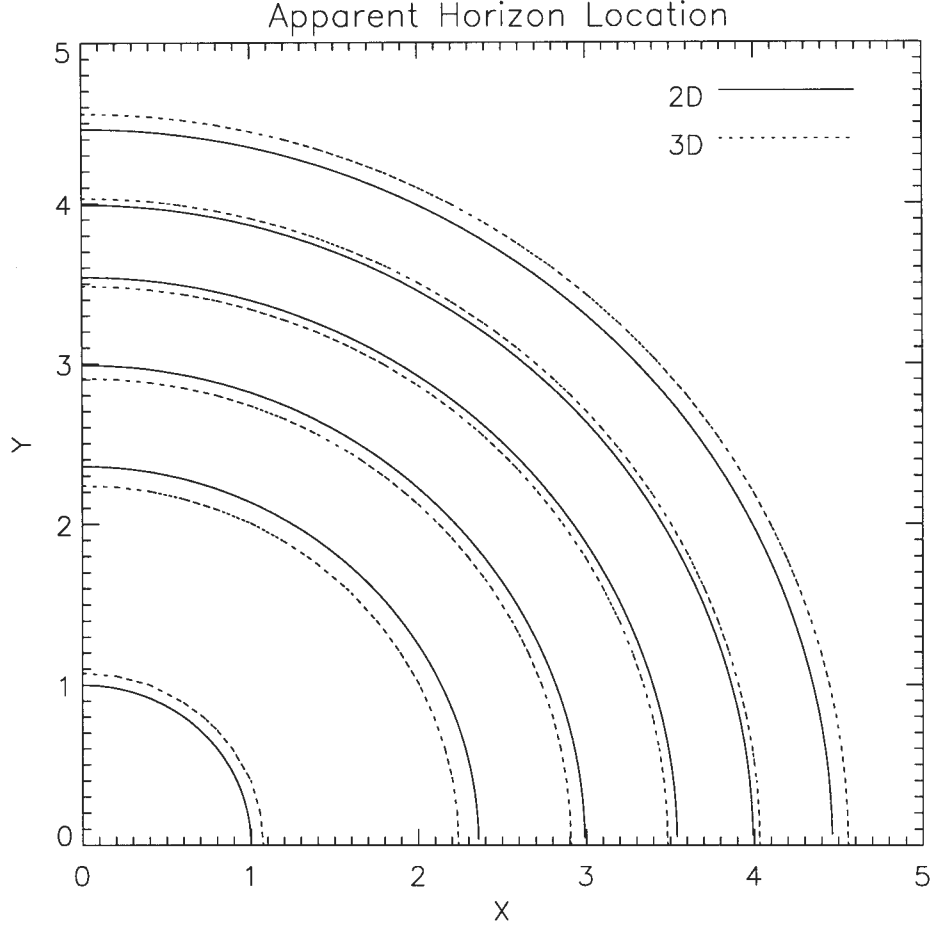


Figure 6.7: Coordinate location of the AH found in the 2D and 3D evolutions of the initial data set $(a, b, w) = (-0.5, 0, 1)$. The surfaces are shown starting at $t = 0$ with time intervals of $1.12M$, where $M = 1.77m$ is the ADM mass of the spacetime. Although we do not expect identical results due to different kinematic conditions in the two cases, the surfaces differ at most by little more than two grid cells. The evolution is performed on a 66^3 grid with $\Delta x = 0.0564M$.

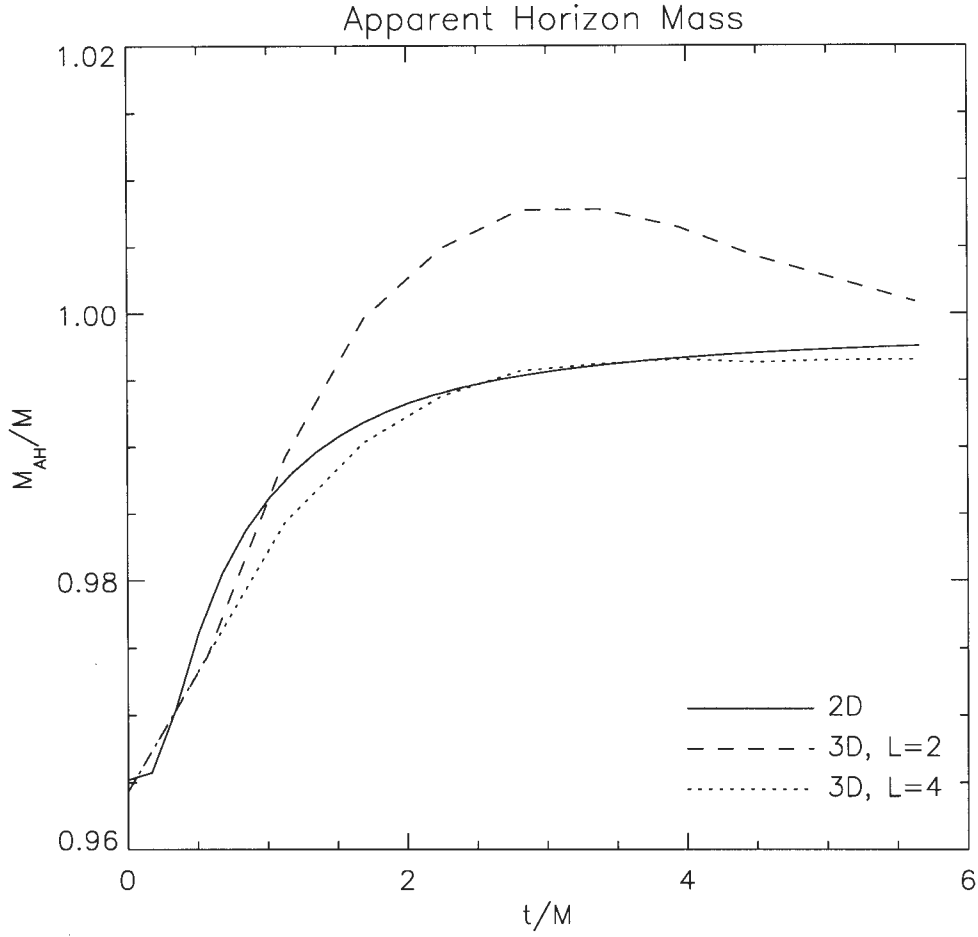


Figure 6.8: Comparison of the apparent horizon masses computed from the 2D and 3D evolutions of the initial data set $(a, b, w) = (-0.5, 0, 1)$. The AH mass increases initially as the Brill wave falls into the black hole. By $t \sim 6M$, the mass approaches $0.997M$ and the 2D and $L = 4$, 3D results differ by just 0.1% at this time. We also show the corresponding masses computed from the surfaces found with an $L = 2$ multipole expansion. The low order expansion is clearly not adequate in resolving the surface at early times when the horizon is most distorted.

of metric functions, horizons, and low amplitude waves on a spherical background. Armed with these tests, we now consider evolutions of initial data sets which are not axisymmetric, *i.e.*, data sets which have non-vanishing azimuthal parameter c . For all results presented in this Section, we have used the “1+ln” slicing, and the initial lapse described above.

6.3.1 Metric Functions

We first use as an illustration the evolution of the data set $(a, b, w, c) = (0.5, 0, 1, 0.5)$. This is identical to a simulation discussed in the last section, but with the azimuthal parameter turned on. We present data from a run done with 200^3 grid points and a resolution of $\Delta x = 0.106M$. With these parameters, the code can run to $t = 27.3M$ before crashing. In Figure 6.9 we show the metric function $\hat{\gamma}_{rr}$ in the first octant of each of the coordinate planes. We see that this function is quite similar to the axisymmetric case in the $x = 0$ plane, but that the azimuthal dependence makes a significant difference in the $y = 0$ and $z = 0$ planes.

We next consider the evolution of the data set $(-0.5, 0, 1, 0.5)$. Data we present here are from a calculation done with 100^3 grid points and a resolution of $\Delta x = 0.0433M$. The ADM mass of this data set is $M = 2.31m$. With these parameters, this data set can be run to $t = 14.9M$ before crashing.

In Figure 6.10 we show the radial metric function $\hat{\gamma}_{rr}$ in each of the coordinate planes at the time $t = 13.0M$. We see that in the $x = 0$ and $z = 0$ planes, this function is quite similar to that obtained during the evolution of the data set $(-0.5, 0, 1, 0)$, shown in Figure 6.4a. However, in the $y = 0$ plane, $\hat{\gamma}_{rr}$ is significantly different from its axisymmetric shape.

In Figure 6.11 we show the lapse function along the x -axis every $1.08M$ in time. The initial profile is initially Schwarzschild-like, as expected. As the evolution proceeds, we see that the lapse collapses outside the throat, as it has for the evolution

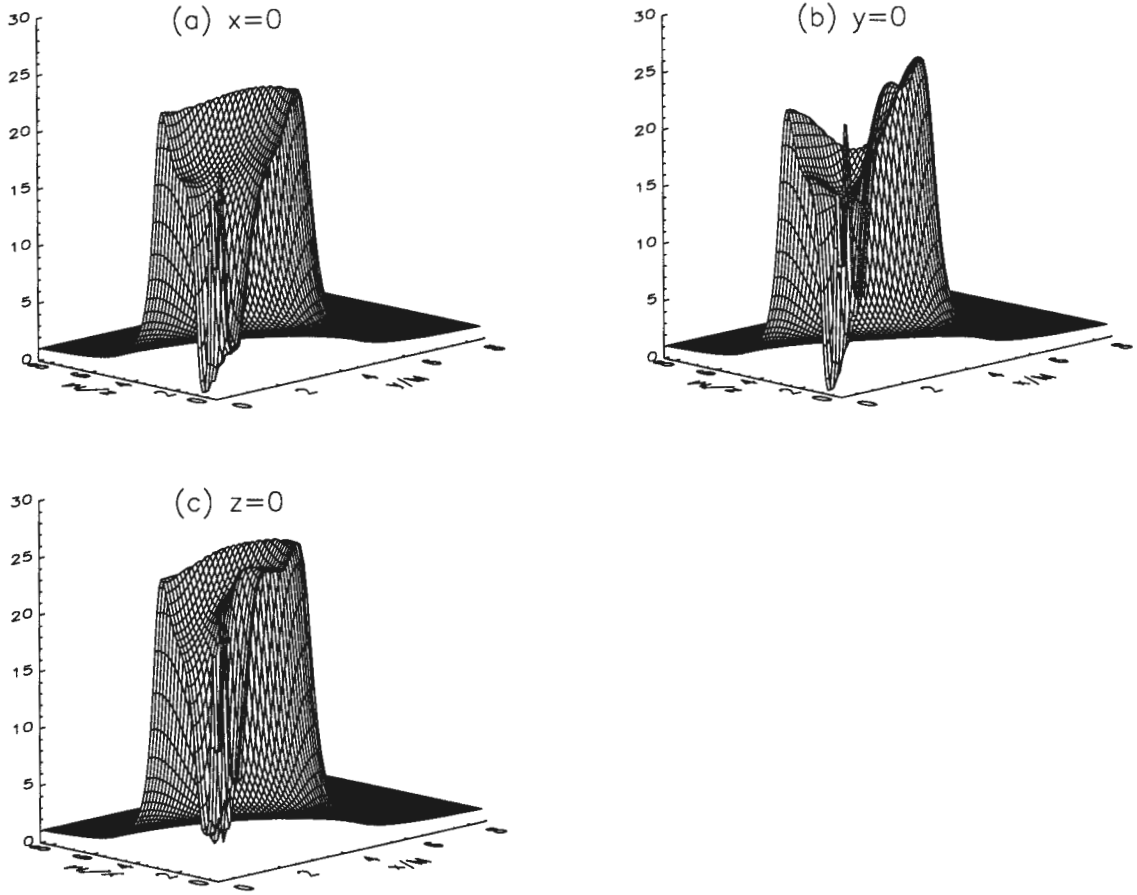


Figure 6.9: We show the metric function $\hat{\gamma}_{rr}$ in the first quadrant of each of the coordinate planes at time $t = 21.2M$ for the evolution of the initial data set $(0.5, 0, 1, 0.5)$. The calculation was done with 200^3 grid points, although data is only shown from the inner 80^3 points to bring out detail. The resolution used was $\Delta x = 0.106M$.

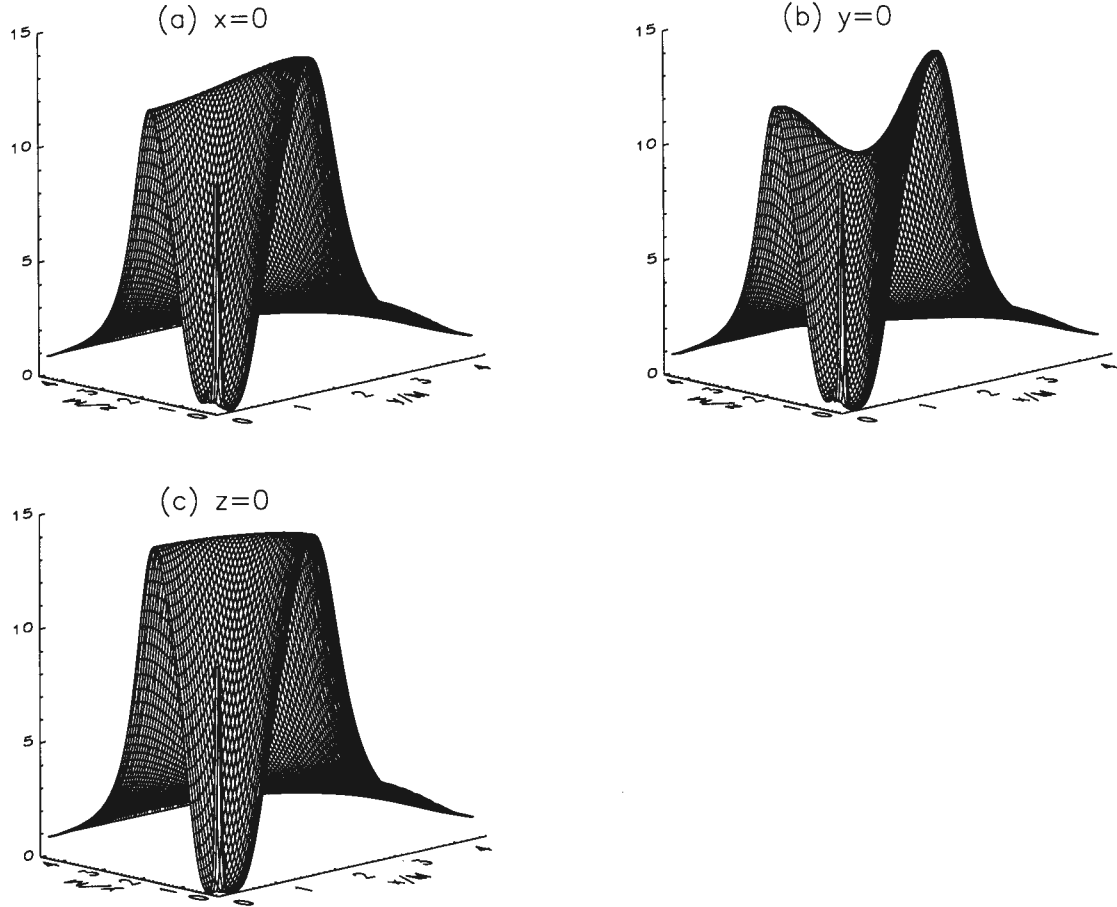


Figure 6.10: We show the metric function $\hat{\gamma}_{rr}$ in the first quadrant of each of the coordinate planes at time $t = 13.0M$ for the evolution of the initial data set $(-0.5, 0, 1, 0.5)$. The calculation was done with 100^3 grid points and a resolution of $\Delta x = 0.0433M$.

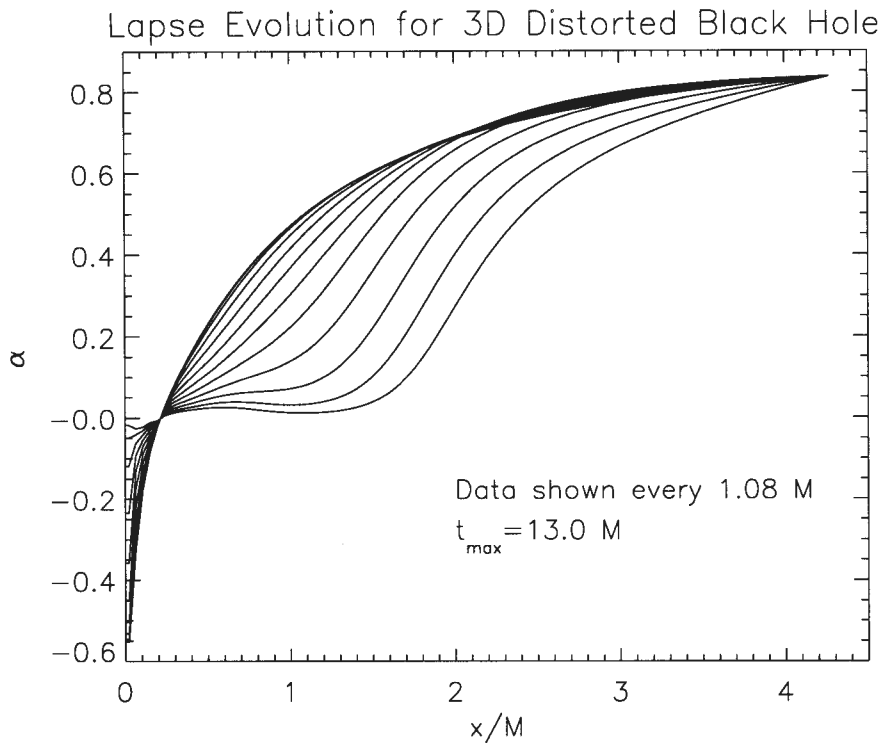


Figure 6.11: We show the lapse function α along the x -axis as a function of time for the evolution of the initial data set $(-0.5, 0, 1, 0.5)$. The calculation was done with 100^3 grid points and a resolution of $\Delta x = 0.0433M$.

of other data sets using “1+ln” slicing.

6.3.2 Radiation Extraction

We now turn to radiation extraction in full 3D black hole evolutions. This is of major importance for the connection between numerical relativity and gravitational wave astronomy. Gravitational wave detectors such as LIGO, VIRGO, and GEO will measure these waves directly, and may depend on numerical relativity to provide templates to both extract the signals from the experimental data and to interpret the results. This is the first time that non-axisymmetric modes of gravitational waves have been extracted in a numerical black hole spacetime. In the preliminary study in this thesis, we are just scratching the surface of this complex subject. Black hole perturbation theory predicts that for each ℓ -mode, there is a series of quasinormal

mode frequencies of the hole, as discussed below in Chapter 7. However, we have previously dealt only with axisymmetric distorted black holes. For full 3D distorted holes, there are quasinormal mode frequencies for each m -mode as well. For rotating black holes, the m -modes are all distinct, and have different frequencies depending on the angular momentum of the hole. However, for non-rotating black holes, it turns out that for a given ℓ -value, all m -modes have the same quasinormal mode frequency. Hence, when we compare our results with the known quasinormal mode frequencies, we expect all m -modes to fit the $m = 0$ quasinormal mode.

We extracted the $\ell = 2$ and $\ell = 4$ Zerilli functions as a function of time at various radii during the evolution of the initial data sets discussed in the previous Section. Due to our chosen symmetry, only even- m modes will be present.

In Figure 6.12 we show the $\ell = 2$ Zerilli functions computed during an evolution of the initial data set $(a, b, w, c) = (0.5, 0, 1, 0.5)$. This computation was done with 200^3 grid points and a resolution of $\Delta x = 0.106M$. The Zerilli functions were extracted at a radius of $r = 8.46M$. For each mode we also show the fit to the two lowest $\ell = 2$ quasinormal modes. The good agreement indicates that these waveforms have the correct frequency. As discussed in Chapter 4, we do not expect to be able to fit quasinormal modes to the waveforms initially, because here the waveform is highly dependent on the particular dynamics of the situation, and the black hole has not yet settled down to a perturbed Schwarzschild black hole.

In Figure 6.13 we show the $\ell = 4$ Zerilli functions extracted during the same evolution. Again, the functions presented were extracted at a radius of $r = 8.46M$. For $\psi^{(40)}$ and $\psi^{(42)}$ we also show the fit of the two lowest $\ell = 4$ quasinormal modes. We were not able to fit a quasinormal mode to any significant part of the $\psi^{(44)}$ waveform. It appears to have the correct wavelength, but the lack of damping is unexpected. However, we note that this waveform has a smaller amplitude than the others, and may be small enough that numerical noise, *e.g.*, noise from the outer

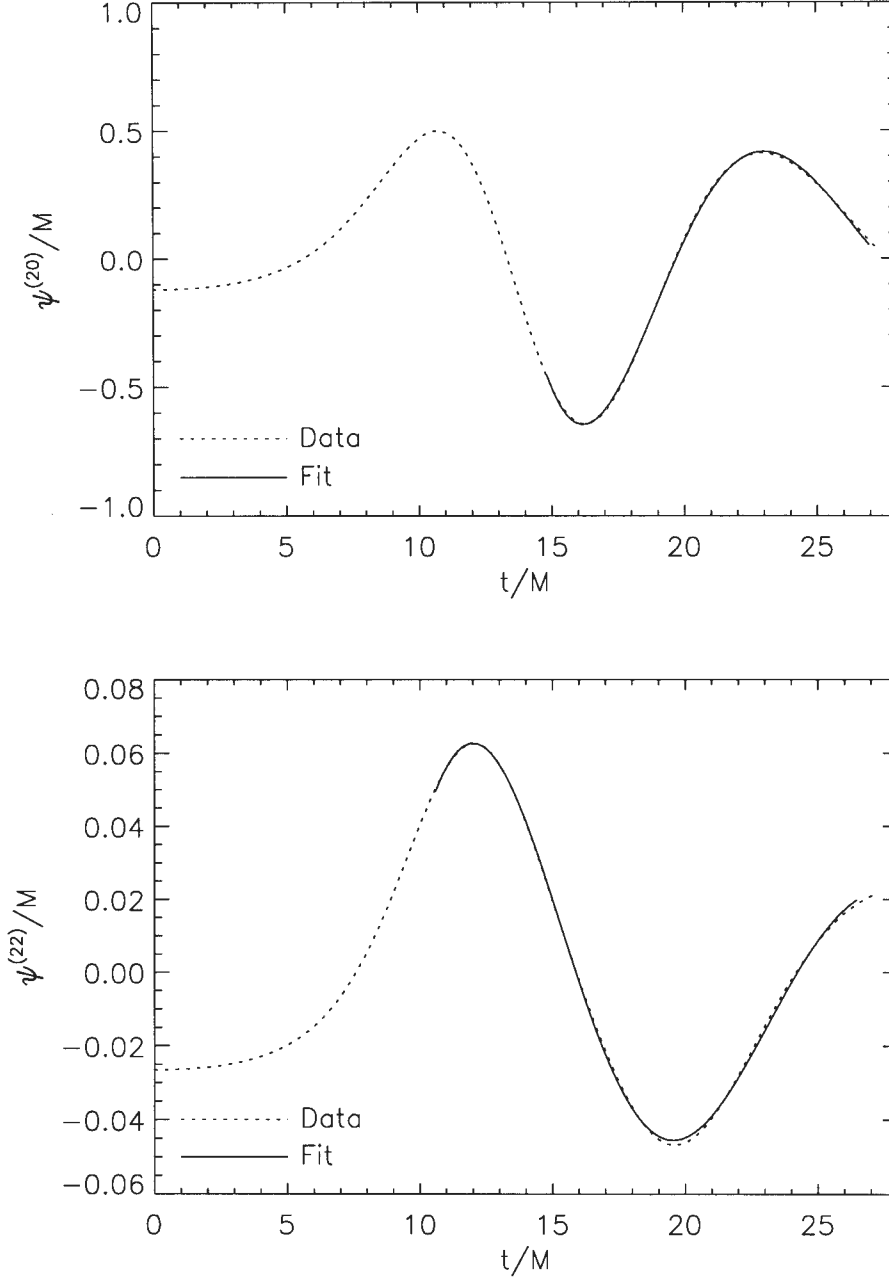


Figure 6.12: We show the $\ell = 2, m = 0$ and $\ell = 2, m = 2$ Zerilli functions extracted at $r = 8.46M$ as a function of time for the evolution of the initial data set $(0.5, 0, 1, 0.5)$. Also plotted are fits of the two lowest $\ell = 2$ quasinormal modes to each waveform. The calculation was done with 200^3 grid points and a resolution of $\Delta x = 0.106M$, placing the outer boundary along an axis at approximately $x = 21.2M$.

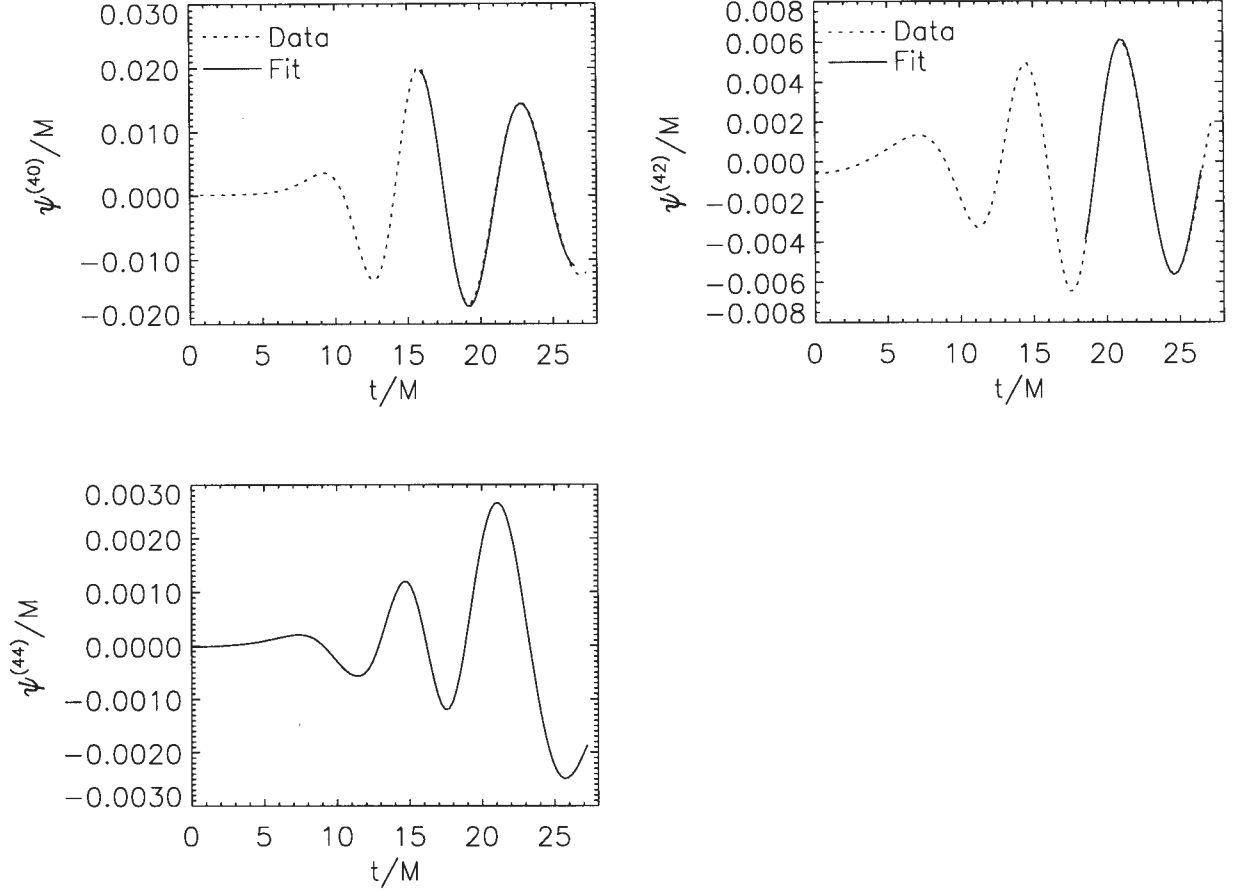


Figure 6.13: We show the $\ell = 4$ Zerilli functions extracted at a radius of $r = 8.46M$ as a function of time for the evolution of the initial data set $(0.5, 0, 1, 0.5)$. For the $m = 0$ and $m = 2$ cases, we also show fits to the two lowest $\ell = 4$ quasinormal modes. The calculation was done with 200^3 grid points and a resolution of $\Delta x = 0.105M$.

boundary, conceals the quasinormal mode signal. This interpretation is supported by the remarkable similarity between this waveform and the analogous waveform in Figure 6.6, which was computed during an evolution of the axisymmetric initial data set $(a, b, w) = (0.5, 0, 1)$.

In Figure 6.14 we show the $\ell = 2$ Zerilli functions extracted during an evolution of the data set $(-0.5, 0, 1, 0.5)$. The extraction was done at a radius of $r = 3.90M$. The calculation was performed with 200^3 grid points and a resolution of $\Delta x = 0.0433M$. Also shown are fits of the lowest two $\ell = 2$ quasinormal modes to each waveform. In

Figure 6.15, we plot the $\ell = 4$ Zerilli functions extracted at the same radius during the same run, and their quasinormal mode fits.

The integral over time of the square of the Zerilli function is a measure of the energy being emitted in gravitational radiation in that mode. We note that all the waveforms extracted during the evolution of the data set with the negative Brill wave amplitude, with the exception of the $\ell = 4$, $m = 4$ waveform, are smaller in amplitude than those extracted during the evolution of the positive Brill wave amplitude data set. This is what we would expect, given that the *MRL* for the positive Brill wave amplitude case is much larger, as can be seen in Figure 5.12. We note, however, that this is not required to be true, as the *MRL* is merely an upper bound on the amount of radiation that can be emitted.

Of potential interest is the study of relative amplitudes and phases of various wave modes during an evolution of a 3D data set. In Figure 6.16, we reproduce the $\ell = 4$ Zerilli functions for the above two evolutions, but for each evolution we plot them on the same graph. We see that for the $a = +0.5$ case, the $m = 0$ waveform is significantly larger than the $m = 2$ waveform, which is in turn larger than the $m = 4$ waveform. This is in contrast to the $a = -0.5$ case, in which the $m = 4$ waveform is larger than the $m = 2$ waveform, and is in fact almost as large as the $m = 0$ waveform. For the $a = +0.5$ case, the $m = 2$ and $m = 4$ waveforms seem to be in phase, lagging the $m = 0$ waveform by about a quarter of a cycle. The $a = -0.5$ waveforms show a different phase behavior: here, the $m = 0$ and $m = 4$ waveforms seem to be in phase with each other, leading the $m = 2$ waveform by about half a cycle. We intend to study this behavior further. In particular, we expect that much insight will come using the perturbative methods which we describe in the next Chapter.

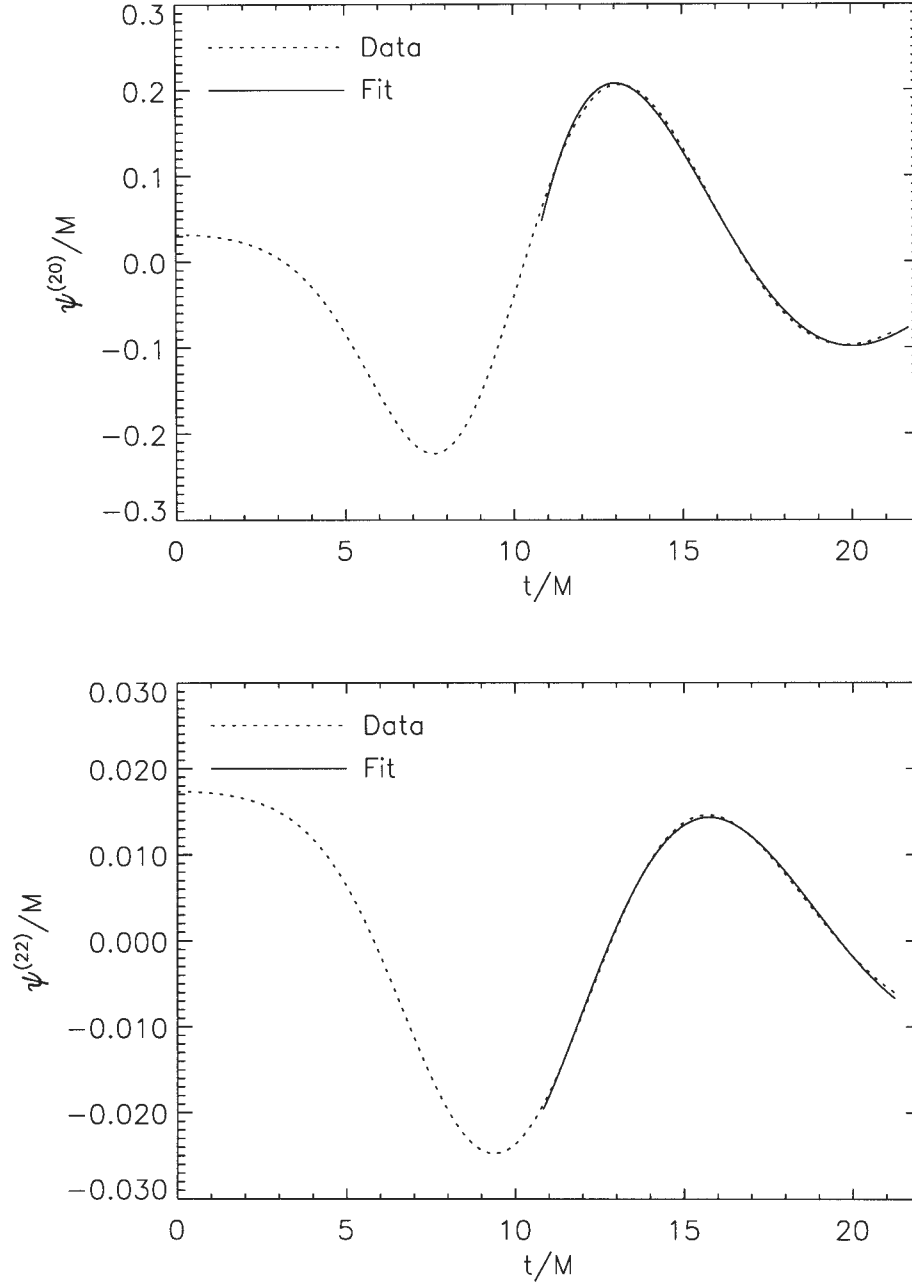


Figure 6.14: We show the $\ell = 2, m = 0$ and $\ell = 2, m = 2$ Zerilli functions extracted at $r = 3.90M$ as a function of time for the evolution of the initial data set $(-0.5, 0, 1, 0.5)$. Also shown are fits to the two lowest $\ell = 2$ quasinormal modes. The calculation was done with 200^3 grid points and a resolution of $\Delta x = 0.0433M$, placing the outer boundary along an axis at approximately $x = 4.33M$.

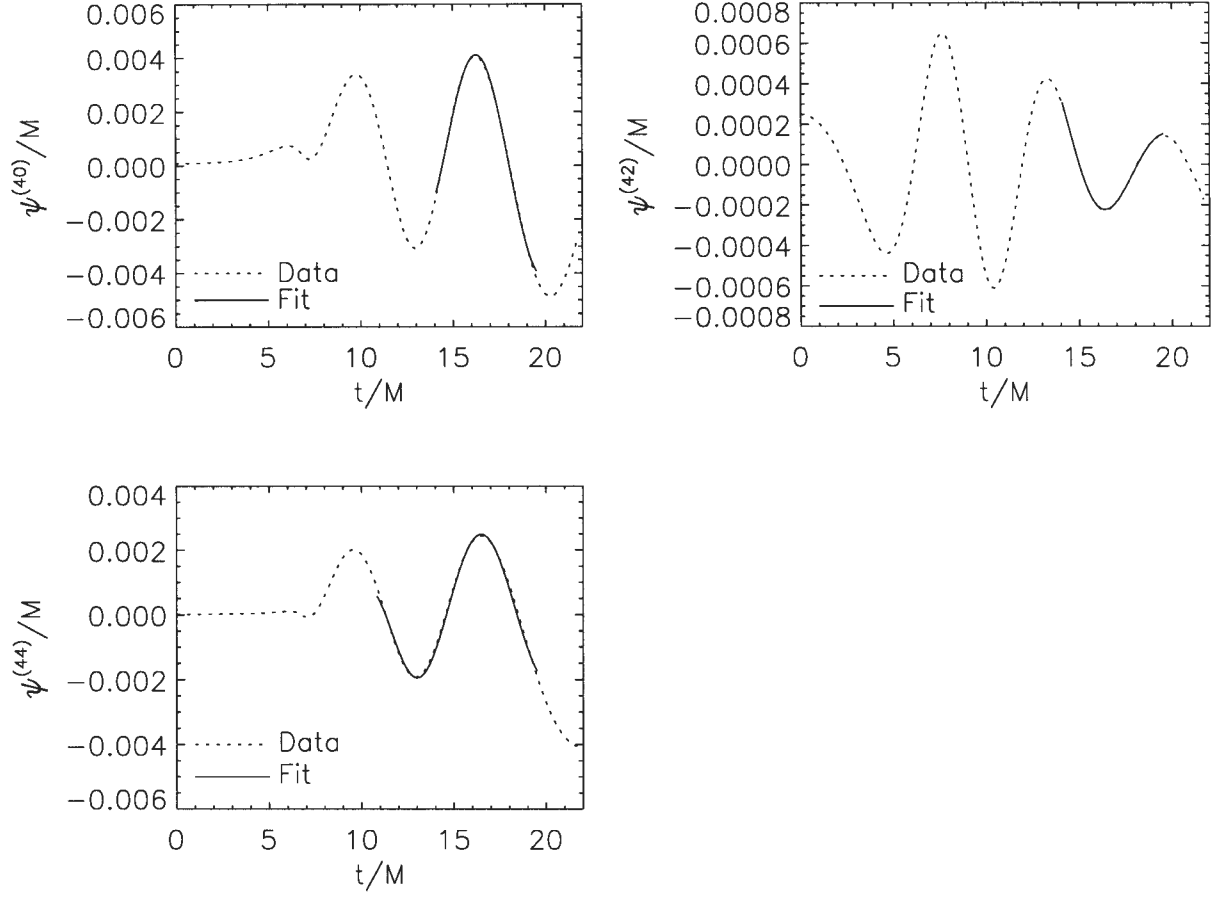


Figure 6.15: We show the $\ell = 4$ Zerilli functions extracted at a radius of $r = 3.90M$ as a functions of time for the evolution of the initial data set $(-0.5, 0, 1, 0.5)$. Also plotted are fits of the two lowest $\ell = 4$ quasinormal modes to each waveform. The calculation was done with 200^3 grid points and a resolution of $\Delta x = 0.0433M$.

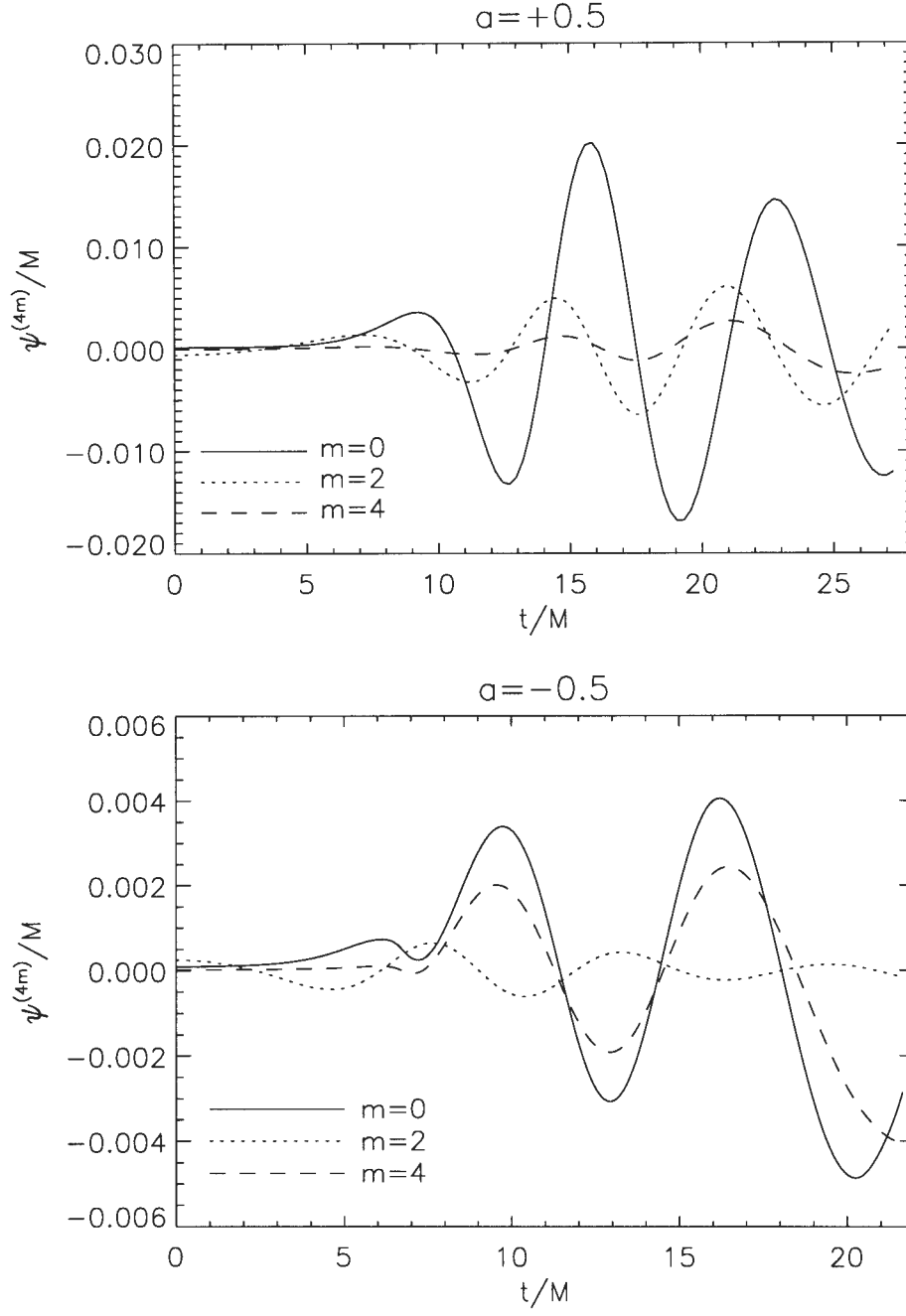


Figure 6.16: We show the $\ell = 4$ Zerilli functions superimposed for each of the previous runs, to illustrate the difference between the relative amplitudes and phases in the two cases.

6.4 Summary

We have presented results from numerical evolutions of Brill wave plus black hole spacetimes in full 3D. We first evolved spacetimes which were inherently axisymmetric. Here we showed that the metric functions and waveforms obtained agreed with results from mature, 2D codes. These results show that one can evolve distorted black holes in 3D and accurately extract the radiation emitted, including higher ℓ -poles. Studies with different resolutions indicated that waveforms can be extracted accurately in the (relatively) far zone, even when the metric functions in the inner regions are not adequately resolved at late times.

We then presented results from evolutions of intrinsically 3D data sets. Here we showed the non-axisymmetric wave modes, which were not present in the axisymmetric case. We showed that the waveforms could be, in most cases, fit to the quasinormal modes expected. A more detailed treatment is in progress, and will be published elsewhere [37, 38, 39].

In the next Chapter, we discuss another means of studying evolution of these data sets: perturbation theory. For adequately small Brill wave amplitude, these data sets should represent perturbations of Schwarzschild. We should be able to evolve the perturbations with Einstein's equations linearized about Schwarzschild. This will allow us to gain a better understanding of the physical system. It will also provide us with a test of our nonlinear numerical code.

Chapter 7

Linear Evolution of Perturbations

In Chapter 6 we presented preliminary results from nonlinear evolutions of 3D distorted black holes. Of particular interest was our ability to measure the non-axisymmetric wave modes. Unfortunately, due to our limited resolution and outer boundary location, we were unable to extract full waveforms before the code crashed. Also, because these were 3D data sets, we could not compare our results with results from a lower dimensional code.

Another way to study the evolution of these data sets is to study them with perturbation theory, in the regime in which the Brill wave amplitude is small enough that they can be considered perturbations of Schwarzschild. In this limit we could gain an understanding of the relative amplitudes and phases of the various wave modes, and we could measure the total energy emitted in each mode. It would also provide a test of our 3D nonlinear evolution code and the wave extraction routine being used. This work is very much in the spirit of the original work of Price and Pullin, and its extension by a series of authors [3, 4, 17, 51, 67], in treating the Misner two-black hole collision as a perturbation of Schwarzschild. In our case, the application is even more direct, as we are considering a perturbed, truly single black hole, as opposed to two nearby holes. This linearized treatment will be an important testbed for many future 3D black hole studies, and is being extended to the full 3D case of Chapter 6 in preparation for publication elsewhere [6].

In this Chapter, we give a brief overview of the theory of Schwarzschild perturbations, and present preliminary results which show agreement between perturbative, linear evolutions and 2D, nonlinear evolutions of axisymmetric initial data sets.

7.1 The Theory of Schwarzschild Perturbations

In this Section we give a very brief overview of the theory of perturbations of the Schwarzschild spacetime. This is a topic which is very rich and has a long history. A more detailed discussion can be found in [41]. In the next Section, we will discuss how we apply the theory to study our data sets.

One begins the analysis by writing the full metric as a sum of the Schwarzschild metric and a small perturbation:

$$g_{\alpha\beta} = \overset{\circ}{g}_{\alpha\beta} + h_{\alpha\beta}. \quad (7.1)$$

One then plugs this expression into the vacuum Einstein equations to get an equation for the perturbation tensor $h_{\alpha\beta}$. One can separate off the angular part of the solution by expanding $h_{\alpha\beta}$ in spherical tensor harmonics, as was originally done by Regge and Wheeler [68]. There are two independent expansions: one, known as even parity, which does not introduce any angular momentum to the hole, and one, known as odd parity, which does. We will concentrate here on even parity perturbations, as these are the ones which will be relevant for studying our data sets. The odd parity perturbations produce equations which are very similar. Also, just for this theoretical overview, we will assume that the solution has a time dependence given by $e^{-i\omega t}$. We will drop this assumption when it comes time to actually apply the theory. Now we are left with equations in the radial variable.

For even parity, after performing a particular gauge transformation, one gets three first-order and three second-order equations for three quantities, and an algebraic relation relating the three quantities. Zerilli [87] found that a certain combination of

these quantities satisfies the linear wave equation

$$\frac{\partial^2 \psi^{(\ell m)}}{\partial r^{*2}} + \omega^2 \psi^{(\ell m)} - V^{(\ell)}(r) \psi^{(\ell m)} = 0, \quad (7.2)$$

where the potential function $V^{(\ell)}(r)$ is given by

$$V^{(\ell)}(r) = \left(1 - \frac{2M}{r}\right) \times \left\{ \frac{1}{\Lambda^2} \left[\frac{72M^3}{r^5} - \frac{12M}{r^3} (\ell - 1)(\ell + 2) \left(1 - \frac{3M}{r}\right) \right] + \frac{\ell(\ell - 1)(\ell + 2)(\ell + 1)}{r^2 \Lambda} \right\}, \quad (7.3)$$

and r^* is the tortoise coordinate defined in Section 2.5. For all ℓ , the potential function is positive and has a peak near $r = 3M$. Regge and Wheeler had found an analogous equation for the odd parity perturbations, which are much simpler than the even parity perturbations, 13 years earlier in 1957 [68].

It has been shown that Schwarzschild is stable to perturbations, and has characteristic oscillation frequencies known as quasinormal modes. These modes are solutions to the Zerilli equation as given in eqn. (7.2) which are completely ingoing at the horizon ($r^* = -\infty$) and completely outgoing at infinity ($r^* = \infty$). For each ℓ -mode, independent of m , there is a fundamental frequency, and a possibly infinite number of overtones.

Moncrief showed that if one performs a coordinate (gauge) transformation of the form $x^\mu \rightarrow x^\mu + \delta x^\mu$, the Zerilli function is only effected to second order in δx^μ [63]. In this way, the Zerilli function is said to be gauge invariant. Further, while Zerilli's result was derived in a particular gauge, Moncrief showed how to compute $\psi^{(\ell m)}$ in a completely gauge independent way. This makes it a good quantity for studying gravitational radiation. This also makes it important for our numerical work, because we do not have to transform our results to some particular gauge developed for other purposes; we simply follow Moncrief's gauge invariant prescription.

In Chapters 4 and 6 we used the gauge-invariant extraction method described in Appendix C to compute the Zerilli function at a constant isotropic radius as a

function of time. In this Chapter, we extract the Zerilli functions as a function of radius on the initial time slice, and evolve it with the Zerilli equation to find the Zerilli function as a function of r and t . Note that to do this, we drop the assumption that the time dependence has the form $e^{-i\omega t}$, giving the Zerilli equation in the form

$$\frac{\partial^2 \psi^{(\ell m)}}{\partial r^2} - \frac{\partial^2 \psi^{(\ell m)}}{\partial t^2} + V^{(\ell)}(r) \psi^{(\ell m)} = 0. \quad (7.4)$$

The code which evolves this equation was written by Gabrielle Allen. We then compare the Zerilli function found in this way with that obtained from a 2D nonlinear evolution code.

7.2 Axisymmetric Tests

We first apply this method to axisymmetric initial data sets. This will allow us to compare the output from the linear code to that obtained with a well-tested 2D code that performs full nonlinear evolutions. This will serve as a test of the initial Zerilli function being given to the linear code, and of the linear evolution code itself. It will also help us determine for which Brill wave amplitudes this procedure breaks down.

The first data set we chose to evolve was $(a, b, w) = (0.05, 1, 1)$, with the power of the sine function n equal to 4. In this case the Brill wave is off the throat. In Figure 7.1 we show the $\ell = 2$ and $\ell = 4$ Zerilli functions as a function of time, at a radius of $r = 15M$. Data are shown from both the linear and 2D nonlinear codes. We see that for both functions, the linear and nonlinear results line up nicely until about $t = 50M$, when a phase shift starts to be significant. This phase shift and widening of the wave at late times is known from previous studies of numerical simulations of distorted black hole spacetimes in axisymmetry [1]. There it was claimed that difficulties evolving the peak in $\hat{\gamma}_{rr}$ led to a spurious increase in the mass of the black hole, which then oscillated at a slightly lower frequency than it should have. We are seeing the same result here.

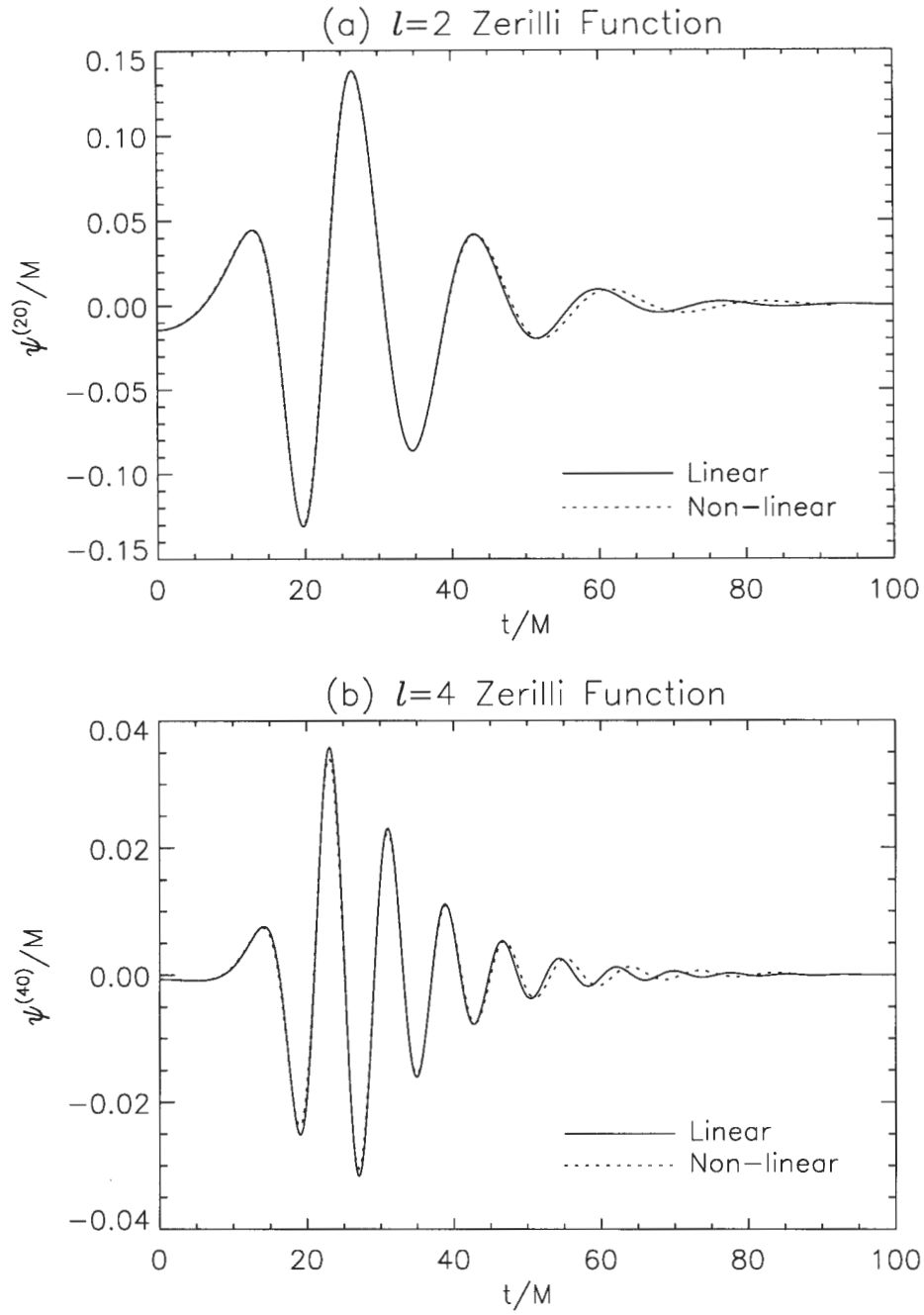


Figure 7.1: We show the (a) $\ell = 2$ and (b) $\ell = 4$ Zerilli functions as a function of time, extracted during linear and 2D nonlinear evolutions of the data set $(a, b, w, n) = (0.05, 1, 1, 4)$. The data were extracted at a radius of $r = 15M$.

The second data set that we chose to evolve was $(a, b, w) = (0.05, 0, 1)$, again with $n = 4$. In this case the Brill wave is on the throat. This is a slightly harder problem. The radial metric function in the Schwarzschild metric becomes infinite at $r = 2M$, which is near the throat. This makes it difficult to extract the Zerilli function in this region accurately. When the Brill wave is off the throat, as it was in the previous case, errors in this initial calculation do not propagate outwards. However, they can when the Brill wave is on the throat. In Figure 7.2 we show the $\ell = 2$ and the $\ell = 4$ waveforms as a function of time extracted at a radius of $r = 15M$. Again, data from both the linear and 2D nonlinear codes are shown. The data line up well until about $t = 50M$, when phase errors begin to be significant.

7.3 Summary

The use of perturbation theory is expected to be a powerful tool with which to study the evolution of these distorted black hole data sets, and to test our nonlinear code. We have shown the feasibility of this method for axisymmetric initial data sets. Work is in progress to extend this method to the study of the 3D distorted data sets. This work will be reported on in a future publication [6].

In the future, we also plan to extend this idea to study the evolution of distorted Kerr black holes, which have non-vanishing angular momentum. This is a harder problem, as it requires going to the more complicated Teukolsky formulation of perturbations of Kerr [79].

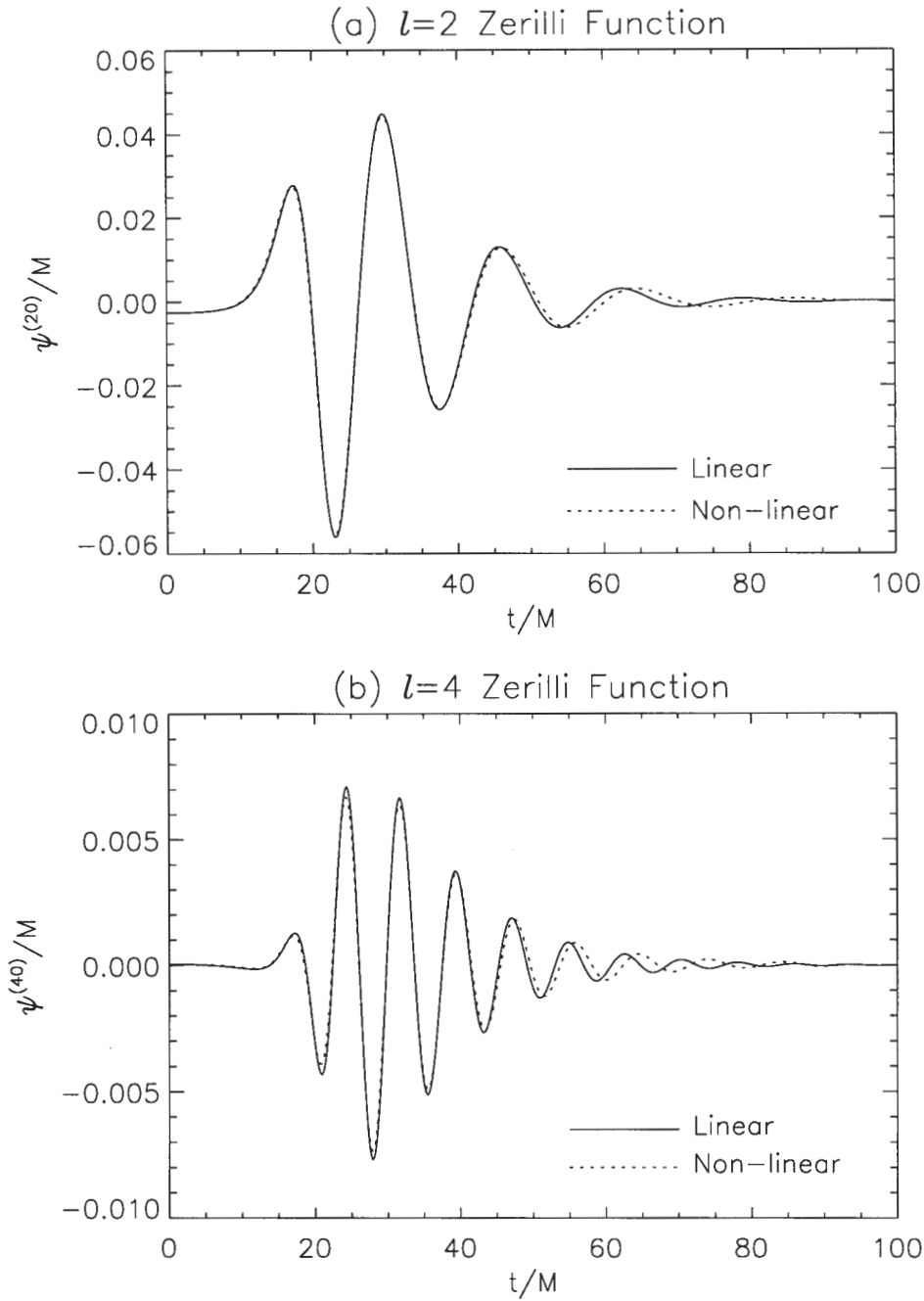


Figure 7.2: We show the (a) $\ell = 2$ and (b) $\ell = 4$ Zerilli functions as a function of time, extracted during linear and 2D nonlinear evolutions of the data set $(a, b, w, n) = (0.05, 0, 1, 4)$. The data were extracted at a radius of $r = 15M$.

Chapter 8

Conclusions and Future Directions

The numerical calculation of the inspiral of two black holes will require a computer code which uses 3D coordinates. We have presented results from a code which solves the ADM form of the Einstein equations in 3D Cartesian coordinates. While the general problem is far from solved, this work represents significant progress towards the final goal.

We first presented results from the 3D evolution of a Schwarzschild black hole. Here we showed that with certain slicing conditions, subject to current computer limitations, the black holes could be evolved to around $t = 50M$ in time.

We next presented results from the evolution of Misner two-black hole initial data sets. Here we showed that the slicing and boundary conditions that worked the best in the Schwarzschild case also work in other spacetimes. We also extracted measures of the gravitational radiation emitted as the holes fall together. These were shown to oscillate at the quasinormal mode frequency of the final black hole, as had been seen in 2D studies.

We then moved on to spacetimes which are truly 3D. First, we discussed the generation of 3D distorted black hole initial data sets. Here we showed that Bernstein's axisymmetric data sets could be extended in a straight-forward way. Properties of these data sets were derived, and shown to be consistent with axisymmetric results.

We then presented full, nonlinear 3D evolutions of these distorted black hole initial

data sets. We first showed that the metric functions and waveforms from evolved axisymmetric data sets agreed with 2D results. The agreement between the well-tested 2D results obtained in spherical polar coordinates and the 3D results obtained in Cartesian coordinates shows that it is possible to evolve accurately distorted black holes in 3D. In particular, the excellent match seen in the gauge invariant waveforms, which are small perturbations on the black hole background, shows that even the radiation from distorted black holes can be studied accurately as well. We then presented results from the evolution of truly 3D data sets. Here we showed all non-trivial $\ell = 2$ and $\ell = 4$ waveforms. These were shown to oscillate at the expected quasinormal mode frequencies.

We presented a method of studying these distorted black hole spacetimes with perturbation theory. It was shown that this method gives waveforms which match the output from 2D nonlinear codes, indicating its feasibility.

While much progress has been made towards understanding how to evolve black hole spacetimes in 3D, it is clear that the standard methods used are not adequate. In all cases studied, the sharp features that develop in the metric functions caused the code to crash, at best at around $t = 50M$. In order to follow a full inspiral, one needs to be able to evolve black hole spacetimes for several hundred M . One needs advanced techniques to either handle these peaks better, or to eliminate them altogether.

There are two main methods currently being investigated by the numerical relativity community whose purpose is to handle the peaks better. The first of these is adaptive mesh refinement (AMR). With this method, one places grid points down in higher density in regions where functions have steep spatial gradients, and in lower density where functions are smooth. One would like to do this adaptively, in order to follow regions of steep gradients as they move during an evolution. Such a scheme has been successful in 1D black hole simulations [59]. We are currently working with

Lee Wild at AEI to investigate such a scheme in 3D, with encouraging preliminary results.

Another method being developed in part to handle sharp features in evolved functions is the formulation of the Einstein equations as a system of first order balance laws, *i.e.*, a system of equations with the form $\partial_t u + \partial_i F^i(u) = S(u)$ [24]. It has been shown that this system is hyperbolic for a large class of gauge conditions, including maximal slicing and the “1+ln” algebraic slicing [25]. This allows us to use many of the shock capturing techniques developed in the field of numerical hydrodynamics to resolve the sharp features. Tests with a 1D code evolving a spherical black hole show that with this formalism, the black hole could be run past $t = 1000M$ [25]. A 3D code using this formalism is under development, and will be reported on soon [60].

While AMR and the hyperbolic formulation allow one to handle the peaks better, eventually these sharp features will crash the code. Because these peaks are artifacts of the singularity-avoiding slicing conditions, one like to get rid of the peaks altogether by excising the singularity region from the domain. This is the idea behind what is know as the apparent horizon boundary conditions (AHBC) [71]. An AHBC uses the shift vector to lock the position of the apparent horizon to a particular coordinate location, or to control its motion. Because of the causal structure of the spacetime, the data outside the apparent horizon does not depend on the data inside the apparent horizon, so that any points lying inside the horizon can be excised from the computational domain. This method eliminates both the sharp transition in the lapse and the peaks that develop in the metric function. 1D studies with this method show evolution is possible past $t = 1000M$ [13]. We are aware of a number of methods to implement this method in 3D [34, 45, 84]. In particular, Daues has achieved stable evolutions of black holes in 3D past $t = 100M$ [45].

Independent of the methods employed, this work also lays some of the groundwork for extension to 3D evolutions of other data sets. In all the cases studied in this thesis,

there was no angular momentum. For his thesis work, Brandt extended Bernstein's non-rotating data sets to include rotation, while maintaining axisymmetry. We have begun to extend Brandt's initial data sets to 3D, but the extension turns out not to be as straight-forward as in the non-rotating case [27]. Also, we expect that the evolution of rotating initial data in 3D will pose new problems. For example, Brandt found it necessary to use the ϕ -component of the shift vector to keep constant-coordinate lines from becoming "tangled up". We will need to develop a condition on the shift vector in Cartesian coordinates that achieves the same result.

Appendix A

Analysis of Numerical Methods

In this Appendix we discuss details of the numerical computations done to produce initial data and to do evolutions.

A.1 Finite Differences and Convergence

Solving a differential equation numerically using finite differencing techniques involves solving an approximation to that equation at a finite number of points within the domain. In this Section we discuss how we obtain the finite-differenced version of the differential equation, and how one measures the difference between the solution to the finite differenced equation and the solution to the original differential equation.

All approximations to the derivatives that appear in our equations are derived from the Taylor series expansion, the first few terms of which are given by

$$f(x_0 + \Delta) = f(x_0) + \Delta \left. \frac{\partial f}{\partial x} \right|_{x=x_0} + \frac{1}{2} \Delta^2 \left. \frac{\partial^2 f}{\partial x^2} \right|_{x=x_0} + O(\Delta^3). \quad (\text{A.1})$$

For example, an approximation to the derivative $\left. \frac{\partial f}{\partial x} \right|_{x=x_0}$ at a point x_0 can be obtained by subtracting the Taylor series expansion for $f(x_0 - \Delta)$ from the above expansion, resulting in

$$\left. \frac{\partial f}{\partial x} \right|_{x=x_0} = \frac{f(x_0 + \Delta) - f(x_0 - \Delta)}{2\Delta} + O(\Delta^2). \quad (\text{A.2})$$

This equation is second order accurate, meaning that the difference between the analytic value of the derivative of f at x_0 and the approximate value is proportional

to the grid spacing Δ to at least the second power. All results presented in this work have been obtained with second order finite difference approximations to the derivatives.

Once one has written a computer program to solve a second-order accurate, finite-differenced equation, it is often useful to determine if the solution produced is in fact accurate to second order. This is a test of the numerical method being used, boundary conditions, and of the accuracy of the coding. Consider a quantity y computed at a point x . Let the analytic value of y at x be denoted y_0 . Consider the numerical value of y at x computed with the three grid resolutions Δx , $\lambda \Delta x$, and $\lambda^2 \Delta x$, where $\lambda < 1$, and is generally chosen to be $\frac{1}{2}$. Then,

$$y_1 = y_0 + k (\Delta x)^\sigma \quad (\text{A.3})$$

$$y_2 = y_0 + k (\lambda \Delta x)^\sigma \quad (\text{A.4})$$

$$y_3 = y_0 + k (\lambda^2 \Delta x)^\sigma, \quad (\text{A.5})$$

where k is a constant, and σ is the rate of convergence, which we will generally expect to be two. The rate of convergence can then be computed with

$$\sigma = \frac{\ln \left(\frac{y_3 - y_2}{y_2 - y_1} \right)}{\ln \lambda}. \quad (\text{A.6})$$

This is the equation that we will use to compute the convergence rate in the sections that follow.

A.2 Initial Data

In Chapter 5, we described the construction of initial data for 3D distorted black holes. To derive initial data, we need to solve the Hamiltonian constraint (5.9) for the conformal factor, which we repeat here for reference:

$$\frac{\partial^2 \tilde{\psi}}{\partial \eta^2} + \frac{\partial^2 \tilde{\psi}}{\partial \theta^2} + \cot \theta \frac{\partial \tilde{\psi}}{\partial \theta} + \csc^2 \theta e^{2q} \frac{\partial^2 \tilde{\psi}}{\partial \phi^2} + 2 \csc^2 \theta e^{2q} \frac{\partial q}{\partial \phi} \frac{\partial \tilde{\psi}}{\partial \phi} = \quad (\text{A.7})$$

$$-\frac{1}{4}\tilde{\psi}\left(\frac{\partial^2 q}{\partial\eta^2} + \frac{\partial^2 q}{\partial\theta^2} + 2e^{2q}\csc^2\theta\frac{\partial^2 q}{\partial\phi^2} + 3e^{2q}\csc^2\theta\left(\frac{\partial q}{\partial\phi}\right)^2 - 1\right).$$

Instead of solving this equation for the conformal factor, we instead solve it for the difference between the conformal factor and the analytic spherical solution, *i.e.*, we solve for the quantity $\tilde{\psi}'$, given by

$$\begin{aligned}\tilde{\psi}' &= \tilde{\psi} - \tilde{\psi}_0 \\ &= \tilde{\psi} - \sqrt{2m} \cosh\left(\frac{\eta}{2}\right).\end{aligned}\tag{A.8}$$

The equation to be solved thus becomes

$$\begin{aligned}\frac{\partial^2 \tilde{\psi}'}{\partial\eta^2} + \frac{\partial^2 \tilde{\psi}'}{\partial\theta^2} + \cot\theta\frac{\partial\tilde{\psi}'}{\partial\theta} + \csc^2\theta e^{2q}\frac{\partial^2 \tilde{\psi}'}{\partial\phi^2} + 2\csc^2\theta e^{2q}\frac{\partial q}{\partial\phi}\frac{\partial\tilde{\psi}'}{\partial\phi} = \\ -\frac{1}{4}(\tilde{\psi}' + \tilde{\psi}_0)\left(\frac{\partial^2 q}{\partial\eta^2} + \frac{\partial^2 q}{\partial\theta^2} + 2e^{2q}\csc^2\theta\frac{\partial^2 q}{\partial\phi^2} + 3e^{2q}\csc^2\theta\left(\frac{\partial q}{\partial\phi}\right)^2 - 1\right).\end{aligned}\tag{A.9}$$

This linear equation is solved with a 3D conjugate gradient solver based on a 2D solver written by John Towns.

The next things to consider are the computational domain and the boundary conditions. As mentioned in Chapter 5, the throat of the black hole is located at $\eta = 0$. We choose this to be our inner boundary in the radial direction. The isometry condition provides a boundary condition there:

$$\left.\frac{\partial\tilde{\psi}}{\partial\eta}\right|_{\eta=0} = 0.\tag{A.10}$$

The outer boundary is chosen to be far enough away that the metric there is essentially flat, and that the outer boundary does not influence the computation adversely in the inner regions. Following Bernstein, we choose η_{max} to be 6, corresponding to a maximum isotropic radial coordinate of $r_{max} \approx 200m$, where m is chosen to be 2. The outer boundary condition on $\tilde{\psi}$ is at first unclear. Consider first the desired behavior of ψ , the conformal factor in the r coordinates. As $r \rightarrow \infty$ for Schwarzschild, $\psi \rightarrow 1$. Thus, setting ψ to 1 at the outer boundary seems like a possibility. However, it has

been shown that this leads to first order errors [86]. Another possibility is to require that ψ fall off as $1/r$, as the Schwarzschild solution does. This leads to the Robin condition suggested by York [86]:

$$\frac{\partial\psi}{\partial r} + \frac{\psi - 1}{r} = 0. \quad (\text{A.11})$$

Transforming this condition to the η -coordinate system gives the following boundary condition on $\tilde{\psi}$:

$$\left(\frac{\partial\tilde{\psi}}{\partial\eta} + \frac{1}{2}\tilde{\psi} - \sqrt{\frac{m}{2}}e^{\eta/2} \right) \Big|_{\eta=\eta_{\max}} = 0. \quad (\text{A.12})$$

This is equivalent to the following condition on $\tilde{\psi}'$:

$$\left(\frac{\partial\tilde{\psi}'}{\partial\eta} + \frac{1}{2}\tilde{\psi}' \right) \Big|_{\eta=\eta_{\max}} = 0. \quad (\text{A.13})$$

This is the condition we use to derive our initial data.

In the angular directions, the symmetries of the chosen q allow us to restrict ourselves to the region $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq \pi/2$. The boundary conditions in these directions are determined by the symmetries of the problem, and are given by

$$\frac{\partial\tilde{\psi}}{\partial\theta} \Big|_{\theta=0} = \frac{\partial\tilde{\psi}}{\partial\theta} \Big|_{\theta=\pi/2} = \frac{\partial\tilde{\psi}}{\partial\phi} \Big|_{\phi=0} = \frac{\partial\tilde{\psi}}{\partial\phi} \Big|_{\phi=\pi/2} = 0 \quad (\text{A.14})$$

Note that the computed function $\tilde{\psi}'$ must also obey these boundary conditions.

As our conjugate gradient code is written with second-order accurate, finite-difference approximations to the derivatives, an important check of the code is to confirm that the numerical solution converges to second order to the continuous solution as resolution is increased. We computed the initial data for the parameter set $(0.5, 0., 1., 0.5)$ at three different resolutions, and computed the convergence rate of the solution for $\tilde{\psi}'$ along the line $\theta = \phi = \pi/4$. In Figure A.1 we show a plot of the convergence rate as a function of η along this line. From this plot we see that the code is indeed converging to second order.

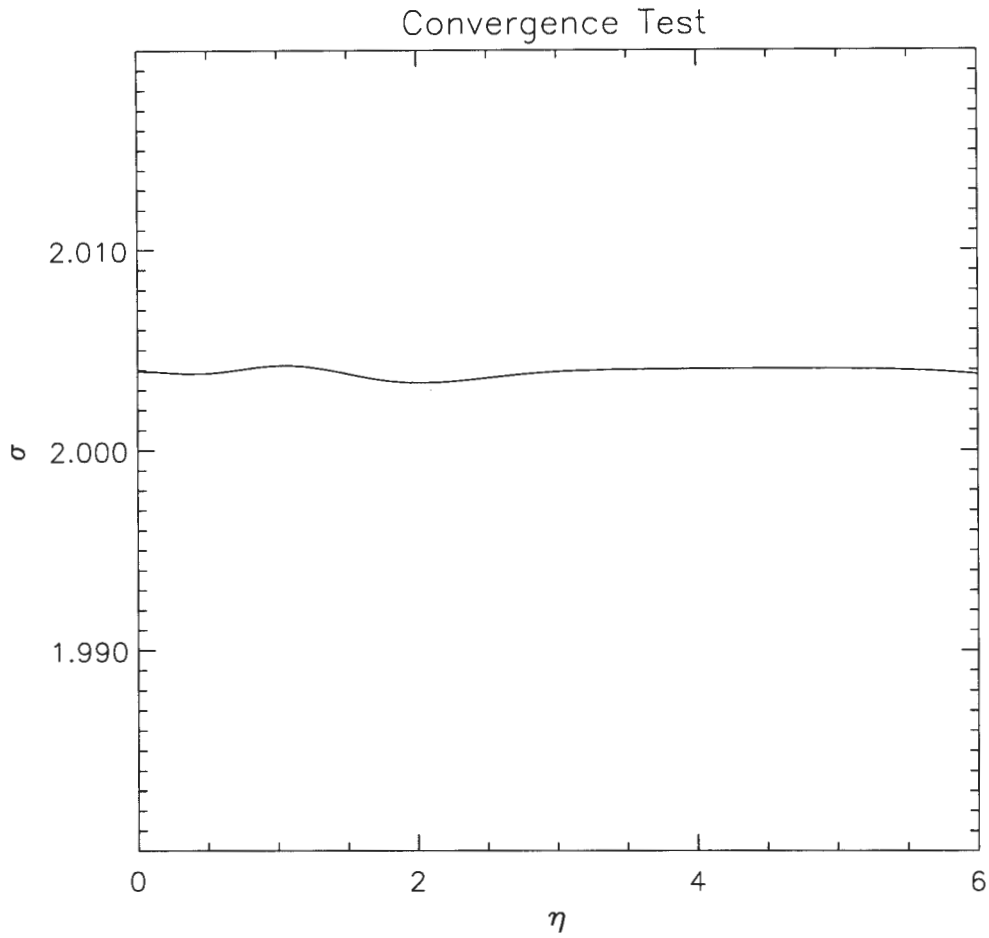


Figure A.1: We show the convergence rate σ for the solution of $\tilde{\psi}'$ along the line $\theta = \phi = \pi/4$. As expected, it is nearly 2 everywhere.

A.3 Evolutions

The metric and extrinsic curvature components are evolved in the 3D code using an explicit staggered leapfrog method. In this method, the extrinsic curvature variables are offset a half step in time relative to the metric variables. Placing the metric variables and the lapse at the full time step and the extrinsic curvature at the half step, we can write the finite differenced forms of (2.5) and (2.6) schematically as

$$\gamma_{i,j,k}^{n+1} = \gamma_{i,j,k}^n - 2\alpha_{i,j,k}^n K_{i,j,k}^{n+1/2} \Delta t \quad (\text{A.15})$$

and

$$K_{i,j,k}^{n+3/2} = K_{i,j,k}^{n+1/2} + \left\{ \alpha_{i,j,k}^{n+1} \left[\left(K_{i,j,k}^{n+1/2} \right)^2 + R_{i,j,k}^{n+1} \right] - \left(\nabla^2 \alpha \right)_{i,j,k}^{n+1} \right\} \Delta t, \quad (\text{A.16})$$

where we have dropped the tensor index notation to prevent confusion with the indices i , j , and k , which are used to locate a quantity on the spatial grid, and where a number of terms have been omitted for clarity.

These equations produce first order errors due to the placement of the lapse function in the metric evolution, and the nonlinear terms $\left(K_{i,j,k}^{n+1/2} \right)^2$ in the extrinsic curvature evolution. We adjust for this problem by extrapolating the necessary variables to the appropriate time slice using the formula

$$K_{i,j,k}^{n+1} = \frac{3}{2} K_{i,j,k}^{n+1/2} - \frac{1}{2} K_{i,j,k}^{n-1/2}. \quad (\text{A.17})$$

(The lapse is extrapolated to the half time step for the metric evolution using an analogous formula.) We have used this method of solution in previous work [9, 14, 21] and found it to work very well.

Spatial first and second derivatives of the metric components present in the terms $(\nabla^2 \alpha)_{i,j,k}$ and $R_{i,j,k}$ are defined using standard second order centered differences.

It turns out that the staggered leapfrog method is incompatible with standard adaptive mesh refinement techniques due to the necessary extrapolations. To this end, we have begun experimenting with a related time stepping scheme known as double leapfrog. In this method, the metric and extrinsic curvature variables are located at the same timestep. Evolving them forward one time step involves evolving them both forward a half time step twice. Schematically, we can write this technique as

$$\gamma_{i,j,k}^{n+1/2} = \gamma_{i,j,k}^{n-1/2} + \dot{\gamma}_{i,j,k}^n \Delta t \quad (\text{A.18})$$

$$K_{i,j,k}^{n+1/2} = K_{i,j,k}^{n-1/2} + \dot{K}_{i,j,k}^n \Delta t \quad (\text{A.19})$$

$$\gamma_{i,j,k}^{n+1} = \gamma_{i,j,k}^n + \dot{\gamma}_{i,j,k}^{n+1/2} \Delta t \quad (\text{A.20})$$

$$K_{i,j,k}^{n+1} = K_{i,j,k}^n + \dot{K}_{i,j,k}^{n+1/2} \Delta t. \quad (\text{A.21})$$

Preliminary results using this method show it to be less diffusive than the staggered leapfrog, and thus less stable at late times when sharp features begin to develop.

Appendix B

Numerical Location of Apparent Horizons in 3D

In this Appendix, we discuss how the apparent horizons presented in this work were computed. As mentioned in Chapter 2, the location of apparent horizons in 3D is a topic of on-going research. Here we concentrate on the method our group has used to date. This method, as applied in Cartesian coordinates, is described in more detail in our paper on this method [11], from which this Appendix is developed.

There are many well developed methods for determining the location of apparent horizons in lower dimensional spacetimes [20, 23, 36, 44, 83], *e.g.*, in axisymmetry. The partial differential equation (PDE) defining a marginally trapped surface reduces to an ordinary differential equation (ODE) in the axisymmetric case, and the symmetry conditions also provide boundary conditions for starting the integration of the ODE. This simplifies the problem tremendously, and enables the construction of efficient methods for finding the apparent horizon (AH). However, as these methods rely strongly on the symmetry assumptions, they are not easily generalizable to 3D; going from 2D to 3D does not amount simply to adding one more spatial dimension. For the general 3D case, there is no symmetry and the AH surface to be determined is a *closed* surface (hence no boundary conditions for starting the integration) described by a non-linear elliptical PDE. At present there are no efficient algorithms for solving such a partial differential problem in general.

We are aware of three independent efforts to determine the AH location in the general 3D case. The first method is based on an expansion of the AH surface in terms of spherical harmonics, with the expansion coefficients determined by an integral equation. The equation is then solved iteratively [57, 65]. Improvements on this scheme are under development [52]. The second method attempts to solve directly the elliptic PDE [56, 81]. The third method, our method, is based on expanding the closed surface in terms of orthogonal functions and using a minimization procedure to determine the expansion coefficients [11, 58]. Variations on this idea were originally explored by Brill and Lindquist [32] and by Eppley [47]. The major advantage of this method is that the numerical solution of the minimization problem is much better understood than the corresponding PDE problem. In reference [58], we demonstrated that our method can be efficiently applied to testbeds made up of analytically given data sets representing time-symmetric slices of spacetime. In reference [19], the convergence of the symmetric trace-free tensor expansion for similar testbeds is studied in detail. In reference [11] we pushed the application of this method in two directions: (1) We applied the method to data sets which were not time-symmetric (non-zero extrinsic curvature), and (2) We applied the method to data which were obtained in actual numerical evolutions of dynamical spacetimes.

B.1 The Method

We now discuss the method itself. Defining s^μ to be the outward-pointing space-like unit normal of a two-sphere $\mathcal{S}$ embedded in a constant time slice Σ with timelike unit normal n^μ , we can construct the outgoing null normal to any point on $\mathcal{S}$ as $k^\mu = n^\mu + s^\mu$. The surface $\mathcal{S}$ is called a *marginally trapped surface* (MTS) if the divergence of the outgoing null vectors $\nabla_\mu k^\mu$ vanishes, or equivalently, if [86]

$$\Theta = D_i s^i + K_{ij} s^i s^j - K = 0, \quad (\text{B.1})$$

where Θ is the expansion of the outgoing rays, D_i the covariant derivative with respect to the 3-metric γ_{ij} , K_{ij} the extrinsic curvature of Σ , and K the trace of K_{ij} . The AH is defined as the outer-most marginally trapped surface.

In our method, the apparent horizon is taken to be the surface defined by $F(x^i) = 0$. The unit normal s^i of this surface is then given by

$$s^i = \gamma^{ij} \partial_j F \left(\gamma^{k\ell} \partial_k F \partial_\ell F \right)^{-1/2}. \quad (\text{B.2})$$

We proceed by expanding F in some set of basis functions, and finding the coefficients of the expansion which make $\Theta = 0$. In practice, we find the set of expansion coefficients $\{a_i\}$ which minimize the function

$$\sum_{\sigma} W_{\sigma} \Theta_{\sigma}^2, \quad (\text{B.3})$$

where the sum is over the value of $W\Theta^2$ at the finite number of points on a given test surface. W is a weight function, which we have taken in all cases to be unity.

An obvious potential difficulty of this method is that there is no guarantee that the minimization procedure will converge, or that it will converge rapidly. For black holes not in highly dynamical situations, we expect the AH surface to be smooth so that one needs only a few terms in the expansion of the surface to find the surface accurately. This is usually the case through most of the simulations, and our method will be more efficient at these times. However, we also expect that for the cases we would like to simulate, there are also periods of time when the black hole (or holes) goes through highly dynamical evolutions, *e.g.*, around the time of the coalescence of two black holes.

The numerical problem is to find a set of parameters $\{a_i\}$ that minimizes (B.3). Minimization techniques (such as conjugate gradient or quasi-Newton methods) that evaluate both the function and all of its various partial derivatives are often preferred,

as a means to increase the convergence rate, to those that do not require derivative information. However, because of the complexity of equation (B.1), in this first generation 3D AH finder we have chosen to implement a multi-dimensional method that does not require knowledge of derivatives, namely a direction set or Powell's method [66]. This method is based on successive line minimizations, whereby the function $\sum \Theta^2$ is successively minimized along different vector directions using the one-dimensional Brent's method with parabolic interpolation. We find Powell's method to be generally robust, with a good convergence rate and computational speed, for surface functions parameterized by fewer than about fifteen or so parameters. Although solutions can still be found for parametrizations of higher order, the computational cost becomes excessive, especially when compared to the evolution cycle time.

For the apparent horizons found on the 3D Cartesian grid, $F(x^i)$ was expanded in symmetric trace-free (STF) tensors:

$$F(x, y, z) = \sum_{i=1}^3 (x^i - x_0^i)^2 - \sum_{\ell=0}^L \mathcal{F}_{K_\ell} N_{K_\ell} = 0, \quad (\text{B.4})$$

where x^i are the Cartesian coordinates and x_0^i the Cartesian points interior to the $F = 0$ surface representing the horizon center. In equation (B.4), we have adopted the notation from reference [82]. The STF tensors $\mathcal{F}_{K_\ell}$ are coordinate independent coefficients, where the subscript K_ℓ is abbreviated notation for the vector product $N_{K_\ell} \equiv n_{k_1} n_{k_2} \dots n_{k_\ell}$, where the n_i are the unit directional vectors

$$n_i = \frac{x^i - x_0^i}{\sqrt{\sum_{j=1}^3 (x^j - x_0^j)^2}}. \quad (\text{B.5})$$

Although STF tensor expressions are available in the literature, we find it convenient to include in our code a routine for the automatic generation of STF tensors to an arbitrarily high order. The procedure takes advantage of the fact that we can associate the n_i with essentially $x - x_0$, $y - y_0$ or $z - z_0$ in three dimensions. We can therefore construct all symmetric and independent permutations of

$$N_{K_\ell} \equiv A_\ell (x - x_0)^\alpha (y - y_0)^\beta (z - z_0)^\gamma, \quad (\text{B.6})$$

subject to the constraint $\alpha + \beta + \gamma = \ell$, where ℓ is the rank of the corresponding symmetric tensor, or equivalently the order of the multipole expansion, and A_ℓ is the normalization factor making the RHS dimensionless. There are $(\ell + 1)(\ell + 2)/2$ such independent combinations. The combinations constructed in this fashion can be supplemented with the $\ell(\ell - 1)/2$ independent conditions imposed on the symmetric permutations to make the rank ℓ tensor trace-free by contracting on any two indices. The second summation in (B.4) can be expanded as

$$f(x, y, z) = \sum_{\ell=0}^L \sum_{\alpha+\beta+\gamma=\ell} C_k < (x - x_0)^{\alpha_k} (y - y_0)^{\beta_k} (z - z_0)^{\gamma_k} >, \quad (\text{B.7})$$

where C_k are coordinate independent coefficients, and $<>$ denotes STF combinations [82]. Partial derivatives $\partial_i F$ and $\partial_i \partial_j F$ needed to evaluate Θ on the trial surfaces are then easily computed from (B.4) and (B.7). The simplicity of the form (B.7) also allows one to easily construct the multipole expansion to take advantage of any symmetries present in the problem. For instance, our current implementation, by setting a flag, can invoke either the even or odd multipoles independently of the other, enforce axisymmetry, or allow the most general parametric expansion.

B.2 Code Tests

The simplest, most basic, test for any apparent horizon finder is the static Schwarzschild initial data described in Section 3.1. The apparent horizon in this case is spherically symmetric and located at $r = M/2$. Although only the $\ell = 0$ term should contribute to the surface that defines the apparent horizon, we tested the solver with a more general multipole expansion with the $m \neq 0$ terms up to and including $L = 6$, a total of 28 coefficients.

The computed surface is plotted in Figure B.1 for four separate cases with various black hole masses. The grid resolution used in each case was $\Delta x = \Delta y = \Delta z = 0.075M$, using 25^3 cells. As expected, the surface is mostly defined by the $\ell = 0$

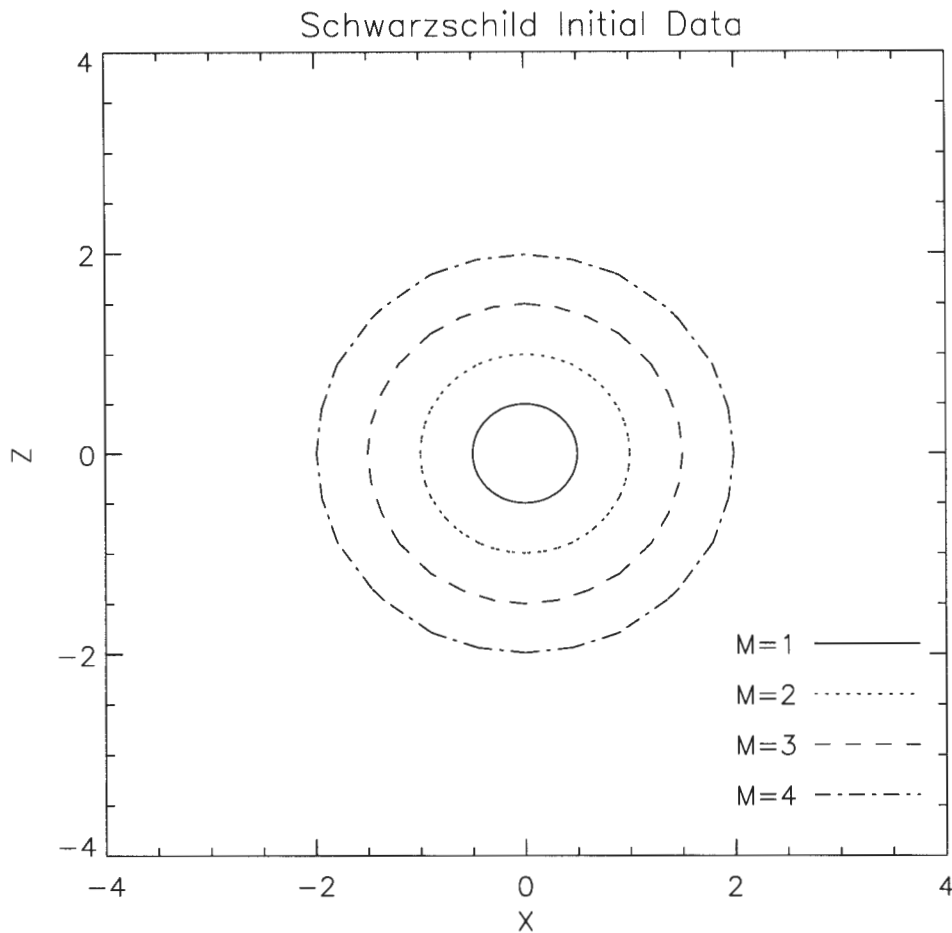


Figure B.1: The coordinate location of the apparent horizon in the x - z plane for Schwarzschild initial data with various masses. The computed surfaces are indistinguishable from the analytic solutions, with deviations of order 0.04%.

contribution: The other higher order terms are small in comparison, roughly a factor of 10^{-8} smaller. We find typical errors in the horizon radius of order 0.04%, and the numerical surfaces are indistinguishable from the analytic solution in Figure B.1. In each of the cases shown here, the finder converged to the correct surface in approximately 30 iterations. However, the number of iterations decreases significantly if fewer parameters are varied.

We also tested the finder on the Misner initial data described in Section 4.1. As discussed in that Section, data sets with $\mu < 1.36$ have a single apparent horizon surrounding the two throats. In these cases, the surface can be distorted significantly.

To find the distorted horizons accurately, then, we need to keep higher order multipole terms, but *a priori* it is not clear how many terms will be required. In Figure B.2, we show the results of systematically increasing the number of axisymmetric expansion terms for the case $\mu = 1.2$. In each calculation we use a 64^3 grid with $\Delta x = 0.1$ and run the code on the 128 node partition of the CM5. We also show the result obtained with our 2D, axisymmetric code described in reference [16], which implements an independent ODE integration method [7]. It is obvious that a high order expansion, up to $L = 6$, is required to accurately describe this surface which has a major to minor axis ratio of 1.5. We expect that even higher multipole expansions would be required for more distorted surfaces.

For $\mu > 1.36$, there are two separate and spherical trapped surfaces on the initial slice, centered off the origin at $z = \pm \coth \mu$. In the cases we have tested ($\mu = 2.0$ and 2.2), the solver is able to locate both the center and radius of the offset horizons to an accuracy of better than 0.06% using general expansions to any order, $L = 0$ to $L = 6$.

We also tested the finder on the black hole plus Brill wave initial data sets described in Chapter 5, restricting to axisymmetry. In this Appendix we use the symbol m_0 , instead of m , to represent the scale parameter, so as to avoid confusion with the spherical harmonic label m . In Figure B.3 we show the coordinate location of the apparent horizon for various parameters of the q -function (5.8). The sequence shown corresponds to different values of the Brill wave amplitude a , for the other parameters having the values $(b, w, n) = (0, 1, 2)$. Results from both the 2D and 3D calculations are shown. In the 3D case, we allowed searches up through $L = 4$, but for this case we include only the axisymmetric terms. In all cases shown, the finder was able to locate the correct surface to within a third of a grid zone in just 6 iterations.

Figure B.4 is a resolution convergence study of the case $a = -0.75$ in which the cell size is varied from $\Delta x = 0.6$, 0.3 and 0.15 with 16^3 , 32^3 and 64^3 grids respectively.

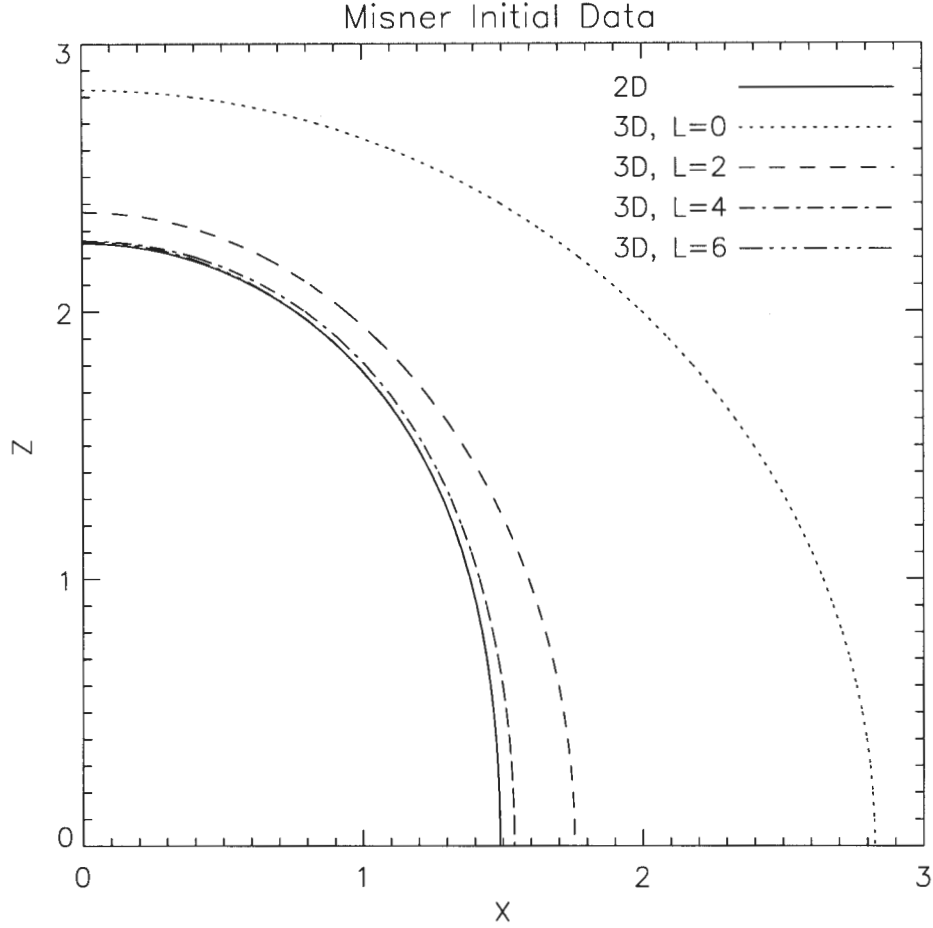


Figure B.2: The coordinate location of the apparent horizon for the $\mu = 1.2$ Misner initial data. The solid line is the apparent horizon computed from our 2D code. The various broken lines are the surfaces obtained from the different multipole order expansions. The surface obtained from the $L = 6$ expansion is indistinguishable from the 2D result in this plot. We note that terms up to $L = 6$ must be included to find the AH accurately.

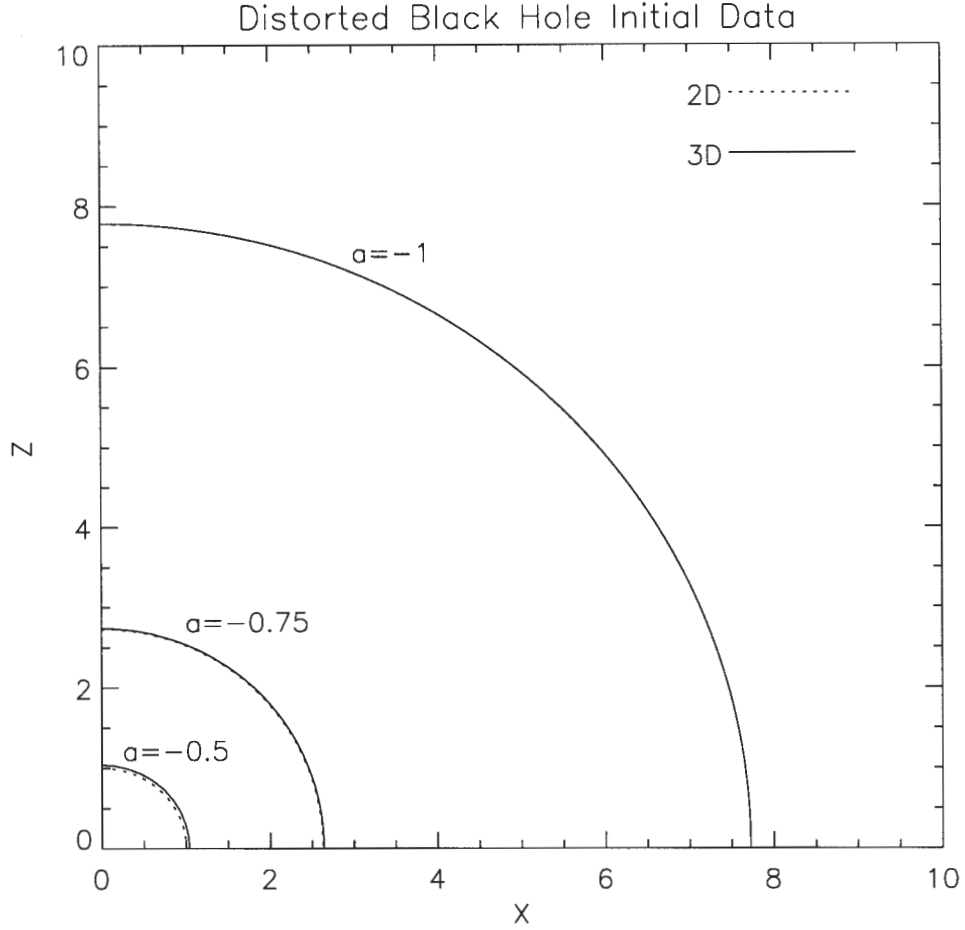


Figure B.3: The coordinate location of the apparent horizon for 2D and 3D Brill wave plus black hole initial data with various values of the Brill wave amplitude. In all runs, the Brill wave is centered at $b = 0$ and has a width of $w = 1$. Note that there is good agreement even though the initial data is not analytic and actually contains highly geometrically distorted black hole horizons.

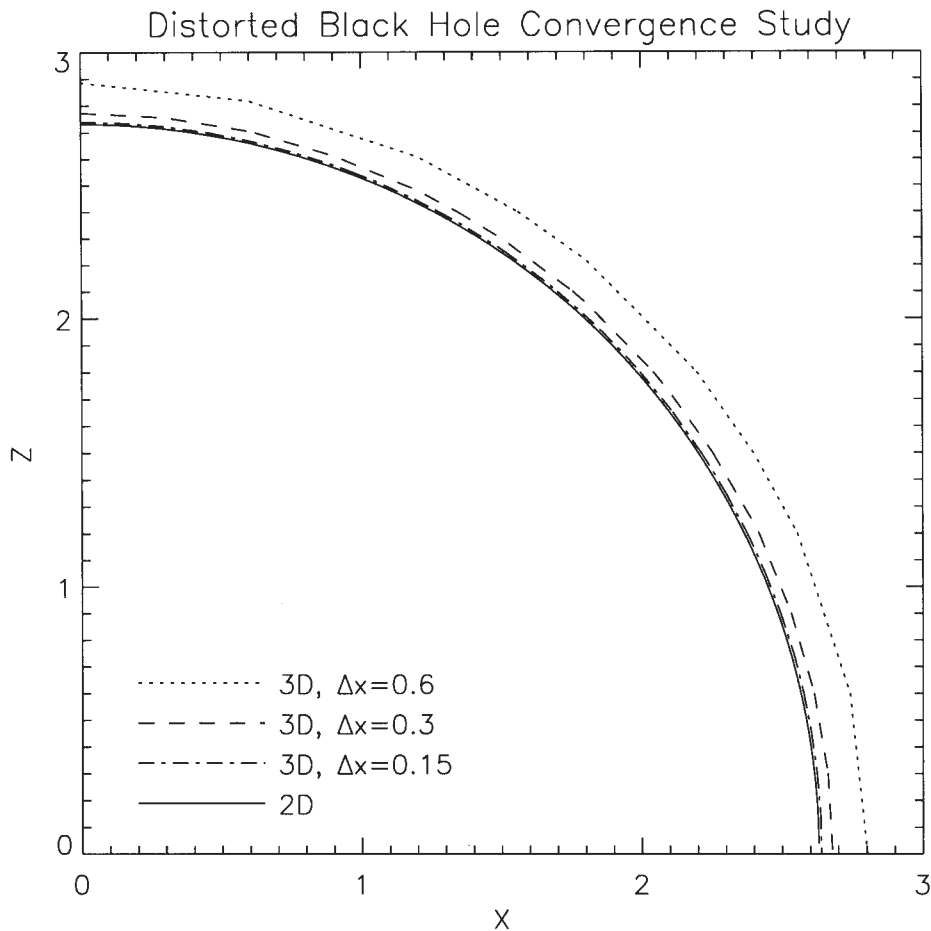


Figure B.4: A resolution study of the Brill wave plus black hole spacetime for the case $(a, \eta_0, \sigma) = (-0.75, 0, 1)$. The surface converges quadratically with grid spacing to the correct location, as represented by an independent 2D calculation shown by the solid line.

The solver clearly converges to the correct surface quadratically with cell size.

For negative values of the Brill wave amplitude parameter a , the horizon is found off the throat, but for positive values below certain limits (depending on the other parameters) the horizon is located on the throat at $r = m_0/2$. We are also able to locate the AH at the same level of accuracy for these cases.

The calculations presented so far have tested the ability of the solver to find single or multiple horizons of spherical and highly distorted black holes in 3D Cartesian coordinates. However, the data in all these cases are time symmetric. The Kerr

initial data set describing a rotating black hole has nontrivial extrinsic curvature, and thus provides another important testbed with a known analytic solution. The 3-metric for this spacetime in Boyer-Lindquist coordinates is given by

$$d\ell^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\left((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta\right) \sin^2 \theta}{\rho^2} d\phi^2, \quad (\text{B.8})$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad (\text{B.9})$$

$$\Delta = r^2 - 2Mr + a^2. \quad (\text{B.10})$$

In these coordinates, the non-vanishing components of the extrinsic curvature are

$$\hat{K}_{r\phi} = aM \left[2r^2 (r^2 + a^2) + \rho^2 (r^2 - a^2) \right] \sin^2 \theta / (r\rho^4), \quad (\text{B.11})$$

$$\hat{K}_{\theta\phi} = -2a^3 Mr \sqrt{\Delta} \cos \theta \sin^3 \theta \rho^{-4}, \quad (\text{B.12})$$

where $\hat{K}_{ij} = \psi^2 K_{ij}$, and

$$\psi^4 = \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\rho^2}. \quad (\text{B.13})$$

To construct this data in 3D, we first transform to an isotropic radial coordinate through the transformation

$$r = \bar{r} \left(1 + \frac{M + a}{2\bar{r}} \right) \left(1 + \frac{M - a}{2\bar{r}} \right), \quad (\text{B.14})$$

as described in [30]. In this coordinate system, the coordinate singularities at $r = M \pm \sqrt{M^2 - a^2}$ disappear. We then convert the metric and extrinsic curvature from isotropic coordinates to Cartesian coordinates. The apparent horizon for the Kerr data is a coordinate sphere, located at

$$\bar{r} = \frac{\sqrt{M^2 - a^2}}{2}. \quad (\text{B.15})$$

In Table B.1 we show results from runs with various values of the rotation parameter a/M and expansion order L . These runs were all done with general expansions,

without restricting to axisymmetry. For the tests performed here, Δx , Δy and Δz were chosen to be approximately one tenth of the analytic AH radius, and the number of cells (32^3) was kept the same in all cases. Only the $\ell = 0$ contributions are shown. As expected, the contributions from higher- ℓ terms are small, ranging from 10^{-9} for $a/M = 0$ to 3×10^{-2} for $a/M = 0.9$. The reason that these terms, in the highly rotating cases, are not as small as in the Schwarzschild case is because the metric is not conformally flat, so the interpolations within the finder are not as accurate. Although we do not display embeddings of these horizons, we note that rotating black hole horizons can be extremely oblate, and for rotation parameters $a/M > 0.866$ a global embedding into Euclidean space is not possible. The horizon finder has no difficulty in locating these surfaces to within a grid zone, although we note, again, the high computational cost for large order multipole expansions as indicated by the $a/M = 0.3$ sequence of calculations.

a/M	L	Δx	r_a	r_n	$\Delta r/\Delta x$	iterations	cpu
0	4	0.1	1.0	1.0002	0.003	30	2.4 hr
0.3	4	0.1	0.954	1.0095	0.556	37	1.0 hr
0.3	2	0.1	0.954	0.9562	0.022	9	6.7 min
0.3	0	0.1	0.954	0.9542	0.002	2	0.5 min
0.9	4	0.039	0.436	0.4748	0.997	27	0.5 hr

Table B.1: Performance measures of the horizon solver applied to the Kerr black hole data with various spin parameters a/M . Here L is the maximum multipole order, Δx is the 3D grid spacing, r_a is the analytical position of the horizon, r_n is the horizon location found by the solver, and $\Delta r/\Delta x$ is the difference between the analytic and numerical locations normalized to the grid spacing. We also show the number of iterations and cpu time required by the solver running on a 32-node partition of the CM5, using a grid with 32^3 cells and a 5×5 mesh for the surface.

The above test cases, although treated in full 3D, have all been axisymmetric. We also tested our finder on data which does not have any particular symmetry, and for which we can derive the correct solution in order to gauge the accuracy of the solver. To this end, we chose to find the apparent horizon on a Schwarzschild initial

slice using a coordinate system in which the horizon surface does not appear to be axisymmetric. The coordinate transformation we use is

$$\bar{r} = r f(\theta, \phi) = r \left(1 + \frac{1}{4} \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) \right)^{-1/2} \quad (\text{B.16})$$

where r is the radial isotropic coordinate. This $f(\theta, \phi)$ has surface modes of $L = 2$. The apparent horizon location is then defined by $\bar{r} = \frac{M}{2} f(\theta, \phi)$.

The solver was allowed to search coefficients with terms up to $L = 2$ and $L = 4$. In all cases for the $L = 2$ tests, the finder successfully found the horizon to high accuracy. Each non-zero coefficient was accurate to better than a tenth of one percent, and the largest value found for a coefficient that was supposed to be zero was less than 2% of the smallest non-zero component. When the finder was generalized to allow searches with coefficients up to $L = 4$, the horizon was again successfully found and to comparable accuracy. However with the large number of search parameters, some care was necessary in choosing the initial search direction of the Powell routine in order to avoid getting trapped at local minima in Θ^2 generated in the discretized representation of the spacetime. When the resolution of spacetime grid is increased, such local minima will disappear/change in location, in sharp contrast to the actual $\Theta^2 = 0$ point (the actual location of the AH, which converges to a fixed location with increased resolution.) In Figure B.5, we show the apparent horizon found and the analytic solution in the first quadrant of each of the coordinate planes. The numerical results shown in this Figure were obtained with a run in which coefficients up to $L = 4$ were allowed to vary.

We now turn our attention to the location of apparent horizons in evolved black hole spacetimes. In general 3D black hole evolutions, all metric and curvature components will be present, and the data will be contaminated with numerical inaccuracies generated during the course of evolution. The horizon finder must work under these conditions for it to be a useful tool in the numerical construction of spacetimes.

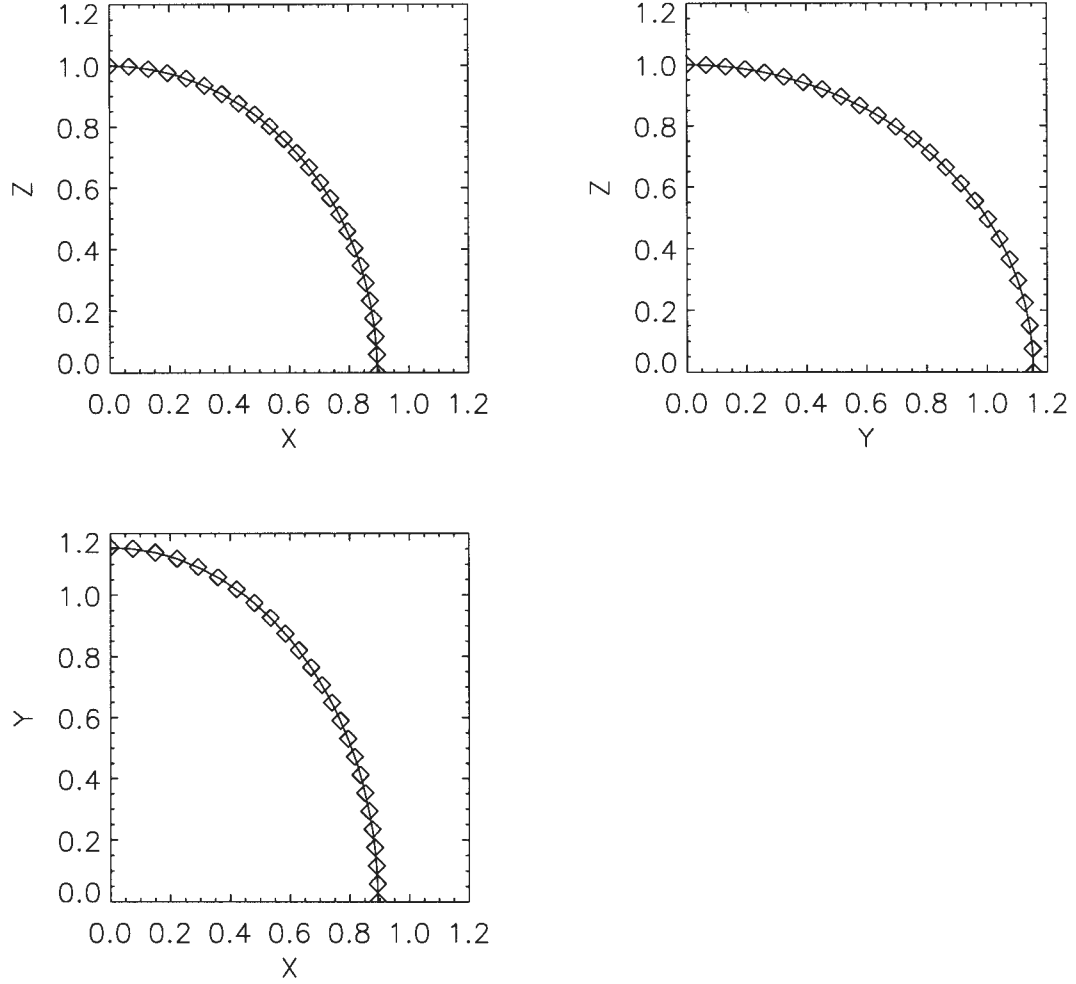


Figure B.5: Apparent horizon location in each of the coordinate planes for the transformed Schwarzschild initial data. The numerical data are represented by solid lines, and the analytic data by symbols. The surface is pure $L = 2$, although the finder was allowed to search through the $L = 4$ coefficients as well. The computed coefficients are found to roughly 0.1% accuracy.

The tests discussed here were carried out using multipole expansions up to and including the $L = 4$ terms. However, in order to save computational time, we restricted the search to axisymmetric surfaces. We have verified that a more general expansion does not change the results significantly.

The results for a 3D Schwarzschild spacetime, evolved with both geodesic and maximal slicings with zero shift, are shown in Figures B.6 and B.7 respectively. A 64^3 (130^3) grid with $\Delta x = 0.15$ (0.2) was used for the geodesic (maximal) case with $\Delta t = 0.25\Delta x$. Only the $\ell = 0$ contribution to the surface is plotted. The other parameters remain small ($< 10^{-3}$) during the entire evolutions, as expected. In both cases, the surface locations are compared against the corresponding results from 1D calculations. The two results agree to a small fraction of a grid zone throughout the evolution. The late time deviation in the maximal run is attributed to errors in the 3D spacetime evolution, which becomes inaccurate for $t > 20M$, although we note that the difference at the end of the run is still only about a grid zone.

A more difficult evolution scenario is to capture the horizon surfaces as two black holes collide and merge. Our 3D code was used to evolve the two black hole Misner data set, described in Section 4.1, for a time sufficiently long that we can test the AH finder on this dynamic spacetime. The results of this test are described in Section 4.3.3.

A final test of the solver in real time evolutions is on an axisymmetric distorted black hole. The results of this test are described in Section 6.2.3.

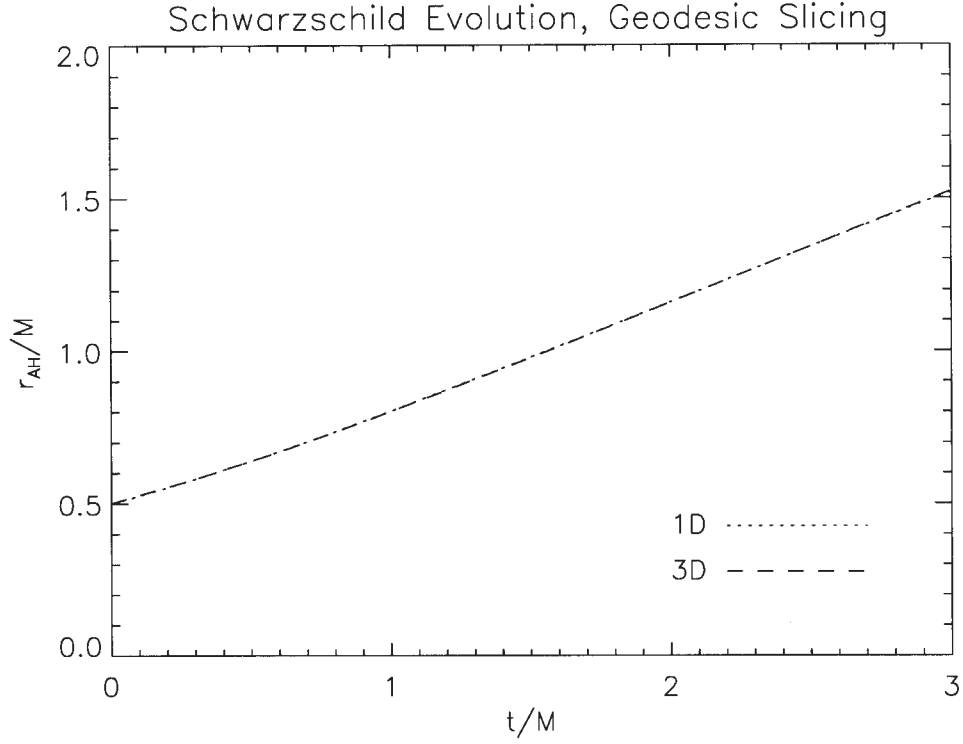


Figure B.6: The apparent horizon location for 1D and 3D Schwarzschild evolutions using geodesic slicing. The 1D data were obtained using 128 grid points and resolution $\Delta r = 0.0375M$, where M is the mass of the black hole. The 3D data were obtained using 64^3 grid points and resolution $\Delta x = 0.075M$. Only the $\ell = 0$ contribution to the 3D apparent horizon is plotted but the other terms in the $L = 4$ series are negligible, as discussed in the text.

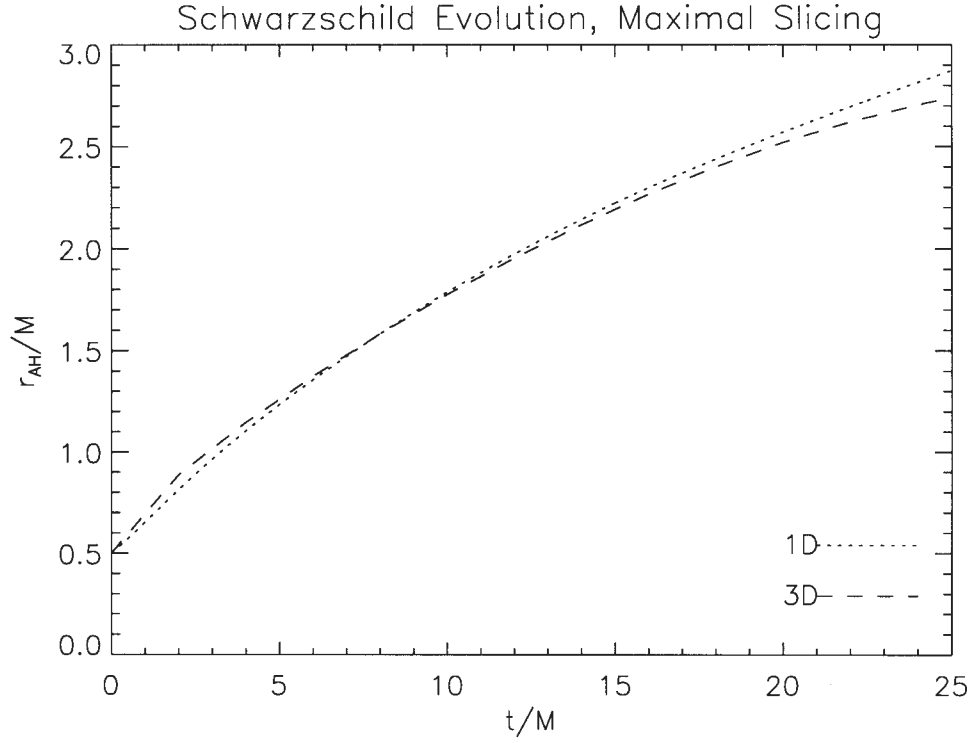


Figure B.7: The apparent horizon location for 1D and 3D Schwarzschild evolutions using maximal slicing. The 1D data were obtained using 130 grid points and resolution $\Delta r = 0.1M$, the 3D data using 130^3 grid zones and resolution $\Delta x = 0.1M$. We note that the agreement is within a grid zone even to the end of the calculation, when the 3D evolution becomes inaccurate.

Appendix C

Gauge Invariant Radiation Extraction in 3D

In this Appendix, we show the details of Moncrief's gauge invariant Zerilli function extraction, and our numerical implementation of it in 3D.

C.1 Constructing the Zerilli Function

As mentioned above, we assume the general metric can be decomposed into its spherical and non-spherical parts:

$$g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu}. \quad (\text{C.1})$$

The spherical part $\overset{\circ}{g}_{\mu\nu}$ will of course be Schwarzschild, but we will in general not know the mass of this Schwarzschild background, or what coordinate system it will be in. However, in general, we know it can be written

$$\overset{\circ}{g}_{\mu\nu} = \begin{pmatrix} -N^2 & 0 & 0 & 0 \\ 0 & A^2 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2 \sin^2 \theta \end{pmatrix} \quad (\text{C.2})$$

where the functions N , A , and R are functions of our coordinate radius and time. Regge and Wheeler showed that $h_{\mu\nu}$ for even-parity perturbations (*i.e.*, perturbations which do not introduce angular momentum) can be written

$$h_{tt} = -N^2 H_0^{(\ell m)} Y_{\ell m} \quad (\text{C.3})$$

$$h_{tr} = H_1^{(\ell m)} Y_{\ell m} \quad (C.4)$$

$$h_{t\theta} = h_0^{(\ell m)} Y_{\ell m, \theta} \quad (C.5)$$

$$h_{t\phi} = h_0^{(\ell m)} Y_{\ell m, \phi} \quad (C.6)$$

$$h_{rr} = A^2 H_2^{(\ell m)} Y_{\ell m} \quad (C.7)$$

$$h_{r\theta} = h_1^{(\ell m)} Y_{\ell m, \theta} \quad (C.8)$$

$$h_{r\phi} = h_1^{(\ell m)} Y_{\ell m, \phi} \quad (C.9)$$

$$h_{\theta\theta} = R^2 K^{(\ell m)} Y_{\ell m} + R^2 G^{(\ell m)} Y_{\ell m, \theta\theta} \quad (C.10)$$

$$h_{\theta\phi} = R^2 G^{(\ell m)} (Y_{\ell m, \theta\phi} - \cot \theta Y_{\ell m, \phi}) \quad (C.11)$$

$$h_{\phi\phi} = R^2 K^{(\ell m)} \sin^2 \theta Y_{\ell m} + R^2 G^{(\ell m)} (Y_{\ell m, \phi\phi} + \sin \theta \cos \theta Y_{\ell m, \theta}) \quad (C.12)$$

The spherical part of the metric, given in the functions N , A , and R , can be obtained by projecting the full metric against Y_{00} , yielding the following expressions¹:

$$N^2 = -\frac{1}{4\pi} \int g_{tt} d\Omega \quad (C.13)$$

$$A^2 = \frac{1}{4\pi} \int g_{rr} d\Omega \quad (C.14)$$

$$R^2 = \frac{1}{8\pi} \int \left(g_{\theta\theta} + \frac{g_{\phi\phi}}{\sin^2 \theta} \right) d\Omega \quad (C.15)$$

Each ℓm -mode of $h_{\mu\nu}$ can then be obtained by projecting the full metric against the appropriate $Y_{\ell m}$:

$$h_1^{(\ell m)} = \frac{1}{\ell(\ell+1)} \int \left(g_{r\theta} Y_{\ell m, \theta}^* + \frac{g_{r\phi}}{\sin^2 \theta} Y_{\ell m, \phi}^* \right) d\Omega \quad (C.16)$$

$$H_2^{(\ell m)} = \frac{1}{A^2} \int g_{rr} Y_{\ell m}^* d\Omega \quad (C.17)$$

$$G^{(\ell m)} = \frac{1}{R^2 \ell(\ell+1)(\ell-1)(\ell+2)} \int \left[\left(g_{\theta\theta} - \frac{g_{\phi\phi}}{\sin^2 \theta} \right) (Y_{\ell m, \theta\theta}^* - \cot \theta Y_{\ell m, \theta}^* - \frac{1}{\sin^2 \theta} Y_{\ell m, \phi\phi}^*) + \frac{4}{\sin^2 \theta} g_{\theta\phi} (Y_{\ell m, \theta\phi}^* - \cot \theta Y_{\ell m, \phi}^*) \right] d\Omega \quad (C.18)$$

$$K^{(\ell m)} = \frac{\ell(\ell+1)}{2} G^{(\ell m)} + \frac{1}{2R^2} \int \left(g_{\theta\theta} + \frac{g_{\phi\phi}}{\sin^2 \theta} \right) Y_{\ell m}^* d\Omega \quad (C.19)$$

¹We thank Gabrielle Allen for pointing out the error in the expression for R^2 in reference [1].

In practice, we do not extract A to compute $H_2^{(\ell m)}$, but rather we assume it to have the form $1 - \frac{2M}{R}$, where we take M to be the ADM mass of the spacetime.

Moncrief showed that the Zerilli function is gauge invariant in the sense discussed in Chapter 7, and can be constructed from the Regge-Wheeler variables as follows:

$$\psi^{(\ell m)} = \sqrt{\frac{2(\ell-1)(\ell+2)}{\ell(\ell+1)}} \frac{4RS^2 k_2^{(\ell m)} + \ell(\ell+1)Rk_1^{(\ell m)}}{\Lambda}, \quad (\text{C.20})$$

where

$$\Lambda \equiv \ell(\ell+1) - 2 + \frac{6M}{R} \quad (\text{C.21})$$

$$k_1^{(\ell m)} \equiv K^{(\ell m)} + SRG_{,R}^{(\ell m)} - 2\frac{S}{R}h_1^{(\ell m)} \quad (\text{C.22})$$

$$k_2^{(\ell m)} \equiv \frac{H_2^{(\ell m)}}{2S} - \frac{1}{2\sqrt{S}} \frac{\partial}{\partial R} \left(\frac{RK^{(\ell m)}}{\sqrt{S}} \right) \quad (\text{C.23})$$

$$S \equiv 1 - \frac{2M}{R}. \quad (\text{C.24})$$

C.2 Implementation in 3D

In order to compute the Regge-Wheeler perturbation functions h_1 , H_2 , G , and K , one needs the spherical metric functions on some 2-sphere. We get these by interpolating the Cartesian metric functions onto a surface of constant coordinate radius, and computing the spherical metric functions from these using the standard transformation. Second order interpolation is used. In order to compute the needed radial derivatives of these functions, we first compute the derivative of the Cartesian metric functions with respect to the coordinate radius on the Cartesian grid. We interpolate these quantities onto the 2-sphere. Then, to get the derivatives with respect to the Schwarzschild radius R , we use the derivative of eqn. C.15 with respect to the coordinate r , giving

$$\frac{\partial R}{\partial r} = \frac{1}{16\pi R} \int \left(g_{\theta\theta,r} + \frac{g_{\phi\phi,r}}{\sin^2 \theta} \right) d\Omega \quad (\text{C.25})$$

The points on the 2-sphere to which we interpolate lie on lines of constant θ and ϕ , staggered across the coordinate axes. Integrations over each of these variables is

performed with the second-order accurate Simpson's rule. Currently, 300 points are used for each angular coordinate.

We are aware of two possible improvements on this scheme. First, in the θ -integration, we could integrate over $\cos\theta$ instead of θ . This would make these integrations more accurate, and would allow us to use fewer points, making the code more efficient. Second, we could interpolate onto a stereographic patch. This would distribute the points more evenly on the surface.

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Vita

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