SOME SIMPLE EXPLICIT RESULTS FOR THE ELASTIC DIELECTRIC PROPERTIES AND STABILITY OF LAYERED COMPOSITES

BY

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THESIS

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ABSTRACT

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A string of partial results — aimed at shedding light on the behavior of dielectric elastomer composites — have been recently reported in the literature for the macroscopic electroelastic response and stability of layered composites with ideal elastic dielectric phases. Such results have been restricted to two phases and plane-strain loading conditions. It is the purpose of this thesis to place on record *simple explicit* expressions for the macroscopic electroelastic response and stability of layered composites with any number of ideal elastic dielectric phases under general electromechanical loading conditions. *Inter alia*, these expressions provide insight into a variety of practical and theoretical issues in relation to the modeling of elastic dielectric composites with anisotropic microstructures, ranging from the choice of invariants to describe their free energy function to the effects of interphasial phenomena.

This thesis also places on record the conditions of ordinary and strong ellipticity for elastic dielectrics in full generality.

To my Parents,

for their unconditional love and endless support

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CHAPTER 1

INTRODUCTION

Over the last decade, a number of experiments have demonstrated that dielectric elastomer composites — comprising, in essence, a mechanically soft matrix of low permittivity filled with high-permittivity inclusions — hold great potential to enable a broad range of new technologies (see, e.g., Huang et al., 2005; Zhang et al., 2007). To aid in the microscopic understanding of this class of emerging materials, various theoretical results have been recently reported in the literature for the macroscopic electroelastic response and stability of layered composites with ideal elastic dielectric phases (see, e.g., DeBotton et al., 2007; Bertoldi and Gei, 2011; Rudykh, 2011; Tian et al., 2012). All such results have been restricted to two phases and plane-strain loading conditions.

The purpose of this thesis is to place on record *simple explicit* expressions for the macroscopic electroelastic response and stability of layered composites with any number of ideal elastic dielectric phases under general electromechanical loading conditions. The stability results presented here pertain exclusively to "macroscopic" stability results[1], as characterized by the loss of strong ellipticity of the free energy function of the composites. In this connection, this thesis also places on record the conditions of ordinary and strong ellipticity for elastic dielectrics in full generality.

The thesis is organized as follows. Chapter 2 formulates the electroelastostatics problem defining the macroscopic response of layered composites with any number of elastic dielectric phases under arbitrarily large electromechanical loads. Section 2.3 presents the conditions of ordinary and strong ellipticity for elastic dielectrics; the derivation of these conditions is deferred to the Appendix. In Chapter 3, the generic problem formulated in Chapter 2 is specialized and solved explicitly for the specific case of layered composites

[1]For completeness, a brief discussion is also included on other types of instabilities, such as "microscopic" instabilities, cavitation, debonding, and electric breakdown.

with ideal elastic dielectric phases. The specialization of the ellipticity conditions of Section 2.3 to such layered composites is worked out in Section 3.2. This section also includes the resulting criticality condition that defines the electromechanical loads at which macroscopic instabilities ensue. Chapter 4 provides a summary of the results generated in Chapter 3 and Section 3.2 in terms of the electric field as the independent electric variable, instead of the electric displacement field. Finally, Chapter 5 provides several remarks on practical and theoretical implications of the main results of this thesis.

CHAPTER 2

MACROSCOPIC RESPONSE OF ELASTIC DIELECTRIC LAYERED COMPOSITES

2.1 Microscopic description of elastic dielectric layered composites

Consider a composite material made up of perfectly bonded layers of an arbitrarily large number M of different phases with initial layering (or lamination) direction $\mathbf{N}$. The domain occupied by the entire composite in its ground state is denoted by Ω_0 and its boundary by $\partial\Omega_0$. Similarly, $\Omega_0^{(r)}$ ($r = 1, 2, ..., M$) denote the domains occupied collectively by the individual phases so that $\Omega_0 = \Omega_0^{(1)} \cup \Omega_0^{(2)} \cup ... \cup \Omega_0^{(M)}$ and their respective initial volume fractions are given by $c_0^{(r)} \doteq |\Omega_0^{(r)}|/|\Omega_0|$. We assume that the distribution of the phases is statistically uniform, the thicknesses of the layers are much smaller than the size of Ω_0, and, for convenience, choose units of length so that Ω_0 has unit volume.

Upon the application of mechanical and electrical stimuli, the initial position vector $\mathbf{X}$ of a material point in Ω_0 moves to a new position specified by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, where $\boldsymbol{\chi}$ is a one-to-one mapping from Ω_0 to the deformed configuration Ω. We assume that $\boldsymbol{\chi}$ is twice continuously differentiable, except possibly on the layer-to-layer interfaces. The associated deformation gradient is denoted by $\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}$ and its determinant by $J = \det \mathbf{F}$.

All M phases in the composite are elastic dielectrics. We find it convenient to characterize their constitutive behaviors in a Lagrangian formulation by "total" free energies $W^{(r)}$ (suitably amended to include contributions from the Maxwell stress) per unit undeformed volume, as introduced by Dorfmann and Ogden (2005). For clarity of presentation, we make use of the deformation gradient $\mathbf{F}$ and Lagrangian electric displacement field $\mathbf{D}$ as the independent variables up until Section 4, where, for completeness, we provide a summary of the results in terms of $\mathbf{F}$ and the Lagrangian electric field $\mathbf{E}$.

Thus, taking $\mathbf{F}$ and $\mathbf{D}$ as the independent variables, the first Piola-Kirchhoff stress tensor $\mathbf{S}$ and Lagrangian electric field $\mathbf{E}$ are simply given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{X}, \mathbf{F}, \mathbf{D}) \qquad \text{and} \qquad \mathbf{E} = \frac{\partial W}{\partial \mathbf{D}}(\mathbf{X}, \mathbf{F}, \mathbf{D}), \tag{2.1}$$

where

$$W(\mathbf{X}, \mathbf{F}, \mathbf{D}) = \sum_{r=1}^{M} \theta_0^{(r)}(\mathbf{X}) W^{(r)}(\mathbf{F}, \mathbf{D}) \tag{2.2}$$

with $\theta_0^{(r)}(\mathbf{X})$ denoting the characteristic function of the regions occupied by phase r: $\theta_0^{(r)}(\mathbf{X}) = 1$ if $\mathbf{X} \in \Omega_0^{(r)}$ and zero otherwise. Note that the total Cauchy stress, Eulerian electric field, and polarization (per unit deformed volume) fields are in turn given by $\mathbf{T} = J^{-1}\mathbf{S}\,\mathbf{F}^T$, $\mathbf{e} = \mathbf{F}^{-T}\mathbf{E}$, and $\mathbf{p} = J^{-1}\mathbf{F}\mathbf{D} - \varepsilon_0\mathbf{e}$, where ε_0 stands for the permittivity of vacuum.

2.2 The macroscopic response

In view of the assumed separation of length scales and statistical uniformity of the microstructure, the above-defined elastic dielectric layered composite — though microscopically heterogeneous — is expected to behave macroscopically as a "homogenous" material. Following Hill (1972), its macroscopic or overall response is defined as the relation between the volume averages of the first Piola-Kirchhoff stress $\mathbf{S}$ and electric field $\mathbf{E}$ and the volume averages of the deformation gradient $\mathbf{F}$ and electric displacement $\mathbf{D}$ over the undeformed configuration Ω_0 under affine boundary conditions. Consistent with our choice of $\mathbf{F}$ and $\mathbf{D}$ as the independent variables, we consider the following boundary conditions

$$\mathbf{x} = \overline{\mathbf{F}}\mathbf{X} \qquad \text{and} \qquad \mathbf{D} \cdot \boldsymbol{\xi} = \overline{\mathbf{D}} \cdot \boldsymbol{\xi} \quad \text{on} \quad \partial\Omega_0, \tag{2.3}$$

where the second-order tensor $\overline{\mathbf{F}}$ and vector $\overline{\mathbf{D}}$ stand for prescribed boundary data, and where $\boldsymbol{\xi}$ is the outward normal to $\partial\Omega_0$. Granted relations (2.3), the divergence theorem warrants that $\int_{\Omega_0} \mathbf{F}(\mathbf{X})\mathrm{d}\mathbf{X} = \overline{\mathbf{F}}$ and $\int_{\Omega_0} \mathbf{D}(\mathbf{X})\mathrm{d}\mathbf{X} = \overline{\mathbf{D}}$ and hence the derivation of the macroscopic response reduces to finding the average stress $\overline{\mathbf{S}} \doteq \int_{\Omega_0} \mathbf{S}(\mathbf{X})\mathrm{d}\mathbf{X}$ and average electric field $\overline{\mathbf{E}} \doteq \int_{\Omega_0} \mathbf{E}(\mathbf{X})\mathrm{d}\mathbf{X}$. In direct analogy with the purely mechanical problem, the result can be expediently written as

$$\overline{\mathbf{S}} = \frac{\partial \overline{W}}{\partial \overline{\mathbf{F}}}(\overline{\mathbf{F}}, \overline{\mathbf{D}}) \qquad \text{and} \qquad \overline{\mathbf{E}} = \frac{\partial \overline{W}}{\partial \overline{\mathbf{D}}}(\overline{\mathbf{F}}, \overline{\mathbf{D}}), \tag{2.4}$$

where

$$\overline{W}(\overline{\mathbf{F}}, \overline{\mathbf{D}}) = \min_{\mathbf{F}} \min_{\mathbf{D}} \int_{\Omega_0} W(\mathbf{X}, \mathbf{F}, \mathbf{D}) \mathrm{d}\mathbf{X} \tag{2.5}$$

corresponds physically to the total electroelastic energy (per unit undeformed volume) stored in the composite material.

A standard calculation — making use of the facts that $\mathbf{F}$ is the gradient of a vector field and that $\mathbf{D}$ is divergence-free — serves to show that the Euler-Lagrange equations associated with the variational problem (2.5) are nothing more than the equations of conservation of linear momentum and Faraday's law. It is also a simple matter to show that $\overline{\mathbf{T}} = \overline{J}^{-1} \overline{\mathbf{S}}\,\overline{\mathbf{F}}^T$, $\overline{\mathbf{e}} = \overline{\mathbf{F}}^{-T} \overline{\mathbf{E}}$, and $\overline{\mathbf{p}} = \overline{J}^{-1} \overline{\mathbf{F}}\,\overline{\mathbf{D}} - \varepsilon_0 \overline{\mathbf{e}}$, where $\overline{\mathbf{T}} \doteq |\Omega|^{-1} \int_\Omega \mathbf{T}(\mathbf{x}) \mathrm{d}\mathbf{x}$, $\overline{\mathbf{e}} \doteq |\Omega|^{-1} \int_\Omega \mathbf{e}(\mathbf{x}) \mathrm{d}\mathbf{x}$, and $\overline{\mathbf{p}} \doteq |\Omega|^{-1} \int_\Omega \mathbf{p}(\mathbf{x}) \mathrm{d}\mathbf{x}$ are the volume averages of the total Cauchy stress $\mathbf{T}$, Eulerian electric field $\mathbf{e}$, and polarization $\mathbf{p}$ over the deformed configuration Ω, and where use has been made of the notation $\overline{J} = \det \overline{\mathbf{F}}$. That is, as a consequence of the above choice of macrovariables, the resulting relations among Lagrangian and Eulerian macroscopic quantities are completely analogous to the corresponding local relations.

For any continuous loading path with the ground state ($\overline{\mathbf{F}} = \mathbf{I}$, $\overline{\mathbf{D}} = \mathbf{0}$ and $\overline{\mathbf{S}} = \mathbf{0}$, $\overline{\mathbf{E}} = \mathbf{0}$) as starting point, the solution of the Euler-Lagrange equations associated with the variational problem (2.5) is *uniquely* given by deformation gradient and electric displacement fields that are *piecewise uniform* up until the onset of a first instability (see, e.g., Geymonat et al., 1993). Namely, up until the onset of a first instability, the effective free energy function (2.5) takes the form

$$\overline{W}(\overline{\mathbf{F}}, \overline{\mathbf{D}}) = \min_{\boldsymbol{\alpha}^{(r)}} \min_{\boldsymbol{\beta}^{(r)}} \sum_{r=1}^{M} c_0^{(r)} W^{(r)}(\overline{\mathbf{F}}^{(r)}, \overline{\mathbf{D}}^{(r)}) \tag{2.6}$$

with

$$\mathbf{F}(\mathbf{X}) = \begin{cases} \overline{\mathbf{F}}^{(1)} = \overline{\mathbf{F}} + \boldsymbol{\alpha}^{(1)} \otimes \mathbf{N} & \text{if } \mathbf{X} \in \Omega_0^{(1)} \\ \overline{\mathbf{F}}^{(2)} = \overline{\mathbf{F}} + \boldsymbol{\alpha}^{(2)} \otimes \mathbf{N} & \text{if } \mathbf{X} \in \Omega_0^{(2)} \\ \vdots & \\ \overline{\mathbf{F}}^{(M)} = \overline{\mathbf{F}} + \boldsymbol{\alpha}^{(M)} \otimes \mathbf{N} & \text{if } \mathbf{X} \in \Omega_0^{(M)} \end{cases} \tag{2.7}$$

and

$$\mathbf{D}(\mathbf{X}) = \begin{cases} \overline{\mathbf{D}}^{(1)} = \overline{\mathbf{D}} + (\mathbf{I} - \mathbf{N} \otimes \mathbf{N})\boldsymbol{\beta}^{(1)} & \text{if } \mathbf{X} \in \Omega_0^{(1)} \\ \overline{\mathbf{D}}^{(2)} = \overline{\mathbf{D}} + (\mathbf{I} - \mathbf{N} \otimes \mathbf{N})\boldsymbol{\beta}^{(2)} & \text{if } \mathbf{X} \in \Omega_0^{(2)} \\ \vdots \\ \overline{\mathbf{D}}^{(M)} = \overline{\mathbf{D}} + (\mathbf{I} - \mathbf{N} \otimes \mathbf{N})\boldsymbol{\beta}^{(M)} & \text{if } \mathbf{X} \in \Omega_0^{(M)} \end{cases}, \qquad (2.8)$$

where $\boldsymbol{\alpha}^{(r)}$ and $\boldsymbol{\beta}^{(r)}$ $(r = 1, 2, ..., M)$ are constant vectors (i.e., independent of $\mathbf{X}$) subject to the constraints[1]

$$\sum_{r=1}^{M} c_0^{(r)} \boldsymbol{\alpha}^{(r)} = \mathbf{0} \qquad \text{and} \qquad \sum_{r=1}^{M} c_0^{(r)} \boldsymbol{\beta}^{(r)} = \mathbf{0}. \qquad (2.9)$$

Note that the Euler-Lagrange equations associated with (2.6) are now a system of nonlinear *algebraic* equations for $\boldsymbol{\alpha}^{(r)}$ and $\boldsymbol{\beta}^{(r)}$ $(r = 1, 2, ..., M)$ corresponding to the continuity of the tractions ($[\![\mathbf{S}]\!]\mathbf{N} = \mathbf{0}$) and the tangential continuity of the electric field ($[\![\mathbf{E}]\!] \wedge \mathbf{N} = \mathbf{0}$) across layer-to-layer interfaces.

For sufficiently large electromechanical loads $\overline{\mathbf{F}}$ and $\overline{\mathbf{D}}$, the piecewise uniform solution (2.7)–(2.8) may bifurcate into multiple solutions with different total electroelastic energies. Physically, such bifurcations correspond to the onset instabilities. Following the work of Triantafyllidis and collaborators (see, e.g., Triantafyllidis and Maker, 1985; Geymonat et al., 1993), it is useful to make the distinction between "microscopic" instabilities, that is, instabilities with wavelengths that are of the order of the thicknesses of the layers, and "macroscopic" instabilities, that is, instabilities with wavelengths that are much larger than the thicknesses of the layers. The computation of microscopic instabilities is in general a difficult task, though, if the spatial distribution of the M different phases happens to be periodic, microscopic instabilities may be computed elegantly with help of the Floquet theory for ordinary differential equations (Triantafyllidis and Maker, 1985). On the other hand, the computation of macroscopic instabilities is a much simpler endeavor, since it is expected to be signaled — much like for the purely mechanical case (Geymonat et al., 1993; Bertoldi and Gei, 2011) — by the loss of strong ellipticity of the effective free energy function (2.6) associated with

[1] Constraints (2.9) warrant that $\int_{\Omega_0} \mathbf{F}(\mathbf{X})\, \mathrm{d}\mathbf{X} = \sum_{r=1}^{M} c_0^{(r)} \overline{\mathbf{F}}^{(r)} = \overline{\mathbf{F}}$ and $\int_{\Omega_0} \mathbf{D}(\mathbf{X})\, \mathrm{d}\mathbf{X} = \sum_{r=1}^{M} c_0^{(r)} \overline{\mathbf{D}}^{(r)} = \overline{\mathbf{D}}$, as required by the affine boundary conditions (2.3).

the "principal" piecewise uniform solution (2.7)–(2.8).

As anticipated in the Introduction, the primary objective of this work is to generate an explicit result for the effective free energy function (2.6) of a layered composite with a specific class of elastic dielectric phases: the so-called ideal elastic dielectrics. And to establish explicit conditions at which such an effective energy loses strong ellipticity, and thus at which macroscopic instabilities may ensue in the composite. Before proceeding with the pertinent calculations, we dedicate the next section to present in full generality the conditions of ordinary and strong ellipticity for elastic dielectrics, as they will be required in later sections.

2.3 The conditions of ordinary and strong ellipticity for elastic dielectrics

Following a derivation akin to the standard derivation in the purely mechanical case (see, e.g., Hill, 1979; Chapter 6.2.7 in Ogden, 1997 and references therein), the "generalized" acoustic tensor of an unconstrained elastic dielectric with *arbitrary* effective free energy function $\overline{W} = \overline{W}(\overline{\mathbf{F}}, \overline{\mathbf{D}})$ can be shown to be given by

$$\mathbf{\Gamma}(\mathbf{u}; \overline{\mathbf{F}}, \overline{\mathbf{D}}) = \mathbf{K} - \frac{2}{(\mathrm{tr}\,\widehat{\mathbf{B}})^2 - \mathrm{tr}\,\widehat{\mathbf{B}}^2}\,\mathbf{R}\left[(\mathrm{tr}\,\widehat{\mathbf{B}})\,\widehat{\mathbf{I}} - \widehat{\mathbf{B}}\right]\mathbf{R}^T, \qquad (2.10)$$

where $\mathbf{K} = \mathbf{K}(\mathbf{u}; \overline{\mathbf{F}}, \overline{\mathbf{D}})$, the acoustic tensor, $\mathbf{R} = \mathbf{R}(\mathbf{u}; \overline{\mathbf{F}}, \overline{\mathbf{D}})$, the electro-acoustic tensor, and the projection tensor $\widehat{\mathbf{I}} = \widehat{\mathbf{I}}(\mathbf{u})$ and projected impermeability tensor $\widehat{\mathbf{B}} = \widehat{\mathbf{B}}(\mathbf{u}; \overline{\mathbf{F}}, \overline{\mathbf{D}})$ are given (in component form) in terms of the unit vector $\mathbf{u}$ by

$$\begin{aligned}
K_{ik} &= \overline{\mathcal{L}}_{ijkl} u_j u_l, \\
R_{ik} &= \overline{\mathcal{M}}_{ijk} u_j, \\
\widehat{I}_{ik} &= \delta_{ik} - u_i u_k, \\
\widehat{B}_{ik} &= (\delta_{ip} - u_i u_p)\overline{\mathcal{B}}_{pq}(\delta_{qk} - u_q u_k)
\end{aligned} \qquad (2.11)$$

with

$$
\begin{aligned}
\overline{\mathcal{L}}_{ijkl} &= \overline{J}^{\,-1}\overline{F}_{ja}\overline{F}_{lb}\frac{\partial^2\overline{W}}{\partial\overline{F}_{ia}\partial\overline{F}_{kb}}(\overline{\mathbf{F}},\overline{\mathbf{D}}),\\
\overline{\mathcal{M}}_{ijk} &= \overline{F}_{ja}\overline{F}^{-1}_{bk}\frac{\partial^2\overline{W}}{\partial\overline{F}_{ia}\partial\overline{D}_b}(\overline{\mathbf{F}},\overline{\mathbf{D}}),\\
\overline{\mathcal{B}}_{ij} &= \overline{J}\,\overline{F}^{-1}_{ai}\overline{F}^{-1}_{bj}\frac{\partial^2\overline{W}}{\partial\overline{D}_a\partial\overline{D}_b}(\overline{\mathbf{F}},\overline{\mathbf{D}})
\end{aligned}
\tag{2.12}
$$

denoting the components of the incremental moduli of the elastic dielectric in updated Lagrangian form, i.e., using as reference configuration the current configuration with deformation gradient $\overline{\mathbf{F}}$ and electric displacement $\overline{\mathbf{D}}$. A brief derivation of the above result is provided in the appendix.

Having established the generalized acoustic tensor (2.10), the definitions of ordinary and strong ellipticity for elastic dielectrics follow readily (see the appendix). Thus, an elastic dielectric with effective free energy function $\overline{W}=\overline{W}(\overline{\mathbf{F}},\overline{\mathbf{D}})$ is said to be *elliptic* if its acoustic tensor is nonsingular:

$$
\det\boldsymbol{\Gamma}(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}})\neq 0 \tag{2.13}
$$

for all unit vectors $\mathbf{u}$ and all $\overline{\mathbf{F}}$, $\overline{\mathbf{D}}$. Furthermore, an elastic dielectric with effective free energy function $\overline{W}=\overline{W}(\overline{\mathbf{F}},\overline{\mathbf{D}})$ is said to be *strongly elliptic* if its acoustic tensor is positive definite:

$$
\mathbf{v}\cdot\boldsymbol{\Gamma}(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}})\mathbf{v}>0 \tag{2.14}
$$

for all unit vectors $\mathbf{u}$, $\mathbf{v}$ and all $\overline{\mathbf{F}}$, $\overline{\mathbf{D}}$. Similar to the purely mechanical case, strong ellipticity (2.14) implies ellipticity (2.13), but the converse is not true in general.

2.3.1 Incompressible materials

When dealing with soft dielectrics, such as elastomers, it is often assumed that they are incompressible. As detailed in the appendix, the generalized acoustic tensor of an incompressible elastic dielectric, with arbitrary effective free energy function of the form $\overline{W}=\overline{W}(\overline{\mathbf{F}},\overline{\mathbf{D}})$ if $\overline{J}=1$ and $\overline{W}=+\infty$

otherwise, adopts the more specialized form

$$\widehat{\mathbf{\Gamma}}(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,) = \widehat{\mathbf{K}} - \frac{2}{(\operatorname{tr}\widehat{\mathbf{B}})^2 - \operatorname{tr}\widehat{\mathbf{B}}^2}\,\widehat{\mathbf{R}}\left[(\operatorname{tr}\widehat{\mathbf{B}})\,\widehat{\mathbf{I}} - \widehat{\mathbf{B}}\right]\widehat{\mathbf{R}}^T \tag{2.15}$$

with $\widehat{\mathbf{K}} = \widehat{\mathbf{I}}\,\mathbf{K}\,\widehat{\mathbf{I}}$ and $\widehat{\mathbf{R}} = \widehat{\mathbf{I}}\,\mathbf{R}\,\widehat{\mathbf{I}}$, where $\mathbf{K}$, $\mathbf{R}$, $\widehat{\mathbf{I}}$, $\widehat{\mathbf{B}}$ are given by expressions (2.11)–(2.12) with $\overline{J} = 1$. It then follows (see the appendix) that an incompressible elastic dielectric is *elliptic* if its acoustic tensor is nonsingular on the two-dimensional space normal to $\mathbf{u}$:

$$\left(\operatorname{tr}\widehat{\mathbf{\Gamma}}(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,)\right)^2 - \operatorname{tr}\widehat{\mathbf{\Gamma}}^2(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,) \neq 0 \tag{2.16}$$

for all unit vectors $\mathbf{u}$ and all $\overline{\mathbf{F}}$ with $\overline{J} = 1$, $\overline{\mathbf{D}}$. Similarly, an incompressible elastic dielectric is *strongly elliptic* if its acoustic tensor is positive definite on the two-dimensional space normal to $\mathbf{u}$:

$$\mathbf{v} \cdot \widehat{\mathbf{\Gamma}}(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,)\mathbf{v} > 0 \tag{2.17}$$

for all unit vectors $\mathbf{u}$, $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ and all $\overline{\mathbf{F}}$ with $\overline{J} = 1$, $\overline{\mathbf{D}}$.

CHAPTER 3

EXPLICIT RESULTS FOR IDEAL DIELECTRIC PHASES

3.1 Local fields, macroscopic response, and incremental moduli

In the sequel, we work out specific results for a layered composite with phases that are ideal elastic dielectrics characterized by the free energy functions

$$W^{(r)}(\mathbf{F},\mathbf{D})=\begin{cases}\dfrac{\mu^{(r)}}{2}\left(\mathbf{F}\cdot\mathbf{F}-3\right)+\dfrac{1}{2\,\varepsilon^{(r)}}\mathbf{F}\mathbf{D}\cdot\mathbf{F}\mathbf{D} & \text{if} \quad J=1\\ +\infty & \text{otherwise}\end{cases} \tag{3.1}$$

$r=1,2,...,M$. In this expression, the material parameters $\mu^{(r)}>0$ and $\varepsilon^{(r)}=(1+\chi^{(r)})\varepsilon_0>0$ stand for the shear modulus and permittivity of phase r in its ground state with $\chi^{(r)}$ denoting its electric susceptibility. The elastic dielectric described by (3.1) is referred to as "ideal" in the sense that it is mechanically a Gaussian rubber whose polarization $\mathbf{p}$ remains linearly proportional to the underlying Eulerian electric field $\mathbf{e}$ independently of the applied deformation: $\mathbf{p}=\mathbf{F}\mathbf{D}-\varepsilon_0\mathbf{F}^{-T}\partial W^{(r)}(\mathbf{F},\mathbf{D})/\partial\mathbf{D}=\varepsilon_0\chi^{(r)}\mathbf{F}^{-T}\mathbf{E}=\varepsilon_0\chi^{(r)}\mathbf{e}$. In addition to its theoretical appeal and mathematical simplicity, the model (3.1) has been shown to describe reasonably well the electromechanical response of a variety of soft dielectrics over small-to-moderate ranges of deformations and large ranges of electric fields (see, e.g., Kofod et al., 2003; Wissler and Mazza, 2007).

Local fields For free energy functions (3.1), the minimizing vectors $\boldsymbol{\alpha}^{(r)}$ and $\boldsymbol{\beta}^{(r)}$ in (2.6) can be determined explicitly. They read as

$$\boldsymbol{\alpha}^{(r)}=\left(\frac{\overline{\mu}_R}{\mu^{(r)}}-1\right)\overline{\mathbf{F}}\mathbf{N}-\frac{\dfrac{\overline{\mu}_R}{\mu^{(r)}}-1}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\overline{\mathbf{F}}^{-T}\mathbf{N} \tag{3.2}$$

and

$$\boldsymbol{\beta}^{(r)} = \left(\frac{\varepsilon^{(r)}}{\overline{\varepsilon}_V} - 1\right)\overline{\mathbf{D}} - \left(\frac{\overline{\mu}_R}{\mu^{(r)}} - 1\right)(\overline{\mathbf{D}}\cdot\mathbf{N})\mathbf{N} + \left(\frac{\overline{\mu}_R}{\mu^{(r)}} - \frac{\varepsilon^{(r)}}{\overline{\varepsilon}_V}\right)\frac{\overline{\mathbf{D}}\cdot\mathbf{N}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\mathbf{N} \tag{3.3}$$

$r = 1, 2, ..., M$, where

$$\overline{\mu}_R = \left(\sum_{r=1}^{M}\frac{c_0^{(r)}}{\mu^{(r)}}\right)^{-1} \tag{3.4}$$

stands for the harmonic average of the shear moduli of the M phases in the layered composite, while

$$\overline{\varepsilon}_V = \sum_{r=1}^{M} c_0^{(r)}\varepsilon^{(r)} \qquad \text{and} \qquad \overline{\varepsilon}_R = \left(\sum_{r=1}^{M}\frac{c_0^{(r)}}{\varepsilon^{(r)}}\right)^{-1} \tag{3.5}$$

stand, respectively, for the arithmetic and harmonic averages of the permittivities.

In view of the above expressions, the local deformation gradients (2.7) and electric displacement fields (2.8) within each of the phases $r = 1, 2, ..., M$ can be compactly written in closed form as

$$\overline{\mathbf{F}}^{(r)} = \overline{\mathbf{F}} + \left(\frac{\overline{\mu}_R}{\mu^{(r)}} - 1\right)\overline{\mathbf{F}}\mathbf{N}\otimes\mathbf{N} - \frac{\dfrac{\overline{\mu}_R}{\mu^{(r)}} - 1}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\overline{\mathbf{F}}^{-T}\mathbf{N}\otimes\mathbf{N} \tag{3.6}$$

and

$$\overline{\mathbf{D}}^{(r)} = \frac{\varepsilon^{(r)}}{\overline{\varepsilon}_V}\overline{\mathbf{D}} - \left(\frac{\overline{\mu}_R}{\mu^{(r)}} - 1\right)(\overline{\mathbf{D}}\cdot\mathbf{N})\mathbf{N} + \left(\frac{\overline{\mu}_R}{\mu^{(r)}} - \frac{\varepsilon^{(r)}}{\overline{\varepsilon}_V}\right)\frac{\overline{\mathbf{D}}\cdot\mathbf{N}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\mathbf{N}. \tag{3.7}$$

Here, it is interesting to remark that the local deformation gradients (3.6) depend on the shear moduli of the phases but *not* on their permittivities. On the other hand, the local electric displacements (3.7) *do* depend on the shear moduli of the phases, in addition, of course, to their permittivities. Note also that corresponding closed-form results for the local stresses and electric fields within each of the phases in the composite are readily computable from knowledge of the local fields (3.6)–(3.7).

The macroscopic response After direct substitution of expressions (3.6)–(3.7) in relation (2.6), some algebraic manipulation, and use of the notation

$$\overline{\mu}_V = \sum_{r=1}^{M} c_0^{(r)} \mu^{(r)} \tag{3.8}$$

for the arithmetic average of the shear moduli, the effective free energy function of the layered composite takes the remarkably simple form

$$\overline{W}(\overline{\mathbf{F}}, \overline{\mathbf{D}}) = \begin{cases} \dfrac{\overline{\mu}_V}{2}\left[\overline{\mathbf{F}} \cdot \overline{\mathbf{F}} - 3\right] + \dfrac{1}{2\overline{\varepsilon}_V} \overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \overline{\mathbf{F}}\,\overline{\mathbf{D}} - \\ \dfrac{\overline{\mu}_V - \overline{\mu}_R}{2}\left[\overline{\mathbf{F}}\mathbf{N} \cdot \overline{\mathbf{F}}\mathbf{N} - \dfrac{1}{\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}}\right] + \\ \dfrac{1}{2}\left(\dfrac{1}{\overline{\varepsilon}_R} - \dfrac{1}{\overline{\varepsilon}_V}\right) \dfrac{\left(\overline{\mathbf{D}} \cdot \mathbf{N}\right)^2}{\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}} & \text{if} \quad \overline{J} = 1 \\ \\ +\infty & \text{otherwise} \end{cases} . \tag{3.9}$$

Interestingly, the result (3.9) depends on the shear moduli and permittivities of the underlying phases only through their arithmetic and harmonic averages $\overline{\mu}_V$, $\overline{\mu}_R$, $\overline{\varepsilon}_V$, $\overline{\varepsilon}_R$, which, again, are given explicitly by expressions (3.8) and (3.4)–(3.5). As expected, the *local* incompressibility of all M phases (3.1) implies that the composite itself is incompressible and thus its effective free energy function (3.9) is subject to the *macroscopic* kinematical constraint $C(\overline{\mathbf{F}}) = \det \overline{\mathbf{F}} - 1 = 0$. Also as expected, the effective free energy function (3.9) is transversely isotropic with the layering direction $\mathbf{N}$ denoting its axis of symmetry. In terms of the standard invariants

$$\begin{aligned} &\overline{I}_1 = \overline{\mathbf{F}} \cdot \overline{\mathbf{F}}, \quad \overline{I}_2 = \overline{\mathbf{F}}^{-T} \cdot \overline{\mathbf{F}}^{-T}, \quad \overline{I}_4 = \overline{\mathbf{F}}\mathbf{N} \cdot \overline{\mathbf{F}}\mathbf{N}, \quad \overline{I}_5 = \overline{\mathbf{F}}^T\overline{\mathbf{F}}\mathbf{N} \cdot \overline{\mathbf{F}}^T\overline{\mathbf{F}}\mathbf{N}, \\ &\overline{I}_6 = \overline{\mathbf{D}} \cdot \overline{\mathbf{D}}, \quad \overline{I}_7 = \overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \overline{\mathbf{F}}\,\overline{\mathbf{D}}, \quad \overline{I}_8 = \overline{\mathbf{F}}^T\overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \overline{\mathbf{F}}^T\overline{\mathbf{F}}\,\overline{\mathbf{D}}, \\ &\overline{I}_9 = \overline{\mathbf{D}} \cdot \mathbf{N}, \quad \overline{I}_{10} = \overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \overline{\mathbf{F}}\mathbf{N}, \end{aligned} \tag{3.10}$$

the finite branch of the effective free energy (3.9) can be rewritten as

$$\overline{W}(\overline{\mathbf{F}},\overline{\mathbf{D}}) = \frac{\overline{\mu}_V}{2}\left[\overline{I}_1 - 3\right] - \frac{\overline{\mu}_V - \overline{\mu}_R}{2}\left[\overline{I}_4 - \frac{1}{\overline{I}_2 - \overline{I}_1\overline{I}_4 + \overline{I}_5}\right] + \frac{\overline{I}_7}{2\overline{\varepsilon}_V} + \frac{\left(\dfrac{1}{\overline{\varepsilon}_R} - \dfrac{1}{\overline{\varepsilon}_V}\right)\overline{I}_9^2}{2\left[\overline{I}_2 - \overline{I}_1\overline{I}_4 + \overline{I}_5\right]}. \tag{3.11}$$

Thus, unlike the local free energy functions (3.1), the resulting effective free energy function (3.11) is *not* of the separable form $\overline{W} = \overline{W}_{elas}(\overline{I}_1, \overline{I}_2, \overline{I}_4, \overline{I}_5) + \overline{W}_{elec}(\overline{I}_6, \overline{I}_7, \overline{I}_8, \overline{I}_9, \overline{I}_{10})$. What is more, the purely mechanical contribution in (3.11) is *not* of the separable form $\overline{W} = \overline{W}_{iso}(\overline{I}_1, \overline{I}_2) + \overline{W}_{ani}(\overline{I}_4, \overline{I}_5)$, which is often assumed in the literature on a purely phenomenological basis.

Having established the explicit result (3.9), it is a simple matter to compute the constitutive relations (2.4) relating the macroscopic first Piola-Kirchhoff stress $\overline{\mathbf{S}}$ and electric field $\overline{\mathbf{E}}$ to the macroscopic deformation gradient $\overline{\mathbf{F}}$ and electric displacement field $\overline{\mathbf{D}}$. They read as

$$\begin{aligned}\overline{\mathbf{S}} &= \frac{\partial \overline{W}}{\partial \overline{\mathbf{F}}} - \overline{q}\,\overline{\mathbf{F}}^{-T} \\ &= \overline{\mu}_V \overline{\mathbf{F}} - \overline{q}\,\overline{\mathbf{F}}^{-T} - (\overline{\mu}_V - \overline{\mu}_R)\overline{\mathbf{F}}\mathbf{N}\otimes\mathbf{N} + \frac{1}{\overline{\varepsilon}_V}\overline{\mathbf{F}}\,\overline{\mathbf{D}}\otimes\overline{\mathbf{D}} + \\ &\quad \frac{1}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}\left[\overline{\mu}_V - \overline{\mu}_R + \left(\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}\right)\left(\overline{\mathbf{D}}\cdot\mathbf{N}\right)^2\right] \times \\ &\quad \overline{\mathbf{F}}^{-T}\mathbf{N}\otimes\overline{\mathbf{F}}^{-1}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\end{aligned} \tag{3.12}$$

and

$$\overline{\mathbf{E}} = \frac{\partial \overline{W}}{\partial \overline{\mathbf{D}}} = \frac{1}{\overline{\varepsilon}_V}\overline{\mathbf{F}}^T\,\overline{\mathbf{F}}\,\overline{\mathbf{D}} + \left(\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}\right)\frac{\overline{\mathbf{D}}\cdot\mathbf{N}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\mathbf{N}, \tag{3.13}$$

where the scalar $\overline{q}$ in (3.12) stands for the Lagrange multiplier associated with the overall incompressibility constraint of the composite. Corresponding simple explicit expressions for the Cauchy stress $\overline{\mathbf{T}}$, Eulerian electric field $\overline{\mathbf{e}}$, and polarization $\overline{\mathbf{p}}$ follow trivially from the connections $\overline{\mathbf{T}} = \overline{\mathbf{S}}\,\overline{\mathbf{F}}^T$, $\overline{\mathbf{e}} = \overline{\mathbf{F}}^{-T}\,\overline{\mathbf{E}}$, and $\overline{\mathbf{p}} = \overline{\mathbf{F}}\,\overline{\mathbf{D}} - \varepsilon_0\overline{\mathbf{e}}$.

The incremental moduli For subsequent use in the analysis of macroscopic stability, we record next the incremental moduli (2.12) associated with the effective free energy function (3.9). In component form, they are given explicitly by

$$\begin{aligned}\overline{\mathcal{L}}_{ijkl} =&\overline{\mu}_V\delta_{ik}\overline{F}_{jb}\overline{F}_{lb} - (\overline{\mu}_V - \overline{\mu}_R)\delta_{ik}\overline{F}_{ja}N_a\overline{F}_{lb}N_b + \frac{1}{\overline{\varepsilon}_V}\delta_{ik}\overline{F}_{ja}\overline{D}_a\overline{F}_{lb}\overline{D}_b+\\ &4\frac{\overline{\mu}_V - \overline{\mu}_R + \left(\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}\right)\left(\overline{\mathbf{D}}\cdot\mathbf{N}\right)^2}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^3}\overline{F}^{-1}_{pi}N_p\overline{F}^{-1}_{rj}N_r\overline{F}^{-1}_{sk}N_s\overline{F}^{-1}_{nl}N_n-\\ &\frac{\overline{\mu}_V - \overline{\mu}_R + \left(\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}\right)\left(\overline{\mathbf{D}}\cdot\mathbf{N}\right)^2}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}\Big\{\delta_{il}\overline{F}^{-1}_{pk}N_p\overline{F}^{-1}_{rj}N_r+\\ &\delta_{jk}\overline{F}^{-1}_{pi}N_p\overline{F}^{-1}_{rl}N_r + \delta_{jl}\overline{F}^{-1}_{pi}N_p\overline{F}^{-1}_{rk}N_r\Big\},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\overline{\mathcal{M}}_{ijk} =& \frac{1}{\overline{\varepsilon}_V}(\delta_{ik}\overline{F}_{ja}\overline{D}_a + \overline{F}_{im}\overline{D}_m\delta_{jk}) +\\ &2\frac{\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}\overline{D}_mN_m\overline{F}^{-1}_{bk}N_b\overline{F}^{-1}_{pi}N_p\overline{F}^{-1}_{rj}N_r,\end{aligned} \tag{3.15}$$

and

$$\overline{\mathcal{B}}_{ij} = \frac{1}{\overline{\varepsilon}_V}\delta_{ij} + \frac{\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\overline{F}^{-1}_{ai}N_a\overline{F}^{-1}_{bj}N_b. \tag{3.16}$$

3.2 Macroscopic stability

Direct use of the incremental moduli (3.14)–(3.16) of the layered composite into the general expressions (2.11) leads to the following results

$$\widehat{\mathbf{K}} = \widehat{\mathbf{I}}\mathbf{K}\widehat{\mathbf{I}} = \left[\overline{\mu}_V \overline{\mathbf{F}}^T\mathbf{u}\cdot\overline{\mathbf{F}}^T\mathbf{u} - (\overline{\mu}_V - \overline{\mu}_R)(\overline{\mathbf{F}}\mathbf{N}\cdot\mathbf{u})^2 + \frac{1}{\overline{\varepsilon}_V}(\overline{\mathbf{F}}\,\overline{\mathbf{D}}\cdot\mathbf{u})^2\right]\widehat{\mathbf{I}} + \frac{\overline{\mu}_V - \overline{\mu}_R + \left(\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}\right)\left(\overline{\mathbf{D}}\cdot\mathbf{N}\right)^2}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}\left(\frac{4(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\mathbf{u})^2}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}} - 1\right)\times \widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\otimes\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}, \tag{3.17}$$

$$\widehat{\mathbf{R}} = \widehat{\mathbf{I}}\mathbf{R}\widehat{\mathbf{I}} = \frac{1}{\overline{\varepsilon}_V}(\overline{\mathbf{F}}\,\overline{\mathbf{D}}\cdot\mathbf{u})\widehat{\mathbf{I}} + 2\left(\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}\right)\frac{(\overline{\mathbf{D}}\cdot\mathbf{N})(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\mathbf{u})}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\otimes\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}, \tag{3.18}$$

$$\widehat{\mathbf{B}} = \widehat{\mathbf{I}}\boldsymbol{\mathcal{B}}\widehat{\mathbf{I}} = \frac{1}{\overline{\varepsilon}_V}\widehat{\mathbf{I}} + \frac{\frac{1}{\overline{\varepsilon}_R} - \frac{1}{\overline{\varepsilon}_V}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\otimes\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}, \tag{3.19}$$

for the acoustic tensor $\widehat{\mathbf{K}}$, electro-elastic acoustic tensor $\widehat{\mathbf{R}}$, and impermeability tensor $\widehat{\mathbf{B}}$, where it is recalled that $\widehat{\mathbf{I}} = \mathbf{I} - \mathbf{u}\otimes\mathbf{u}$. The corresponding generalized acoustic tensor (2.15) takes then the form

$$\widehat{\mathbf{\Gamma}}(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,) = \left\{\overline{\mu}_V \overline{\mathbf{F}}^T\mathbf{u}\cdot\overline{\mathbf{F}}^T\mathbf{u} - (\overline{\mu}_V-\overline{\mu}_R)(\overline{\mathbf{F}}\mathbf{N}\cdot\mathbf{u})^2\right\}\widehat{\mathbf{I}}+$$
$$\left\{4(\overline{\mu}_V-\overline{\mu}_R)\frac{(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\mathbf{u})^2}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^3} - \frac{\overline{\mu}_V-\overline{\mu}_R+\left(\dfrac{1}{\overline{\varepsilon}_R}-\dfrac{1}{\overline{\varepsilon}_V}\right)\left(\overline{\mathbf{D}}\cdot\mathbf{N}\right)^2}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}+\right.$$
$$\frac{4\left(\dfrac{1}{\overline{\varepsilon}_R}-\dfrac{1}{\overline{\varepsilon}_V}\right)}{1+\dfrac{\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\left(\dfrac{\overline{\varepsilon}_V}{\overline{\varepsilon}_R}-1\right)}\left[\frac{(\overline{\mathbf{D}}\cdot\mathbf{N})^2(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\mathbf{u})^2}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^3}+\right.$$
$$\left.\left.\frac{(\overline{\mathbf{F}}\,\overline{\mathbf{D}}\cdot\mathbf{u})^2}{4\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}} - \frac{(\overline{\mathbf{D}}\cdot\mathbf{N})(\overline{\mathbf{F}}\,\overline{\mathbf{D}}\cdot\mathbf{u})(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\mathbf{u})}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}\right]\right\}\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\otimes\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}. \tag{3.20}$$

Now, it is not difficult to deduce that (3.20) has two non-trivial eigenvalues on the two-dimensional space normal to $\mathbf{u}$ given by

$$\widehat{\gamma}_1(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,) = \overline{\mu}_V\overline{\mathbf{F}}^T\mathbf{u}\cdot\overline{\mathbf{F}}^T\mathbf{u} - (\overline{\mu}_V-\overline{\mu}_R)(\overline{\mathbf{F}}\mathbf{N}\cdot\mathbf{u})^2 \tag{3.21}$$

and

$$
\begin{aligned}
\widehat{\gamma}_2(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,) = & \overline{\mu}_V \overline{\mathbf{F}}^T\mathbf{u}\cdot\overline{\mathbf{F}}^T\mathbf{u} - (\overline{\mu}_V-\overline{\mu}_R)\,(\overline{\mathbf{F}}\mathbf{N}\cdot\mathbf{u})^2 + \\
& 4\,(\overline{\mu}_V-\overline{\mu}_R)\,\frac{\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^3}(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\mathbf{u})^2 - \\
& (\overline{\mu}_V-\overline{\mu}_R)\,\frac{\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2} - \\
& \left(\frac{1}{\overline{\varepsilon}_R}-\frac{1}{\overline{\varepsilon}_V}\right)\frac{\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}\left(\overline{\mathbf{D}}\cdot\mathbf{N}\right)^2 + \\
& \frac{4\left(\dfrac{1}{\overline{\varepsilon}_R}-\dfrac{1}{\overline{\varepsilon}_V}\right)\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}+\left(\dfrac{\overline{\varepsilon}_V}{\overline{\varepsilon}_R}-1\right)\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\widehat{\mathbf{I}}\,\overline{\mathbf{F}}^{-T}\mathbf{N}}\times \\
& \left[\frac{(\overline{\mathbf{D}}\cdot\mathbf{N})^2(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\mathbf{u})^2}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}+\frac{1}{4}(\overline{\mathbf{F}}\,\overline{\mathbf{D}}\cdot\mathbf{u})^2 - \right. \\
& \left.\frac{(\overline{\mathbf{D}}\cdot\mathbf{N})(\overline{\mathbf{F}}\,\overline{\mathbf{D}}\cdot\mathbf{u})(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\mathbf{u})}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\right]. \qquad (3.22)
\end{aligned}
$$

Accordingly, the ordinary ellipticity condition (2.16) reduces in this case to the product of the above two eigenvalues being non-zero:

$$\widehat{\gamma}_1(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,)\widehat{\gamma}_2(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,)\neq 0. \qquad (3.23)$$

Moreover, the strong ellipticity condition (2.17) reduces to the above two eigenvalues being strictly positive:

$$\widehat{\gamma}_1(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,)>0 \qquad \text{and} \qquad \widehat{\gamma}_2(\mathbf{u};\overline{\mathbf{F}},\overline{\mathbf{D}}\,)>0. \qquad (3.24)$$

Clearly, $\widehat{\gamma}_1>0$ for all unit vectors $\mathbf{u}$, deformation gradients $\overline{\mathbf{F}}$, and electric displacements $\overline{\mathbf{D}}$, since $\overline{\mu}_V\geq\overline{\mu}_R>0$ and $||\mathbf{N}||=1$. Moreover, $\widehat{\gamma}_2>0$ for all unit vectors $\mathbf{u}$ when $\overline{\mathbf{F}}=\mathbf{I}$ and $\overline{\mathbf{D}}=\mathbf{0}$, but need not be positive more generally. Thus, the overall behavior of the layered composite is strongly elliptic in its ground state when $\overline{\mathbf{F}}=\mathbf{I}$, $\overline{\mathbf{D}}=\mathbf{0}$ (and $\overline{\mathbf{S}}=\mathbf{0}$, $\overline{\mathbf{E}}=\mathbf{0}$). But — in spite of the fact that the underlying phases (3.1) are strongly elliptic — it may lose ordinary and strong ellipticity at sufficiently large electromechanical loads. More specifically, the set of critical pairs $(\overline{\mathbf{F}}, \overline{\mathbf{D}})$ and associated critical vectors $\mathbf{u}$ at which the layered composite first loses ellipticity is characterized by the equation $\widehat{\gamma}_2 = 0$. After some algebraic manipulation of expression (3.22) to reveal explicitly its dependence on $\mathbf{u}$, this criticality equation can be rewritten as

$$
\begin{aligned}
&\frac{\overline{\mu}_R}{\overline{\mu}_V} - 1 - \left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right) \frac{(\overline{\mathbf{D}} \cdot \mathbf{N})^2}{\overline{\varepsilon}_R \overline{\mu}_V} + \frac{4\left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)}{(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N})^3} (\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^6 + \\
&\frac{\left(\frac{5\overline{\varepsilon}_R}{\overline{\varepsilon}_V} - 9\right)\left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right) + \left(\frac{3\overline{\varepsilon}_R}{\overline{\varepsilon}_V} - \frac{\overline{\varepsilon}_V}{\overline{\varepsilon}_R} - 2\right)\frac{(\overline{\mathbf{D}} \cdot \mathbf{N})^2}{\overline{\mu}_V \overline{\varepsilon}_V}}{(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N})^2} (\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^4 - \\
&\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)\left[\overline{\mathbf{F}}^T\mathbf{u} \cdot \overline{\mathbf{F}}^T\mathbf{u} - \left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)(\overline{\mathbf{F}}\mathbf{N} \cdot \mathbf{u})^2 + \frac{(\overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \mathbf{u})^2}{\overline{\mu}_V \overline{\varepsilon}_V}\right] \times \\
&(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^2 + \frac{4\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)\overline{\mathbf{D}} \cdot \mathbf{N}}{\overline{\mu}_V \overline{\varepsilon}_V \overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}} (\overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \mathbf{u})(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^3 - \\
&\frac{4\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)\overline{\mathbf{D}} \cdot \mathbf{N}}{\overline{\mu}_V \overline{\varepsilon}_V} (\overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \mathbf{u})(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u}) + \\
&\frac{\left(6 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)\left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right) - \left(\frac{3\overline{\varepsilon}_R}{\overline{\varepsilon}_V} - \frac{2\overline{\varepsilon}_V}{\overline{\varepsilon}_R} - 1\right)\frac{(\overline{\mathbf{D}} \cdot \mathbf{N})^2}{\overline{\mu}_V \overline{\varepsilon}_V}}{\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}} (\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^2 + \\
&\left[\overline{\mathbf{F}}^T\mathbf{u} \cdot \overline{\mathbf{F}}^T\mathbf{u} - \left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)(\overline{\mathbf{F}}\mathbf{N} \cdot \mathbf{u})^2 + \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V^2 \overline{\mu}_V}\left(\frac{\overline{\varepsilon}_V}{\overline{\varepsilon}_R} - 1\right)(\overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \mathbf{u})^2\right] \times \\
&\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N} = 0. \qquad (3.25)
\end{aligned}
$$

In terms of $\mathbf{u}$, as expected from the definition of the generalized acoustic tensor (2.15), equation (3.25) is a polynomial of degree 6. Consequently, an explicit formula for the entire set of critical pairs $(\overline{\mathbf{F}}, \overline{\mathbf{D}})$ that is separate of the associated critical vectors $\mathbf{u}$ at which the layered composite loses ellipticity is unlikely to exist. At any rate, it is an easy task to generate such a set numerically: starting at $(\overline{\mathbf{F}} = \mathbf{I}, \overline{\mathbf{D}} = \mathbf{0})$ and marching along any path of choice in the $(\overline{\mathbf{F}}, \overline{\mathbf{D}})$–space, equation (3.25) can be checked at every marching increment — via a scanning process — for the existence of unit vectors $\mathbf{u}$ for which it holds. Once the set of critical pairs $(\overline{\mathbf{F}}, \overline{\mathbf{D}})$ has been computed, the corresponding critical stresses $\overline{\mathbf{S}}$ and critical electric fields $\overline{\mathbf{E}}$ can be readily determined from expressions (3.12) and (3.13), thus providing a complete characterization of the macroscopic stability of the layered composite.

3.2.1 Some special loading conditions

There are special loading conditions of practical relevance for which condition (3.25) renders separate explicit formulae for the critical pairs $(\overline{\mathbf{F}}, \overline{\mathbf{D}})$ and critical vectors $\mathbf{u}$ at which the layered composite loses ellipticity. That is the case, for instance, for the experimentally standard conditions of uniaxial electric displacement combined with mechanical stretches in the transverse directions (see, e.g., Section 2.25 in Stratton, 1941; Pelrine et al., 1998). In the present notation, this corresponds to setting

$$\overline{\mathbf{F}} = \overline{\lambda}_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \overline{\lambda}_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{1}{\overline{\lambda}_1 \overline{\lambda}_2} \mathbf{e}_3 \otimes \mathbf{e}_3 \qquad \text{and} \qquad \overline{\mathbf{D}} = \overline{D}_3 \mathbf{e}_3, \tag{3.26}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ stand for the Cartesian laboratory axes and $\overline{\lambda}_1 > 0$, $\overline{\lambda}_2 > 0$, $\overline{D}_3$ are loading parameters denoting the applied transverse stretches and electric displacement.

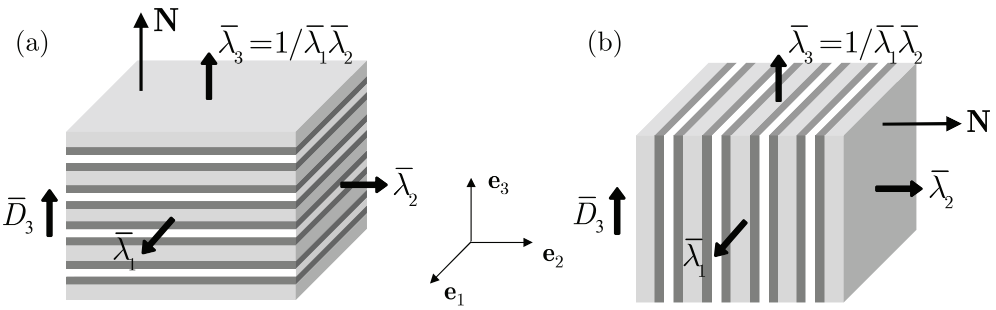

Figure 3.1: Schematics of the loading conditions (3.26) applied to the layered composite for the cases when the layers are (a) transverse ($\mathbf{N} = \mathbf{e}_3$) and (b) aligned ($\mathbf{N} = \mathbf{e}_2$) with the applied electric displacement.

Now, for the case schematically depicted in Fig. 3.1(a) when the layers are transverse to the applied electric displacement $(3.26)_2$,

$$\mathbf{N} = \mathbf{e}_3, \tag{3.27}$$

it is not difficult to deduce from (3.25) that the critical stretches $\overline{\lambda}_1$, $\overline{\lambda}_2$, electric displacement $\overline{D}_3$, and associated vector $\mathbf{u}$ at which ellipticity is lost are simply given by

$$\overline{\lambda}_1^2\overline{\lambda}_2^4 = 1 - \frac{\overline{\mu}_R}{\overline{\mu}_V} + \frac{\overline{D}_3^2}{\overline{\varepsilon}_R\overline{\mu}_V}\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right) \qquad \text{and} \qquad \mathbf{u} = \pm\mathbf{e}_2 \tag{3.28}$$

if $\overline{\lambda}_1 > \overline{\lambda}_2$, and by

$$\overline{\lambda}_1^4\overline{\lambda}_2^2 = 1 - \frac{\overline{\mu}_R}{\overline{\mu}_V} + \frac{\overline{D}_3^2}{\overline{\varepsilon}_R\overline{\mu}_V}\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right) \qquad \text{and} \qquad \mathbf{u} = \pm\mathbf{e}_1 \tag{3.29}$$

if, on the other hand, $\overline{\lambda}_1 < \overline{\lambda}_2$. Furthermore, for the case schematically depicted in Fig. 3.1(b) when the layers are aligned[1] with the applied electric displacement $(3.26)_2$,

$$\mathbf{N} = \mathbf{e}_2, \tag{3.30}$$

the critical stretches $\overline{\lambda}_1$, $\overline{\lambda}_2$, electric displacement $\overline{D}_3$, and associated vector $\mathbf{u}$ can be shown to be given by

$$\overline{\lambda}_1^2\overline{\lambda}_2^4 = \left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)^{-1} + \frac{\overline{D}_3^2}{\overline{\mu}_V\overline{\varepsilon}_V}\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)\left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)^{-1} \qquad \text{and} \qquad \mathbf{u} = \pm\mathbf{e}_3 \tag{3.31}$$

if $\overline{\lambda}_1^4\overline{\lambda}_2^2 > 1 + \dfrac{\overline{D}_3^2}{\overline{\mu}_V\overline{\varepsilon}_V}\left(1 - \dfrac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)$, whereas they are given by

$$\frac{\overline{\lambda}_1^2}{\overline{\lambda}_2^2} = 1 - \frac{\overline{\mu}_R}{\overline{\mu}_V} \qquad \text{and} \qquad \mathbf{u} = \pm\mathbf{e}_1 \tag{3.32}$$

if, on the other hand, $\overline{\lambda}_1^4\overline{\lambda}_2^2 < 1 + \dfrac{\overline{D}_3^2}{\overline{\mu}_V\overline{\varepsilon}_V}\left(1 - \dfrac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)$. The more general case when the layers are not transverse nor aligned with the applied electric displacement does not seem to admit separate results for the critical loading $\overline{\lambda}_1$, $\overline{\lambda}_2$, $\overline{D}_3$ and associated vector $\mathbf{u}$. Howbeit, as explained above, it is straightforward to carry out such calculations numerically.

For the case of two phases ($M = 2$) and plane-strain loading conditions ($\overline{\lambda}_1 = 1$), the "macroscopic failure surfaces" defined by $(3.28)_1$ and $(3.31)_1$ reduce to those found earlier by Bertoldi and Gei (2011) and Rudykh (2011). If, in addition, no electric displacement is applied ($\overline{D}_3 = 0$), these surfaces

[1]Choosing $\mathbf{N} = \mathbf{e}_1$ is of course equivalent to choosing $\mathbf{N} = \mathbf{e}_2$.

reduce further to the classical result of Triantafyllidis and Maker (1985).

CHAPTER 4

THE $\overline{\mathbf{F}}$ AND $\overline{\mathbf{E}}$ FORMULATION

Depending on the specific problem at hand, it may be more convenient to utilize the Lagrangian electric field $\overline{\mathbf{E}}$ as the independent electric variable instead of $\overline{\mathbf{D}}$. In the sequel, for the sake of completeness, we summarize the explicit results generated in Chapters 4 and 5 for ideal dielectric layered composites in terms of this alternative variable.

The macroscopic response By introducing the partial Legendre transform

$$\begin{aligned}\overline{W}^*(\overline{\mathbf{F}},\overline{\mathbf{E}}) &= -\sup_{\overline{\mathbf{D}}}\left\{\overline{\mathbf{D}}\cdot\overline{\mathbf{E}}-\overline{W}(\overline{\mathbf{F}},\overline{\mathbf{D}})\right\}\\ &= \begin{cases}\dfrac{\overline{\mu}_V}{2}\left[\overline{\mathbf{F}}\cdot\overline{\mathbf{F}}-3\right]-\dfrac{\overline{\varepsilon}_V}{2}\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}-\\ \dfrac{\overline{\mu}_V-\overline{\mu}_R}{2}\left[\overline{\mathbf{F}}\mathbf{N}\cdot\overline{\mathbf{F}}\mathbf{N}-\dfrac{1}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\right]+ & \text{if}\quad \overline{J}=1\\ \dfrac{\overline{\varepsilon}_V-\overline{\varepsilon}_R}{2}\dfrac{\left(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}} & \\ +\infty & \text{otherwise}\end{cases}\end{aligned} \tag{4.1}$$

of the effective free energy function (3.9), it follows that the macroscopic first Piola-Kirchhoff stress $\overline{\mathbf{S}}$ and electric displacement field $\overline{\mathbf{D}}$ for the layered composite with M ideal elastic dielectric phases can be written in terms of the macroscopic deformation gradient $\overline{\mathbf{F}}$ and electric field $\overline{\mathbf{E}}$ simply as

$$\begin{aligned}\overline{\mathbf{S}} &= \frac{\partial \overline{W}^*}{\partial \overline{\mathbf{F}}} - \overline{q}\,\overline{\mathbf{F}}^{-T} \\ &= \overline{\mu}_V \overline{\mathbf{F}} - \overline{q}\,\overline{\mathbf{F}}^{-T} - (\overline{\mu}_V - \overline{\mu}_R)\overline{\mathbf{F}}\mathbf{N}\otimes\mathbf{N} + \overline{\varepsilon}_V \overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\otimes\overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} + \\ &\quad \frac{\overline{\mu}_V - \overline{\mu}_R + (\overline{\varepsilon}_V - \overline{\varepsilon}_R)\left(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}{\left(\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}\right)^2}\overline{\mathbf{F}}^{-T}\mathbf{N}\otimes\overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\mathbf{N} - \\ &\quad \frac{(\overline{\varepsilon}_V - \overline{\varepsilon}_R)\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}} \times \\ &\quad \left[\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\otimes\overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\mathbf{N} + \overline{\mathbf{F}}^{-T}\mathbf{N}\otimes\overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\right] \end{aligned} \tag{4.2}$$

and

$$\overline{\mathbf{D}} = -\frac{\partial \overline{W}^*}{\partial \overline{\mathbf{E}}} = \overline{\varepsilon}_V \overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} - \frac{(\overline{\varepsilon}_V - \overline{\varepsilon}_R)\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}{\overline{\mathbf{F}}^{-T}\mathbf{N}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}}\overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\mathbf{N}, \tag{4.3}$$

where, again, the parameters $\overline{\mu}_V$, $\overline{\mu}_R$, $\overline{\varepsilon}_V$, $\overline{\varepsilon}_R$ are given by expressions (3.8), (3.4)–(3.5) and the scalar $\overline{q}$ in (4.2) denotes the Lagrange multiplier associated with the overall incompressibility constraint. As for the $\overline{\mathbf{F}}$ and $\overline{\mathbf{D}}$ version of the result, corresponding expressions for the Cauchy stress $\overline{\mathbf{T}}$, Eulerian electric displacement $\overline{\mathbf{d}}$, and polarization $\overline{\mathbf{p}}$ follow trivially from the connections $\overline{\mathbf{T}} = \overline{\mathbf{S}}\,\overline{\mathbf{F}}^T$, $\overline{\mathbf{d}} = \overline{\mathbf{F}}\,\overline{\mathbf{D}}$, and $\overline{\mathbf{p}} = \overline{\mathbf{d}} - \varepsilon_0 \overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}$.

In terms of the standard invariants

$$\begin{aligned} &\overline{I}_1 = \overline{\mathbf{F}}\cdot\overline{\mathbf{F}}, \quad \overline{I}_2 = \overline{\mathbf{F}}^{-T}\cdot\overline{\mathbf{F}}^{-T}, \quad \overline{I}_4 = \overline{\mathbf{F}}\mathbf{N}\cdot\overline{\mathbf{F}}\mathbf{N}, \quad \overline{I}_5 = \overline{\mathbf{F}}^T\overline{\mathbf{F}}\mathbf{N}\cdot\overline{\mathbf{F}}^T\overline{\mathbf{F}}\mathbf{N}, \\ &\overline{I}_6^* = \overline{\mathbf{E}}\cdot\overline{\mathbf{E}}, \quad \overline{I}_7^* = \overline{\mathbf{F}}\,\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}\,\overline{\mathbf{E}}, \quad \overline{I}_8^* = \overline{\mathbf{F}}^T\overline{\mathbf{F}}\,\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^T\overline{\mathbf{F}}\,\overline{\mathbf{E}}, \\ &\overline{I}_9^* = \overline{\mathbf{E}}\cdot\mathbf{N}, \quad \overline{I}_{10}^* = \overline{\mathbf{F}\mathbf{E}}\cdot\overline{\mathbf{F}}\mathbf{N}, \end{aligned} \tag{4.4}$$

it is interesting to recognize that the finite branch of the effective free energy function (4.1) reads as

$$\overline{W}^*(\overline{\mathbf{F}},\overline{\mathbf{E}}) = \frac{\overline{\mu}_V}{2}\left[\overline{I}_1 - 3\right] - \frac{\overline{\mu}_V - \overline{\mu}_R}{2}\left[\overline{I}_4 - \frac{1}{\overline{I}_2 - \overline{I}_1\overline{I}_4 + \overline{I}_5}\right] - \frac{\overline{\varepsilon}_V}{2}\left[\overline{I}_2\overline{I}_6^* - \overline{I}_1\overline{I}_7^* + \overline{I}_8^*\right] + \frac{(\overline{\varepsilon}_V - \overline{\varepsilon}_R)}{8} \times \frac{\left[\overline{I}_1\overline{I}_4\overline{I}_6^* + \overline{I}_1\overline{I}_7^* - \overline{I}_2\overline{I}_6^* - \overline{I}_2\overline{I}_9^{*\,2} - \overline{I}_4\overline{I}_7^* - \overline{I}_5\overline{I}_6^* - \overline{I}_8^* + \overline{I}_{10}^{*\,2}\right]^2}{\left[\overline{I}_2 - \overline{I}_1\overline{I}_4 + \overline{I}_5\right]\overline{I}_9^{*\,2}}, \tag{4.5}$$

which, much like its $\overline{\mathbf{F}}$ and $\overline{\mathbf{D}}$ counterpart (3.11), is *not* of the separable form $\overline{W}^* = \overline{W}_{elas}(\overline{I}_1, \overline{I}_2, \overline{I}_4, \overline{I}_5) + \overline{W}^*_{elec}(\overline{I}_6^*, \overline{I}_7^*, \overline{I}_8^*, \overline{I}_9^*, \overline{I}_{10}^*)$. What is more, the functional dependence on the electric invariants $\overline{I}_6^*$ through $\overline{I}_{10}^*$ is admittedly cumbersome. This suggests that the standard set of invariants (4.4) may not be the most appropriate choice to model transversely isotropic elastic dielectrics (see, e.g., Bustamante, 2009). The alternative set

$$\begin{aligned}
&\overline{I}_1 = \overline{\mathbf{F}}\cdot\overline{\mathbf{F}}, \quad \overline{I}_2 = \overline{\mathbf{F}}^{-T}\cdot\overline{\mathbf{F}}^{-T}, \quad \overline{I}_4 = \overline{\mathbf{F}}\mathbf{N}\cdot\overline{\mathbf{F}}\mathbf{N}, \quad \overline{I}_5 = \overline{\mathbf{F}}^T\overline{\mathbf{F}}\mathbf{N}\cdot\overline{\mathbf{F}}^T\overline{\mathbf{F}}\mathbf{N},\\
&\overline{J}_6^* = \overline{\mathbf{E}}\cdot\overline{\mathbf{E}}, \quad \overline{J}_7^* = \overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}, \quad \overline{J}_8^* = \overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^{-1}\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}},\\
&\overline{J}_9^* = \overline{\mathbf{E}}\cdot\mathbf{N}, \quad \overline{J}_{10}^* = \overline{\mathbf{F}}^{-T}\overline{\mathbf{E}}\cdot\overline{\mathbf{F}}^{-T}\mathbf{N}
\end{aligned} \tag{4.6}$$

may prove more appropriate, as it leads to the more compact form

$$\overline{W}^*(\overline{\mathbf{F}},\overline{\mathbf{E}}) = \frac{\overline{\mu}_V}{2}\left[\overline{I}_1 - 3\right] - \frac{\overline{\mu}_V - \overline{\mu}_R}{2}\left[\overline{I}_4 - \frac{1}{\overline{I}_2 - \overline{I}_1\overline{I}_4 + \overline{I}_5}\right] - \frac{\overline{\varepsilon}_V}{2}\overline{J}_7^* + \frac{(\overline{\varepsilon}_V - \overline{\varepsilon}_R)\overline{J}_{10}^{*\,2}}{2\left[\overline{I}_2 - \overline{I}_1\overline{I}_4 + \overline{I}_5\right]}. \tag{4.7}$$

Macroscopic stability Making use of expression (4.3) in equation (3.25) allows to rewrite the criticality condition for the loss of ellipticity of the layered composite in terms of the electric field $\overline{\mathbf{E}}$ instead of $\overline{\mathbf{D}}$. The result reads as

$$
\begin{aligned}
&\frac{\overline{\mu}_R}{\overline{\mu}_V} - 1 - \left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right) \frac{\overline{\varepsilon}_R(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N})^2}{\overline{\mu}_V} + \\
&\frac{4\left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)}{(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N})^3} (\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^6 + \\
&\frac{\left(\frac{5\overline{\varepsilon}_R}{\overline{\varepsilon}_V} - 9\right)\left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right) + \left(\frac{3\overline{\varepsilon}_R^2}{\overline{\varepsilon}_V^2} - \frac{2\overline{\varepsilon}_R}{\overline{\varepsilon}_V} - 1\right) \frac{\overline{\varepsilon}_V(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N})^2}{\overline{\mu}_V}}{(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N})^2} \times \\
&(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^4 - \\
&\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)\left[\overline{\mathbf{F}}^T\mathbf{u} \cdot \overline{\mathbf{F}}^T\mathbf{u} - \left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)(\overline{\mathbf{F}}\mathbf{N} \cdot \mathbf{u})^2 + \frac{\overline{\varepsilon}_V(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \mathbf{u})^2}{\overline{\mu}_V}\right] \times \\
&(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^2 + \\
&\frac{2\overline{\varepsilon}_R}{\overline{\mu}_V}\left(\frac{\overline{\varepsilon}_V}{\overline{\varepsilon}_R} - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N})(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \mathbf{u})\left[\frac{(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^2}{\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}} - 1\right] \times \\
&(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u}) + \\
&\frac{\left(6 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)\left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right) - \left(\frac{4\overline{\varepsilon}_R^2}{\overline{\varepsilon}_V^2} - \frac{3\overline{\varepsilon}_R}{\overline{\varepsilon}_V} - 1\right) \frac{\overline{\varepsilon}_V(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N})^2}{\overline{\mu}_V}}{\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}} \times \\
&(\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \mathbf{u})^2 + \\
&\left[\overline{\mathbf{F}}^T\mathbf{u} \cdot \overline{\mathbf{F}}^T\mathbf{u} - \left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)(\overline{\mathbf{F}}\mathbf{N} \cdot \mathbf{u})^2 + \frac{\overline{\varepsilon}_R}{\overline{\mu}_V}\left(\frac{\overline{\varepsilon}_V}{\overline{\varepsilon}_R} - 1\right)(\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \mathbf{u})^2\right] \times \\
&\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N} = 0. \qquad (4.8)
\end{aligned}
$$

For electromechanical loadings analogous to (3.26) with

$$
\overline{\mathbf{F}} = \overline{\lambda}_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \overline{\lambda}_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{1}{\overline{\lambda}_1 \overline{\lambda}_2} \mathbf{e}_3 \otimes \mathbf{e}_3 \qquad \text{and} \qquad \overline{\mathbf{E}} = \overline{E}_3 \mathbf{e}_3, \qquad (4.9)
$$

the general condition (4.8) renders results for the critical pairs $(\overline{\mathbf{F}}, \overline{\mathbf{E}})$ that are separate from the associated critical vectors $\mathbf{u}$. Indeed, for the case with $\mathbf{N} = \mathbf{e}_3$ when the layers are transverse to the applied electric field, it follows from (4.8) that the critical stretches $\overline{\lambda}_1$, $\overline{\lambda}_2$, electric field $\overline{E}_3$, and associated vector $\mathbf{u}$ at which the layered composite loses ellipticity under loading conditions of the form (4.9) are given by

$$\overline{\lambda}_1^2\overline{\lambda}_2^4 = 1 - \frac{\overline{\mu}_R}{\overline{\mu}_V} + \frac{\overline{\varepsilon}_R\overline{\lambda}_1^4\overline{\lambda}_2^4\overline{E}_3^2}{\overline{\mu}_V}\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right) \qquad \text{and} \qquad \mathbf{u} = \pm\mathbf{e}_2 \tag{4.10}$$

if $\overline{\lambda}_1 > \overline{\lambda}_2$, and by

$$\overline{\lambda}_1^4\overline{\lambda}_2^2 = 1 - \frac{\overline{\mu}_R}{\overline{\mu}_V} + \frac{\overline{\varepsilon}_R\overline{\lambda}_1^4\overline{\lambda}_2^4\overline{E}_3^2}{\overline{\mu}_V}\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right) \qquad \text{and} \qquad \mathbf{u} = \pm\mathbf{e}_1 \tag{4.11}$$

if $\overline{\lambda}_1 < \overline{\lambda}_2$. Similarly, for the case with $\mathbf{N} = \mathbf{e}_2$ when the layers are aligned with the applied electric field, it follows from (4.8) that the critical stretches $\overline{\lambda}_1$, $\overline{\lambda}_2$, electric field $\overline{E}_3$, and associated vector $\mathbf{u}$ at which ellipticity is lost are given by

$$\overline{\lambda}_1^2\overline{\lambda}_2^4 = \left(1 - \frac{\overline{\mu}_R}{\overline{\mu}_V}\right)^{-1}\left[1 + \frac{\overline{\varepsilon}_V\overline{\lambda}_1^4\overline{\lambda}_2^4\overline{E}_3^2}{\overline{\mu}_V}\left(1 - \frac{\overline{\varepsilon}_R}{\overline{\varepsilon}_V}\right)\right] \qquad \text{and} \qquad \mathbf{u} = \pm\mathbf{e}_3 \tag{4.12}$$

if $\overline{\lambda}_1^4\overline{\lambda}_2^2 > 1 + \dfrac{\overline{\lambda}_1^4\overline{\lambda}_2^4\overline{E}_3^2}{\overline{\mu}_V}(\overline{\varepsilon}_V - \overline{\varepsilon}_R)$, whereas they are given by

$$\frac{\overline{\lambda}_1^2}{\overline{\lambda}_2^2} = 1 - \frac{\overline{\mu}_R}{\overline{\mu}_V} \qquad \text{and} \qquad \mathbf{u} = \pm\mathbf{e}_1 \tag{4.13}$$

if $\overline{\lambda}_1^4\overline{\lambda}_2^2 < 1 + \dfrac{\overline{\lambda}_1^4\overline{\lambda}_2^4\overline{E}_3^2}{\overline{\mu}_V}(\overline{\varepsilon}_V - \overline{\varepsilon}_R)$.

CHAPTER 5

CONCLUDING REMARKS

The exact homogenization results (3.11), (4.5), (4.7) provide insight into the identification of the invariants that dominate the effective free energy functions of elastic dielectrics with transversely isotropic microstructures. Indeed, when using the electric displacement field $\overline{\mathbf{D}}$ as the independent electric variable, the result (3.11) suggests that the dominant electric invariants are simply the standard $\overline{I}_7 = \overline{\mathbf{F}}\,\overline{\mathbf{D}} \cdot \overline{\mathbf{F}}\,\overline{\mathbf{D}} = \overline{\mathbf{d}} \cdot \overline{\mathbf{d}}$ and $\overline{I}_9 = \overline{\mathbf{D}} \cdot \mathbf{N} = \overline{\mathbf{d}} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}$. When using the electric field $\overline{\mathbf{E}}$ as the independent electric variable, on the other hand, the result (4.5) appears to suggest that all standard electric invariants $(4.4)_5$-$(4.4)_{10}$ are required and equally important in the modeling of transversely isotropic elastic dielectrics. Rewriting the effective free energy function in the form (4.7) reveals, however, that only the alternative invariants $\overline{J}_7^* = \overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} = \overline{\mathbf{e}} \cdot \overline{\mathbf{e}}$ and $\overline{J}_{10}^* = \overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N} = \overline{\mathbf{e}} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}$ may be of importance. Further studies of elastic dielectrics with richer anisotropic microstructures are under way to corroborate these initial findings (Lopez-Pamies, 2014).

While only results for the onset of macroscopic instabilities have been reported here, the results (3.6) and (3.7) for the local deformation gradient and electric displacement fields within each of the phases in the layered composite allow to readily compute the onset of other types of instabilities. These include geometric instabilities of microscopic wavelengths (for the case when the phases are periodically distributed), as well as material instabilities such as cavitation, debonding, and electric breakdown (see, e.g., Michel et al., 2010; Lopez-Pamies et al., 2011). Focusing on the latter, for instance, electric breakdown is often presumed to initiate when the magnitude of the electric field at a material point reaches a critical value, E_{cr} say (see, e.g., Plante and Dubowsky, 2006; Moscardo et al., 2008). By combining the results (4.3) and (3.7), the local electric field within each of the phases $r = 1, 2, ..., M$ in the layered composite can be simply written as

$$\overline{\mathbf{E}}^{(r)} = \overline{\mathbf{E}} + \left(\frac{\overline{\mu}_R}{\mu^{(r)}} - 1\right)(\overline{\mathbf{E}} \cdot \mathbf{N})\mathbf{N} + \left(\frac{\overline{\varepsilon}_R}{\varepsilon^{(r)}} - \frac{\overline{\mu}_R}{\mu^{(r)}}\right) \frac{\overline{\mathbf{F}}^{-T}\overline{\mathbf{E}} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}}{\overline{\mathbf{F}}^{-T}\mathbf{N} \cdot \overline{\mathbf{F}}^{-T}\mathbf{N}} \mathbf{N}. \quad (5.1)$$

Thus, the onset of electric breakdown in the layered composite could be readily computed by monitoring failure of the conditions $||\overline{\mathbf{E}}^{(r)}|| < E_{cr}$ for all $r = 1, 2, ..., M$.

We conclude by remarking that the results (3.9) and (3.25) — or equivalently (4.1) and (4.8) — can be utilized to shed light on the intricate effects that interphases can have on the macroscopic response and stability of dielectric elastomer composites (Lewis, 2004; Roy et al., 2005). To see this, it suffices to consider the case of a three-phase layered composite ($M = 3$) where layers of material $r = 1$ are contiguous only to layers of material $r = 3$ sandwiching layers of material $r = 2$, in such a manner that material $r = 3$ acts as an interphase between materials $r = 1$ and $r = 2$. Because the results (3.9) and (3.25) depend on the shear modulus $\mu^{(3)}$ and permittivity $\varepsilon^{(3)}$ of the interphase via the arithmetic and harmonic averages (3.8), (3.4)–(3.5), large or small values of $\mu^{(3)}$ and $\varepsilon^{(3)}$ can have a substantial effect on these results even when the volume fraction $c_0^{(3)}$ of the interphase is small. The interested reader is referred to Bertoldi and Lopez-Pamies (2012) for a parametric analysis of the effects of $\mu^{(3)}$ and $c_0^{(3)}$ in the purely mechanical context.

APPENDIX A

DERIVATION OF THE ELLIPTICITY CONDITIONS

In this appendix, we provide a brief derivation of the ellipticity conditions (2.13)–(2.14) and (2.16)–(2.17) for elastic dielectric presented in Section 2.3. Before proceeding with the technical details, it is fitting to mention that a condition equivalent to (2.14) for the mathematically identical problem of magnetoelasticity was apparently first put forward by Kankanala and Triantafyllidis (2004). A condition equivalent to (2.17), also within the problem of magnetoelasticity, was later given by Destrade and Ogden (2011) for incompressible materials. This latter condition has been recently adapted for elastic dielectrics by Bertoldi and Gei (2011) and Rudykh (2011) for the special case of plane-strain loading conditions.

When describing the constitutive response of an elastic dielectric by its "total" free energy function[1] $W = W(\mathbf{F}, \mathbf{D})$, as introduced by Dorfmann and Ogden (2005), the conservation of momentum and Maxwell equations governing its incremental quasi-static response at finitely deformed states can be expediently written in updated Lagrangian form as (Ogden, 2009)

$$\operatorname{div}\dot{\mathbf{S}} = \mathbf{0}, \quad \operatorname{curl}\dot{\mathbf{E}} = \mathbf{0}, \quad \operatorname{div}\dot{\mathbf{D}} = 0. \tag{A.1}$$

Here, the div and curl operators are with respect to $\mathbf{x}$, and the infinitesimal increments in the first Piola-Kirchhoff stress $\dot{\mathbf{S}}$ and Lagrangian electric field $\dot{\mathbf{E}}$ are given in component form by

$$\dot{S}_{ij} = \mathcal{L}_{ijkl}\dot{x}_{k,l} + \mathcal{M}_{ijk}\dot{D}_k, \qquad \dot{E}_i = \mathcal{M}_{jki}\dot{x}_{j,k} + \mathcal{B}_{ij}\dot{D}_j, \tag{A.2}$$

in terms of the infinitesimal increments in the deformation $\dot{\mathbf{x}}$ and Lagrangian

[1] The use of bars over the variables W, $\mathbf{F}$, $\mathbf{D}$, $\mathbf{S}$, $\mathbf{E}$ to indicate macroscopic quantities is unnecessary here, and hence dropped for simplicity.

electric displacement $\dot{\mathbf{D}}$, where

$$\begin{aligned}\mathcal{L}_{ijkl} &= J^{-1}F_{ja}F_{lb}\frac{\partial^2 W}{\partial F_{ia}\partial F_{kb}}(\mathbf{F},\mathbf{D}),\\ \mathcal{M}_{ijk} &= F_{ja}F_{bk}^{-1}\frac{\partial^2 W}{\partial F_{ia}\partial D_b}(\mathbf{F},\mathbf{D}),\\ \mathcal{B}_{ij} &= J\,F_{ai}^{-1}F_{bj}^{-1}\frac{\partial^2 W}{\partial D_a\partial D_b}(\mathbf{F},\mathbf{D})\end{aligned}\tag{A.3}$$

stand for the incremental moduli.

In order to deduce the conditions of ordinary and strong ellipticity — akin to the purely mechanical problem (see, e.g., Hill, 1979; Chapter 6.2.7 in Ogden, 1997 and references therein) — we now consider the underlying configuration in the incremental problem (A.1) to be uniform, with homogeneous deformation gradient $\mathbf{F}$ and electric displacement field $\mathbf{D}$, and seek solutions of the form

$$\dot{\mathbf{x}} = \mathbf{v} f(\mathbf{u}\cdot\mathbf{x}), \qquad \dot{\mathbf{D}} = \mathbf{w}\, g(\mathbf{u}\cdot\mathbf{x}),\tag{A.4}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are constant unit vectors while f and g are scalar-valued functions of their argument. Direct use of expressions (A.4) allows to rewrite equations (A.1) as

$$\mathbf{K}\mathbf{v} f'' + \mathbf{R}\mathbf{w}\, g' = \mathbf{0}, \quad \widehat{\mathbf{I}}\left[\mathbf{R}^T\mathbf{v} f'' + \boldsymbol{\mathcal{B}}\mathbf{w}\, g'\right] = \mathbf{0}, \qquad \widehat{\mathbf{I}}\mathbf{w} = \mathbf{w},\tag{A.5}$$

where we have introduced the acoustic tensor $\mathbf{K}$ and electro-acoustic tensor $\mathbf{R}$, given in component form by

$$K_{ik} = \mathcal{L}_{ijkl}u_j u_l, \quad R_{ik} = \mathcal{M}_{ijk}u_j,\tag{A.6}$$

and made use of the notation $f'' = \mathrm{d}^2 f(s)/\mathrm{d}s^2$, $g' = \mathrm{d}g(s)/\mathrm{d}s$, and

$$\widehat{\mathbf{I}} = \mathbf{I} - \mathbf{u}\otimes\mathbf{u}.\tag{A.7}$$

Solving equations $(\mathrm{A.5})_2$ and $(\mathrm{A.5})_3$ for $\mathbf{w}$ yields

$$\mathbf{w} = -\frac{2}{(\mathrm{tr}\widehat{\mathbf{B}})^2 - \mathrm{tr}\widehat{\mathbf{B}}^2}\left[(\mathrm{tr}\widehat{\mathbf{B}})\widehat{\mathbf{I}} - \widehat{\mathbf{B}}\right]\mathbf{R}^T\mathbf{v}\,\frac{f''}{g'},\tag{A.8}$$

where

$$\widehat{\mathbf{B}} = \widehat{\mathbf{I}}\,\boldsymbol{\mathcal{B}}\,\widehat{\mathbf{I}}. \tag{A.9}$$

Substituting (A.8) in $(\text{A.5})_1$ yields in turn the following equation for $\mathbf{v}$:

$$\boldsymbol{\Gamma}(\mathbf{u};\mathbf{F},\mathbf{D})\mathbf{v} = \mathbf{0}, \tag{A.10}$$

where

$$\boldsymbol{\Gamma}(\mathbf{u};\mathbf{F},\mathbf{D}) = \mathbf{K} - \frac{2}{(\operatorname{tr}\widehat{\mathbf{B}})^2 - \operatorname{tr}\widehat{\mathbf{B}}^2}\,\mathbf{R}\left[(\operatorname{tr}\widehat{\mathbf{B}})\,\widehat{\mathbf{I}} - \widehat{\mathbf{B}}\right]\mathbf{R}^T \tag{A.11}$$

corresponds to the generalized acoustic tensor of the elastic dielectric. Clearly, non-trivial solutions of the form (A.4) to the incremental problem (A.1) *cannot* exist if

$$\det\boldsymbol{\Gamma}(\mathbf{u};\mathbf{F},\mathbf{D}) \neq 0 \tag{A.12}$$

for all unit vectors $\mathbf{u}$ and all $\mathbf{F}$, $\mathbf{D}$. Noting that $\boldsymbol{\Gamma}^T = \boldsymbol{\Gamma}$, a stronger condition that prevents non-trivial solutions of the form (A.4) is given by

$$\mathbf{v}\cdot\boldsymbol{\Gamma}(\mathbf{u};\mathbf{F},\mathbf{D})\mathbf{v} > 0 \tag{A.13}$$

for all unit vectors $\mathbf{u}$, $\mathbf{v}$ and all $\mathbf{F}$, $\mathbf{D}$. Conditions (A.12) and (A.13) constitute, respectively, the conditions of ordinary and strong ellipticity for unconstrained elastic dielectrics.

Incompressible materials For the case of incompressible materials, when the free energy function of the elastic dielectric is of the constrained form $W = W(\mathbf{F},\mathbf{D})$ if $J = 1$ and $W = +\infty$ otherwise, the vector $\mathbf{v}$ in (A.4) is constrained according to

$$\widehat{\mathbf{I}}\mathbf{v} = \mathbf{v}, \tag{A.14}$$

and the incremental equations (A.1) specialize further to

$$\widehat{\mathbf{I}}\,[\mathbf{K}\mathbf{v}f'' + \mathbf{R}\mathbf{w}\,g'] = \mathbf{0}, \quad \widehat{\mathbf{I}}\left[\mathbf{R}^T\mathbf{v}f'' + \boldsymbol{\mathcal{B}}\mathbf{w}\,g'\right] = \mathbf{0}, \qquad \widehat{\mathbf{I}}\mathbf{w} = \mathbf{w} \tag{A.15}$$

with $J=1$ in (A.3). Substituting relations (A.14) and $(A.15)_3$ in equation $(A.15)_2$ and then solving for $\mathbf{w}$ leads to the explicit result

$$\mathbf{w}=-\frac{2}{(\mathrm{tr}\widehat{\mathbf{B}})^2-\mathrm{tr}\widehat{\mathbf{B}}^2}\left[(\mathrm{tr}\widehat{\mathbf{B}})\widehat{\mathbf{I}}-\widehat{\mathbf{B}}\right]\widehat{\mathbf{R}}^T\mathbf{v}\,\frac{f''}{g'}, \tag{A.16}$$

where $\widehat{\mathbf{R}}=\widehat{\mathbf{I}}\,\mathbf{R}\,\widehat{\mathbf{I}}$ and $\widehat{\mathbf{B}}$ is recalled to be given by (A.9). Substituting (A.14) and (A.16) in $(A.15)_1$ leads in turn to the following equation for $\mathbf{v}$:

$$\widehat{\mathbf{\Gamma}}(\mathbf{u};\mathbf{F},\mathbf{D}\,)\mathbf{v}=\mathbf{0}, \tag{A.17}$$

where

$$\widehat{\mathbf{\Gamma}}(\mathbf{u};\mathbf{F},\mathbf{D}\,)=\widehat{\mathbf{K}}-\frac{2}{(\mathrm{tr}\,\widehat{\mathbf{B}})^2-\mathrm{tr}\,\widehat{\mathbf{B}}^2}\,\widehat{\mathbf{R}}\left[(\mathrm{tr}\,\widehat{\mathbf{B}})\,\widehat{\mathbf{I}}-\widehat{\mathbf{B}}\right]\widehat{\mathbf{R}}^T \tag{A.18}$$

with $\widehat{\mathbf{K}}=\widehat{\mathbf{I}}\,\mathbf{K}\,\widehat{\mathbf{I}}$. Expression (A.18) corresponds to the generalized acoustic tensor of the incompressible elastic dielectric. Clearly, $\widehat{\mathbf{\Gamma}}^T=\widehat{\mathbf{\Gamma}}$ and thus $\widehat{\mathbf{\Gamma}}$ has two non-trivial real eigenvalues on the two-dimensional space normal to $\mathbf{u}$. Since, according to (A.14), the vector $\mathbf{v}$ also lies on the two-dimensional space normal to $\mathbf{u}$, it follows from (A.17) that non-trivial solutions of the localization type (A.4) to the incremental problem (A.1) *cannot* exist if

$$\left(\mathrm{tr}\,\widehat{\mathbf{\Gamma}}(\mathbf{u};\mathbf{F},\mathbf{D})\right)^2-\mathrm{tr}\,\widehat{\mathbf{\Gamma}}^2(\mathbf{u};\mathbf{F},\mathbf{D})\neq 0 \tag{A.19}$$

for all unit vectors $\mathbf{u}$ and all $\mathbf{F}$ with $J=1$, $\mathbf{D}$. A stronger condition that prevents the existence of non-trivial solutions is given by

$$\mathbf{v}\cdot\widehat{\mathbf{\Gamma}}(\mathbf{u};\mathbf{F},\mathbf{D})\mathbf{v}>0 \tag{A.20}$$

for all unit vectors $\mathbf{u}$, $\mathbf{v}$ such that $\mathbf{u}\cdot\mathbf{v}=0$ and all $\mathbf{F}$ with $J=1$, $\mathbf{D}$. Conditions (A.19) and (A.20) constitute, respectively, the conditions of ordinary and strong ellipticity for incompressible elastic dielectrics.

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