Joseph O. Droffin

UNIVERSITY OF ILLINOIS BULLETIN

ISSUED TWICE A WEEK

Vol. XXXV | June 17, 1938 | No. 84

[Entered as second-class matter December 11, 1912, at the post office at Urbana, Illinois, under the Act of August 24, 1912. Acceptance for mailing at the special rate of postage provided for in section 1103, Act of October 3, 1917, authorized July 31, 1918.]

304

A DISTRIBUTION PROCEDURE FOR THE ANALYSIS OF SLABS CONTINUOUS OVER FLEXIBLE BEAMS

A REPORT OF AN INVESTIGATION

CONDUCTED BY

THE ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS

IN COÖPERATION WITH

THE UNITED STATES BUREAU OF PUBLIC ROADS

AND

THE ILLINOIS DIVISION OF HIGHWAYS

BY

NATHAN M. NEWMARK

PUBLISHED BY THE UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

THE Engineering Experiment Station was established by act of the Board of Trustees of the University of Illinois on December 8, 1903. It is the purpose of the Station to conduct investigations and make studies of importance to the engineering, manufacturing, railway, mining, and other industrial interests of the State.

The management of the Engineering Experiment Station is vested in an Executive Staff composed of the Director and his Assistant, the Heads of the several Departments in the College of Engineering, and the Professor of Chemical Engineering. This Staff is responsible for the establishment of general policies governing the work of the Station, including the approval of material for publication. All members of the teaching staff of the College are encouraged to engage in scientific research, either directly or in coöperation with the Research Corps, composed of full-time research assistants, research graduate assistants, and special investigators.

To render the results of its scientific investigations available to the public, the Engineering Experiment Station publishes and distributes a series of bulletins. Occasionally it publishes circulars of timely interest, presenting information of importance, compiled from various sources which may not readily be accessible to the clientele of the Station, and reprints of articles appearing in the technical press written by members of the staff and others.

The volume and number at the top of the front cover page are merely arbitrary numbers and refer to the general publications of the University. *At the top of the inner title page* is given the number of the Engineering Experiment Station bulletin, circular, or reprint which should be used in referring to these publications.

For copies of publications or for other information address

THE ENGINEERING EXPERIMENT STATION,
UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS

UNIVERSITY OF ILLINOIS
ENGINEERING EXPERIMENT STATION

BULLETIN No. 304

A DISTRIBUTION PROCEDURE FOR THE ANALYSIS OF SLABS CONTINUOUS OVER FLEXIBLE BEAMS

A REPORT OF AN INVESTIGATION

CONDUCTED BY

THE ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS

IN COÖPERATION WITH

THE UNITED STATES BUREAU OF PUBLIC ROADS

AND

THE ILLINOIS DIVISION OF HIGHWAYS

BY

NATHAN M. NEWMARK
RESEARCH ASSISTANT PROFESSOR OF CIVIL ENGINEERING

PUBLISHED BY THE UNIVERSITY OF ILLINOIS
ENGINEERING EXPERIMENT STATION

PRICE: ONE DOLLAR

3000—6-38—14492

UNIVERSITY
OF ILLINOIS
PRESS

CONTENTS

LIST OF FIGURES

LIST OF TABLES

A DISTRIBUTION PROCEDURE FOR THE ANALYSIS OF SLABS CONTINUOUS OVER FLEXIBLE BEAMS

I. Introduction

1. *Object and Scope of Investigation.*—The purpose of this bulletin is to explain a method of analysis for certain types of continuous slabs subjected to concentrated or distributed loads. In order to study such problems as the effect of wheel loads on a slab continuous over steel stringers, or on a single-span bridge-slab with integral curbs or other stiffening beams at the edges, it is necessary to have a rapid method of analysis. Of course, analytical procedures do not answer all the questions that arise in connection with the design of a slab. Certainly the procedure presented herein is not intended directly for use as a design procedure. The analysis applies to a structure with certain ideal properties, but does not deal with such problems as the ultimate strength of the slab, effects of temperature, shrinkage, and exposure to weather.

The method of analysis is applicable to any rectangular slab simply supported on two opposite edges, with any manner of support on the other two edges, and continuous over any number and spacing of rigid or flexible simple beams transverse to the simply-supported edges. The slab may have sections of different depths or with different moduli of elasticity provided that, in each rectangular section formed by parts of the simply-supported edges and transverse lines thereto, the depth and elastic moduli are constant. It is assumed that the supporting beams offer only vertical support to the slab along a line. Under certain conditions the torsional restraint offered by the beams may be taken into account. However, in every case, it must be assumed that there are no horizontal shearing forces or frictional forces on horizontal planes between the slab and the supporting beams, or, in other words, that there is no resultant horizontal direct stress on any vertical cross-section in the slab or in a beam. An approximation to "T beam action" may possibly be made by modifying the actual stiffness of the supporting beam.

The analysis is exact in the sense that it leads to formulas in terms of infinite series that satisfy the fundamental equations of the ordinary theory of flexure of slabs* for homogeneous, elastic, and isotropic material. The formulas so obtained are the same as those given by others for certain special cases that have been analyzed. Generally

*See Appendix A for a discussion of the ordinary theory of flexure of slabs.

the distribution procedure is convenient for obtaining numerical results rather than formulas, and it is intended for this use.

The essential features of the procedure are similar to, and derived from, the moment-distribution method of analysis developed by Hardy Cross.* Obviously other analytical procedures applicable to continuous beams or continuous frames may be applied to continuous slabs in much the same way as the present analysis. Marcus† pointed out that an equation analogous to the three-moment equation could be applied to slabs continuous over rigid beams, and made analyses for a number of cases. An analysis of slabs continuous over rigid beams was also given by Galerkin‡.

A method of solution in which simultaneous equations are used to determine the constants entering into the formulas for the deflections, and consequently the moments, shears, etc., for some particular cases, is given by Weber.¶ Formulas for various quantities such as deflections, moments in the slab, and moments in supporting beams for a number of cases of rectangular slabs supported on two or three flexible beams, are given by Jensen.§ The distribution procedure described in the present report bears somewhat the same relation to other procedures and the formulas derived thereby as the moment-distribution procedure for continuous frames bears to the slope-deflection method, or the equation of three moments, and formulas for moment coefficients.

Numerical values of the quantities necessary for the analysis by the distribution procedure are tabulated in Section 22 and are explained in Chapter II. These quantities—stiffness, carry-over factors, fixed-end moments, and fixed-end reactions—are used in exactly the same way as in the analysis of continuous beams. The loading on the slab is divided into components in a manner described in Section 5, and each component is treated separately. One obtains, finally for each panel, component edge moments and edge deflections. The effects of these on the interior of the slab are considered as corrections to the effect of the load on a simply-supported panel of the slab, as is outlined in Section 17. The resultant effect in the interior or at an edge is found by superposing all the component effects.

*Hardy Cross, "Analysis of Continuous Frames by Distributing Fixed-End Moments," Trans. Am. Soc. C. E., Vol. 96, 1932, p. 1-156.
Hardy Cross and Newlin D. Morgan, "Continuous Frames of Reinforced Concrete," 1932, John Wiley and Sons, New York, Chapter IV.

†H. Marcus, "Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten," 1932, Julius Springer, Berlin, p. 201-240.

‡B. G. Galerkin, "Elastic Thin Plates," (in Russian), 1933, Gosstrojisdat, Leningrad, U.S.S.R., p. 43-74.

¶Erhard Weber, "Die Berechnung rechteckiger Platten, die durch elastische Träger unterstützt sind," Ingenieur-Archiv, Vol. 8, No. 5, 1937, p. 311-325.

§V. P. Jensen, "Solutions for Certain Rectangular Slabs Continuous Over Flexible Supports," Univ. of Ill., Eng. Exp. Sta., Bul. 303, 1938.

In general the distribution procedure for any component of the loading converges more rapidly than the corresponding distribution in a continuous beam, but the procedure must usually be applied for several components in order to get the effect of the whole loading.

Illustrative examples showing the way in which the procedure may be used, and the manner of obtaining the required numerical constants from the tables given in Section 22, are presented in Chapter VI.

The use of the procedure in studying influence surfaces for slabs is pointed out in Chapter VIII, and certain qualitative aspects of the analysis of slabs are discussed in Chapter IX.

2. *Acknowledgment.*—The work reported herein was done as a part of an investigation of the effect of concentrated loads on reinforced concrete bridge slabs being carried on by the Engineering Experiment Station of the University of Illinois in coöperation with the United States Bureau of Public Roads and the Illinois Division of Highways. The investigation is conducted under the joint administration of the Department of Theoretical and Applied Mechanics, of which PROFESSOR F. B. SEELY is head, and the Department of Civil Engineering, of which PROFESSOR W. C. HUNTINGTON is head. All the work of the Experiment Station is under the administrative supervision of DEAN M. L. ENGER, Director of the Engineering Experiment Station.

The work of the investigation is under the general direction of F. E. RICHART, Research Professor of Engineering Materials, University of Illinois, and is guided by an Advisory Committee whose personnel is as follows:

E. F. KELLEY, Chief, Division of Tests, and A. L. GEMENY, Senior Structural Engineer, representing the United States Bureau of Public Roads; ERNST LIEBERMAN, Chief Highway Engineer, and A. BENESCH, Engineer of Grade Separations, representing the Illinois Division of Highways; PROFESSOR RICHART and the writer representing the University of Illinois. Acting as consultants to the Advisory Committee are T. C. SHEDD, Professor of Structural Engineering, and W. M. WILSON, Research Professor of Structural Engineering, both of the University of Illinois. Until July, 1937, HARDY CROSS, Chairman of the Department of Civil Engineering, Yale University, and formerly Professor of Structural Engineering at the University of Illinois, was also a consultant to the committee.

The development of the analytical procedure described herein grew out of discussions with PROFESSOR CROSS, and out of work done during 1935-1936 in coöperation with H. M. WESTERGAARD, Gordon McKay

Professor of Civil Engineering and Dean of the Graduate School of Engineering of Harvard University. Acknowledgment of their part in this work is gratefully made.

The writer is also indebted to members of the staff and of the Advisory Committee, and particularly to PROFESSOR RICHART and to V. P. JENSEN, Special Research Assistant Professor of Theoretical and Applied Mechanics, for their encouragement and advice.

Detailed numerical computations for this investigation have been made by H. A. LEPPER, JR., E. D. OLSON, W. A. RENNER, and C. P. SIESS, Research Graduate Assistants, and by L. R. MARCUS, test assistant.

3. *Notation.*—The following notation is used throughout this bulletin:

x, y = horizontal rectangular coordinates. The origin of coordinates is always at a simply supported edge of the slab. The y axis is always parallel to the pair of simply supported edges of the slab.

z = vertical coordinate, positive downward.

u, v = particular values of x and y, respectively.

a, b = span lengths in the directions of x and y, respectively, of a panel of the slab.

n = integer representing number of arcs of a sine wave of deflection, moment, shear, etc., alternately plus and minus, across the span of length a.

$s = a/n$.

w = deflection of the slab or of a beam, positive downward.

h = thickness of the slab in a particular panel.

$I = h^3/12$ = moment of inertia per unit width of the slab in a particular panel.

E = modulus of elasticity of the material of the slab in a particular panel.

μ = Poisson's ratio of lateral contraction for the material of the slab in a particular panel.

$N = EI/(1-\mu^2)$ = measure of stiffness of an element of the slab in a particular panel.

E = modulus of elasticity of the material in a beam.

I = moment of inertia of the cross-section of a beam.

EI = measure of stiffness of an element of a beam.

P = concentrated load applied to the slab, positive when acting downward.

F = load per unit of length distributed over a line, positive when acting downward.

p = distributed load per unit of area, positive when acting downward.

V_x = vertical shear per unit of length acting on a cross-section of the slab normal to the x axis, positive when acting upward on the part of the slab, bounded by the cross-section, having the greater values of x, as shown in Fig. 1. This definition corresponds to that of shear in a beam.

V_y = vertical shear per unit of length acting on a cross-section of the slab normal to the y axis, positive when acting upward on the part of the slab, bounded by the cross-section, having the greater values of y, as shown in Fig. 1.

R_x, R_y = vertical reactions per unit of length defined in the same manner as V_x and V_y.

R_c = concentrated reaction at a corner, as shown in Fig. 1.

M_x, M_y = bending moments in the direction of x or y, respectively, per unit of length, acting on a section normal to the x axis or y axis, respectively, positive when producing compression at the top of the slab, as shown in Fig. 1.

M_{xy} = twisting moment in the directions of x and y per unit of length, positive when tending to produce compression at the top of the slab in the direction of the line $x = y$, as shown in Fig. 1.

M_{beam} = bending moment in a beam, positive when producing compression at the top of the beam.

F = load per unit of length acting on a beam, positive when acting downward.

R = vertical reaction per unit of length acting at an edge parallel to the x axis of a panel of the slab, positive when acting downward. See Fig. 3.

M = the bending moment M_y at the edge of a panel of the slab. See Fig. 3.

$\mathsf{R}^F, \mathsf{M}^F$ = values of R and M for a given loading when the edges of the panel of the slab parallel to the x axis are fixed against deflection and rotation.

Φ = rotation of an edge, numerically equal to the slope in the y direction of an edge parallel to the x axis of a panel of the slab, positive when in the direction of positive M at that edge; also rotation of a cross-section of a beam about a longitudinal axis, positive as indicated in particular cases.

Δ = deflection w of an edge parallel to the x axis of a panel of the slab, or deflection of a beam, positive downward.

$G = \mathsf{E}/[2(1+\mu)]$ = modulus of elasticity in shear of the material in a beam.

M_T = total twisting moment or torque acting in a cross-section of a beam, positive when acting counter to the sense of positive Φ on the part of the beam, bounded by the cross-section, lying on the side of the larger values of x.

$m = -\dfrac{dM_T}{dx}$ = moment per unit of length, causing twisting of a beam, distributed along the longitudinal axis of a beam, positive when M_T decreases with x, or positive in the same sense as positive Φ.

$\theta = \dfrac{d\Phi}{dx}$ = relative rotation per unit of length of cross-sections of the beam, positive when the total rotation Φ of a cross-section increases with x.

$J = \dfrac{M_T}{G\theta}$ = measure of torsional rigidity of a beam.

K, S, Q, T, k, q, t = elastic constants for a panel of the slab defined in Section 6.

K', T' = modified elastic constants for a panel of a slab, defined in Section 14.

U, L = elastic constants relating deflection and load, and rotation and moment, respectively, for a beam; defined in Section 8.

C_K, C_Q, C_T, C_S = dimensionless coefficients used in Equation (31).

C_M, C_R, c_m, c_r = dimensionless coefficients used in Section 9, and defined in Appendix B, Sections 3, 4, and 7.

m_x, m_y, m_{xy}, C_w = dimensionless coefficients used in Section 20, and defined in Appendix B, Section 5.

C, C_{xy} = dimensionless coefficients used in Section 19, and defined in Appendix B, Section 6.

$-\mathsf{M}_w$ = an edge moment acting with an edge deflection, defined in Equation (75), and in Appendix B, Section 6.

F_0, M_0, w_0, etc. = coefficients of sin $n\pi x/a$ or of cos $n\pi x/a$ in the expressions for F, M, w, etc., when F, M, w, etc., vary as the ordinates to a sine or cosine curve.

$$\beta = \frac{\pi b}{s} = \frac{n\pi b}{a} .$$

$$\alpha = \frac{\pi v}{s} = \frac{n\pi v}{a} .$$

$$\eta = \frac{\pi y}{s} = \frac{n\pi y}{a} .$$

$$\xi = \frac{\pi x}{s} = \frac{n\pi x}{a} .$$

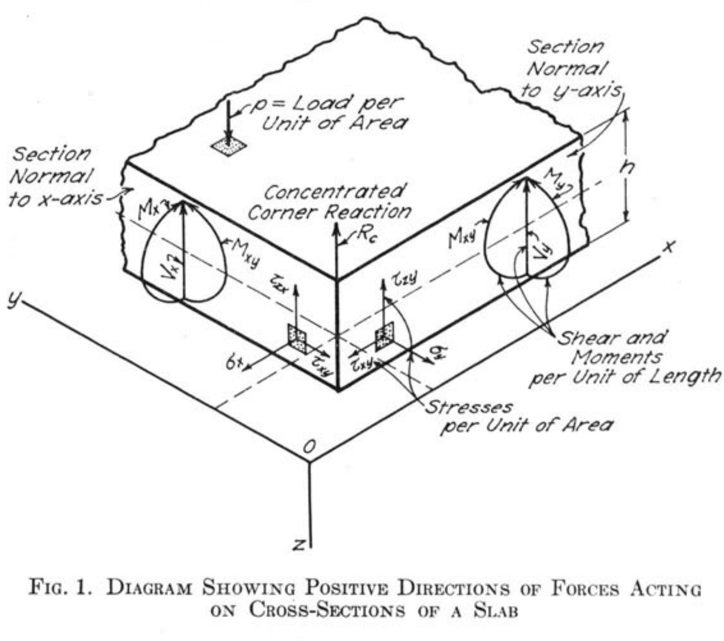


FIG. 1. DIAGRAM SHOWING POSITIVE DIRECTIONS OF FORCES ACTING ON CROSS-SECTIONS OF A SLAB

II. Method of Analysis

4. *Basis of Method of Analysis.*—The distribution procedure is based on a resolution of the loading applied to the slab into components each of which can be handled separately in a simple manner. The effect of the total load is found by superposition of the effects of the component loadings, within the limitations of the fundamental equations derived by means of the ordinary theory of flexure for slabs.

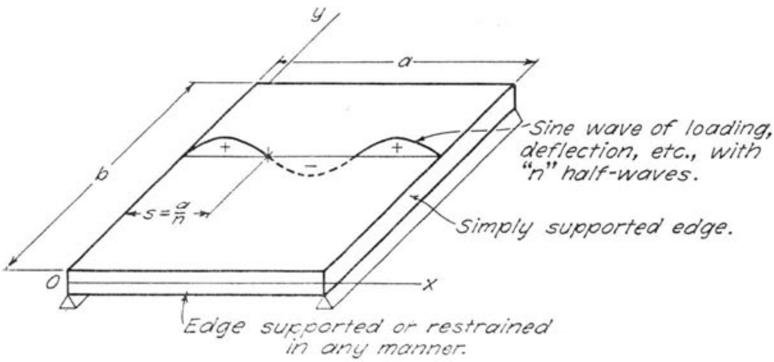


Fig. 2. Rectangular Panel of Slab Considered in the Analysis

Consider the slab shown in Fig. 2, with span a in the x direction, and with the two sides parallel to the axis of y simply supported. Let the other two sides be supported or restrained in some manner to be determined later. Let the deflection of the slab be given by the equation

$$w = Y_n\left(\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a} \tag{1}$$

in which Y_n is a function of $n\pi y/a$, and consequently is a function of y only. Such a deflection is, for constant y, a sine curve of n half-waves, alternately upward and downward, the maximum ordinates varying with y only. Along the lines $x = 0$, $x = a/n$, $x = 2a/n$, $x = 3a/n$, etc., the deflections are zero.

With the notation $s = a/n$, Equation (1) may be rewritten as

$$w = Y\left(\frac{\pi y}{s}\right)\sin\frac{\pi x}{s} \tag{2}$$

All the derivatives of w with respect to y or x will have the form of a function of $\pi y/s$ multiplied by $\sin \pi x/s$ or by $\cos \pi x/s$. The moments, shears, reactions, and the loading found from the fundamental relations of the ordinary theory of flexure for slabs are stated in Section 1 of Appendix B. It is noted that the bending moments, per unit of length, M_x and M_y, the shears and reactions, per unit of length, V_y and R_y, and the load p, are all of the same form as w and involve a function of y only, multiplied by a sine curve with a half-wave-length of s in the x direction; and the twisting moment M_{xy}, and the shears and reactions V_x and R_x involve a function of y only, multiplied by a cosine curve with a half-wave-length of s in the x direction.

Then it may be concluded that a load on the slab of the form

$$p_n = p_0 \sin \frac{\pi x}{s} \tag{3}$$

in which p_0 is a function of y only, will produce deflections of the kind given in Equation (2) when the conditions of support or restraint at the ends of the panel involve deflections, slopes, moments, and reactions that are distributed as sine curves of half-wave-length s.

Whatever the loading on the slab, if it can be resolved into component loadings p_n of the form

$$p_n = p_{0n} \sin \frac{n\pi x}{a} \tag{4}$$

where p_{0n} is a function of y only, each component may be treated separately and the effects—deflections, slopes, moments, shears, reactions—of each component may be determined from consideration of a single sine wave of loading on a slab of span s where $s = a/n$. The total load p may be expressed in the form of the trigonometric series

$$p = \sum_{n=1,2,3,\cdots} p_{0n} \sin \frac{n\pi x}{a} \tag{5}$$

and the effects of the total load may be obtained by adding all the component effects.

A strip of the slab of width a/n may be considered as a unit with supports parallel to the y axis at the sides of the strip. The load on the strip may be taken as a single sine wave with the maximum intensity of loading at the center of the strip. The actual slab may be considered as made up of n such strips with equal loads alternately upward and downward. It is necessary to handle only the one problem of a single sine wave of loading on a slab of width $s = a/n$. By taking different values of s one then obtains effects for $n = 1$, 2, 3, etc.

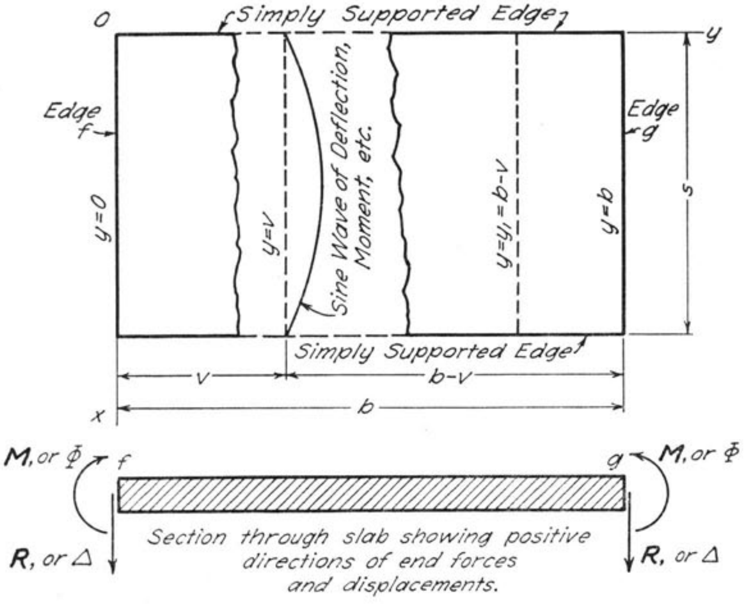


FIG. 3. SLAB WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED, SHOWING POSITIVE DIRECTIONS OF FORCES ON AND DISPLACEMENTS OF THE OTHER EDGES

Consider a panel of a slab as in Fig. 3, with two opposite edges simply supported and the other two edges, f and g, fixed. A sine wave of loading of the type given by Equation (3)

$$p = p_0 \sin \frac{\pi x}{s}$$

will cause moments and reactions at the fixed edges each distributed as a sine wave. Similarly, consider a rotation of one fixed edge of the slab with magnitude of rotation varying as a sine wave,

$$\Phi = \Phi_0 \sin \frac{\pi x}{s} \tag{6}$$

in which Φ is equal to the numerical value of the slope at any point along the edge of the panel, positive as defined in Fig. 3. According to the ordinary theory of flexure of slabs, such a rotation will produce moments at the edge rotated, and at the far edge, with magnitude varying as the ordinates to a sine curve also. In a manner entirely analogous to that used in the analysis of continuous beams by moment distribution, one may define stiffness of the edge of the slab against rotation as the ratio of the edge moment to the edge rotation, and one may define a carry-over factor for moment as the ratio of the moment at the far edge to that at the edge rotated, for sine waves of rotation.

In addition one may define other elastic constants for the slab in order to compute edge reactions, and also in order to handle edge displacements of the panel. These elastic constants are all dependent on the shape of the slab, the quantity N, and the span length between fixed supports, and their values are defined and discussed in Section 6.

It is noted that there are no statical relations between the various elastic constants for a slab as there are for a girder. For example, when the edge of the slab is rotated as is indicated by Equation (6), the vertical reactions at the edges cannot be computed by statics from the moments at the edges. However, the elastic constants for the slab automatically take into account the behavior of the slab in transmitting effects laterally to the simply-supported edges.

For each sine wave of loading on a supporting beam of span s in the x direction, it can be shown from the relations between the deflections, moments, and loads on a beam that the deflection of the beam is also a sine curve.

Since the loading and the deflections in the x direction are all given as sine waves for both slabs and supporting beams, the deflections along all sections parallel to the y axis are similar in shape. Consequently, it is possible to set up an analogous continuous girder in the y direction for each sine wave component of loading in the x direction on a continuous slab. The spans of the slab in the y direction between flexible or rigid supporting beams correspond to the spans of the girder between yielding or rigid supports. The stiffnesses and carry-over factors for the analogous continuous girder are those values for the slab as discussed in the foregoing. The essential difference between the analogous continuous girder and an actual continuous girder, however, is that the reactions and shears cannot be found by statics from the moments and the loading, but must be found by use of the elastic constants.

The distribution process may be explained as follows: For a particular sine wave component of loading on the continuous slab all the "joints" or lines along which there are supporting beams (or in special cases, changes in section, or free edges) are considered locked against rotation or deflection. The fixed-end moments and reactions for the loading or other conditions are found from tabulated values given in Section 22. In general, each joint will be statically unbalanced. Then the joints are released successively, (or simultaneously if desired), and the changes in fixed-end forces recorded. The release of a joint balances the forces acting on that joint. The release of any joint introduces additional unbalanced moments and reactions at neighboring joints. These are recorded by use of carry-over factors. The process is carried on until the unbalanced forces remaining at all joints are only negligible in amount.

Then the moments and deflections at the various supports may be computed. The loads acting on the supporting beams and the moments in the beams may be readily found. The effects in the interior of the slab are determined by superposing the following quantities:

(1) the effect of the load acting on a panel simply supported on all edges;

(2) the effects of the moments at each continuous edge;

(3) the effects of the deflections at each continuous edge.

This procedure may be varied in detail, and several possibilities are discussed in Section 17. Numerical values for performing this operation are tabulated in Section 22.

For the whole loading, one adds together the effects for all the sine wave components of the loading to get the total effect.

Any type of loading can be treated by this procedure. The manner of determining the components for some of the simpler loadings is discussed in the next section.

Illustrative examples are given in Chapter VI, where the various steps outlined are performed numerically.

It is evident that the distribution procedure is merely one of a number of possible methods of analysis for the continuous slab. In order to satisfy statical conditions, the moments normal to the joint in the members meeting at any joint must be in equilibrium, and the reactions acting on all the members at the joint must be in equilibrium. In order to satisfy conditions of continuity, the slopes normal to the joint of all members at a joint must be equal in magnitude, and the deflections of all members at the joint must be equal. These conditions may be expressed as equations in various ways, and the equations may be solved by any of a number of procedures. With the distribution procedure one does not express the equations explicitly. Instead, one satisfies the conditions by a process of successive approximations. The method is not an approximate method, since the computations may in general be carried to any desired degree of accuracy.

5. *Type of Load Components Considered.*—In general, subject to certain mathematical limitations, any distribution of loading on the rectangular slab shown in Fig. 2 can be expressed as a trigonometric series

$$p = \sum_{n=1,2,3,\cdots} p_{0n} \sin \frac{n\pi x}{a} \tag{7}$$

in which p_{0n} is a function of y only (and may be a trigonometric series itself).

A particular type of distributed load that is of interest is a loading that is independent of y; that is, a load distributed over the whole span b of the slab with the same intensity, p, at all points having the same value of x. With some minor restrictions such a load can be expressed as a Fourier series of sines similar to Equation (7), but with p_{0n} independent of y, and depending only on n. The constants p_{0n} can be determined by means of the relation

$$p_{0n} = \frac{2}{a} \int_0^a p \sin \frac{n\pi x}{a}\, dx. \tag{8}$$

The following special cases are shown in Fig. 4:

(a) A uniform load of intensity p on a strip of width $2i$, from $x = u - i$ to $x = u + i$, extending the length of the panel in the y direction. The component loads determined from (8) are

$$p_{0n} = \frac{4p}{\pi n} \sin \frac{n\pi i}{a} \sin \frac{n\pi u}{a} \tag{9}$$

and the total loading is, therefore,

$$p = \frac{4p}{\pi} \sum_{n=1,2,3,\cdots} \frac{1}{n} \sin \frac{n\pi i}{a} \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a} \tag{10}$$

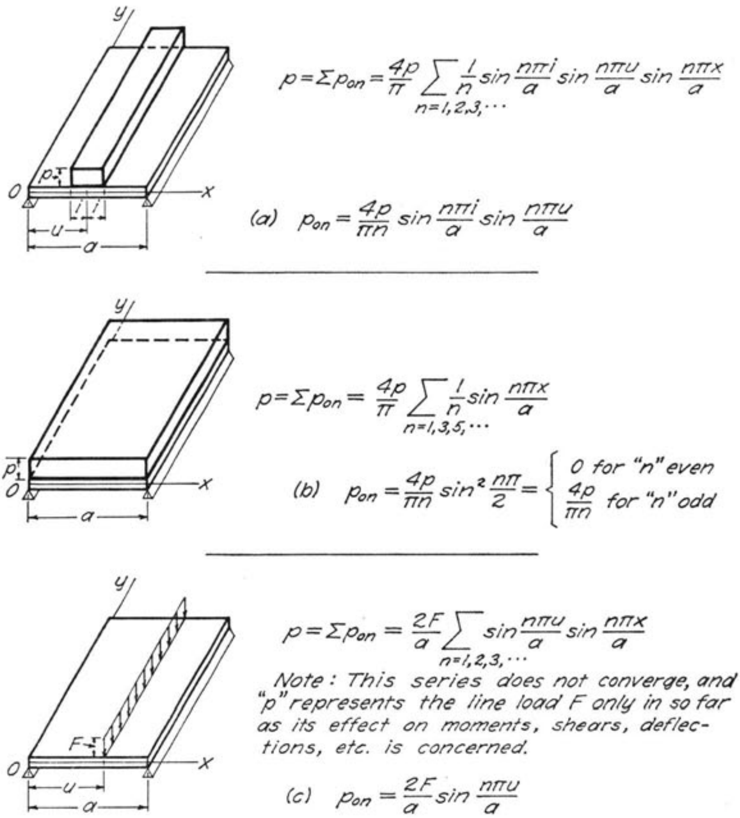


Fig. 4. Types of Distributed Loads and Line Loads and Their Component Distributed Loads

(b) A uniform load of intensity p over the whole panel. The components and total loading are obtained from (9) and (10) by setting $i = u = a/2$. One finds

$$p_{0n} = \frac{4p}{\pi n} \text{ for } n \text{ odd}, \qquad p_{0n} = 0 \text{ for } n \text{ even} \tag{11}$$

and

$$p = \frac{4p}{\pi} \sum_{n=1,3,5,\cdots} \frac{1}{n} \sin \frac{n\pi x}{a} \tag{12}$$

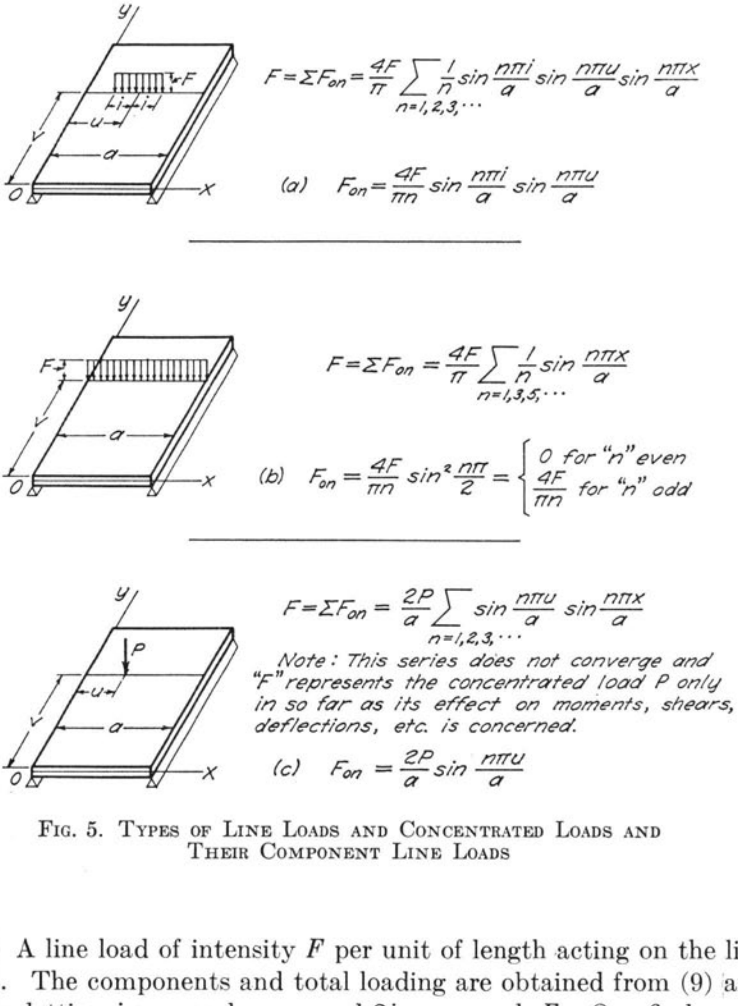


FIG. 5. TYPES OF LINE LOADS AND CONCENTRATED LOADS AND THEIR COMPONENT LINE LOADS

(c) A line load of intensity F per unit of length acting on the line $x = u$. The components and total loading are obtained from (9) and (10) by letting i approach zero, and $2ip$ approach F. One finds

$$p_{0n} = \frac{2F}{a} \sin \frac{n\pi u}{a} \tag{13}$$

and

$$p = \frac{2F}{a} \sum_{n=1,2,3,\cdots} \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a} \tag{14}$$

The trigonometric series for this case does not converge, and consequently is not equivalent to the load. The series, however, represents the load in so far as its effect on deflections, moments, shears, etc., are concerned.

Another type of loading of particular importance is that of a load distributed in some manner along a line such as the line $y = v$ in Fig. 5. With certain restrictions such a loading can be expressed in the form of a Fourier series

$$F = \sum_{n=1,2,3,\cdots} F_{0n} \sin \frac{n\pi x}{a} \tag{15}$$

where F_{0n} is a constant depending only on n and can be evaluated by means of the relation

$$F_{0n} = \frac{2}{a} \int_0^a F \sin \frac{n\pi x}{a}\, dx \tag{16}$$

which is similar to Equation (8).

The following special cases are of interest, and are shown in Fig. 5: (each of the cases can be obtained from the corresponding case of Fig. 4)

(a) A line load of intensity F uniformly distributed over a section of length $2i$, from $x = u - i$ to $x = u + i$ on the line $y = v$. The component loads determined from (16) are

$$F_{0n} = \frac{4F}{\pi n} \sin \frac{n\pi i}{a} \sin \frac{n\pi u}{a} \tag{17}$$

and the total loading is, therefore,

$$F = \frac{4F}{\pi} \sum_{n=1,2,3,\cdots} \frac{1}{n} \sin \frac{n\pi i}{a} \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a} \tag{18}$$

(b) A line load of intensity F uniformly distributed over the whole line $y = v$. The results are

$$\left.\begin{aligned} F_{0n} &= \frac{4F}{\pi n} \text{ for } n \text{ odd}, \qquad F_{0n} = 0 \text{ for } n \text{ even} \\ F &= \frac{4F}{\pi} \sum_{n=1,3,5,\cdots} \frac{1}{n} \sin \frac{n\pi x}{a} \end{aligned}\right\} \tag{19}$$

(c) A concentrated load of magnitude P at the point $x = u$ on the line $y = v$. One finds

$$\left.\begin{aligned} F_{0n} &= \frac{2P}{a} \sin \frac{n\pi u}{a} \\ F &= \frac{2P}{a} \sum_{n=1,2,3,\cdots} \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a} \end{aligned}\right\} \tag{20}$$

The trigonometric series for this case does not converge, and consequently is not equivalent to the concentrated load. The series, however, represents the effects of the load in so far as deflections, moments, shears, etc., are concerned.

Other types of loading may be divided into components in a similar manner.

6. *Elastic Constants for a Panel of a Slab.*—Consider the slab shown in Fig. 3 with two opposite edges simply supported and the edge g fixed. Let the edge f be subjected to a rotation without deflection such that the magnitude of the rotation is given by the relation

$$\Phi = \Phi_0 \sin \frac{\pi x}{s}$$

The positive direction of the rotation Φ, and the positive directions of the deflection Δ, the edge moments M, and the edge reactions R, are shown in Fig. 3. The edge moments and reaction, and the deflection of the slab along a line parallel to the simply-supported edges are shown in Fig. 6 (a). The edge moment on the edge f is distributed as a sine curve, and may be written in the form*

$$\mathsf{M}_f = K\Phi \tag{21}$$

The quantity K is then a measure of the resistance to rotation of one edge of the slab when the far edge is fixed, and is called herein the "flexural stiffness" of the slab. This is analogous to the definition of the term "stiffness" used in the moment-distribution procedure for beams.

For the same conditions the edge moment on the edge g may be written† as

$$\mathsf{M}_g = -kK\Phi = -k\mathsf{M}_f \tag{22}$$

*Appendix B, Equations (B16) and (B17).
†Appendix B, Equations (B20) and (B21).

Then the quantity " $-k$" is the ratio of the edge moments at the fixed edge to the edge moment producing rotation at the rotated edge for a sine wave of rotation, and is called herein the "flexural carry-over factor."

The quantities K and k may be used just as the corresponding quantities are used for a beam. However, in a beam, the reactions at the edges for rotation of one edge may be found from the moments by statics. In the slab the reactions must be defined by additional elastic constants, as follows:

The vertical reaction on the edge is distributed as a sine curve, and may be written* as

$$R_f = Q\Phi \tag{23}$$

and the distributed vertical reaction on the edge g may be written† as

$$R_g = -qQ\Phi = -qR_f \tag{24}$$

The quantity Q may be called a "flexure-shear stiffness" and the quantity " $-q$" a "flexure-shear carry-over factor."

Now let the edge f be subjected to a deflection without rotation such that the magnitude of the deflection is given by the relation:

$$\Delta = \Delta_0 \sin \frac{\pi x}{s} \tag{25}$$

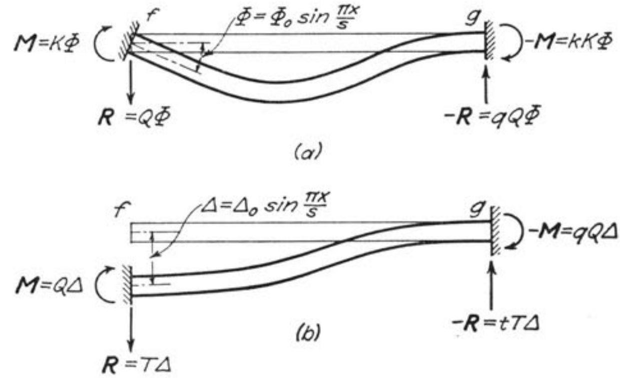
The deflection of the slab, and the edge moments and reactions are shown in Fig. 6 (b). The reaction on the edge f is distributed as a sine curve, and may be written‡ as

$$R_f = T\Delta \tag{26}$$

and T is called the "shear stiffness" of the slab. The reaction on the edge g may be written¶ as

$$R_g = -tT\Delta = -tR_f \tag{27}$$

*Appendix B, Equations (B18) and (B19).
†Appendix B, Equations (B22) and (B23).
‡Appendix B, Equations (B28) and (B29).
¶Appendix B, Equations (B31) and (B32).



Sections on lines parallel to y-axis of slab shown in Fig. 3 with (a) rotation of one edge and (b) deflection of one edge.

Summary of Elastic Constants for a Slab
Rotations, deflections, moments, and reactions distributed along the edges as sine waves.

Movement of Center of Edge "f"	At Center of Edge "f"		At Center of Edge "g"	
	Moment **M**	Reaction **R**	Moment **M**	Reaction **R**
$\Phi_o = 1$	K	Q	$-kK$	$-qQ$
$\Delta_o = 1$	Q	T	$-qQ$	$-tT$

FIG. 6. DEFINITION OF ELASTIC CONSTANTS FOR A PANEL OF A SLAB

and "$-t$" may be termed the "shear carry-over factor" for the slab. The moment on the edge f is written as

$$\mathsf{M}_f = Q\Delta \tag{28}$$

and the moment on the edge g is written as

$$\mathsf{M}_g = -qQ\Delta = -q\mathsf{M}_f \tag{29}$$

where the quantities Q and q are the same as in Equations (23) and (24), as should be evident from considerations of symmetry and Maxwell's Theorem of reciprocal deflections.

The notation for the various stiffness and carry-over factors is summarized in Fig. 6.

It is of interest to define another quantity, measuring the stiffness against rotation of an edge of a slab when the far edge is simply supported instead of fixed. The magnitude of this quantity, S, may be derived as follows:

For a unit rotation Φ_0 at edge f the center moments are K and $-kK$ at f and g, respectively, when the edge g is fixed. Now release the moment at g at the same time maintaining the edge f in a fixed condition. The additional center moments will be kK and $-k^2K$ at the edges g and f, respectively. The total center moments will be $(1 - k^2)K$ and zero at f and g, respectively, for a unit rotation Φ_0 of f. Consequently the flexural stiffness for the far end hinged is

$$S = (1 - k^2)\, K \tag{30}$$

This is exactly the same relation that is found in the case of a beam.

7. *Numerical Values of Elastic Constants for a Panel of a Slab.*—The value of each of the stiffnesses, K, S, Q, and T, may be expressed in terms of a numerical coefficient which is a function of the shape of the slab, the quantity N, and the span length b, as follows:

$$\left.\begin{aligned} K &= C_K \frac{N}{b} \;, & S &= C_S \frac{N}{b} \\ T &= C_T \frac{N}{b^3} \;, & Q &= C_Q \frac{N}{b^2} \end{aligned}\right\} \tag{31}$$

Values of the various dimensionless coefficients in the expressions of Equations (31) for the stiffness of a panel of a slab are given in Table 1 in Section 22 for various values of the ratio of the sides of the panel.

The value of Q depends on the magnitude of Poisson's ratio, μ. The magnitude of C_Q varies linearly with μ. Consequently if values of C_Q are known for any two particular values of μ, C_Q may be found by interpolation or extrapolation for any other value of μ.

It is noted that for $b/s = 0$, or for s infinitely large, the values of the various stiffness factors become the same as for a girder of span b where N is equivalent to EI for the girder. For b/s very large, the values of the various stiffness factors approach those for an infinitely long slab and the dimensionless coefficients, for b/s greater than 3.4, are given by means of the formulas listed below Table 1 which apply to an infinitely long slab.

As an example of the use of Table 1, if a slab of dimensions $s = a/n = 10$ ft., $b = 5$ ft. $= 60$ in., has a moment of inertia per unit width of 20.0 in.^{3} and a modulus of elasticity of 3.6 $(10)^6$ lb. in.$^{-2}$, the values of K and Q for $\mu = 0$ are obtained as follows:

$$N = 3.6\ (10)^6 \text{ lb. in.}^{-2}\ (20.0 \text{ in.}^3) = 72\ (10)^6 \text{ lb. in.}$$

$$\frac{b}{s} = 0.5$$

Then, to a degree of precision entirely unnecessary except for purposes of illustration, one has the following results:

$$K = 4.66875 \frac{72\ (10)^6 \text{ lb. in.}}{60 \text{ in.}} = 5.60250\ (10)^6 \text{ lb.}$$

$$Q = 6.77209 \frac{72\ (10)^6 \text{ lb. in.}}{3600 \text{ in.}^2} = 13.54418\ (10)^4 \text{ lb. in.}^{-1}$$

For $\mu = 0.10$, and all other properties of the slab the same as before,

$$K = \frac{5.60250\ (10)^6 \text{ lb.}}{1 - 0.01} = 5.65909\ (10)^6 \text{ lb.}$$

$$Q = 7.01883 \frac{72\ (10)^6 \text{ lb. in.}}{0.99\ (3600) \text{ in.}^2} = 14.17945\ (10)^4 \text{ lb. in.}^{-1}$$

Values of the various carry-over factors are given in Table 2 for various ratios of the sides of the panel. For s infinitely large, or for $b/s = 0$, the carry-over factors are the same as for a girder. The magnitude of the carry-over factors diminishes rapidly as b/s increases.

It is noted that the value of q for values of μ other than 0, 0.1, 0.2, and 0.3, may be obtained by use of the relation that the reciprocal of q varies linearly with μ. It is further noted that for any value of μ, the product of C_Q and q is independent of μ. Consequently, one may also use the relation

$$q \cdot C_Q = q^{\mu=0} \cdot C_Q^{\mu=0} \tag{32}$$

to find the value of q for other values of μ than those listed in Table 2.

8. *Elastic Constants for a Beam.*—Consider a simply-supported beam of span s with a deflection given by the equation

$$\Delta = \Delta_0 \sin \frac{\pi x}{s} \tag{33}$$

Then, by means of the relation between load and deflection of a beam,

$$\frac{d^4\Delta}{dx^4} = \frac{F}{EI} \tag{34}$$

one finds the following value for the load on the beam:

$$F = U\Delta \tag{35}$$

where

$$U = \pi^4 \frac{EI}{s^4} = 97.4091 \frac{EI}{s^4} \tag{36}$$

The quantity U is a measure of the stiffness against deflection of a beam, and corresponds to the quantity T for the edge of the slab. One may call U the "load stiffness" of a beam.

Now consider the case of a beam offering resistance to rotation about a longitudinal axis; that is, a beam with torsional stiffness. For the present study it must be assumed that the ends of the beam do not rotate about the longitudinal axis of the beam, and consequently are subjected to concentrated torques which resist the distributed moment m tending to rotate the central portions of the beam.

Let the rotations of the cross-sections of the beam be given by the relation

$$\Phi = \Phi_0 \sin \frac{\pi x}{s} \tag{37}$$

The angle of twist per unit of length, θ, or relative angle of rotation, is

$$\theta = \frac{d\Phi}{dx} = \frac{\pi}{s} \Phi_0 \cos \frac{\pi x}{s} \tag{38}$$

The torque M_T acting on any cross-section is, then,

$$M_T = GJ\theta = GJ\frac{\pi}{s}\Phi_0 \cos\frac{\pi x}{s} \tag{39}$$

Consequently the distributed moment m is given by the equation

$$m = -\frac{dM_T}{dx} = GJ\frac{\pi^2}{s^2}\Phi_0 \sin\frac{\pi x}{s} = L\Phi \tag{40}$$

where

$$L = \pi^2\frac{GJ}{s^2} = 9.86960\frac{GJ}{s^2} \tag{41}$$

The quantity L is a measure of the stiffness of the beam against rotation about a longitudinal axis, and corresponds to the quantity K for the edge of the slab. One may call L the "torque stiffness" of the beam.

9. *Fixed-End Moments and Reactions.*—Consider a load on the line $y = v$ of the slab shown in Fig. 3, where the edges f and g are fixed. Let the line load be given by the relation

$$F = F_0 \sin\frac{\pi x}{s} \tag{42}$$

With this loading, the moment at the fixed edge f is of the form*

$$M^F = -C_M Fb = -C_M bF_0 \sin\frac{\pi x}{s} \tag{43}$$

where C_M is a dimensionless numerical coefficient depending on the shape of the slab and the position of the load. Numerical values of C_M for various ratios of b to s, for v at successive tenth points of the span b, are given in Table 3. The values in Table 3 may be interpreted as influence ordinates for the maximum value in a sine wave of fixed-end moment in a slab, when $b = 1$, due to a sine wave of loading

*Appendix B, Equations (B37) and (B38).

of maximum intensity equal to unity. It will be seen that for $b/s = 0$, or for s infinitely large, the influence line for the slab becomes the same as for a fixed-end beam. For $v/b = 0$ or 1 the fixed-end moments are zero, of course. For b/s greater than 4.0 the fixed-end moment may be obtained by use of the formulas applying to the case of a slab having b infinitely large. Formulas for this case are given in Table 12, and some numerical values are given in Table 13.

For the loading given by Equation (42), the vertical reaction at the fixed edge f is of the form*

$$R^F = -C_R F = -C_R F_0 \sin \frac{\pi x}{s} \tag{44}$$

where C_R is a numerical coefficient depending on the shape of the slab and the position of the load. Numerical values of C_R are given in Table 4 for various ratios of b to s, for v at successive tenth points of the span b. The numerical values in Table 4 may be interpreted as influence ordinates for the maximum value in a sine wave of fixed-end reaction due to a sine wave of loading with maximum intensity of unity. It is noted that for $b/s = 0$, or for s infinitely large, the influence line for the slab becomes the same as for a fixed-end beam. For $v/b = 0$, C_R is 1, and for $v/b = 1$, C_R is zero, of course. For b/s greater than 4.0 or 5.0, the magnitude of the fixed-end reaction may be obtained by considering b infinitely large. Formulas for this case are given in Table 12, and numerical values in Table 13.

Tables 3 and 4 may be used to obtain the components of fixed-edge moments and reactions for any line loading that can be expressed as a trigonometric series of sines in the form of Equation (15).

Now consider a loading distributed over the whole area of the panel where the intensity of load is expressed in the form

$$p = p_0 \sin \frac{\pi x}{s} \tag{45}$$

in which p_0 is a constant. The moment at the fixed edge f is of the form†

$$M^F = -c_m p b^2 = -c_m b^2 p_0 \sin \frac{\pi x}{s} \tag{46}$$

*Appendix B, Equations (B41) and (B42).
†Appendix B, Equations (B60) and (B61).

and the reaction at the fixed edge f is of the form*

$$R^F = -c_r p b = -c_r b p_0 \sin \frac{\pi x}{s} \tag{47}$$

In these equations c_m and c_r are dimensionless numerical coefficients depending only on the shape of the slab.

Numerical values of c_m and c_r for various ratios of b to s are given in Table 5. It will be noted that for s infinitely large, $b/s = 0$, and the fixed-end moment at the center of the edge has the value

$$M_0^F = -\frac{1}{12} p_0 b^2$$

which is the same as for a fixed-end beam of length b with a uniform load of p_0 per unit length. Similarly the fixed-end reaction at the center of the edge for the same conditions has the value

$$R_0^F = -\frac{1}{2} p_0 b$$

which is the same as for the fixed-end beam. For b/s greater than 5.0 the formulas stated below Table 5, which apply to the case of b infinitely large, may be used.

Table 5 may be used to obtain the components of fixed-edge moments and reactions for any loading over the whole area of the slab that varies only with x but is independent of y, if the loading can be expressed as a trigonometric sine series of the form of Equation (7).

III. Details of the Distribution Procedure

10. *Distribution of Moments.*—When the beams supporting the edges of the various panels of the slab are infinitely stiff, or if they are temporarily prevented from deflection as one step in the calculations, the detailed procedure for distributing moments for each sine wave component of loading, on a slab continuous in one direction, is exactly the same as the procedure for a continuous girder. When the

*Appendix B, Equations (B62) and (B63).

non-deflecting beams have torsional resistance, the analogous continuous girder may be considered as supported on non-deflecting supports which offer a resistance to rotation, that is, on columns. In either case, the quantities necessary for the analysis are the fixed-end moments due to the loading or to other conditions, the flexural stiffness K for each panel, the torque stiffness L for each beam, and the flexural carry-over factor, $-k$, for each panel.

One divides the given loading on the slab into sine wave components in the x direction by means of the formulas given in Figs. 4 and 5. For line loads one finds the fixed-end moments from Table 3; for distributed loads, from Table 5. For edge deflections or rotations one finds the fixed-end moments by means of the elastic constants as shown in Fig. 6, with tabulated values in Tables 1 and 2.

The flexural stiffnesses, K, of the panels of the slab are determined from Table 1, and the absolute values of the carry-over factors, k, from Table 2. The torque stiffnesses, L, for the beams are computed by means of Equation (41). For beams which may be considered to offer no resistance to rotation, L is zero.

The sign convention used herein for moments is the so-called "designer's convention" in which positive moment produces compression at the top fibers of the slab. The flexural carry-over factor for the slab is consequently negative in sign and always less in absolute value than 0.5, the value for a beam. Hence, the distribution converges more rapidly for the slab than for a beam. The flexural stiffness for the slab is always greater than $4\ N/b$, which corresponds to $4EI/L$ for a fixed-end beam.

As in the case of continuous girders or frames, the unbalanced moment at a joint is "distributed" in proportion to the flexural stiffness of the members meeting at the joint. Therefore, one may take any arbitrary values of N, in proportion to the real values for the various panels, without affecting the results. As a matter of fact, in the whole procedure for continuous slabs one may use, for supporting beams and slabs, any proportional values of modulus of elasticity in all calculations, and one will obtain the same proportional values of deflection and slope, but the correct magnitude of forces and moments.

In general, the fixed-end moments at a "joint" (that is, the junction of two panels of the slab along the line of a supporting beam) will not be statically balanced. The "unbalanced" moment is the algebraic difference in moment at the two sides of the joint. The distribution is so made as to equalize the moments at the right and left of the joint, both in magnitude and in sign. The torsional moments in the supporting beams are most conveniently handled in the manner indicated by Cross* for the treatment of moments in columns of a continuous frame. Or one may arbitrarily assign a positive direction of rotation and of moment for each beam, and then consider the beam as being on the same side of the joint as the panel of the slab at the joint which has the same positive direction of rotation and of moment.

If a panel is simply supported at one end, it is convenient to use, for the other end, the stiffness with the far end hinged, S, where

$$S = (1 - k^2)\,K$$

as is derived in Section 6. The moment at a fixed edge, when the far edge is simply supported, is most easily obtained by correcting the fixed-end moment for both ends fixed by a single distribution to eliminate the moment at the simply-supported edge (in which the distributed moment is equal but opposite in sign to the fixed-end moment), and a carry-over of $-k$ times the distributed moment to the fixed edge. This procedure is the same as for a corresponding beam.

For certain conditions of symmetry or antisymmetry one may use modified elastic constants as a short-cut, as is explained in Section 14. One may also use modified elastic constants for the actual condition of a far edge, or any of the "direct distribution" procedures for distributing moments, but it seems hardly worth while to discuss this aspect of the subject here.

For other details of the moment-distribution procedure, the reader is referred to the various articles by Cross.†

11. *Rotation of Joints and Change in Reactions Due to Distribution of Moments.*—Consider the slab shown in Fig. 3 as representing an intermediate panel in a series of continuous slabs, where the edges f and g are continuous with other panels. For some sine-wave condition of loading or of edge distortions let the sine waves of fixed-end moment at f and at g in the panel fg be denoted by M_f^F and M_g^F, respectively. After distribution, without deflection of the "joints," let the sine wave of final end moment at f and at g in the panel fg be denoted by M_f and M_g, respectively. Then define the change in moment at f, $\delta\mathsf{M}_f$, in the panel fg due to the sine waves of rotation of f and g during the distribution process, as

$$\delta\mathsf{M}_f = \mathsf{M}_f - \mathsf{M}_f^F \tag{48}$$

---

*See, for example, Hardy Cross, "Analysis of Continuous Frames by Distributing Fixed-End Moments," Trans. Am. Soc. C. E., Vol. 96, 1932, p. 1-156.

Hardy Cross and Newlin D. Morgan, "Continuous Frames of Reinforced Concrete," 1932, John Wiley and Sons, New York, Chapter IV.

†See preceding footnote.

Similarly, define $\delta\mathbf{M}_g$ as

$$\delta\mathbf{M}_g = \mathbf{M}_g - \mathbf{M}_g^F \tag{49}$$

One may readily verify the following equations for the sine waves of rotation of f and g, Φ_f and Φ_g respectively, by observing that they give the proper changes in moment at f and g in the panel fg.

$$\left.\begin{aligned}\Phi_f &= \frac{\delta\mathbf{M}_f + k\delta\mathbf{M}_g}{(1 - k^2)\,K} = \frac{\delta\mathbf{M}_f + k\delta\mathbf{M}_g}{S}\\ \Phi_g &= \frac{\delta\mathbf{M}_g + k\delta\mathbf{M}_f}{S}\end{aligned}\right\} \tag{50}$$

These equations may be used as a check on the moment distribution since the rotations of the two edges of the panels meeting at a joint must be equal in magnitude but opposite in sign.

If there is a beam with torsional stiffness at the joint, one may compute the rotation of the joint more simply in terms of the change in distributed torsional moment, δm, in the beam, and the torque stiffness L:

$$\Phi = \frac{\delta m}{L} \tag{51}$$

In the case of a continuous girder it is possible to compute the change in reaction (or shear) at the ends of each panel or span by statics, from the changes in moment at the ends of the span. In the case of the continuous slab, however, one cannot compute the changes in reaction at the ends of the panel by statics. The elastic constants Q and q defined in Section 6 may be used to determine the changes in reaction at the ends of a panel of a slab from the rotations of the ends in the following manner:

With the rotations at f and g in the panel fg defined as Φ_f and Φ_g, one has, from the definitions of the elastic constants, the relation for the change in reaction $\delta\mathbf{R}$ due to rotation

$$\left.\begin{aligned}\delta\mathbf{R}_f &= Q\Phi_f - qQ\Phi_g\\ \delta\mathbf{R}_g &= Q\Phi_g - qQ\Phi_f\end{aligned}\right\} \tag{52}$$

Another form of Equation (52) may be obtained by substituting the values of Φ_f and Φ_g from (50) into (52):

$$\delta R_f = (1 - kq) \frac{Q}{S} \delta M_f - (q - k) \frac{Q}{S} \delta M_g \tag{53}$$

Equation (53) is convenient to use in determining the changes in reaction due to making a fixed end hinged.

12. *Distribution of Reactions.*—When the "joints," corresponding to real or imaginary supporting beams and the edges of the panels of the slab adjacent thereto, are permitted to deflect, but are not permitted to rotate during the distribution, one may distribute reactions for each sine-wave component of loading in the continuous slab in much the same manner as one distributes moments. An analogous continuous girder may be set up which may be considered as supported on springs or yielding supports, but with rotation of the joints prevented.

The quantities necessary for the analysis are the fixed-end reactions due to the loading or other conditions, the shear stiffness T for each panel, the load stiffness U for each beam, and the shear carry-over factor, $-t$, for each panel.

One divides the given loading on the slab into sine wave components in the x direction by means of the formulas given in Figs. 4 and 5. For line loads one finds the fixed-end reactions from Table 4; for distributed loads, from Table 5. For edge deflections or rotations one finds the fixed-end reactions by means of the elastic constants as shown in Fig. 6, with numerical values given in Tables 1 and 2. The shear stiffnesses, T, are determined from Table 1, and the absolute values of the shear carry-over factors, t, from Table 2. The load stiffnesses, U, for the beams are computed by means of Equation (36).

The reactions at the end of a panel of a slab, and the distributed reaction or load on a beam, are defined as positive downward. If the algebraic sum of the reactions at a joint is not zero the joint will not be in statical equilibrium. The "unbalanced" reaction at a joint is, then, the algebraic sum of the reactions in the two panels of the slab and in the beam at a joint. A sine wave of deflection of the joint, without rotation, will produce sine waves of reaction in the panels of the slab proportional to the values of T, and in the beam proportional to U. When the joint is permitted to deflect an amount sufficient to make the sum of the end reactions at the joint zero, the total end reaction due to the deflection must equal the unbalanced reaction, and the change in reaction in each member will be proportional to the value of T or U for that member. One may state this in another manner: the unbalanced reactions are distributed to the connecting members at a joint in the ratio of their shear stiffness or load stiffness. A sine wave of deflection of a particular joint, without rotation, produces a sine wave of reaction at the far edge of the panel of a slab equal to the reaction "distributed" to the edge of the panel at the joint multiplied by the shear carry-over factor, $-t$.

The procedure for distributing reactions without permitting rotations is consequently as follows:

(a) Find the reactions in the members meeting at all joints in the structure when the joints are held so that they cannot deflect nor rotate. These are called fixed-end reactions, and may be due to sine waves of loading (or also to sine waves of displacement, in the general case).

(b) The algebraic sum of the fixed-end reactions at any joint is called the unbalanced reaction. Distribute the unbalanced reaction among the connecting members in proportion to the elastic constant defined for each member as the shear or load stiffness, T for slabs, U for beams.

(c) Multiply the reaction distributed to each member at a joint by the shear carry-over factor for that member, and introduce this "carried-over" reaction at the other end of the member.

(d) The reactions "carried over" are now new fixed-end reactions and must be distributed. This process is continued until the unbalanced reactions remaining at any joint are negligible. The sum of all the reactions, fixed, distributed, and carried over to each end of each slab, and in the beam, are added together to get the true sine-wave component of reaction at the end of a particular member.

It is evident, then, that for each sine wave of loading on the continuous slab, one may distribute reactions (when rotation of the joints is prevented) in a manner similar to that used for the distribution of moments. The sign convention is different for reaction distribution than for moment distribution, since the unbalanced reaction at a joint is distributed in such a way as to make the sums of the joint reactions equal to zero. The shear carry-over factors for the slab are always negative in sign, and always less in absolute value than 1.0.

The convergence of the process for distributing reactions may be considerably slower than for distributing moments, since the shear carry-over factors may be almost equal to unity. However, one may hasten the convergence in distributing reactions by the procedure of writing in arbitrary deflections of all joints, generally equal deflections so as to simplify the calculations, in such a way as very nearly to balance the joint before formal distribution is started.* This has the effect of putting most of the reaction into the supporting beam where it belongs. It is noted that, for equal deflections of the two edges of the panel of the slab, the shear stiffness becomes $(1 - t)T$ instead of T. Where t is nearly equal to one, the quantity $(1 - t)T$ is very small. Where t is small it is not necessary to write in fixed-end reactions corresponding to uniform deflections, since the ordinary procedure will converge rapidly enough.

One may use modified elastic constants for certain symmetrical or antisymmetrical conditions, as in the procedure for distributing moments. One may also adapt any of the so-called "direct distribution" procedures to the problem of distributing reactions.

13. *Deflection of Joints and Change in Moments Due to Distribution of Reactions.*—One may compute the change in moment at the ends of each panel or span due to distributing reactions without rotation of the joints in a manner entirely similar to the procedure described in Section 11 for computing the change in reactions due to distributing moments without deflection of the joints. The following notation and equations follow directly from the discussion in Section 11:

$\mathbf{R}^F$ = sine wave of fixed-end reaction due to sine wave of loading or of edge distortion

$\mathbf{R}$ = final reaction after distribution, without rotation of the joints

$$\delta\mathbf{R} = \mathbf{R} - \mathbf{R}^F \tag{54}$$

$$\Delta_f = \frac{\delta\mathbf{R}_f + t\delta\mathbf{R}_g}{(1 - t^2)\,T} \tag{55}$$

$\delta\mathbf{F}$ = change in load (or reaction) on the beam due to the distribution

$$\Delta = \frac{\delta\mathbf{F}}{U} \tag{56}$$

*This procedure is similar to that suggested by Cross for the analysis of wind stress in a multi-story building. See Hardy Cross and Newlin D. Morgan, "Continuous Frames of Reinforced Concrete," 1932, John Wiley and Sons, New York, p. 228-232.

Equation (55) will generally be used to find the deflection of a joint only where an imaginary beam with zero stiffness has been introduced as an intermediate step in the calculations.

The change in moment at end f of panel fg due to the deflections without rotation of f and g may be stated in terms of the elastic constants defined in Fig. 6:

$$\delta M_f = Q\Delta_f - qQ\Delta_g \tag{57}$$

14. *Use of Modified Elastic Constants.*—The sine wave of rotation Φ_f at end f of a panel fg continuous with other panels, is expressed in terms of the sine waves of change in moment, δM_f and δM_g at f and g, by Equation (50), which may be restated as follows:

$$\Phi_f = \frac{\delta M_f + k\delta M_g}{(1 - k^2)\,K} = \frac{\delta M_f}{K}\,\frac{1 + k\dfrac{\delta M_g}{\delta M_f}}{1 - k^2} \tag{58}$$

The ratio of the change in moment δM_f to the rotation Φ_f, when end g also rotates according to some definite rule, may be defined as the modified flexural stiffness K', where, for the end f, K'_f is defined by the relation, obtained from (58),

$$K'_f = \frac{\delta M_f}{\Phi_f} = K\,\frac{1 - k^2}{1 + k\dfrac{\delta M_g}{\delta M_f}} \tag{59}$$

The use of this formula permits various shortcuts in the analysis, as in the case of continuous beams. The more or less obvious cases of general utility are summarized here.

(a) End g is hinged, or $\delta M_g = 0$:

$$K'_f = K\,(1 - k^2) \tag{60}$$

(b) The deformation of span fg is symmetrical, or $\delta M_g = \delta M_f$:

$$K' = K\,(1 - k) \tag{61}$$

(c) The deformation of span fg is antisymmetrical, or $\delta M_g = -\delta M_f$:

$$K' = K\,(1 + k) \tag{62}$$

Similar relations are readily derived for modified shear stiffness T' by an obvious substitution of δR for δM, t for k, and T for K:

$$T'_f = \frac{\delta R_f}{\Delta_f} = T \frac{1 - t^2}{1 + t \dfrac{\delta R_g}{\delta R_f}} \tag{63}$$

For symmetry or equal deflections at f and g, $\delta R_g = \delta R_f$, and

$$T' = T\,(1 - t) \tag{64}$$

For antisymmetry, or for equal and opposite deflections at f and g, $\delta R_g = -\delta R_f$, and

$$T' = T\,(1 + t) \tag{65}$$

Other types of modified stiffness are useful in particular cases. These are generally more easily derived by use of numerical values in a preliminary distribution of moments or reactions, or both, than by substitution in a formula. Examples are shown in Fig. 7 of the method of obtaining the modified shear stiffness for a panel with one end hinged. If the hinged end is deflected, one finds

$$T' = T - \frac{Q^2}{K} \tag{66}$$

and if the fixed end is deflected,

$$T' = T - q^2 \frac{Q^2}{K} \tag{67}$$

In either case, the reaction at the far end is

$$-tT + q \frac{Q^2}{K} \tag{68}$$

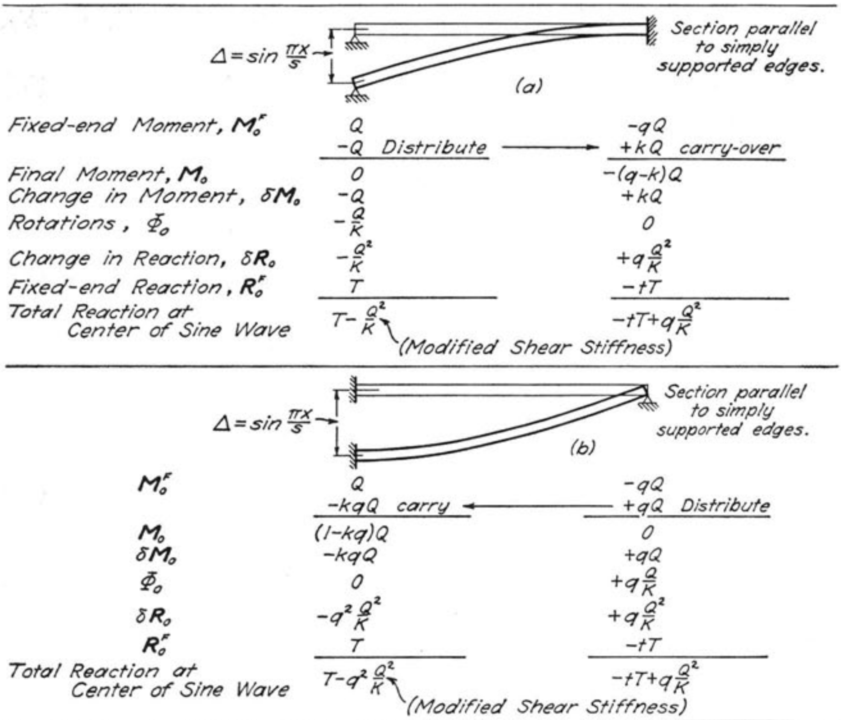


FIG. 7. MODIFIED SHEAR STIFFNESS FOR A PANEL WITH ONE END HINGED

15. *Types of Distribution Procedures Available.*—The most workable procedure for the analysis of slabs continuous over flexible beams is similar in many respects to the "Indirect Method"* used by Cross for the study of problems involving sidesway or deflection of supports. An outline of the procedure for slabs is given in the following:

Indirect Method

(1) Assume temporarily that all joints in the continuous slab do not deflect. Distribute moments due to the sine waves of loading as in Section 10, and find the total unbalanced reactions at each of the joints by the procedure described in Section 11.

(2) Assume some arbitrary pattern of deflections of the joints, say a deflection of one joint with no deflection of the others. Write fixed-end moments in all members, and distribute without further deflection. Then find the unbalanced reactions acting at all the joints.

*Hardy Cross and Newlin D. Morgan, "Continuous Frames of Reinforced Concrete," 1932, John Wiley and Sons, New York, p. 105-117, and especially Figs. 27 and 28.

(3) Repeat step (2) to get as many sets of independent displacements as there are flexible beams in the structure.

(4) Combine proportions of the sets of independent displacements in such a manner that the sum of the combined displacements and the original loading will correspond to zero unbalanced reactions at the joints.

The procedure works best when the particular displacements in steps (2) and (3) are taken in such a way as to correspond most nearly to sets of loading conditions giving only one joint loading at a time.

The procedure will always work and yield a solution. For a large number of flexible beams it generally involves a solution of simultaneous equations. In general, the equations may be solved by successive approximations very readily. Where the panels are of such shape that the distance between joints, b, is less than 0.1 to 0.2 of the span s, when the supporting beams are also very flexible, some care is necessary in choosing the independent displacements if it is desired to use successive approximations in combining them. Otherwise the method is entirely automatic.

A very straightforward method for the analysis of slabs continuous over flexible beams is a procedure of successively distributing shears and reactions as outlined in the following:

Successive Distribution

(1) Consider all joints locked against rotation and deflection. Compute fixed-end moments and fixed-end reactions due to the given sine waves of loading or other conditions of deformation.

(2) Distribute the unbalanced moments, and carry over, and repeat this process until the unbalanced moments at all joints are negligible, all the while preventing deflection of the joints, as in Section 10.

(3) Compute the changes in fixed-end reactions due to the distribution of moments, and add these to the original fixed-end reactions, by the procedure described in Section 12.

(4) Distribute the unbalanced reactions without permitting rotation of the joints and carry over and repeat this process until the unbalanced reactions at all joints are negligible, as in Section 12.

(5) Compute the changes in fixed-end moment due to distributing reactions, by the procedure described in Section 13, and distribute these moments as in step (2).

Repeat this process until all unbalanced moments and reactions are negligible. The order of the various steps may be interchanged. One may distribute the reactions first and then the moments. The convergence will generally be slow unless the beams are quite stiff compared with the slab, or unless the shear carry-over factors for the slab are small. The procedure works best for slabs where the ratio of the distance between joints b to the span between simply-supported edges s is greater than about 0.5 or where the quantity U for the beam at any joint is at least of the same magnitude as the quantities T for the panels of the slab adjacent to the joint.

In Chapter VI, which contains several illustrative problems, examples are given of each of the foregoing procedures for the analysis.

Other procedures are possible for the analysis of slabs continuous over flexible beams. For example, one may write simultaneous equations, in terms of the displacements Δ and the rotations Φ of the various joints, which express the statical equilibrium of the joint, in a manner analogous to the slope-deflection procedure for continuous frames. The elastic constants for such equations may be obtained from those given in Section 6.

IV. Manner of Using Results of the Distribution Procedure

16. *Effects on Supporting Beams.*—In the case of a slab continuous over rigid supports, for each value of n, the moments $\mathsf{M} = M_y$ in the slab are found at the supporting beams directly from the analysis. The values of the reactions R at the edges of the slab can be obtained by adding the fixed-end reactions and the reactions due to rotations of the edges. Since the sum of all the reactions at a joint must equal zero, one can obtain the reaction F on the supporting beam at a joint.

In the case of a slab continuous over flexible beams, one obtains for each value of n either the moments M and the deflections Δ at the various joints, or the moments M and the reactions R and F, depending on the particular procedure used. The reaction on the beam and the deflection of the beam are related by the simple equation, for each sine wave of deflection or reaction,

$$\Delta = \frac{\mathsf{F}}{U} \tag{69}$$

The loading on a beam, F, is related to the moment in the beam, M_{beam}, by the equation, derived from statics,

$$\mathsf{F} = -\frac{d^2 M_{\text{beam}}}{dx^2} \tag{70}$$

Consequently, for each value of n, one obtains the result

$$M_{\text{beam}} = \frac{s^2}{\pi^2}\,\mathsf{F} = \frac{s^2}{\pi^2}\,\mathsf{F}_0 \sin\frac{\pi x}{s} \tag{71}$$

The resultant deflection, resultant load, or resultant moment in the beam is obtained simply by adding all the component sine waves of deflection, load, or moment due to the various sine-wave components of loading or distortion, or other causes, on the structure.

17. *Deflections and Moments in the Panels of the Slab.*—The deflections, moments, shears, etc., in the interior of a panel of the slab, simply supported on two opposite sides and supported by beams or continuous with other panels on the other two sides, such as f and g in Fig. 3, may be obtained by superposition of the following effects:

(1) the effect of the load acting on a simply supported rectangular slab;

and, for each value of n,

(2) the effect of the deflection Δ_f at the edge f,
(3) the effect of the deflection Δ_g at the edge g,
(4) the effect of the moment M_f at the edge f,
(5) the effect of the moment M_g at edge g.

The loading, of course, is known, and the edge deflections and edge moments on the panel are obtained directly from the results of the distribution procedure.

In general it is somewhat simpler to consider a fictitious moment acting with the deflection at an edge. Then the effects superposed are:

(1) the effect of the loading on a simply supported slab;

and, for each value of n,

(2) the effect of the deflection Δ_f of edge f plus the effect of a fictitious moment $-\mathsf{M}_{wf}$ at edge f proportional to the deflection
(3) the effect of Δ_g and $-\mathsf{M}_{wg}$ at edge g
(4) the effect of the moment M_f acting at edge f minus the moment $-\mathsf{M}_{wf}$ introduced in step (2), or the effect of $\mathsf{M}_f + \mathsf{M}_{wf}$
(5) the effect of $\mathsf{M}_g + \mathsf{M}_{wg}$ at edge g.

The quantities added together may, if desired, be different from those outlined. For example, one may add the effect of the loading on a slab with two edges fixed and two edges simply supported, the effect of deflections of the fixed edges, and the effect of rotations of the fixed edges. In whatever manner the computation is carried out the facility with which results are obtained will depend upon the information available for the various quantities added together.

For a particular span in a continuous girder one may compute effects within the span by statics from the edge moments and reactions, but in a continuous slab one cannot compute effects within a span by statics. Numerical coefficients for the effects within a panel of the continuous slab may be obtained, however, and used as described in Sections 19 and 20.

Illustrations of the procedure are described in Chapter VI.

18. *Concentrated or Uniform Load on a Simply Supported Rectangular Slab.*—The effect of a loading on the deflections, moments, shears, reactions, and other quantities of interest in a simply-supported rectangular slab have been treated by a number of writers. The effect of a uniform load on a simply-supported rectangular slab is discussed and summarized with tables and curves in works by A. Nádai, H. M. Westergaard, H. Leitz, and others.* A summary of the most important results, that is, the bending moments M_x and M_y at the center of the panel, and the twisting moments at the corners, is given by Westergaard.†

The effect of a concentrated load on a rectangular slab may be obtained by adding the effects due to a number of upward and downward concentrated loads on a slab infinitely long, as is shown by Westergaard.‡ Numerical values for use in this computation are given by Westergaard in tables and curves.¶

The various numerical results referred to are given for different values of Poisson's ratio. The following relations§ may be found useful for the determination of the effects for various values of

*A. Nádai, "Die elastischen Platten," 1925, Julius Springer, Berlin, p. 120-130.
H. M. Westergaard and W. A. Slater, "Moments and Stresses in Slabs," Proc. American Concrete Institute, Vol. 17, 1921, p. 415-538, see p. 431.
H. Leitz, "Die Berechnung der frei aufliegenden, rechteckigen Platten," Forscherarbeiten auf dem Gebiete des Eisenbetons, Vol. 23, 1914, p. 1-59.
H. Hencky, "Der Spannungszustand in rechteckigen Platten," 1913, R. Oldenbourg, Berlin.

†H. M. Westergaard and W. A. Slater, "Moments and Stresses in Slabs," Proc. American Concrete Institute, Vol. 17, 1921, p. 415-538, see page 431.

‡H. M. Westergaard, "Computation of Stresses in Bridge Slabs Due to Wheel Loads," Public Roads, Vol. 11, No. 1, 1930, p. 1-23, see p. 17 and 19.

¶See preceding reference, p. 8-9, and 12-15. It is noted that the values given by Westergaard are for Poisson's ratio $\mu = 0.15$.

§See, for example, H. M. Westergaard, "Formulas for the Design of Rectangular Floor Slabs and the Supporting Girders," Proc. American Concrete Institute, Vol. 22, 1926, p. 26-43, especially p. 29.

Poisson's ratio. These relations apply only to rectangular slabs where each of the four edges is held simply supported or fixed:

$$\left.\begin{aligned} M_x^{\mu=0} &= \frac{M_x - \mu M_y}{1 - \mu^2} \\ M_y^{\mu=0} &= \frac{M_y - \mu M_x}{1 - \mu^2} \\ M_{xy}^{\mu=0} &= \frac{1 + \mu}{1 - \mu^2} M_{xy} \\ w^{\mu=0} &= \frac{1}{1 - \mu^2} w \end{aligned}\right\} \tag{72}$$

$$\left.\begin{aligned} M_x &= M_x^{\mu=0} + \mu M_y^{\mu=0} \\ M_y &= M_y^{\mu=0} + \mu M_x^{\mu=0} \\ M_{xy} &= (1 - \mu)\, M_{xy}^{\mu=0} \\ w &= (1 - \mu^2)\, w^{\mu=0} \end{aligned}\right\} \tag{73}$$

19. *Deflection of One Edge of a Panel Combined With a Moment at the Edge.*—Let the slab shown in Fig. 3 be simply supported on the two sides parallel to the y axis and also on edge g. Consider a deflection Δ of edge f

$$\Delta = \Delta_0 \sin \frac{\pi x}{s} \tag{74}$$

combined with an edge moment $-\mathsf{M}_w$ at edge f

$$-\mathsf{M}_w = -\mathsf{M}_{0w} \sin \frac{\pi x}{s} = -(1 - \mu) \frac{N\pi^2}{s^2} \Delta_0 \sin \frac{\pi x}{s} \tag{75}$$

The bending moments M_x and M_y in the interior of the panel on the line $y = v$ are given by the equation

$$M_x = -M_y = C\mathsf{M}_w \tag{76}$$

where C is a dimensionless coefficient depending on the dimensions of the slab and on the point considered. The twisting moment M_{xy} on the line $y = v$ is given by the equation

$$M_{xy} = C_{xy}(1 - \mu)\frac{N\pi^2}{s^2}\Delta_0 \cos\frac{\pi x}{s} = C_{xy}\mathsf{M}_{0w}\cos\frac{\pi x}{s} \qquad (77)$$

where C_{xy} is a dimensionless coefficient also. The deflection w on the line $y = v$ is given by the expression

$$w = C\Delta \qquad (78)$$

where C has the same value as in Equation (76).

Equations (76), (77) and (78) are derived from corresponding expressions in Section 6 of Appendix B, where formulas are given for the effect of a deflection and edge moment applied at edge g.

It is noted that the bending moments M_x and M_y and the deflection w at a point in a panel of a slab due to an edge deflection combined with an edge moment, according to Equations (74) and (75), are stated in terms of a coefficient depending on the distance v in the y direction from the distorted edge to the point considered. These effects are independent of the sense of positive direction of y. However, the manner of definition of M_{xy} involves a positive direction of the y axis; therefore, one must take due account of the positive directions of the axes in determining M_{xy} due to distortions of both edges f and g of the slab.

With the notation and directions of Fig. 3, the twisting moment M_{xy} on the line $y = v$, due to distortions of the edge $y = b$ corresponding to Equations (74) and (75), is of the same form as Equation (77) for the line $y = b - v$, but is opposite in sign to Equation (77).

Numerical values of C are given in Table 6 for various values of b/s and for v at successive tenth points of the span b. For $v = 0$, C is 1.0, and for $v = b$, C is zero. The values of C for b/s greater than 4.0 are practically the same as for b infinitely large, and may be obtained from Tables 12 and 13.

Numerical values of C_{xy} are given in Table 7. For b/s greater than 4.0 the values of C_{xy} are practically the same as the values of C. For b/s very small, less than 0.1, the value of C_{xy} approaches the quantity $\dfrac{s}{\pi b}$.

It is noted that the edge moment $-\mathsf{M}_w$ is a fictitious moment that is applied with the edge deflection merely to simplify the calculations and to make it easier to handle Poisson's ratio. One may state the effects due to a deflection alone without edge moment as the sum of three tabular values given in Chapter V, but it seems unwise to do this. It is evident from the equations that all moments are stated in terms of the fictitious edge moments, and the deflections in terms of the actual edge deflections. The effects due to the actual deflection plus the fictitious edge moment must be combined with the effects due to the actual edge moment minus the fictitious edge moment in order to get the true effects due to edge deflection and edge moment.

20. *Moment Without Deflection of One Edge of a Panel.*—Consider the panel of the slab shown in Fig. 3 where the edge g and the edges parallel to the y axis are simply supported, and the edge f is supported without deflection, but subjected to the moment M, where

$$\mathsf{M} = \mathsf{M}_0 \sin \frac{\pi x}{s} \tag{79}$$

The bending moments M_x and M_y and the twisting moment M_{xy} on the line $y = v$ may be stated in the form:

$$\left.\begin{aligned} M_x &= (m_x + \mu m_y)\, \mathsf{M}_0 \sin \frac{\pi x}{s} \\ M_y &= (m_y + \mu m_x)\, \mathsf{M}_0 \sin \frac{\pi x}{s} \end{aligned}\right\} \tag{80}$$

$$M_{xy} = (1 - \mu)\, m_{xy} \mathsf{M}_0 \cos \frac{\pi x}{s} \tag{81}$$

and the deflection w on the line $y = v$ may be stated as

$$w = C_w \frac{b^2}{N} \mathsf{M}_0 \sin \frac{\pi x}{s} \tag{82}$$

where m_x, m_y, m_{xy}, and C_w are dimensionless coefficients. The various coefficients are independent of Poisson's ratio; consequently values of two coefficients are combined in order to obtain bending moments.

Equations (79) to (82), inclusive, are derived from corresponding expressions in Section 5 of Appendix B.

As in Section 19, the bending moments M_x and M_y and the deflection w at a point in a panel due to a moment applied at one edge are stated in terms of a coefficient depending on the distance v in the y direction from the edge at which the moment is applied, to the point considered. The effects are independent of the direction of positive y, or, in other words, m_x, m_y, and C_w are independent of the sign of v.

However, the twisting moment M_{xy} on the line $y = v$ due to a sine wave of moment applied at $y = b$ is of the same form as Equation (81) for the line $y = b - v$, but is opposite in sign to (81). In other words, m_{xy} is of the same sign as v, where v is the distance from the edge at which moment is applied to the point considered.

Numerical values of m_x, m_y, m_{xy}, and C_w are given in Tables 8, 9, 10, and 11, respectively, for various values of b/s and for v at successive tenth points of the span b. For $v = 0$, the coefficients are $m_x = 0$, $m_y = 1$, $C_w = 0$; and for $v = b$, the coefficients m_x, m_y, and C_w are zero. For b/s greater than 4.0 the values of all these coefficients are practically equal to the corresponding coefficients for b infinitely large, for which case equations and numerical values are given in Tables 12 and 13.

The use of some of the various coefficients mentioned herein is demonstrated in Chapter VI in the illustrative problems.

21. *Dimensional Relations.*—Certain dimensional relations are of interest in that they permit extension of results obtained for one particular size of structure to another of similar proportions. For example, the slabs shown in Fig. 8 (a) and (b) are similar. It is noted that all horizontal dimensions are changed for Fig. 8 (b) in the proportion r, the span between simple supports becoming ra instead of a. Further, the values of the quantity N, which is a measure of the stiffness of elements of the slab, must be in the same proportion among the different panels in (b) as they are in (a), and the values of Poisson's ratio μ must be equal, panel for panel, in (b) and (a). If the quantities N are the same for corresponding panels in (a) and in (b) of Fig. 8 the quantity EI for corresponding beams in Fig. 8 (b) must be changed in the proportion r, the same proportion as the linear dimensions, for similarity to obtain.

When the two structures are loaded at the same points or along the same lines, or loaded similarly, and if the total loads are the same the following relations hold: the deflections in (b) will be r^2 times those in (a) at corresponding points; the slopes in (b) will consequently be r times the corresponding slopes in (a); and the curvatures and twists in (b) will be equal to the curvatures and twists in (a). Then the

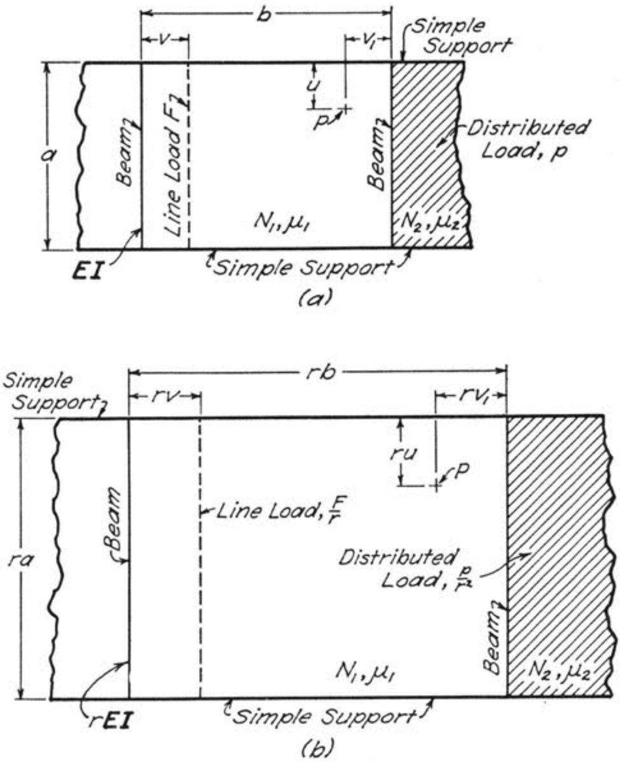


FIG. 8. SIMILAR SLABS AND LOADING CONDITIONS

bending moments and twisting moments per unit of length will be equal at corresponding points in the two structures. The shear per unit of length in (b) will be $1/r$ times the corresponding shears in (a), and the reactions in (b) will have the same relations to the reactions in (a) as the shears have. Consequently, line loads per unit of length in (b) must be $1/r$ times the corresponding line loads in (a), and total loads must be equal in the two structures. From this requirement one readily derives the fact that distributed loads per unit of area in (b) must be $1/r^2$ times the corresponding quantities in (a).

Now consider the slab shown in Fig. 8 (a) only. Let the quantity N be changed for each panel in the same proportion; then the quantity EI for the various beams must be changed in the same proportion also. With the loads unchanged, the moments, reactions, and shears will be unchanged. However, the slopes and the deflections will be changed so as to vary as $1/N$ at corresponding points.

The use of these principles permits one to write moment coefficients and reaction coefficients for given loadings on similar structures from the results obtained in a particular numerical case. That is, if in a particular type of structure (say, for example, a single panel of a slab simply supported on three sides, and supported on the fourth side by a flexible beam) one finds for a load of 2000 lb. at a given point a moment of 100 lb. at the load point in the x direction, one may write the general relationship that in a similar structure the moment under the load in the x direction is 1/20th the load.

V. Tabulated Numerical Constants

22. *Tabulated Numerical Constants for Use in the Distribution Procedure.*—Tables 1 to 13, inclusive, contain numerical values for the constants entering into the analysis by means of the distribution procedure. The numerical values given are reported generally to six or more significant figures for the larger numbers. This is, of course, considerably greater accuracy than is required or justified in any analysis of slabs. One may take only as many significant figures from the table as are desired, however, in an analysis.

For a panel of a slab, the various stiffnesses are obtained from Table 1; the carry-over factors from Table 2; the fixed-end moments for line loads from Table 3; the fixed-end reactions for line loads from Table 4; the fixed-end moments and fixed-end reactions for distributed loads from Table 5.

Values of bending moments, twisting moments, and deflections within a panel of a slab due to a deflection combined with a moment at one edge are obtained from Tables 6 and 7; and those due to a moment without deflection at one edge, from Tables 8, 9, 10, and 11.

Table 12 contains a summary of the various formulas applying to a slab infinitely long in one direction. These formulas are obtained by letting b approach infinity in the corresponding expressions for the finite slab. Some numerical values of the dimensionless coefficients stated in Table 12 are given in Table 13.

TABLE 1

NUMERICAL VALUES OF DIMENSIONLESS COEFFICIENTS IN THE EXPRESSIONS FOR THE STIFFNESSES OF A PANEL OF A SLAB

The various stiffnesses are defined in Fig. 6, and are expressed as follows:

$$K = C_K \frac{N}{b}, \quad S = C_S \frac{N}{b}, \quad T = C_T \frac{N}{b^3}, \quad Q^{\mu=0} = C_Q^{\mu=0} \frac{N}{b^2}, \quad Q^{\mu=0.2} = C_Q^{\mu=0.2} \frac{N}{b^2}.$$

Ratio of Sides of Panel b/s	C_K	C_S	C_T	$C_Q^{\mu=0}$	$C_Q^{\mu=0.2}$
0.0	4.00000	3.00000	12.00000	6.00000	6.00000
0.1	4.02634	3.03944	12.24043	6.02022	6.03996
0.2	4.10565	3.15729	13.00444	6.08658	6.16554
0.3	4.23864	3.35198	14.41982	6.21566	6.39331
0.4	4.42617	3.62049	16.69787	6.43331	6.74914
0.5	4.66875	3.95801	20.12906	6.77209	7.26557
0.6	4.96608	4.35792	25.07433	7.26783	7.97844
0.7	5.31663	4.81204	31.95119	7.95581	8.92303
0.8	5.71739	5.31138	41.21523	8.86708	10.13039
0.9	6.16397	5.84689	53.33967	10.02565	11.62453
1.0	6.65086	6.41019	68.7962	11.44703	13.42095
1.1	7.17191	6.99406	88.0405	13.13818	15.52663
1.2	7.72088	7.59266	111.5041	15.09880	17.94124
1.3	8.29187	8.20150	139.5926	17.32325	20.65918
1.4	8.87967	8.81726	172.6888	19.80277	23.67166
1.5	9.47993	9.43761	211.1584	22.52733	26.96865
1.6	10.08916	10.06094	255.3563	25.48704	30.54027
1.7	10.70472	10.68617	305.6321	28.67304	34.37768
1.8	11.32463	11.31259	362.3342	32.07791	38.47342
1.9	11.94748	11.93976	425.8127	35.69572	42.82157
2.0	12.57228	12.56738	496.4207	39.52192	47.41761
2.1	13.19837	13.19529	574.5147	43.55316	52.25815
2.2	13.82528	13.82336	660.4544	47.78701	57.34078
2.3	14.45272	14.45153	754.6025	52.22176	62.66380
2.4	15.08049	15.07976	857.3244	56.85623	68.22601
2.5	15.70848	15.70803	968.987	61.68962	74.02662
2.6	16.33659	16.33632	1089.959	66.72139	80.06509
2.7	16.96479	16.96462	1220.610	71.95119	86.34108
2.8	17.59303	17.59293	1361.310	77.37879	92.85433
2.9	18.22130	18.22125	1512.431	83.00405	99.60472
3.0	18.84960	18.84956	1674.343	88.8269	106.5921
3.1	19.47790	19.47788	1847.419	94.8471	113.8165
3.2	20.10621	20.10619	2032.029	101.0649	121.2779
3.3	20.73452	20.73451	2228.546	107.4801	128.9761
3.4	21.36283	21.36283	2437.342	114.0927	136.9112
3.5	21.99115*	$C_S = C_K$	2658.789*	120.9027*	145.0832*
3.6	22.61947	$C_S = C_K$	2893.258	127.9101	153.4921
3.7	23.24779	$C_S = C_K$	3141.122	135.1149	162.1379
3.8	23.87610	$C_S = C_K$	3402.753	142.5171	171.0205
3.9	24.50442	$C_S = C_K$	3678.523	150.1167	180.1400
4.0	25.13274	$C_S = C_K$	3968.803	157.9137	189.4964
4.2	26.38938	$C_S = C_K$	4594.386	174.0998	208.9198
4.4	27.64602	$C_S = C_K$	5282.477	191.0755	229.2906
4.6	28.90265	$C_S = C_K$	6036.054	208.8408	250.6090
4.8	30.15929	$C_S = C_K$	6858.092	227.3957	272.8748
5.0	31.41593	$C_S = C_K$	7751.57	246.7401	296.0881
5.2	32.67256	$C_S = C_K$	8719.46	266.8741	320.2489
5.4	33.92920	$C_S = C_K$	9764.74	287.7977	345.3572
5.6	35.18584	$C_S = C_K$	10890.40	309.5108	371.4130
5.8	36.44247	$C_S = C_K$	12099.39	332.0135	398.4162
6.0	37.69911	$C_S = C_K$	13394.71	355.3058	426.3669

*Note: For values of b/s greater than 3.4 the following expressions are correct to the number of significant figures given in the table:

$$C_K = C_S = 2\pi \frac{b}{s} = 6.283185 \frac{b}{s}$$

$$C_T = 2\pi^3 \frac{b^3}{s^3} = 62.01255 \frac{b^3}{s^3}$$

$$C_Q = (1+\mu)\,\pi^2 \frac{b^2}{s^2} = (1+\mu)\, 9.869604 \frac{b^2}{s^2}$$

TABLE 2
NUMERICAL VALUES OF CARRY-OVER FACTORS FOR A PANEL OF A SLAB
The various carry-over factors are defined in Fig. 6.

Ratio of Sides of Panel b/s	k	t	$q^{\mu=0}$	$q^{\mu=0.1}$	$q^{\mu=0.2}$	$q^{\mu=0.3}$
0.0	0.500000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.495087	0.999602	0.999866	0.998229	0.996598	0.994972
0.2	0.480613	0.994009	0.997894	0.991464	0.985115	0.978847
0.3	0.457369	0.972670	0.989736	0.975791	0.962234	0.949048
0.4	0.426646	0.925569	0.969406	0.946181	0.924042	0.902916
0.5	0.390170	0.849907	0.931263	0.898525	0.868011	0.839501
0.6	0.349949	0.751965	0.872354	0.831695	0.794657	0.760777
0.7	0.308070	0.643339	0.794036	0.748535	0.707966	0.671568
0.8	0.266482	0.535479	0.701815	0.655145	0.614295	0.578240
0.9	0.226806	0.436383	0.603461	0.558895	0.520459	0.486970
1.0	0.190226	0.350080	0.506572	0.466362	0.432067	0.402469
1.1	0.157474	0.277610	0.416860	0.382126	0.352735	0.327542
1.2	0.128865	0.218251	0.337621	0.308575	0.284131	0.263276
1.3	0.104396	0.170459	0.270050	0.246332	0.226443	0.209527
1.4	0.083833	0.132447	0.213905	0.194869	0.178945	0.165426
1.5	0.066809	0.102482	0.168139	0.153052	0.140449	0.129764
1.6	0.052889	0.079019	0.131359	0.119512	0.109624	0.101248
1.7	0.041627	0.060747	0.102116	0.092877	0.085171	0.078645
1.8	0.032598	0.046577	0.079055	0.071889	0.065914	0.060856
1.9	0.025413	0.035630	0.060988	0.055453	0.050839	0.046934
2.0	0.019734	0.027200	0.046908	0.042648	0.039097	0.036092
2.1	0.015270	0.020724	0.035983	0.032714	0.029989	0.027683
2.2	0.011779	0.015763	0.027537	0.025035	0.022949	0.021184
2.3	0.009060	0.011971	0.021029	0.019118	0.017525	0.016177
2.4	0.006952	0.009078	0.016028	0.014571	0.013357	0.012330
2.5	0.005322	0.006874	0.012195	0.011087	0.010163	0.009381
2.6	0.004065	0.005199	0.009264	0.008422	0.007720	0.007126
2.7	0.003099	0.003928	0.007027	0.006388	0.005856	0.005405
2.8	0.002359	0.002964	0.005322	0.004839	0.004435	0.004094
2.9	0.001792	0.002234	0.004026	0.003660	0.003355	0.003097
3.0	0.001360	0.001683	0.003042	0.002766	0.002535	0.002340
3.1	0.001030	0.001266	0.002296	0.002087	0.001913	0.001766
3.2	0.000780	0.000952	0.001731	0.001574	0.001443	0.001332
3.3	0.000589	0.000715	0.001304	0.001185	0.001087	0.001003
3.4	0.000445	0.000537	0.000981	0.000892	0.000818	0.000755
3.5	0.000335	0.000402	0.000738	0.000671	0.000615	0.000568
3.6	0.000253	0.000302	0.000554	0.000504	0.000462	0.000426
3.7	0.000190	0.000226	0.000416	0.000378	0.000347	0.000320
3.8	0.000143	0.000169	0.000312	0.000284	0.000260	0.000240
3.9	0.000107	0.000127	0.000234	0.000213	0.000195	0.000180
4.0	0.000081	0.000095	0.000175	0.000159	0.000146	0.000135
4.2	0.000045	0.000053	0.000098	0.000089	0.000082	0.000076
4.4	0.000025	0.000029	0.000055	0.000050	0.000046	0.000042
4.6	0.000014	0.000016	0.000031	0.000028	0.000026	0.000024
4.8	0.000008	0.000009	0.000017	0.000015	0.000014	0.000013
5.0	0.000004	0.000005	0.000009	0.000009	0.000008	0.000007
5.2	0.000002	0.000003	0.000005	0.000005	0.000004	0.000004
5.4	0.000001	0.000002	0.000003	0.000003	0.000002	0.000002
5.6	0.000001	0.000001	0.000002	0.000001	0.000001	0.000001
5.8	0.000000	0.000000	0.000001	0.000001	0.000001	0.000001
6.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 3

Numerical Values of the Coefficient in the Expression for Fixed-End Moment in a Slab Due to a Line Load

The moment at a fixed edge for a loading $F = F_0 \sin \pi x/s$ on a line a distance v from the edge, when the opposite edge is also fixed, is expressed as $M^F = -C_M \cdot Fb$. The values of C_M are tabulated for various values of v/b, where b is the span between fixed edges.

b/s	Values of C_M								
	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$
0.0	0.081000	0.128000	0.147000	0.144000	0.125000	0.096000	0.063000	0.032000	0.009000
0.1	0.080901	0.127714	0.146550	0.143469	0.124486	0.095583	0.062724	0.031865	0.008965
0.2	0.080605	0.126853	0.145199	0.141876	0.122941	0.094329	0.061893	0.031457	0.008860
0.3	0.080109	0.125416	0.142947	0.139219	0.120364	0.092232	0.060500	0.030771	0.008681
0.4	0.079412	0.123405	0.139799	0.135509	0.116763	0.089297	0.058541	0.029799	0.008426
0.5	0.078516	0.120829	0.135779	0.130779	0.112171	0.085544	0.056025	0.028542	0.008092
0.6	0.077425	0.117714	0.130939	0.125097	0.106656	0.081031	0.052986	0.027012	0.007680
0.7	0.076152	0.114105	0.125364	0.118575	0.100336	0.075852	0.049485	0.025237	0.007198
0.8	0.074714	0.110066	0.119172	0.111370	0.093373	0.070146	0.045616	0.023262	0.006655
0.9	0.073133	0.105679	0.112510	0.103672	0.085964	0.064082	0.041496	0.021147	0.006067
1.0	0.071437	0.101038	0.105542	0.095691	0.078327	0.057848	0.037257	0.018960	0.005454
1.1	0.069655	0.096239	0.098434	0.087637	0.070679	0.051630	0.033030	0.016772	0.004836
1.2	0.067815	0.091374	0.091340	0.079700	0.063213	0.045597	0.028936	0.014648	0.004232
1.3	0.065944	0.086525	0.084394	0.072042	0.056091	0.039882	0.025072	0.012639	0.003657
1.4	0.064063	0.081758	0.077698	0.064783	0.049429	0.034588	0.021507	0.010787	0.003123
1.5	0.062191	0.077125	0.071329	0.058006	0.043302	0.029772	0.018285	0.009114	0.002640
1.6	0.060342	0.072662	0.065335	0.051756	0.037747	0.025460	0.015423	0.007631	0.002210
1.7	0.058525	0.068393	0.059741	0.046052	0.032771	0.021653	0.012919	0.006338	0.001833
1.8	0.056747	0.064330	0.054554	0.040887	0.028356	0.018329	0.010757	0.005227	0.001509
1.9	0.055013	0.060479	0.049769	0.036240	0.024471	0.015455	0.008909	0.004283	0.001234
2.0	0.053325	0.056838	0.045372	0.032079	0.021072	0.012989	0.007346	0.003490	0.001002
2.1	0.051684	0.053403	0.041341	0.028369	0.018115	0.010887	0.006034	0.002830	0.000810
2.2	0.050091	0.050168	0.037655	0.025069	0.015552	0.009105	0.004939	0.002284	0.000651
2.3	0.048545	0.047123	0.034288	0.022141	0.013338	0.007601	0.004032	0.001837	0.000520
2.4	0.047046	0.044260	0.031217	0.019547	0.011429	0.006336	0.003283	0.001472	0.000415
2.5	0.045592	0.041569	0.028417	0.017252	0.009788	0.005275	0.002668	0.001176	0.000329
2.6	0.044182	0.039040	0.025866	0.015223	0.008378	0.004387	0.002164	0.000937	0.000260
2.7	0.042816	0.036664	0.023543	0.013431	0.007169	0.003646	0.001753	0.000745	0.000205
2.8	0.041493	0.034432	0.021427	0.011848	0.006132	0.003028	0.001418	0.000590	0.000161
2.9	0.040209	0.032335	0.019501	0.010451	0.005244	0.002513	0.001145	0.000467	0.000126
3.0	0.038966	0.030367	0.017748	0.009218	0.004484	0.002085	0.000924	0.000369	0.000099
3.1	0.037761	0.028518	0.016152	0.008131	0.003834	0.001729	0.000746	0.000291	0.000077
3.2	0.036593	0.026781	0.014700	0.007171	0.003278	0.001434	0.000601	0.000229	0.000060
3.3	0.035461	0.025150	0.013378	0.006325	0.002802	0.001189	0.000484	0.000180	0.000047
3.4	0.034365	0.023619	0.012174	0.005578	0.002395	0.000985	0.000390	0.000142	0.000036
3.5	0.033302	0.022180	0.011080	0.004919	0.002047	0.000816	0.000313	0.000111	0.000028
3.6	0.032272	0.020829	0.010083	0.004338	0.001750	0.000677	0.000252	0.000087	0.000022
3.7	0.031274	0.019561	0.009176	0.003826	0.001495	0.000561	0.000203	0.000068	0.000017
3.8	0.030307	0.018370	0.008351	0.003374	0.001278	0.000464	0.000163	0.000054	0.000013
3.9	0.029369	0.017251	0.007600	0.002976	0.001092	0.000385	0.000131	0.000042	0.000010
4.0	0.028461	0.016201	0.006916	0.002625	0.000934	0.000319	0.000105	0.000033	0.000008
4.2	0.026728	0.014287	0.005728	0.002041	0.000682	0.000219	0.000068	0.000020	0.000005
4.4	0.025100	0.012600	0.004744	0.001588	0.000498	0.000150	0.000044	0.000012	0.000003
4.6	0.023571	0.011112	0.003929	0.001235	0.000364	0.000103	0.000028	0.000007	0.000002
4.8	0.022136	0.009800	0.003254	0.000960	0.000266	0.000071	0.000018	0.000005	0.000001
5.0	0.020788	0.008643	0.002695	0.000747	0.000194	0.000048	0.000012	0.000003	0.000001
5.2	0.019522	0.007622	0.002232	0.000581	0.000142	0.000033	0.000008	0.000002	0.000000
5.4	0.018333	0.006722	0.001849	0.000452	0.000104	0.000023	0.000005	0.000001	0.000000
5.6	0.017217	0.005928	0.001531	0.000351	0.000076	0.000016	0.000003	0.000001	0.000000
5.8	0.016168	0.005228	0.001268	0.000273	0.000055	0.000011	0.000002	0.000000	0.000000
6.0	0.015184	0.004611	0.001050	0.000213	0.000040	0.000007	0.000001	0.000000	0.000000

Table 4

Numerical Values of the Coefficient in the Expression for Fixed-End Reaction in a Slab Due to a Line Load

The reaction at a fixed edge for a loading $F = F_0 \sin \pi x/s$ on a line a distance v from the edge, when the opposite edge is also fixed, is expressed as $R^F = -C_R \cdot F$. The values of C_R are tabulated for various values of v/b, where b is the span between fixed edges.

b/s	Values of C_R								
	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$
0.0	0.972000	0.896000	0.784000	0.648000	0.500000	0.352000	0.216000	0.104000	0.028000
0.1	0.971934	0.895843	0.783816	0.647874	0.499987	0.352102	0.216166	0.104147	0.028063
0.2	0.971715	0.895304	0.783154	0.647356	0.499801	0.352277	0.216565	0.104533	0.028234
0.3	0.971278	0.894192	0.781695	0.646044	0.499018	0.352145	0.216915	0.105000	0.028465
0.4	0.970526	0.892213	0.778959	0.643334	0.497004	0.351139	0.216796	0.105310	0.028685
0.5	0.969338	0.889015	0.774374	0.638516	0.493019	0.348596	0.215709	0.105181	0.028804
0.6	0.967586	0.884236	0.767368	0.630889	0.486335	0.343864	0.213164	0.104328	0.028735
0.7	0.965153	0.877556	0.757458	0.619882	0.476369	0.336420	0.208761	0.102516	0.028399
0.8	0.961942	0.868745	0.744331	0.605151	0.462785	0.325963	0.202259	0.099592	0.027742
0.9	0.957895	0.857696	0.727889	0.586638	0.445553	0.312467	0.193620	0.095515	0.026745
1.0	0.952990	0.844430	0.708264	0.564579	0.424953	0.296182	0.183011	0.090360	0.025419
1.1	0.947244	0.829088	0.685784	0.539457	0.401526	0.277587	0.170770	0.084296	0.023810
1.2	0.940705	0.811901	0.660922	0.511927	0.375982	0.257315	0.157349	0.077561	0.021982
1.3	0.933439	0.793157	0.634226	0.482726	0.349107	0.236059	0.143247	0.070425	0.020014
1.4	0.925526	0.773162	0.606263	0.452588	0.321669	0.214493	0.128952	0.063152	0.017983
1.5	0.917044	0.752217	0.577563	0.422186	0.294357	0.193212	0.114892	0.055980	0.015960
1.6	0.908070	0.730599	0.548596	0.392089	0.267739	0.172697	0.101413	0.049099	0.014005
1.7	0.898673	0.708547	0.519753	0.362755	0.242249	0.153307	0.088768	0.042651	0.012162
1.8	0.888915	0.686265	0.491345	0.334521	0.218191	0.135278	0.077121	0.036729	0.010463
1.9	0.878846	0.663918	0.463610	0.307625	0.195756	0.118747	0.066560	0.031382	0.008923
2.0	0.868512	0.641641	0.436721	0.282215	0.175040	0.103763	0.057109	0.026626	0.007551
2.1	0.857950	0.619541	0.410801	0.258373	0.156067	0.090312	0.048746	0.022447	0.006345
2.2	0.847193	0.597701	0.385928	0.236124	0.138805	0.078336	0.041418	0.018817	0.005297
2.3	0.836268	0.576190	0.362151	0.215455	0.123189	0.067745	0.035049	0.015693	0.004396
2.4	0.825201	0.555060	0.339490	0.196325	0.109126	0.058434	0.029554	0.013027	0.003628
2.5	0.814013	0.534351	0.317952	0.178675	0.096510	0.050289	0.024840	0.010769	0.002980
2.6	0.802725	0.514096	0.297524	0.162433	0.085230	0.043192	0.020820	0.008868	0.002437
2.7	0.791354	0.494320	0.278189	0.147521	0.075171	0.037032	0.017407	0.007278	0.001984
2.8	0.779917	0.475043	0.259917	0.133855	0.066223	0.031701	0.014520	0.005955	0.001609
2.9	0.768430	0.456277	0.242677	0.121352	0.058280	0.027099	0.012088	0.004858	0.001301
3.0	0.756907	0.438034	0.226433	0.109932	0.051241	0.023136	0.010045	0.003953	0.001048
3.1	0.745361	0.420319	0.211146	0.099513	0.045013	0.019731	0.008334	0.003209	0.000841
3.2	0.733805	0.403137	0.196776	0.090020	0.039510	0.016810	0.006904	0.002599	0.000674
3.3	0.722251	0.386488	0.183282	0.081381	0.034654	0.014308	0.005712	0.002101	0.000538
3.4	0.710710	0.370372	0.170623	0.073525	0.030373	0.012167	0.004720	0.001695	0.000429
3.5	0.699191	0.354785	0.158758	0.066390	0.026604	0.010339	0.003896	0.001366	0.000341
3.6	0.687705	0.339723	0.147647	0.059914	0.023289	0.008780	0.003213	0.001098	0.000270
3.7	0.676261	0.325180	0.137250	0.054041	0.020375	0.007450	0.002647	0.000882	0.000214
3.8	0.664867	0.311148	0.127529	0.048720	0.017815	0.006318	0.002179	0.000707	0.000169
3.9	0.653531	0.297619	0.118446	0.043902	0.015569	0.005354	0.001792	0.000567	0.000133
4.0	0.642260	0.284584	0.109966	0.039542	0.013600	0.004536	0.001473	0.000453	0.000105
4.2	0.619941	0.259955	0.094673	0.032037	0.010362	0.003249	0.000993	0.000289	0.000065
4.4	0.597958	0.237174	0.081390	0.025915	0.007882	0.002323	0.000668	0.000184	0.000040
4.6	0.576354	0.216149	0.069876	0.020932	0.005985	0.001658	0.000448	0.000117	0.000024
4.8	0.555163	0.196782	0.059916	0.016884	0.004539	0.001182	0.000300	0.000074	0.000015
5.0	0.534416	0.178974	0.051316	0.013601	0.003437	0.000841	0.000201	0.000047	0.000009
5.2	0.514137	0.162629	0.043903	0.010943	0.002600	0.000598	0.000134	0.000029	0.000005
5.4	0.494346	0.147648	0.037522	0.008795	0.001964	0.000424	0.000090	0.000018	0.000003
5.6	0.475058	0.133937	0.032038	0.007062	0.001482	0.000301	0.000060	0.000011	0.000002
5.8	0.456287	0.121406	0.027330	0.005664	0.001117	0.000213	0.000040	0.000007	0.000001
6.0	0.438040	0.109966	0.023295	0.004539	0.000841	0.000151	0.000026	0.000005	0.000001

TABLE 5

NUMERICAL VALUES OF THE COEFFICIENTS IN THE EXPRESSIONS FOR FIXED-END MOMENT AND REACTION IN A SLAB DUE TO A DISTRIBUTED LOAD

The moment M^F and the reaction R^F at a fixed edge for a loading $p = p_0 \sin \pi x/s$ over the whole slab, when the opposite edge is also fixed, are expressed as $M^F = -c_m \cdot b^2 p$, and $R^F = -c_r \cdot bp$, where b is the span between fixed edges, and p_0 is a constant.

b/s	c_m	c_r	b/s	c_m	c_r
0.0	0.0833333	0.500000	2.5	0.0160149	0.252908
0.1	0.0830590	0.499993	2.6	0.0148501	0.243587
0.2	0.0822345	0.499894	2.7	0.0138013	0.234862
0.3	0.0808570	0.499475	2.8	0.0128550	0.226692
0.4	0.0789279	0.498396	2.9	0.0119993	0.219035
0.5	0.0764603	0.496255	3.0	0.0112237	0.211850
0.6	0.0734864	0.492650	3.1	0.0105191	0.205102
0.7	0.0700623	0.487250	3.2	0.0098775	0.198755
0.8	0.0662685	0.479848	3.3	0.0092919	0.192777
0.9	0.0622055	0.470398	3.4	0.0087562	0.187141
1.0	0.0579852	0.459013	3.5	0.0082650	0.181818
1.1	0.0537202	0.445947	3.6	0.0078137	0.176785
1.2	0.0495136	0.431552	3.7	0.0073980	0.172021
1.3	0.0454517	0.416224	3.8	0.0070145	0.167503
1.4	0.0415996	0.400358	3.9	0.0066599	0.163215
1.5	0.0380011	0.384315	4.0	0.0063315	0.159140
1.6	0.0346801	0.368398	4.2	0.0057433	0.151568
1.7	0.0316446	0.352847	4.4	0.0052332	0.144682
1.8	0.0288902	0.337835	4.6	0.0047882	0.138393
1.9	0.0264042	0.323481	4.8	0.0043975	0.132628
2.0	0.0241687	0.309852	5.0	0.0040528	0.127323
2.1	0.0221630	0.296982	5.2	0.0037471	0.122427
2.2	0.0203654	0.284874	5.4	0.0034747	0.117892
2.3	0.0187547	0.273512	5.6	0.0032309	0.113682
2.4	0.0173108	0.262870	5.8	0.0030119	0.109762
			6.0	0.0028145	0.106103

Note: For values of b/s greater than 6.0 the following expressions are correct to the number of significant figures given in the table:

$$c_m = \frac{1}{\pi^2} \frac{s^2}{b^2} = 0.101321 \frac{s^2}{b^2},$$

$$c_r = \frac{2}{\pi} \frac{s}{b} = 0.636620 \frac{s}{b}.$$

Table 6

Numerical Values of the Coefficients in the Expressions for Bending Moments Per Unit of Length and for Deflections in a Simply-Supported Panel of a Slab Due to a Deflection of One Edge of Panel Combined With a Moment on That Edge

A deflection $\Delta = \Delta_0 \sin \pi x/s$ combined with a moment $-\mathsf{M}_w = -\dfrac{(1-\mu)\,N\pi^2}{s^2} \cdot \Delta_0 \sin \pi x/s$, is applied at the edge, on the line $y = 0$, of a slab simply supported on all the other edges. The bending moments M_x and M_y on the line $y = v$ are given by the equation $M_x = -M_y = C \cdot \mathsf{M}_w$. The deflection w on the line $y = v$ is given by the equation $w = C \cdot \Delta$. The span of the panel of the slab in the y direction is b.

b/s	Values of C								
	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$
0.0	0.900000	0.800000	0.700000	0.600000	0.500000	0.400000	0.300000	0.200000	0.100000
0.1	0.897208	0.795302	0.694180	0.593744	0.493894	0.394532	0.295559	0.196877	0.098390
0.2	0.889075	0.781660	0.677333	0.575681	0.476302	0.378803	0.282801	0.187916	0.093773
0.3	0.876263	0.760316	0.651127	0.547727	0.449195	0.354656	0.263270	0.174224	0.086726
0.4	0.859700	0.732995	0.617879	0.512533	0.415292	0.324617	0.239076	0.157314	0.078040
0.5	0.840388	0.701555	0.580068	0.472922	0.377470	0.291350	0.212434	0.138771	0.068538
0.6	0.819255	0.667705	0.539949	0.431435	0.338296	0.257212	0.185294	0.119979	0.058939
0.7	0.797069	0.632840	0.499340	0.390086	0.299773	0.224016	0.159137	0.101984	0.049783
0.8	0.774413	0.598000	0.459559	0.350299	0.263283	0.192985	0.134941	0.085466	0.041418
0.9	0.751700	0.563896	0.421472	0.312968	0.229650	0.164815	0.113243	0.070785	0.034023
1.0	0.729208	0.530979	0.385589	0.278568	0.199268	0.139798	0.094238	0.058057	0.027653
1.1	0.707110	0.499509	0.352155	0.247278	0.172225	0.117946	0.077893	0.047235	0.022274
1.2	0.685512	0.469609	0.321243	0.219076	0.148414	0.099097	0.064031	0.038174	0.017807
1.3	0.664470	0.441320	0.292810	0.193823	0.127615	0.082992	0.052404	0.030680	0.014144
1.4	0.644013	0.414628	0.266752	0.171314	0.109554	0.069330	0.042735	0.024542	0.011173
1.5	0.624150	0.389485	0.242925	0.151317	0.093936	0.057805	0.034750	0.019555	0.008784
1.6	0.604877	0.365829	0.221175	0.133590	0.080475	0.048124	0.028190	0.015530	0.006878
1.7	0.586188	0.343588	0.201341	0.117901	0.068898	0.040017	0.022825	0.012299	0.005366
1.8	0.568069	0.322685	0.183267	0.104031	0.058958	0.033246	0.018452	0.009717	0.004173
1.9	0.550506	0.303046	0.166804	0.091778	0.050435	0.027602	0.014898	0.007661	0.003237
2.0	0.533483	0.284598	0.151813	0.080960	0.043133	0.022903	0.012016	0.006030	0.002504
2.1	0.516986	0.267271	0.138166	0.071411	0.036882	0.018996	0.009683	0.004739	0.001933
2.2	0.500998	0.250997	0.125743	0.062986	0.031532	0.015751	0.007797	0.003719	0.001489
2.3	0.485504	0.235713	0.114436	0.055552	0.026956	0.013056	0.006275	0.002916	0.001146
2.4	0.470489	0.221359	0.104145	0.048995	0.023042	0.010821	0.005048	0.002283	0.000880
2.5	0.455938	0.207879	0.094779	0.043210	0.019695	0.008967	0.004059	0.001787	0.000674
2.6	0.441837	0.195220	0.086255	0.038109	0.016834	0.007429	0.003263	0.001397	0.000516
2.7	0.428172	0.183331	0.078497	0.033609	0.014388	0.006155	0.002622	0.001092	0.000395
2.8	0.414930	0.172167	0.071437	0.029641	0.012297	0.005099	0.002107	0.000853	0.000302
2.9	0.402097	0.161682	0.065012	0.026141	0.010510	0.004224	0.001692	0.000665	0.000230
3.0	0.389661	0.151836	0.059164	0.023054	0.008983	0.003499	0.001359	0.000519	0.000176
3.1	0.377610	0.142589	0.053843	0.020332	0.007677	0.002898	0.001092	0.000405	0.000134
3.2	0.365931	0.133906	0.049000	0.017931	0.006561	0.002400	0.000877	0.000316	0.000102
3.3	0.354614	0.125751	0.044593	0.015813	0.005607	0.001988	0.000704	0.000246	0.000078
3.4	0.343647	0.118093	0.040582	0.013946	0.004792	0.001647	0.000565	0.000192	0.000059
3.5	0.333018	0.110901	0.036932	0.012299	0.004096	0.001364	0.000454	0.000149	0.000045
3.6	0.322719	0.104148	0.033610	0.010847	0.003500	0.001130	0.000364	0.000116	0.000034
3.7	0.312738	0.097805	0.030587	0.009566	0.002992	0.000936	0.000292	0.000091	0.000026
3.8	0.303066	0.091849	0.027836	0.008436	0.002557	0.000775	0.000235	0.000071	0.000020
3.9	0.293693	0.086255	0.025333	0.007440	0.002185	0.000642	0.000188	0.000055	0.000015
4.0	0.284610	0.081003	0.023054	0.006561	0.001867	0.000531	0.000151	0.000043	0.000011
4.2	0.267277	0.071437	0.019094	0.005103	0.001364	0.000365	0.000097	0.000026	0.000006
4.4	0.251000	0.063001	0.015813	0.003969	0.000996	0.000250	0.000063	0.000016	0.000004
4.6	0.235715	0.055561	0.013097	0.003087	0.000728	0.000172	0.000040	0.000010	0.000002
4.8	0.221360	0.049000	0.010847	0.002401	0.000531	0.000118	0.000026	0.000006	0.000001
5.0	0.207880	0.043214	0.008983	0.001867	0.000388	0.000081	0.000017	0.000003	0.000001
5.2	0.195220	0.038111	0.007440	0.001452	0.000284	0.000055	0.000011	0.000002	0.000000
5.4	0.183331	0.033610	0.006162	0.001130	0.000207	0.000038	0.000007	0.000001	0.000000
5.6	0.172167	0.029641	0.005103	0.000879	0.000151	0.000026	0.000004	0.000001	0.000000
5.8	0.161682	0.026141	0.004227	0.000683	0.000110	0.000018	0.000003	0.000000	0.000000
6.0	0.151836	0.023054	0.003500	0.000531	0.000081	0.000012	0.000002	0.000000	0.000000

Table 7

Numerical Values of the Coefficient for the Twisting Moment M_{xy} Per Unit of Length in a Simply-Supported Panel of a Slab Due to a Deflection of One Edge of Panel Combined With a Moment on That Edge

A deflection $\Delta = \Delta_0 \sin \pi x/s$ combined with a moment $-M_w = -\frac{(1-\mu)N\pi^2}{s^2} \cdot \Delta_0 \sin \pi x/s$, is applied at the edge, on the line $y = 0$, of a slab simply supported on all the other edges. The twisting moment M_{xy} on the line $y = v$ is given by the equation $M_{xy} = C_{xy} \cdot \frac{(1-\mu)N\pi^2}{s^2} \Delta_0 \cos \pi x/s$. The span of the panel of the slab in the y direction is b.

b/s	Values of C_{xy}					
	$v/b = 0$	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$
0.1	3.287136	3.257337	3.230753	3.207359	3.187130	3.170046
0.2	1.795676	1.736349	1.683878	1.638058	1.598706	1.565668
0.3	1.358034	1.269683	1.192618	1.126154	1.069701	1.022757
0.4	1.176285	1.059590	0.959649	0.874883	0.803950	0.745730
0.5	1.090331	0.946084	0.825229	0.724777	0.642245	0.575592
0.6	1.047196	0.876241	0.736512	0.623029	0.531749	0.459417
0.7	1.024905	0.828099	0.671503	0.547512	0.450107	0.374557
0.8	1.013210	0.791397	0.619836	0.487635	0.386397	0.309696
0.9	1.007025	0.761021	0.576263	0.437880	0.334737	0.258534
1.0	1.003742	0.734331	0.537993	0.395192	0.291716	0.217269
1.1	1.001995	0.709928	0.503490	0.357780	0.255224	0.183452
1.2	1.001064	0.687062	0.471870	0.324538	0.223880	0.155419
1.3	1.000567	0.665323	0.442604	0.294742	0.196728	0.131987
1.4	1.000303	0.644483	0.415357	0.267884	0.173072	0.112282
1.5	1.000161	0.624408	0.389900	0.243589	0.152380	0.095639
1.6	1.000086	0.605020	0.366065	0.221564	0.134233	0.081538
1.7	1.000046	0.586266	0.343721	0.201569	0.118290	0.069561
1.8	1.000025	0.568112	0.322761	0.183400	0.104266	0.059372
1.9	1.000013	0.550530	0.303089	0.166882	0.091921	0.050694
2.0	1.000007	0.533496	0.284623	0.151859	0.081046	0.043295
2.1	1.000004	0.516993	0.267285	0.138193	0.071463	0.036983
2.2	1.000002	0.501002	0.251005	0.125759	0.063017	0.031595
2.3	1.000001	0.485506	0.235717	0.114445	0.055571	0.026995
2.4	1.000001	0.470490	0.221361	0.104150	0.049006	0.023066

b/s	Values of C_{xy}				
	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$	$v/b = 1.0$
0.1	3.156092	3.145253	3.137519	3.132881	3.131336
0.2	1.538813	1.518035	1.503252	1.494405	1.491460
0.3	0.984905	0.955807	0.935206	0.922918	0.918834
0.4	0.699301	0.663930	0.639057	0.624288	0.619391
0.5	0.523171	0.483685	0.456158	0.439909	0.434537
0.6	0.403457	0.361875	0.333189	0.316376	0.310838
0.7	0.317195	0.275234	0.246637	0.230016	0.224564
0.8	0.252659	0.211666	0.184114	0.168253	0.163075
0.9	0.203136	0.164086	0.138242	0.123523	0.118745
1.0	0.164442	0.127979	0.104251	0.090898	0.086590
1.1	0.133807	0.100301	0.078893	0.067001	0.063190
1.2	0.109309	0.078919	0.059879	0.049450	0.046133
1.3	0.089568	0.062298	0.045564	0.036536	0.033687
1.4	0.073566	0.049311	0.034750	0.027020	0.024602
1.5	0.060533	0.039120	0.026557	0.020000	0.017968
1.6	0.049881	0.031095	0.020333	0.014817	0.013123
1.7	0.041149	0.024756	0.015593	0.010985	0.009585
1.8	0.033975	0.019735	0.011977	0.008150	0.007001
1.9	0.028071	0.015751	0.009211	0.006052	0.005113
2.0	0.023205	0.012583	0.007093	0.004497	0.003735
2.1	0.019191	0.010060	0.005468	0.003343	0.002728
2.2	0.015876	0.008048	0.004219	0.002488	0.001993
2.3	0.013137	0.006442	0.003259	0.001852	0.001455
2.4	0.010873	0.005159	0.002519	0.001380	0.001063

Table 7.—(Concluded)

Numerical Values of the Coefficient for the Twisting Moment M_{xy} Per Unit of Length in a Simply-Supported Panel of a Slab Due to a Deflection of One Edge of Panel Combined With a Moment on That Edge

A deflection $\Delta = \Delta_0 \sin \pi x/s$ combined with a moment $-M_w = -\frac{(1-\mu) N\pi^2}{s^2} \cdot \Delta_0 \sin \pi x/s$, is applied at the edge, on the line $y = 0$, of a slab simply supported on all the other edges. The twisting moment M_{xy} on the line $y = v$ is given by the equation $M_{xy} = C_{xy} \cdot \frac{(1-\mu) N\pi^2}{s^2} \Delta_0 \cos \pi x/s$. The span of the panel of the slab in the y direction is b.

b/s	Values of C_{xy}					
	$v/b = 0$	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$
2.5	1.000000	0.455939	0.207880	0.094782	0.043217	0.019711
2.6	1.000000	0.441837	0.195220	0.086256	0.038113	0.016844
2.7	1.000000	0.428172	0.183332	0.078498	0.033612	0.014394
2.8	1.000000	0.414930	0.172167	0.071437	0.029642	0.012301
2.9	1.000000	0.402097	0.161682	0.065012	0.026142	0.010512
3.0	1.000000	0.389661	0.151836	0.059165	0.023054	0.008984
3.1	1.000000	0.377610	0.142589	0.053843	0.020332	0.007678
3.2	1.000000	0.365931	0.133906	0.049000	0.017931	0.006562
3.3	1.000000	0.354614	0.125751	0.044593	0.015813	0.005608
3.4	1.000000	0.343647	0.118093	0.040582	0.013946	0.004793
3.5	1.000000	0.333018	0.110901	0.036932	0.012299	0.004096
3.6	1.000000	0.322719	0.104148	0.033610	0.010847	0.003500
3.7	1.000000	0.312738	0.097805	0.030587	0.009566	0.002992
3.8	1.000000	0.303066	0.091849	0.027836	0.008436	0.002557
3.9	1.000000	0.293693	0.086255	0.025333	0.007440	0.002185
4.0	1.000000	0.284610	0.081003	0.023054	0.006561	0.001867
4.2	1.000000	0.267277	0.071437	0.019094	0.005103	0.001364
4.4	1.000000	0.251000	0.063001	0.015813	0.003969	0.000996
4.6	1.000000	0.235715	0.055561	0.013097	0.003087	0.000728
4.8	1.000000	0.221360	0.049000	0.010847	0.002401	0.000531
5.0	1.000000	0.207880	0.043214	0.008983	0.001867	0.000388
5.2	1.000000	0.195220	0.038111	0.007440	0.001452	0.000284
5.4	1.000000	0.183331	0.033610	0.006162	0.001130	0.000207
5.6	1.000000	0.172167	0.029641	0.005103	0.000879	0.000151
5.8	1.000000	0.161682	0.026141	0.004227	0.000683	0.000110
6.0	1.000000	0.151836	0.023054	0.003500	0.000531	0.000081

b/s	Values of C_{xy}				
	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$	$v/b = 1.0$
2.5	0.009000	0.004133	0.001948	0.001028	0.000776
2.6	0.007451	0.003312	0.001508	0.000767	0.000567
2.7	0.006169	0.002655	0.001168	0.000572	0.000414
2.8	0.005108	0.002128	0.000905	0.000427	0.000303
2.9	0.004229	0.001707	0.000701	0.000319	0.000221
3.0	0.003502	0.001369	0.000544	0.000239	0.000161
3.1	0.002900	0.001098	0.000422	0.000178	0.000118
3.2	0.002402	0.000881	0.000327	0.000133	0.000086
3.3	0.001989	0.000707	0.000254	0.000100	0.000063
3.4	0.001647	0.000567	0.000197	0.000075	0.000044
3.5	0.001364	0.000455	0.000153	0.000056	0.000034
3.6	0.001130	0.000365	0.000119	0.000042	0.000025
3.7	0.000936	0.000293	0.000092	0.000031	0.000018
3.8	0.000775	0.000235	0.000072	0.000024	0.000013
3.9	0.000642	0.000189	0.000056	0.000018	0.000010
4.0	0.000532	0.000151	0.000043	0.000013	0.000007
4.2	0.000365	0.000097	0.000026	0.000007	0.000004
4.4	0.000250	0.000063	0.000016	0.000004	0.000002
4.6	0.000172	0.000040	0.000010	0.000002	0.000001
4.8	0.000118	0.000026	0.000006	0.000001	0.000001
5.0	0.000081	0.000017	0.000003	0.000001	0.000000
5.2	0.000055	0.000011	0.000002	0.000000	0.000000
5.4	0.000038	0.000007	0.000001	0.000000	0.000000
5.6	0.000026	0.000004	0.000001	0.000000	0.000000
5.8	0.000018	0.000003	0.000000	0.000000	0.000000
6.0	0.000012	0.000002	0.000000	0.000000	0.000000

TABLE 8

NUMERICAL VALUES OF THE COEFFICIENTS FOR THE BENDING MOMENTS M_x AND M_y PER UNIT OF LENGTH IN A SIMPLY-SUPPORTED PANEL OF A SLAB DUE TO A MOMENT APPLIED AT ONE EDGE OF PANEL

For a moment $\mathsf{M} = \mathsf{M}_0 \sin \pi x/s$ applied at an edge, on the line $y = 0$, of a slab simply supported on all four sides, the moments M_x and M_y on the line $y = v$, at a distance v from the edge on which the moment M is applied, are given by the equations $M_x = (m_x + \mu m_y)\ \mathsf{M}$, and $M_y = (m_y + \mu m_x)\ \mathsf{M}$. The span of the panel of the slab in the y direction is b.

b/s	Values of m_x								
	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.002771	0.004659	0.005767	0.006195	0.006044	0.005410	0.004393	0.003088	0.001592
0.2	0.010611	0.017752	0.021875	0.023409	0.022761	0.020321	0.016465	0.011557	0.005952
0.3	0.022281	0.036964	0.045213	0.048071	0.046484	0.041315	0.033358	0.023355	0.012010
0.4	0.036205	0.059370	0.071869	0.075722	0.072657	0.064165	0.051548	0.035962	0.018453
0.5	0.050914	0.082266	0.098271	0.102334	0.097210	0.085137	0.067951	0.047182	0.024142
0.6	0.065317	0.103681	0.121874	0.125112	0.117389	0.101758	0.080559	0.055609	0.028353
0.7	0.078762	0.122489	0.141312	0.142653	0.131903	0.112944	0.088547	0.060692	0.030811
0.8	0.090962	0.138268	0.156181	0.154677	0.140634	0.118715	0.092016	0.062546	0.031592
0.9	0.101875	0.151052	0.166701	0.161622	0.144195	0.119768	0.091627	0.061686	0.030975
1.0	0.111587	0.161119	0.173412	0.164276	0.143539	0.117094	0.088275	0.058785	0.029321
1.1	0.120235	0.168836	0.176955	0.163520	0.139687	0.111722	0.082865	0.054515	0.026987
1.2	0.127960	0.174571	0.177954	0.160184	0.133573	0.104575	0.076197	0.049459	0.024279
1.3	0.134888	0.178652	0.176957	0.154981	0.125982	0.096408	0.068907	0.044076	0.021438
1.4	0.141125	0.181356	0.174420	0.148491	0.117534	0.087799	0.061476	0.038703	0.018636
1.5	0.146751	0.182908	0.170712	0.141167	0.108696	0.079171	0.054238	0.033569	0.015988
1.6	0.151831	0.183492	0.166126	0.133359	0.099809	0.070813	0.047411	0.028815	0.013564
1.7	0.156416	0.183257	0.160894	0.125326	0.091113	0.062912	0.041121	0.024517	0.011396
1.8	0.160547	0.182325	0.155201	0.117265	0.082769	0.055579	0.035433	0.020703	0.009496
1.9	0.164257	0.180798	0.149191	0.109314	0.074878	0.048867	0.030361	0.017368	0.007854
2.0	0.167573	0.178763	0.142983	0.101577	0.067501	0.042791	0.025889	0.014487	0.006455
2.1	0.170522	0.176294	0.136669	0.094123	0.060665	0.037340	0.021985	0.012024	0.005274
2.2	0.173124	0.173456	0.130324	0.087000	0.054375	0.032485	0.018602	0.009936	0.004287
2.3	0.175399	0.170306	0.124009	0.080239	0.048622	0.028185	0.015689	0.008179	0.003469
2.4	0.177367	0.166893	0.117771	0.073856	0.043387	0.024397	0.013196	0.006709	0.002796
2.5	0.179045	0.163263	0.111650	0.067858	0.038641	0.021074	0.011071	0.005486	0.002245
2.6	0.180448	0.159456	0.105676	0.062245	0.034356	0.018170	0.009268	0.004474	0.001796
2.7	0.181593	0.155505	0.099872	0.057010	0.030498	0.015639	0.007743	0.003640	0.001433
2.8	0.182496	0.151445	0.094257	0.052142	0.027035	0.013440	0.006457	0.002954	0.001139
2.9	0.183167	0.147302	0.088843	0.047629	0.023933	0.011534	0.005377	0.002393	0.000904
3.0	0.183623	0.143101	0.083641	0.043454	0.021161	0.009885	0.004470	0.001934	0.000715
3.1	0.183876	0.138867	0.078656	0.039600	0.018689	0.008462	0.003711	0.001561	0.000565
3.2	0.183937	0.134617	0.073890	0.036051	0.016489	0.007236	0.003078	0.001258	0.000445
3.3	0.183819	0.130369	0.069346	0.032788	0.014533	0.006181	0.002549	0.001012	0.000350
3.4	0.183532	0.126140	0.065021	0.029792	0.012797	0.005275	0.002109	0.000814	0.000275
3.5	0.183086	0.121942	0.060913	0.027047	0.011258	0.004498	0.001744	0.000653	0.000215
3.6	0.182493	0.117788	0.057019	0.024534	0.009897	0.003832	0.001440	0.000524	0.000169
3.7	0.181762	0.113688	0.053332	0.022238	0.008693	0.003262	0.001188	0.000419	0.000132
3.8	0.180901	0.109650	0.049847	0.020142	0.007630	0.002775	0.000980	0.000336	0.000103
3.9	0.179919	0.105682	0.046557	0.018231	0.006693	0.002359	0.000807	0.000268	0.000080
4.0	0.178825	0.101791	0.043456	0.016491	0.005867	0.002003	0.000665	0.000214	0.000062
4.2	0.176332	0.094259	0.037790	0.013467	0.004499	0.001443	0.000450	0.000136	0.000038
4.4	0.173479	0.087087	0.032788	0.010973	0.003443	0.001037	0.000304	0.000087	0.000023
4.6	0.170320	0.080294	0.028390	0.008922	0.002629	0.000744	0.000204	0.000055	0.000014
4.8	0.166902	0.073891	0.024535	0.007241	0.002004	0.000532	0.000137	0.000035	0.000008
5.0	0.163268	0.067880	0.021166	0.005867	0.001524	0.000380	0.000092	0.000022	0.000005
5.2	0.159458	0.062259	0.018231	0.004745	0.001158	0.000271	0.000062	0.000014	0.000003
5.4	0.155507	0.057019	0.015680	0.003833	0.000878	0.000193	0.000041	0.000009	0.000002
5.6	0.151446	0.052148	0.013467	0.003091	0.000665	0.000137	0.000028	0.000005	0.000001
5.8	0.147302	0.047632	0.011552	0.002490	0.000503	0.000098	0.000018	0.000003	0.000001
6.0	0.143102	0.043456	0.009897	0.002004	0.000380	0.000069	0.000012	0.000002	0.000000

TABLE 9

NUMERICAL VALUES OF THE COEFFICIENTS FOR THE BENDING MOMENTS M_x AND M_y PER UNIT OF LENGTH IN A SIMPLY-SUPPORTED PANEL OF A SLAB DUE TO A MOMENT APPLIED AT ONE EDGE OF PANEL

For a moment $M = M_0 \sin \pi x/s$ applied at an edge, on the line $y = 0$, of a slab simply supported on all four sides, the moments M_x and M_y on the line $y = v$, at a distance v from the edge on which the moment M is applied, are given by the equations $M_x = (m_x + \mu m_y)\ M$, and $M_y = (m_y + \mu m_x)\ M$. The span of the panel of the slab in the y direction is b.

b/s	Values of m_y								
	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$
0.0	0.900000	0.800000	0.700000	0.600000	0.500000	0.400000	0.300000	0.200000	0.100000
0.1	0.894437	0.790642	0.688413	0.587549	0.487851	0.389121	0.291166	0.193789	0.096798
0.2	0.878463	0.763908	0.655458	0.552272	0.453541	0.358483	0.266336	0.176359	0.087821
0.3	0.853983	0.723352	0.605914	0.499655	0.402710	0.313341	0.229912	0.150869	0.074717
0.4	0.823496	0.673624	0.546010	0.436812	0.342635	0.260452	0.187527	0.121352	0.059587
0.5	0.789474	0.619289	0.481797	0.370589	0.280260	0.206213	0.144483	0.091588	0.044396
0.6	0.753938	0.564024	0.418075	0.306323	0.220907	0.155454	0.104734	0.064369	0.030586
0.7	0.718307	0.510351	0.358028	0.247433	0.167870	0.111073	0.070590	0.041292	0.018972
0.8	0.683451	0.459732	0.303378	0.195622	0.122649	0.074270	0.042925	0.022920	0.009826
0.9	0.649825	0.412844	0.254771	0.151346	0.085456	0.045047	0.021616	0.009099	0.003049
1.0	0.617620	0.369860	0.212177	0.114292	0.055729	0.022704	0.005964	-0.000728	-0.001668
1.1	0.586875	0.330672	0.175200	0.083757	0.032539	0.006224	-0.004973	-0.007280	-0.004713
1.2	0.557552	0.295038	0.143289	0.058891	0.014841	-0.005478	-0.012165	-0.011285	-0.006473
1.3	0.529582	0.262668	0.115853	0.038841	0.001633	-0.013416	-0.016503	-0.013396	-0.007294
1.4	0.502888	0.233272	0.092332	0.022823	-0.007980	-0.018469	-0.018741	-0.014161	-0.007463
1.5	0.477399	0.206577	0.072214	0.010149	-0.014759	-0.021366	-0.019488	-0.014014	-0.007204
1.6	0.453046	0.182337	0.055050	0.000231	-0.019334	-0.022689	-0.019220	-0.013285	-0.006686
1.7	0.429772	0.160331	0.040447	-0.007425	-0.022215	-0.022895	-0.018296	-0.012218	-0.006030
1.8	0.407522	0.140360	0.028066	-0.013233	-0.023810	-0.022332	-0.016981	-0.010986	-0.005322
1.9	0.386249	0.122248	0.017613	-0.017536	-0.024443	-0.021265	-0.015463	-0.009706	-0.004617
2.0	0.365910	0.105835	0.008830	-0.020617	-0.024368	-0.019888	-0.013874	-0.008457	-0.003950
2.1	0.346464	0.090977	0.001497	-0.022711	-0.023783	-0.018344	-0.012302	-0.007285	-0.003341
2.2	0.327874	0.077540	-0.004581	-0.024015	-0.022843	-0.016734	-0.010805	-0.006217	-0.002798
2.3	0.310105	0.065407	-0.009572	-0.024687	-0.021667	-0.015129	-0.009414	-0.005263	-0.002324
2.4	0.293122	0.054465	-0.013626	-0.024862	-0.020345	-0.013577	-0.008148	-0.004426	-0.001916
2.5	0.276893	0.044615	-0.016871	-0.024648	-0.018946	-0.012108	-0.007012	-0.003700	-0.001570
2.6	0.261389	0.035764	-0.019421	-0.024136	-0.017522	-0.010740	-0.006005	-0.003077	-0.001280
2.7	0.246579	0.027826	-0.021375	-0.023401	-0.016110	-0.009484	-0.005121	-0.002548	-0.001038
2.8	0.232434	0.020722	-0.022820	-0.022502	-0.014738	-0.008341	-0.004351	-0.002101	-0.000838
2.9	0.218930	0.014380	-0.023832	-0.021488	-0.013423	-0.007310	-0.003684	-0.001727	-0.000674
3.0	0.206038	0.008734	-0.024477	-0.020400	-0.012179	-0.006386	-0.003111	-0.001415	-0.000540
3.1	0.193734	0.003723	-0.024813	-0.019269	-0.011012	-0.005564	-0.002620	-0.001156	-0.000431
3.2	0.181994	-0.000711	-0.024890	-0.018120	-0.009927	-0.004836	-0.002201	-0.000942	-0.000343
3.3	0.170795	-0.004618	-0.024753	-0.016974	-0.008925	-0.004193	-0.001846	-0.000766	-0.000273
3.4	0.160115	-0.008047	-0.024439	-0.015846	-0.008006	-0.003628	-0.001544	-0.000622	-0.000216
3.5	0.149932	-0.011041	-0.023981	-0.014748	-0.007163	-0.003134	-0.001290	-0.000504	-0.000171
3.6	0.140226	-0.013641	-0.023408	-0.013688	-0.006397	-0.002702	-0.001076	-0.000407	-0.000135
3.7	0.130976	-0.015883	-0.022744	-0.012673	-0.005702	-0.002326	-0.000896	-0.000329	-0.000106
3.8	0.122165	-0.017801	-0.022010	-0.011706	-0.005074	-0.002000	-0.000745	-0.000265	-0.000083
3.9	0.113773	-0.019427	-0.021224	-0.010791	-0.004508	-0.001717	-0.000619	-0.000213	-0.000065
4.0	0.105784	-0.020788	-0.020402	-0.009929	-0.003999	-0.001472	-0.000513	-0.000172	-0.000051
4.2	0.090945	-0.022822	-0.018696	-0.008364	-0.003135	-0.001078	-0.000352	-0.000110	-0.000031
4.4	0.077521	-0.024085	-0.016975	-0.007004	-0.002447	-0.000787	-0.000241	-0.000071	-0.000019
4.6	0.065395	-0.024732	-0.015293	-0.005835	-0.001901	-0.000572	-0.000164	-0.000045	-0.000012
4.8	0.054459	-0.024860	-0.013688	-0.004840	-0.001472	-0.000415	-0.000111	-0.000029	-0.000007
5.0	0.044611	-0.024666	-0.012183	-0.003999	-0.001136	-0.000300	-0.000075	-0.000018	-0.000004
5.2	0.035762	-0.024148	-0.010791	-0.003293	-0.000874	-0.000216	-0.000051	-0.000012	-0.000002
5.4	0.027824	-0.023408	-0.009518	-0.002703	-0.000671	-0.000155	-0.000034	-0.000007	-0.000001
5.6	0.020721	-0.022506	-0.008364	-0.002213	-0.000514	-0.000111	-0.000023	-0.000005	-0.000001
5.8	0.014380	-0.021491	-0.007325	-0.001807	-0.000393	-0.000080	-0.000016	-0.000003	-0.000001
6.0	0.008734	-0.020402	-0.006397	-0.001472	-0.000300	-0.000057	-0.000010	-0.000002	-0.000000

Table 10

Numerical Values of the Coefficient for the Twisting Moment M_{xy} Per Unit of Length in a Simply-Supported Panel of a Slab Due to a Moment Applied at One Edge of Panel

For a moment $M = M_0 \sin \pi x/s$ applied at an edge, on the line $y = 0$, of a slab simply supported on all four sides, the twisting moment M_{xy} on the line $y = v$, at a distance v from the edge on which the moment M is applied, is given by the equation $M_{xy} = (1 - \mu)\, m_{xy} \cdot M_0 \cos \pi x/s$. The span of the panel of the slab in the direction of the y axis is b.

b/s	Values of m_{xy}					
	$v/b = 0$	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	−0.103361	−0.073608	−0.047143	−0.023914	−0.003875	0.013015
0.2	−0.199006	−0.140031	−0.088468	−0.043907	−0.005991	0.025586
0.3	−0.281171	−0.193934	−0.119715	−0.057171	−0.005152	0.037304
0.4	−0.347091	−0.232821	−0.139009	−0.062589	−0.001011	0.047822
0.5	−0.396865	−0.256904	−0.146735	−0.060639	0.006001	0.056877
0.6	−0.432536	−0.268223	−0.144786	−0.052833	0.014969	0.064299
0.7	−0.457002	−0.269611	−0.135683	−0.041079	0.024837	0.070017
0.8	−0.473186	−0.263903	−0.121895	−0.027192	0.034657	0.074043
0.9	−0.483579	−0.253506	−0.105487	−0.012642	0.043711	0.076464
1.0	−0.490094	−0.240258	−0.088005	0.001516	0.051538	0.077422
1.1	−0.494098	−0.225469	−0.070513	0.014606	0.057904	0.077096
1.2	−0.496520	−0.210015	−0.053689	0.026252	0.062744	0.075683
1.3	−0.497966	−0.194460	−0.037930	0.036293	0.066114	0.073384
1.4	−0.498820	−0.179154	−0.023436	0.044711	0.068140	0.070394
1.5	−0.499320	−0.164296	−0.010285	0.051576	0.068985	0.066895
1.6	−0.499610	−0.149999	0.001524	0.057006	0.068828	0.063048
1.7	−0.499778	−0.136319	0.012039	0.061145	0.067843	0.058990
1.8	−0.499874	−0.123277	0.021331	0.064143	0.066196	0.054839
1.9	−0.499928	−0.110873	0.029485	0.066148	0.064034	0.050689
2.0	−0.499959	−0.099097	0.036590	0.067299	0.061486	0.046614
2.1	−0.499977	−0.087930	0.042735	0.067723	0.058666	0.042672
2.2	−0.499987	−0.077351	0.048003	0.067537	0.055666	0.038904
2.3	−0.499993	−0.067339	0.052477	0.066843	0.052564	0.035338
2.4	−0.499996	−0.057869	0.056230	0.065731	0.049423	0.031992

b/s	Values of m_{xy}				
	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$	$v/b = 1.0$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.026788	0.037472	0.045089	0.049652	0.051173
0.2	0.051080	0.070696	0.084594	0.092888	0.095645
0.3	0.070995	0.096554	0.114470	0.125083	0.128599
0.4	0.085605	0.113669	0.133018	0.144352	0.148085
0.5	0.094899	0.122305	0.140751	0.151376	0.154845
0.6	0.099501	0.123828	0.139634	0.148507	0.151365
0.7	0.100335	0.120062	0.132200	0.138733	0.140789
0.8	0.098361	0.112797	0.120884	0.124894	0.126096
0.9	0.094426	0.103530	0.107673	0.109282	0.109678
1.0	0.089213	0.093383	0.094006	0.093524	0.093229
1.1	0.083241	0.083126	0.080820	0.078651	0.077807
1.2	0.076890	0.073249	0.068658	0.065229	0.063984
1.3	0.070432	0.064034	0.057784	0.053494	0.051986
1.4	0.064060	0.055624	0.048273	0.043471	0.041818
1.5	0.057905	0.048067	0.040089	0 035062	0.033359
1.6	0.052056	0.041355	0.033132	0.028104	0.026424
1.7	0.046569	0.035447	0.027276	0.022410	0.020804
1.8	0.041477	0.030283	0.022380	0.017790	0.016295
1.9	0.036794	0.025795	0.018312	0.014070	0.012705
2.0	0.032519	0.021914	0.014948	0.011092	0.009866
2.1	0.028645	0.018572	0.012176	0.008720	0.007635
2.2	0.025154	0.015704	0.009901	0.006838	0.005889
2.3	0.022026	0.013251	0.008037	0.005351	0.004530
2.4	0.019236	0.011159	0.006514	0.004180	0.003476

TABLE 10.—(CONCLUDED)

NUMERICAL VALUES OF THE COEFFICIENT FOR THE TWISTING MOMENT M_{xy} PER UNIT OF LENGTH IN A SIMPLY-SUPPORTED PANEL OF A SLAB DUE TO A MOMENT APPLIED AT ONE EDGE OF PANEL

For a moment $M = M_0 \sin \pi x/s$ applied at an edge, on the line $y = 0$, of a slab simply supported on all four sides, the twisting moment M_{xy} on the line $y = v$, at a distance v from the edge on which the moment M is applied, is given by the equation $M_{xy} = (1 - \mu)\, m_{xy} \cdot M_0 \cos \pi x/s$. The span of the panel of the slab in the direction of the y axis is b.

b/s	Values of m_{xy}					
	$v/b = 0$	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$
2.5	-0.499998	-0.048920	0.059333	0.064280	0.046294	0.028876
2.6	-0.499999	-0.040468	0.061851	0.062560	0.043216	0.025993
2.7	-0.499999	-0.032491	0.063843	0.060631	0.040222	0.023339
2.8	-0.500000	-0.024969	0.065363	0.058543	0.037332	0.020909
2.9	-0.500000	-0.017881	0.066462	0.056340	0.034565	0.018693
3.0	-0.500000	-0.011207	0.067184	0.054060	0.031931	0.016679
3.1	-0.500000	-0.004929	0.067572	0.051735	0.029437	0.014857
3.2	-0.500000	0.000971	0.067664	0.049391	0.027087	0.013212
3.3	-0.500000	0.006512	0.067494	0.047050	0.024882	0.011731
3.4	-0.500000	0.011708	0.067094	0.044730	0.022820	0.010402
3.5	-0.500000	0.016577	0.066492	0.042448	0.020898	0.009212
3.6	-0.500000	0.021134	0.065714	0.040213	0.019111	0.008147
3.7	-0.500000	0.025393	0.064785	0.038038	0.017456	0.007198
3.8	-0.500000	0.029368	0.063725	0.035928	0.015924	0.006352
3.9	-0.500000	0.033073	0.062554	0.033891	0.014511	0.005601
4.0	-0.500000	0.036521	0.061290	0.031929	0.013210	0.004933
4.2	-0.500000	0.042693	0.058540	0.028243	0.010916	0.003817
4.4	-0.500000	0.047979	0.055586	0.024881	0.008989	0.002945
4.6	-0.500000	0.052462	0.052513	0.021841	0.007379	0.002265
4.8	-0.500000	0.056222	0.049391	0.019111	0.006041	0.001738
5.0	-0.500000	0.059328	0.046273	0.016675	0.004933	0.001330
5.2	-0.500000	0.061848	0.043204	0.014511	0.004019	0.001016
5.4	-0.500000	0.063842	0.040213	0.012599	0.003268	0.000775
5.6	-0.500000	0.065362	0.037327	0.010916	0.002652	0.000590
5.8	-0.500000	0.066461	0.034562	0.009439	0.002149	0.000448
6.0	-0.500000	0.067184	0.031929	0.008147	0.001738	0.000340

b/s	Values of m_{xy}				
	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$	$v/b = 1.0$
2.5	0.016759	0.009381	0.005273	0.003260	0.002661
2.6	0.014568	0.007872	0.004263	0.002538	0.002032
2.7	0.012637	0.006595	0.003442	0.001974	0.001550
2.8	0.010941	0.005517	0.002777	0.001533	0.001179
2.9	0.009456	0.004608	0.002237	0.001190	0.000896
3.0	0.008158	0.003844	0.001801	0.000922	0.000680
3.1	0.007028	0.003203	0.001449	0.000714	0.000515
3.2	0.006046	0.002665	0.001164	0.000553	0.000390
3.3	0.005194	0.002215	0.000935	0.000427	0.000295
3.4	0.004456	0.001839	0.000750	0.000330	0.000222
3.5	0.003819	0.001525	0.000601	0.000255	0.000168
3.6	0.003269	0.001264	0.000481	0.000197	0.000126
3.7	0.002795	0.001046	0.000385	0.000152	0.000095
3.8	0.002388	0.000865	0.000308	0.000117	0.000072
3.9	0.002037	0.000715	0.000246	0.000090	0.000054
4.0	0.001738	0.000590	0.000197	0.000070	0.000040
4.2	0.001261	0.000402	0.000125	0.000041	0.000023
4.4	0.000912	0.000272	0.000080	0.000024	0.000013
4.6	0.000658	0.000184	0.000051	0.000014	0.000007
4.8	0.000473	0.000124	0.000032	0.000009	0.000004
5.0	0.000340	0.000084	0.000020	0.000005	0.000002
5.2	0.000244	0.000056	0.000013	0.000003	0.000001
5.4	0.000174	0.000038	0.000008	0.000002	0.000001
5.6	0.000124	0.000025	0.000005	0.000001	0.000000
5.8	0.000089	0.000017	0.000003	0.000001	0.000000
6.0	0.000063	0.000011	0.000002	0.000000	0.000000

Table 11

Numerical Values for the Coefficient in the Expression for Deflection of a Simply-Supported Panel of a Slab Due to a Moment Applied at One Edge of Panel

For a moment $M = M_0 \sin \pi x/s$ applied at an edge, on the line $y = 0$, of a slab simply supported on all four sides, the deflection w on the line $y = v$, at a distance v from the edge on which the moment M is applied, is given by the equation $w = C_w \frac{b^2}{N} M$. The span of the panel of the slab in the direction of the y axis is b.

b/s	Values of C_w								
	$v/b = 0.1$	$v/b = 0.2$	$v/b = 0.3$	$v/b = 0.4$	$v/b = 0.5$	$v/b = 0.6$	$v/b = 0.7$	$v/b = 0.8$	$v/b = 0.9$
0.0	0.0285000	0.0480000	0.0595000	0.0640000	0.0625000	0.0560000	0.0455000	0.0320000	0.0165000
0.1	0.0280775	0.0472092	0.0584331	0.0627718	0.0612340	0.0548169	0.0445079	0.0312868	0.0161275
0.2	0.0268784	0.0449664	0.0554101	0.0592949	0.0576533	0.0514733	0.0417059	0.0292734	0.0150761
0.3	0.0250834	0.0416142	0.0509005	0.0541183	0.0523318	0.0465119	0.0375536	0.0262926	0.0135205
0.4	0.0229269	0.0375967	0.0455118	0.0479514	0.0460104	0.0406330	0.0326435	0.0227730	0.0116853
0.5	0.0206348	0.0333413	0.0398277	0.0414743	0.0393976	0.0345048	0.0275396	0.0191223	0.0097843
0.6	0.0183834	0.0291807	0.0343012	0.0352126	0.0330388	0.0286394	0.0226733	0.0156511	0.0079798
0.7	0.0162862	0.0253280	0.0292202	0.0294975	0.0272747	0.0233542	0.0183095	0.0125497	0.0063711
0.8	0.0144006	0.0218897	0.0247257	0.0244876	0.0222644	0.0187943	0.0145674	0.0099019	0.0050014
0.9	0.0127434	0.0188947	0.0208523	0.0202169	0.0180370	0.0149815	0.0114614	0.0077162	0.0038746
1.0	0.0113062	0.0163248	0.0175703	0.0166446	0.0145435	0.0118641	0.0089441	0.0059562	0.0029708
1.1	0.0100680	0.0141378	0.0148176	0.0136926	0.0116969	0.0093552	0.0069389	0.0045649	0.0022598
1.2	0.0090035	0.0122832	0.0125212	0.0112709	0.0093985	0.0073581	0.0053613	0.0034800	0.0017083
1.3	0.0080870	0.0107108	0.0106092	0.0092917	0.0075531	0.0057800	0.0041312	0.0026425	0.0012853
1.4	0.0072954	0.0093751	0.0090166	0.0076761	0.0060758	0.0045387	0.0031780	0.0020007	0.0009634
1.5	0.0066084	0.0082367	0.0076874	0.0063570	0.0048947	0.0035652	0.0024424	0.0015117	0.0007200
1.6	0.0060093	0.0072623	0.0065750	0.0052781	0.0039503	0.0028027	0.0018764	0.0011405	0.0005368
1.7	0.0054838	0.0064248	0.0056408	0.0043938	0.0031943	0.0022056	0.0014417	0.0008596	0.0003996
1.8	0.0050206	0.0057017	0.0048534	0.0036671	0.0025883	0.0017381	0.0011081	0.0006474	0.0002969
1.9	0.0046102	0.0050744	0.0041873	0.0030681	0.0021016	0.0013715	0.0008521	0.0004875	0.0002204
2.0	0.0042447	0.0045281	0.0036218	0.0025730	0.0017098	0.0010839	0.0006558	0.0003670	0.0001635
2.1	0.0039178	0.0040504	0.0031400	0.0021625	0.0013938	0.0008579	0.0005051	0.0002763	0.0001212
2.2	0.0036242	0.0036312	0.0027282	0.0018213	0.0011383	0.0006800	0.0003894	0.0002080	0.0000898
2.3	0.0033595	0.0032619	0.0023752	0.0015368	0.0009313	0.0005398	0.0003005	0.0001567	0.0000665
2.4	0.0031200	0.0029357	0.0020716	0.0012992	0.0007632	0.0004292	0.0002321	0.0001180	0.0000492
2.5	0.0029026	0.0026467	0.0018100	0.0011001	0.0006264	0.0003416	0.0001795	0.0000889	0.0000364
2.6	0.0027046	0.0023900	0.0015839	0.0009330	0.0005149	0.0002723	0.0001389	0.0000671	0.0000269
2.7	0.0025239	0.0021613	0.0013881	0.0007924	0.0004239	0.0002174	0.0001076	0.0000506	0.0000199
2.8	0.0023585	0.0019572	0.0012181	0.0006739	0.0003494	0.0001737	0.0000835	0.0000382	0.0000147
2.9	0.0022067	0.0017746	0.0010704	0.0005738	0.0002883	0.0001390	0.0000648	0.0000288	0.0000109
3.0	0.0020672	0.0016110	0.0009416	0.0004892	0.0002382	0.0001113	0.0000503	0.0000218	0.0000081
3.1	0.0019387	0.0014641	0.0008293	0.0004175	0.0001970	0.0000892	0.0000391	0.0000165	0.0000060
3.2	0.0018200	0.0013320	0.0007311	0.0003567	0.0001631	0.0000716	0.0000305	0.0000124	0.0000044
3.3	0.0017103	0.0012130	0.0006452	0.0003051	0.0001352	0.0000575	0.0000237	0.0000094	0.0000033
3.4	0.0016086	0.0011056	0.0005699	0.0002611	0.0001122	0.0000462	0.0000185	0.0000071	0.0000024
3.5	0.0015143	0.0010086	0.0005038	0.0002237	0.0000931	0.0000372	0.0000144	0.0000054	0.0000018
3.6	0.0014267	0.0009209	0.0004458	0.0001918	0.0000774	0.0000300	0.0000113	0.0000041	0.0000013
3.7	0.0013452	0.0008414	0.0003947	0.0001646	0.0000643	0.0000241	0.0000088	0.0000031	0.0000010
3.8	0.0012693	0.0007694	0.0003498	0.0001413	0.0000535	0.0000195	0.0000069	0.0000024	0.0000007
3.9	0.0011985	0.0007040	0.0003101	0.0001214	0.0000446	0.0000157	0.0000054	0.0000018	0.0000005
4.0	0.0011324	0.0006446	0.0002752	0.0001044	0.0000372	0.0000127	0.0000042	0.0000014	0.0000004
4.2	0.0010128	0.0005414	0.0002171	0.0000774	0.0000258	0.0000084	0.0000026	0.0000008	0.0000002
4.4	0.0009079	0.0004558	0.0001716	0.0000574	0.0000180	0.0000054	0.0000016	0.0000005	0.0000001
4.6	0.0008155	0.0003845	0.0001359	0.0000427	0.0000126	0.0000036	0.0000010	0.0000003	0.0000001
4.8	0.0007340	0.0003249	0.0001079	0.0000318	0.0000088	0.0000023	0.0000006	0.0000002	0.0000000
5.0	0.0006617	0.0002751	0.0000858	0.0000238	0.0000062	0.0000015	0.0000004	0.0000001	0.0000000
5.2	0.0005975	0.0002333	0.0000683	0.0000178	0.0000043	0.0000010	0.0000002	0.0000001	0.0000000
5.4	0.0005403	0.0001981	0.0000545	0.0000133	0.0000031	0.0000007	0.0000001	0.0000000	0.0000000
5.6	0.0004893	0.0001685	0.0000435	0.0000100	0.0000021	0.0000004	0.0000001	0.0000000	0.0000000
5.8	0.0004437	0.0001435	0.0000348	0.0000075	0.0000015	0.0000003	0.0000001	0.0000000	0.0000000
6.0	0.0004028	0.0001223	0.0000279	0.0000056	0.0000011	0.0000002	0.0000000	0.0000000	0.0000000

TABLE 12

FORMULAS FOR THE VARIOUS DIMENSIONLESS COEFFICIENTS FOR A RECTANGULAR SLAB WITH THE DIMENSION IN THE DIRECTION OF A PAIR OF SIMPLY-SUPPORTED EDGES INFINITELY LONG

The various stiffnesses, defined in Fig. 6 are expressed as follows:

$$K = 2\pi \frac{N}{s} = 6.283185 \frac{N}{s} \qquad T = 2\pi^3 \frac{N}{s^3} = 62.01255 \frac{N}{s^3}$$

$$Q = (1 + \mu)\,\pi^2 \frac{N}{s^2} = 9.869604\,(1 + \mu) \frac{N}{s^2}$$

For a distributed load $p = p_0 \sin \frac{\pi x}{s}$ on the whole slab, where p_0 is a constant, the moments and reactions at the edge $y = 0$, if the edge is fixed, are

$$\mathsf{M}^F = -\frac{1}{\pi^2} ps^2 = -0.101321\ ps^2 \qquad \mathsf{R}^F = -\frac{2}{\pi} ps = -0.636620\ ps$$

For a line load $F = F_0 \sin \frac{\pi x}{s}$ on the line $y = v$, the moments and reactions at the edge $y = 0$, if the edge is fixed, are

$$\mathsf{M}^F = -C \cdot Fv \quad \text{where} \quad C = e^{-\frac{\pi v}{s}}$$

$$\mathsf{R}^F = -C_R \cdot F \quad \text{where} \quad C_R = \left(1 + \frac{\pi v}{s}\right) e^{-\frac{\pi v}{s}}$$

For an edge moment $\mathsf{M} = \mathsf{M}_0 \sin \frac{\pi x}{s}$ on the edge $y = 0$, the bending moments M_x and M_y, the twisting moment M_{xy}, and the deflection w on the line $y = v$ are

$$M_x = (m_x + \mu m_y)\,\mathsf{M} \qquad M_y = (m_y + \mu m_x)\,\mathsf{M}$$

$$M_{xy} = (1 - \mu)\,m_{xy} \cdot \mathsf{M}_0 \cos \frac{\pi x}{s} \qquad w = C_w \frac{s^2}{N} \cdot \mathsf{M}$$

where $m_x = \frac{1}{2} \frac{\pi v}{s} e^{-\frac{\pi v}{s}}$, $m_y = C - m_x$, $m_{xy} = -\frac{1}{2} C + m_x$, $C_w = \frac{1}{\pi^2} m_x$

For an edge deflection $\Delta = \Delta_0 \sin \frac{\pi x}{s}$ on the edge $y = 0$ combined with an edge moment $-\mathsf{M}_w = -\frac{(1 - \mu)\,N\pi^2}{s^2} \Delta_0 \sin \frac{\pi x}{s}$ on the edge $y = 0$, the bending moments M_x and M_y, the twisting moment M_{xy}, and the deflection w on the line $y = v$ are

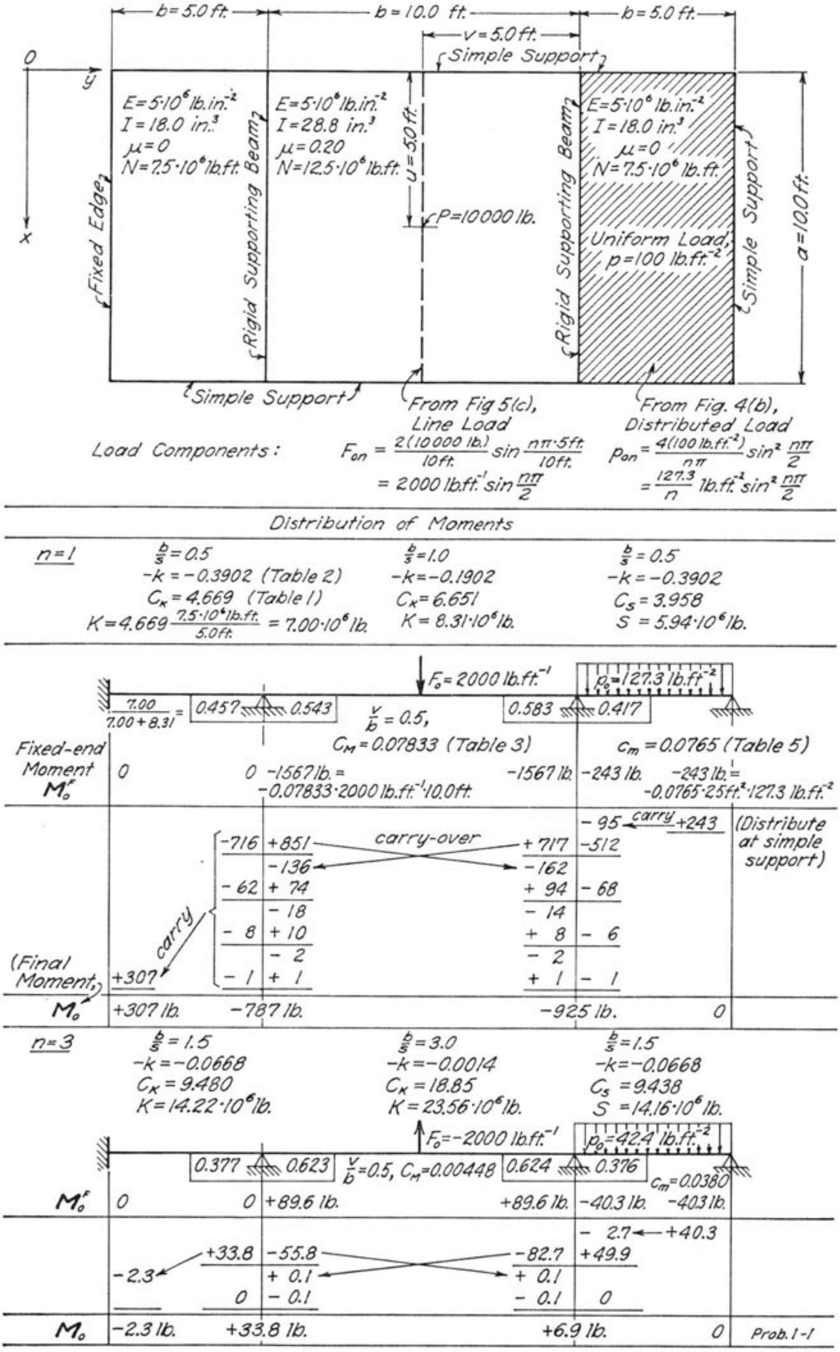
$$M_x = -M_y = C \cdot \mathsf{M}_w, \qquad M_{xy} = C \frac{(1 - \mu)\,N\pi^2}{s^2} \Delta_0 \cos \frac{\pi x}{s}, \qquad w = C \cdot \Delta$$

TABLE 13

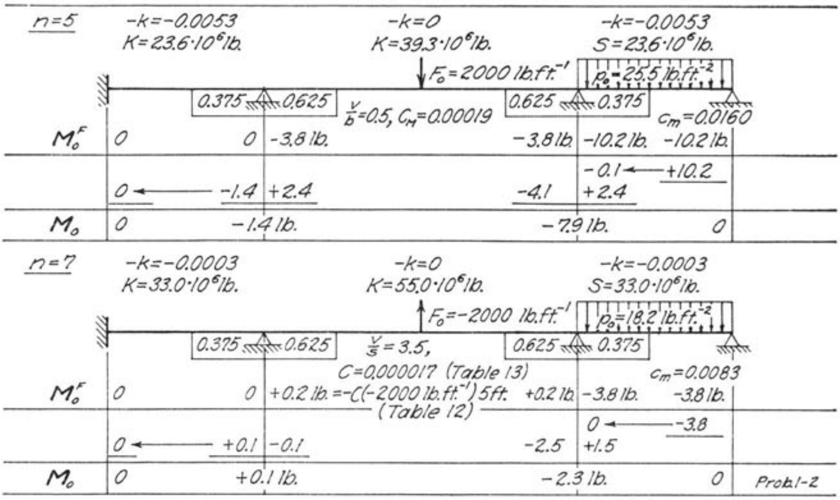
NUMERICAL VALUES FOR THE VARIOUS DIMENSIONLESS COEFFICIENTS DEFINED IN TABLE 12 FOR AN INFINITELY LONG SLAB

v/s	C	C_R	m_x	m_y	m_{xy}	C_w
0.0	1.000000	1.000000	0.000000	1.000000	−0.500000	0.0000000
0.1	0.730403	0.959865	0.114731	0.615671	−0.250470	0.0116247
0.2	0.533488	0.868689	0.167600	0.365888	−0.099144	0.0169815
0.3	0.389661	0.756908	0.183623	0.206038	−0.011207	0.0186049
0.4	0.284610	0.642260	0.178825	0.105784	0.036521	0.0181188
0.5	0.207880	0.534416	0.163268	0.044611	0.059328	0.0165425
0.6	0.151836	0.438040	0.143102	0.008734	0.067184	0.0144993
0.7	0.110901	0.354786	0.121942	−0.011041	0.066492	0.0123553
0.8	0.081003	0.284584	0.101791	−0.020788	0.061290	0.0103136
0.9	0.059165	0.226448	0.083642	−0.024477	0.054060	0.0084747
1.0	0.043214	0.178974	0.067880	−0.024666	0.046273	0.0068777
1.1	0.031564	0.140639	0.054538	−0.022974	0.038756	0.0055258
1.2	0.023054	0.109966	0.043456	−0.020402	0.031929	0.0044030
1.3	0.016839	0.085610	0.034385	−0.017547	0.025966	0.0034840
1.4	0.012299	0.066393	0.027047	−0.014748	0.020898	0.0027404
1.5	0.008983	0.051316	0.021166	−0.012183	0.016675	0.0021446
1.6	0.006561	0.039543	0.016491	−0.009929	0.013210	0.0016709
1.7	0.004792	0.030388	0.012798	−0.008005	0.010401	0.0012967
1.8	0.003500	0.023295	0.009897	−0.006397	0.008147	0.0010028
1.9	0.002557	0.017818	0.007631	−0.005074	0.006352	0.0007731
2.0	0.001867	0.013601	0.005867	−0.003999	0.004933	0.0005944
2.1	0.001364	0.010363	0.004499	−0.003135	0.003817	0.0004559
2.2	0.000996	0.007882	0.003443	−0.002447	0.002945	0.0003488
2.3	0.000728	0.005986	0.002629	−0.001901	0.002265	0.0002664
2.4	0.000531	0.004539	0.002004	−0.001472	0.001738	0.0002030
2.5	0.000388	0.003437	0.001524	−0.001136	0.001330	0.0001545
2.6	0.000284	0.002600	0.001158	−0.000874	0.001016	0.0001173
2.7	0.000207	0.001964	0.000878	−0.000671	0.000775	0.0000890
2.8	0.000151	0.001482	0.000665	−0.000514	0.000590	0.0000674
2.9	0.000110	0.001117	0.000503	−0.000393	0.000448	0.0000510
3.0	0.000081	0.000841	0.000380	−0.000300	0.000340	0.0000385
3.1	0.000059	0.000633	0.000287	−0.000228	0.000258	0.0000291
3.2	0.000043	0.000476	0.000216	−0.000173	0.000195	0.0000219
3.3	0.000031	0.000357	0.000163	−0.000132	0.000147	0.0000165
3.4	0.000023	0.000268	0.000123	−0.000100	0.000111	0.0000124
3.5	0.000017	0.000201	0.000092	−0.000075	0.000084	0.0000093
3.6	0.000012	0.000151	0.000069	−0.000057	0.000063	0.0000070
3.7	0.000009	0.000113	0.000052	−0.000043	0.000048	0.0000053
3.8	0.000007	0.000085	0.000039	−0.000032	0.000036	0.0000040
3.9	0.000005	0.000062	0.000029	−0.000024	0.000027	0.0000030

Illustrative Problem 1



Illustrative Problem 1 (continued)



VI. Illustrative Problems

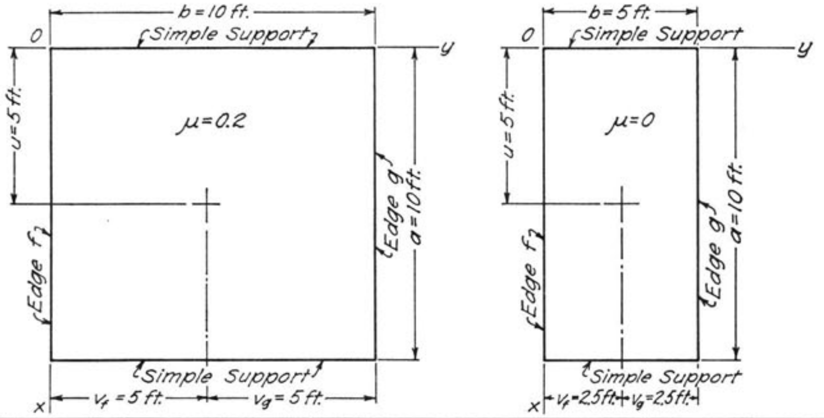
23. *Illustrative Problem 1.*—This problem involves a slab continuous over unyielding supports, with one end fixed. To illustrate certain features of the distribution procedure, differences in depth or moment of inertia and differences in Poisson's ratio are considered. The center panel is loaded at its mid-point by a concentrated load, and one end panel is uniformly loaded.

The load components are found from Figs. 4 and 5, and are stated on the computation sheets. The problem is self-explanatory to anyone familiar with the process of distributing moments in a continuous girder. The work is carried out for four terms in the series. For practical purposes only the first term, $n = 1$, or at most the first two terms, need have been considered. Even terms, $n = 2$, 4, etc., do not appear since the load is symmetrical about the center line of the structure parallel to the y axis.

The analogous continuous girder is shown for each value of n. For example, for $n = 1$, the carry-over factor for the end panels is -0.3902, and the stiffness of the end panels is $7.00 \cdot 10^6$ lb. with the far end fixed, or $5.94 \cdot 10^6$ lb. with the far end hinged. The distribution of an unbalanced moment at the joint to the left of the center is in the proportion of 8.31 to 7.00 for center and end panels, respectively. The distribution factors at the joint are, consequently, $\dfrac{7.00}{7.00 + 8.31} = 0.457$ to the left, and $\dfrac{8.31}{7.00 + 8.31} = 0.543$ to the

Illustrative Problem 1, (continued)

Bending moments per unit of length, M_x and M_y, at centers of loaded panels.



(1)–Due to edge moments only:

Square panel ($\mu = 0.2$)	Panel $b = 5$ ft. ($\mu = 0$)
$\frac{b}{s} = 1.0$	$\frac{b}{s} = 0.5$
$M_{of} = -787$ lb. $\quad M_{og} = -925$ lb.	$M_{of} = -925$ lb. $\quad M_{og} = 0$
$n = 1$: $\frac{v_f}{b} = \frac{v_g}{b} = 0.5$	$\frac{v_f}{b} = \frac{v_g}{b} = 0.5$
$m_x = 0.1435$ (Table 8)	$m_x = 0.0972$
$m_y = 0.0557$ (Table 9)	$m_y = 0.2803$
$m_x + \mu m_y = 0.1546$	$m_x + \mu m_y = 0.0972$
$m_y + \mu m_x = 0.0844$	$m_y + \mu m_x = 0.2803$
$M_{ox} = 0.1546(-787-925)$ lb. $= -265$ lb.	$M_{ox} = 0.0972(-925$ lb.$) = -90$ lb.
$M_{oy} = 0.0844(-787-925)$ lb. $= -145$ lb.	$M_{oy} = 0.2803(-925$ lb.$) = -259$ lb.
$n = 3$: $\frac{b}{s} = 3.0$	$\frac{b}{s} = 1.5$
$M_{of} = +33.8$ lb. $\quad M_{og} = +6.9$ lb.	$M_{of} = +6.9$ lb. $\quad M_{og} = 0$
$\frac{v_f}{b} = \frac{v_g}{b} = 0.5$	$\frac{v_f}{b} = \frac{v_g}{b} = 0.5$
$m_x = 0.0212$	$m_x = 0.1087$
$m_y = -0.0122$	$m_y = -0.0148$
$m_x + \mu m_y = 0.0188$	
$m_y + \mu m_x = -0.0080$	
$M_{ox} = +0.8$ lb.	$M_{ox} = +0.7$ lb.
$M_{oy} = -0.3$ lb.	$M_{oy} = -0.1$ lb.

(2)–Due to loading acting on a simply supported rectangular panel:

Concentrated Load

Use coefficients given by Westergaard, (Explained in text).

Consider load distributed uniformly over a circle with diameter "c", where $c = 0.10a$, and take depth of slab, "h", as $h = 0.05a$.

For $\mu = 0.15$, Westergaard's tables and the equations applying to a square slab, give the result:

$M_x = M_y = 0.3184P - 0.0490P = 0.2694P.$

For $\mu = 0.20$, from equations (72) & (73):

$M_x = M_y = +\frac{1.20}{1.15}\, 0.2694(10000 \text{ lb.}) = +281.1$ lb.

Uniform Load

Use coefficients given by Westergaard (Explained in text).

The moment coefficients, for $\mu = 0$, for a slab with the short side 0.5 times the long side are determined as follows, in terms of the short span "b", and the load per unit area, "p":

In the direction of the long span,

$M_x = +0.0174\,pb^2,$

and in the direction of the short span,

$M_y = +0.0964\,pb^2.$

With $p = 100$ lb. ft.$^{-2}$, $b = 5$ ft.,

$M_x = +43.5$ lb. and

$M_y = +241$ lb.

Prob. 1-3

Illustrative Problem 1 (concluded)

Total Moments at supports and at center of loaded panels.

(Moments to nearest lb.) $P = 10\,000$ lb. $p = 100$ lb.ft.$^{-2}$

	$M = M_y$	$M = M_y$	M_x	M_y	$M = M_y$	M_x	M_y	$M = M_y$
Moments due to Continuity:								
$n=1$, coef. of $\sin \frac{\pi x}{a}$	+307 lb.	−787 lb.	−265 lb.	−145 lb.	−925 lb.	−90 lb.	−259 lb.	0
$n=3$, coef. of $\sin \frac{3\pi x}{a}$	− 2	+ 34	+ 1	0	+ 7	+ 1	0	0
$n=5$, coef. of $\sin \frac{5\pi x}{a}$	0	− 1			− 8			0
$n=7$, coef. of $\sin \frac{7\pi x}{a}$	0	0			− 2			0
Moments at $x = \frac{a}{2}$:								
Due to Continuity-	+309	−822	−266	−145	−938	−91	−259	0
Due to loads acting on simply supported panels-	0	0	+2811	+2811	0	+44	+241	0
Total Moments at $x = \frac{a}{2}$	+309 lb.	−822 lb.	+2545 lb.	+2666 lb.	−938 lb.	−47 lb.	−18 lb.	$0_{1\text{-}4}$

right. The final joint moments are stated for each value of n. The computations were made with a slide rule.

The computation is shown for the moments M_x and M_y at the center of each loaded panel. The effects of the edge moments are first considered. Only the first term in the series is of practical significance in this case. The moments computed in step (1) may be considered as corrections to the moments in a simply-supported panel due to the loading.

The moments at the load point in the center panel due to the loading acting on a simply-supported panel are computed from coefficients given by Westergaard.* From the tabulated coefficients, for a square slab loaded at the center with a load uniformly distributed over a circle with diameter $c = 0.10a$, where a, the span, is $20h$, or twenty times the slab thickness, one finds, under the load,

$$M_x = M_y = 0.3184P - 0.0490P = 0.2694P$$

for $\mu = 0.15$. This value can be corrected to correspond to $\mu = 0$ by Equations (72), and subsequently, to $\mu = 0.20$ by Equations (73). One finds, for $\mu = 0.20$ and $P = 10\ 000$ lb.,

$$M_x = M_y = \frac{1.20}{1.15}\,2694\,\text{lb.} = 2811\,\text{lb.}\dagger$$

*H. M. Westergaard, "Computation of Stresses in Bridge Slabs Due to Wheel Loads," Public Roads, Vol. 11, No. 1, 1930, p. 1 to 23. See especially p. 19 for a square slab with a concentrated load, and Table 1, p. 9 for the moment under a concentrated load. Note that the numerical values are given for $\mu = 0.15$.

†Moments in a slab are always stated in terms of bending moment per unit length; consequently this equation may be interpreted as $M_x = 2811$ ft. lb. per foot, or 2811 in. lb. per inch. An adequate statement is merely $M_x = 2811$ lb., as above.

The moment at the center of the uniformly-loaded end panel is determined from coefficients plotted by Westergaard* for a simply-supported rectangular slab. It is noted that in this particular problem the long span is in the x direction and the short span, which is half the long span, is in the y direction. One finds

$$M_x = 0.0174\, pb^2 = 43.5 \text{ lb.}$$

$$M_y = 0.0964\, pb^2 = 241 \text{ lb.}$$

The moments due to the loading on simply-supported panels are total moments, not components. The correction moments due to continuity are components of a sine series. For example, at the joint to the right of the center of the structure the moments $\mathsf{M} = M_y$ are

$$M_y = \left(-925 \sin\frac{\pi x}{a} + 7 \sin\frac{3\pi x}{a} - 8 \sin\frac{5\pi x}{a} - 2 \sin\frac{7\pi x}{a} + \ldots\right) \text{lb.}$$

For $x = \frac{a}{2}$, one finds

$$M_y = [-925\,(1) + 7\,(-1) - 8\,(1) - 2\,(-1) + \ldots] \text{ lb.}$$

$$= -938 \text{ lb.}$$

For $x = \frac{a}{4}$, one finds

$$M_y = 0.7071\,[-925\,(1) + 7\,(1) - 8\,(-1) - 2\,(-1) + \ldots] \text{ lb.}$$

$$= -642 \text{ lb.}$$

The other components are added in the same manner. Numerical values are reported on the computation sheets only for $x = a/2$.

24. *Illustrative Problem 2.*—This is the same structure as in Problem 1 except that there is a flexible supporting beam instead of a

*H. M. Westergaard and W. A. Slater, "Moments and Stresses in Slabs," Proc. American Concrete Institute, Vol. 17, 1921, p. 415-538. See Fig. 3, p. 431 for moment coefficients for $\mu = 0$.

rigid support at the right of the center panel. The problem is solved, as indicated on the computation sheets, by finding the total unbalanced reaction at the joint from Problem 1, and then adding a deflection of the joint of such magnitude as to make the unbalanced reaction zero.

The computation of unbalanced reaction from the results of Problem 1 is shown for $n = 1$. A deflection of $1000 \cdot 10^{-6}$ ft. is introduced at the joint, the resulting fixed-end moments are distributed, and the unbalanced reaction computed. It is found that the actual deflection Δ_0 is $726 \cdot 10^{-6}$ ft. Then the actual moments for the structure with the flexible beam are found by adding to the moments in Problem 1, 0.726 times the moments due to the deflection introduced in step (2). The results of the computations are recorded for $n = 3$, 5, and 7 also.

The corrective moments M_x and M_y at the center of the loaded panels are computed as in Problem 1. For the center panel, for $n = 1$, the center deflection of the left edge is zero, and of the right edge, $+726 \cdot 10^{-6}$ ft. The center moment at the left edge is -1016 lb., and at the right edge, $+110$ lb. A fictitious edge moment $-\mathsf{M}_{wg}$ is introduced at the right edge. The magnitude of $-\mathsf{M}_{0wg}$ is -716 lb. Since the effects of the deflection and of this moment are combined, one must add to these effects the effect of an edge moment acting at the right edge, M'_{0g}, of magnitude $+110$ lb. $-(-716)$ lb. or $+826$ lb. Then M_{0x} is computed in the following manner:

$$
\begin{aligned}
M_{0x} &= C\,(\mathsf{M}_{0wg}) + (m_x + \mu m_y)\,(\mathsf{M}'_{0g}) + (m_x + \mu m_y)\,(\mathsf{M}_{0f}) \\
&= 0.1993\,(716\text{ lb.}) + 0.1546\,(-1016\text{ lb.}) + 0.1546\,(826\text{ lb.}) \\
&= +113.2\text{ lb.}
\end{aligned}
$$

It is noted that $M_x = M_{0x} \sin \pi x/a$. The other quantities are computed in the same way.

The moments due to the loading on simply-supported panels are the same as in Problem 1. The component moments are added together in the same fashion as in Problem 1, to obtain the total moments at the points considered.

25. *Illustrative Problem 3.*—This problem represents a single span slab with curbs which act as flexible supporting beams offering torsional restraint to the slab. A uniform load is considered. In view

Illustrative Problem 2

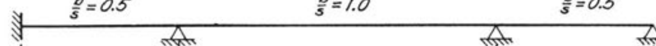
Rigid Supporting Beam	Structure and Loading same as in Problem 1, except for flexible beam.	Flexible Supporting Beam $E = 30 \cdot 10^6$ lb. in.$^{-2}$ $I = 120$ in.^{4}

For flexible beam

$$U = \frac{97.41\,\mathbf{EI}}{s^4}, \quad \text{(Equation 36)},$$
$$= \frac{97.41(30\cdot 10^6 \text{lb.in.}^{-2})(120 \text{ in.}^4)}{(10 \text{ ft.})^4} n^4$$
$$= 35.07 n^4 \cdot 10^6 \text{lb.in.}^2\text{ft.}^{-4}$$
$$= 0.2435 n^4 \cdot 10^6 \text{lb.ft.}^{-2}$$

Indirect Method of Analysis

Step (1)–Unbalanced reaction for no deflection of flexible beam



$n=1$	$\frac{b}{s}=0.5$		$\frac{b}{s}=1.0$		$\frac{b}{s}=0.5$		
	Elastic constants same as for other end panel.		$K=8.31\cdot10^6$ lb. $S=8.01\cdot10^6$ lb. $Q=1.678\cdot10^6$ lb.ft.$^{-1}$ $T=0.860\cdot10^6$ lb.ft.$^{-2}$	$k=0.1902$ $q=0.4321$ $t=0.3501$	$K=7.00\cdot10^6$ lb. $S=5.94\cdot10^6$ lb. $Q=2.032\cdot10^6$ lb.ft.$^{-1}$ $T=1.208\cdot10^6$ lb.ft.$^{-2}$		$k=0.3902$ $q=0.9313$ $t=0.8499$
Final moments, $\mathbf{M}_o$, for no deflection of supports, Problem 1	+307 lb.	−787 lb.	−787 lb.	−925 lb.	−925 lb.	0	
Fixed-end moments, $\mathbf{M}_o^f$, in Problem 1	0	0	−1567 lb.	−1567 lb.	−243 lb.	−243 lb.	
Change in moment, $\delta\mathbf{M}_o$, due to rotations only	+307 lb.	−787 lb.	+780 lb.	+642 lb.	−682 lb.	+243 lb.	
Rotation of joints, Φ_o; $\Phi_{of} = \frac{\delta M_{of} + k\delta M_{og}}{S}$	0	$-112.4\cdot10^{-6}$	$+112.5\cdot10^{-6} = \frac{+780 \text{ lb.} + 0.1902(+642 \text{ lb.})}{8.01\cdot10^6 \text{ lb.}}$	$+98.6\cdot10^{-6}$	$-98.8\cdot10^{-6}$	$-3.9\cdot10^{-6}$	

Note: Rotations are equal and opposite in sign as required for continuity.

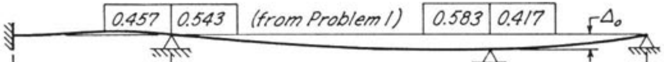
Change in reaction, $\delta\mathbf{R}_o$, due to rotations		$+98.6\cdot10^{-6}(Q) =$ $+112.5\cdot10^{-6}(-qQ) =$	+165 lb.ft.$^{-1}$ −82 lb.ft.$^{-1}$	−201 lb.ft.$^{-1}$ $= -98.8\cdot10^{-6}(Q)$ +7 lb.ft.$^{-1}$ $= -3.9\cdot10^{-6}(-qQ)$	
Fixed-end reaction, $\mathbf{R}_o^f$, due to loads		$\frac{v}{b}=0.5$, $C_R=0.4250$ (Table 4) $-0.4250\cdot2000$ lb.ft.$^{-1}$ =	−850 lb.ft.$^{-1}$	−316 lb.ft.$^{-1}$ = $-0.4963\cdot5$ ft$\cdot127.3$ lb.ft.$^{-2}$ $c_r=0.4963$ (Table 5)	
Total unbalanced reaction at joint, $\Sigma\mathbf{R}_o$			−1277 lb.ft.$^{-1}$		

Prob 2, p1.

Illustrative Problem 2 (continued)

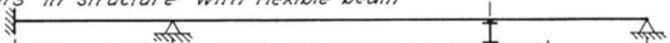
Step (2)-Unbalanced reaction due to deflection of flexible beam, $\Delta_o = 1000 \cdot 10^{-6}$ ft.

$n=1$



Distribution factors		0.457	0.543 (from Problem 1)	0.583	0.417	
M_o^F due to $\Delta_o = 1000 \cdot 10^{-6}$ ft.	0	0	-725 lb. $= -qQ\Delta_o$	$Q\Delta_o = +1678$ lb.	$+2032$ lb.	-1892 lb. $= -qQ\Delta_o$
					$-738 \leftarrow$	$+1892$
		-331	$+394$	-224	$+160$	
			$+43$	-75		
		$+20$	-23	$+44$	-31	
			-8	$+4$		
	$+123$	-4	$+4$	-2	$+2$	
M_o	$+123$ lb.	-315 lb.		$+1425$ lb.		0
δM_o	$+123$ lb.	-315 lb.	$+410$ lb.	-253 lb.	-607 lb	$+1892$ lb.
Φ_o	0	$-45.0 \cdot 10^{-6}$	$+45.2 \cdot 10^{-6}$	$-21.9 \cdot 10^{-6}$	$+22.0 \cdot 10^{-6}$	$+278.6 \cdot 10^{-6}$
Change in reaction, δR_o, due to rotations				$-21.9 \cdot 10^{-6}(Q) = -37$ lb.ft.$^{-1}$ $+45.2 \cdot 10^{-6}(-qQ) = -33$ lb.ft.$^{-1}$	$+45$ lb.ft.$^{-1} = +22.0 \cdot 10^{-6}(Q)$ -527 lb.ft.$^{-1} = +278.6 \cdot 10^{-6}(-qQ)$	
Fixed-end reaction, R_o^F, due to deflection Δ_o			in slab — in beam —	$T \cdot \Delta_o = +860$ lb.ft.$^{-1}$ $U\Delta_o = +244$ lb.ft.$^{-1}$	$+1208$ lb.ft.$^{-1} = T \cdot \Delta_o$	
Total unbalanced reaction at joint, ΣR_o				$+1760$ lb.ft.$^{-1}$		

Step (3) - Moments in structure with flexible beam



$n=1$

Total unbalanced reaction at joint for rigid support, $\Sigma R_o = -1277$ lb.ft.$^{-1}$

ΣR_o for $\Delta_o = 1000 \cdot 10^{-6}$ ft. $= +1760$ lb.ft.$^{-1}$

Δ_o for $\Sigma R_o = 0$ $= +726 \cdot 10^{-6}$ ft. $= -\frac{-1277 \text{ lb.ft.}^{-1}}{1760 \text{ lb.ft.}^{-1}} \cdot 1000 \cdot 10^{-6}$ ft.

M_o for rigid support (Problem 1)	$+307$ lb.	-787 lb.	-925 lb.	0
M_o for $\Delta_o = 726 \cdot 10^{-6}$ ft.	$+89$ lb.	-229 lb. $= -315$ lb. $\frac{726 \cdot 10^{-6} \text{ ft.}}{1000 \cdot 10^{-6} \text{ ft.}}$	$+1035$ lb.	0
M_o for structure with flexible beam	$+396$ lb.	-1016 lb.	$+110$ lb.	0

$n=3$

ΣR_o for rigid support -5.5 lb.ft.$^{-1}$

ΣR_o for $\Delta_o = 1.00 \cdot 10^{-6}$ ft. $+52.2$ lb.ft.$^{-1}$

Δ_o for $\Sigma R_o = 0$ $+0.105 \cdot 10^{-6}$ ft.

M_o for rigid support	-2.3 lb.	$+33.8$ lb.	$+6.9$ lb.	0
M_o for $\Delta_o = 0.105 \cdot 10^{-6}$ ft.	0	0	$+1.0$ lb.	0
M_o for structure with flexible beam	-2.3 lb.	$+33.8$ lb.	$+7.9$ lb.	0

Prob. 2-2

Illustrative Problem 2 (continued)

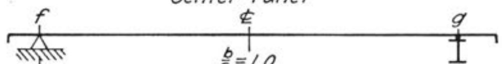
$\underline{n=5}$		ΣR_o for rigid support ΣR_o for $\Delta_o = 1.00\cdot10^{-6}$ft. Δ_o for $\Sigma R_o = 0$	-41.1 lb.ft.$^{-1}$ $+301.8$ lb.ft.$^{-1}$ $+0.136\cdot10^{-6}$ft.	
M_o for rigid support M_o for $\Delta_o = 0.136\cdot10^{-6}$ft.	0 0	-1.4 lb. 0	-7.9 lb. $+3.5$ lb.	0 0
M_o for structure with flexible beam	0	-1.4 lb.	-4.4 lb.	0
$\underline{n=7}$		ΣR_o for rigid support ΣR_o for $\Delta_o = 1.00\cdot10^{-6}$ft. Δ_o for $\Sigma R_o = 0$	-17.9 lb.ft.$^{-1}$ $+995$ lb.ft.$^{-1}$ $+0.018\cdot10^{-6}$ft.	
M_o for rigid support M_o for $\Delta_o = 0.018\cdot10^{-6}$ft.	0 0	$+0.1$ lb. 0	-2.3 lb. $+0.9$ lb.	0 0
M_o for structure with flexible beam	0	$+0.1$ lb.	-1.4 lb.	0

Moments per unit of length, M_x and M_y, at centers of loaded panels:–

Due to edge moments and edge deflections:

$\underline{n=1}$

Center Panel



$\frac{b}{s} = 1.0$

$\Delta_{of} = 0$

$M_{of} = -1016$ lb.

$\Delta_{og} = +726\cdot10^{-6}$ft.

$M_{og} = +110$ lb.

$$-\frac{(1-0.20)(12.5\cdot10^{6}\,\text{lb.ft.})(9.87)}{(10\,\text{ft.})^2}(726\cdot10^{-6}\text{ft.}) = -M_{owg} = -716\text{ lb. (equation 75)}$$

M'_{og} $+826$ lb.

$\frac{v_f}{b} = \frac{v_g}{b} = 0.5$

$m_x + \mu m_y = 0.1546$ (See Problem 1)

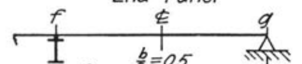
$m_y + \mu m_x = 0.0844$

$C = 0.1993$ (Table 6)

$M_{ox} = 0.1993(716\text{ lb.}) + 0.1546(-1016\text{ lb.} + 826\text{ lb.}) = +113.2\text{ lb.}$

$M_{oy} = -0.1993(716\text{ lb.}) + 0.0844(-1016\text{ lb.} + 826\text{ lb.}) = -158.6\text{ lb.}$

End Panel



$\frac{b}{s} = 0.5$

$\Delta_{of} = +726\cdot10^{-6}$ft.

$M_{of} = +110$ lb.

$-M_{owf}$ -537 lb.

M'_{of} $+647$ lb.

$\Delta_{og} = 0$

$M_{og} = 0$

$\frac{v_f}{b} = \frac{v_g}{b} = 0.5$

$m_x + \mu m_y = 0.0972$

$m_y + \mu m_x = 0.2803$

$C = 0.3775$

$M_{ox} = 0.3775(537\text{ lb.}) + 0.0972(647\text{ lb.}) = +265.7\text{ lb.}$

$M_{oy} = -0.3775(537\text{ lb.}) + 0.2803(647\text{ lb.}) = -21.3\text{ lb.}$

Prob2-3

Illustrative Problem 2 (concluded)

$\underline{n=3}$

Center Panel

f: $\Delta_{of}=0$, $M_{of}=+33.8$ lb.

℄: $\frac{b}{s}=3.0$

g: $\Delta_{og}=+0.105\cdot10^{-6}$ ft., $M_{og}=+7.9$ lb., $-M_{owg}=-0.9$ lb., $M'_{og}=+8.8$ lb.

$\frac{v_f}{b}=\frac{v_g}{b}=0.5$

$m_x+\mu m_y=0.0188$
$m_y+\mu m_x=-0.0080$
$C=0.0090$
$M_{ox}=+0.8$ lb.
$M_{oy}=-0.3$ lb.

End Panel

f: $\Delta_{of}=+0.105\cdot10^{-6}$ ft., $M_{of}=+7.9$ lb., $-M_{owf}=-0.7$ lb., $M'_{of}=+8.6$ lb.

℄: $\frac{b}{s}=1.5$

g: $\Delta_{og}=0$, $M_{og}=0$

$\frac{v_f}{b}=\frac{v_g}{b}=0.5$

$m_x+\mu m_y=0.1087$
$m_y+\mu m_x=-0.0148$
$C=0.0939$
$M_{ox}=+1.0$ lb.
$M_{oy}=-0.2$ lb.

Moments due to the loading acting on simply supported rectangular panels are the same as in Problem 1.

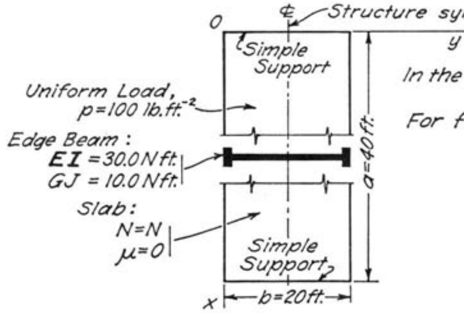
Total moments at supports and at center of loaded panels.

$P=10\,000$ lb. $\quad p=100$ lb. ft.$^{-2}$

	$M=M_y$	$M=M_y$	℄ M_x	℄ M_y	$M=M_y$	℄ M_x	℄ M_y	$M=M_y$
Moments due to continuity:								
$n=1$, coefficients of $\sin\frac{\pi x}{a}$	+396 lb.	−1016 lb.	+113 lb.	−159 lb.	+110 lb.	+266 lb.	−21 lb.	0
$n=3$, coefficients of $\sin\frac{3\pi x}{a}$	−2	+34	+1	0	+8	+1	0	0
$n=5$, coefficients of $\sin\frac{5\pi x}{a}$	0	−1			−4			0
$n=7$, coefficients of $\sin\frac{7\pi x}{a}$	0	0			−1			0
Moments at $x=\frac{a}{2}$ due to continuity	+398	−1051	+112	−159	+99	+265	−21	0
Moments at $x=\frac{a}{2}$ due to load acting on simply supported panels	0	0	+2811	+2811	0	+44	+241	0
Total moments at $x=\frac{a}{2}$	+398 lb.	−1051 lb.	+2923 lb.	+2652 lb.	+99 lb.	+309 lb.	+220 lb.	0

Prob. 2, -4

Illustrative Problem 3.



In the computations take $N = 100$ lb.ft.

For flexible edge beam:

$$U = \frac{97.41\,EI}{s^4} \quad \text{(Equation 36)}$$
$$= 0.1142\,n^4\ \text{lb.ft.}^{-2}$$
$$L = \frac{9.870\,GJ}{s^2} \quad \text{(Equation 41)}$$
$$= 6.17\,n^2\ \text{lb.}$$

---

(1)–Effect of load with no deflection of edge beams

$n=1$, $\frac{b}{s}=0.5$

$K = 23.34$ lb. $k = 0.3902$
$S = 19.79$ lb. $q = 0.9313$
$Q = 1.693$ lb.ft.$^{-1}$ $t = 0.8499$
$T = 0.2516$ lb.ft.$^{-2}$

(2)–Effect of deflection of both edge beams $\Delta_o = 10\,000$ ft.

Edge beam: $L = 6.17$ lb., $K(1-k) = 14.24$ lb.; $p_o = 127.3$ lb.ft.$^{-2}$

(1) Edge beam 0.302	(1) 0.698		(2) Edge beam 0.302	(2) 0.698
0	−3893 lb.	Fixed-end moment, M_o^F	0	+1164 lb. $= Q(1-q)\Delta_o$
−1176	+2717	Distribute	+352	−812
−1176 lb.		Final moment, M_o	+352 lb.	
−1176 lb.		Change in torsional moment in edge beam, δm_o	+352 lb.	
−190.6	+190.6	Rotation of joint, $\Phi_o = \frac{\delta m_o}{L}$	+57.0	−57.0
0	+22 lb.ft.$^{-1}$ = +190.6·$Q(1-q)$	Reaction due to rotation	0	−7 lb.ft.$^{-1}$
0	−1264 lb.ft.$^{-1}$	Fixed-end reaction, $U\cdot\Delta_o$ = +1142 lb.ft.$^{-1}$	+378 lb.ft.$^{-1}$ $= T(1-t)\cdot\Delta_o$	
−1242 lb.ft.$^{-1}$		Total unbalanced reaction at joint	+1513 lb.ft.$^{-1}$	

---

(3)–Load on structure with flexible edge beams.

(Structure symmetrical about center-line.)

$n=1$

Actual deflection, Δ_o ... $8210\text{ ft.} = -\frac{-1242\text{ lb.ft.}^{-1}}{1513\text{ lb.ft.}^{-1}}\cdot 10\,000\text{ ft.}$

M_o for $\Delta_o = 0$ ...... −1176 lb.
M_o due to $\Delta_o = 8210$ ft. .. + 289 lb.
M_o ... − 887 lb. (for structure with flexible beams)

$n=3$		$n=5$	
Δ_o	26.0 ft.	Δ_o	1.43 ft.
M_o for $\Delta_o=0$	−359 lb.	M_o for $\Delta_o=0$	−108 lb.
M_o due to $\Delta_o=26.0$ ft.	+68 lb.	M_o due to $\Delta_o=1.43$ ft.	+14 lb.
M_o	−291 lb.	M_o	−94 lb.

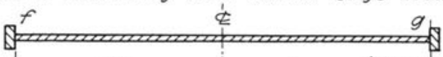
$n=7$		$n=9$	
Δ_o	0.202 ft.	Δ_o	0.0463 ft.
M_o for $\Delta_o=0$	−44 lb.	M_o for $\Delta_o=0$	−22 lb.
M_o due to $\Delta_o=0.202$ ft.	+4 lb.	M_o due to $\Delta_o=0.0463$ ft.	+2 lb.
M_o	−40 lb.	M_o	−20 lb.

3-1

Illustrative Problem 3 (concluded)

Twisting moment M_{xy} per unit of length at edge "f", $y=0$.

(1)–Due to continuity with flexible edge beams:



$-M_{ow} = -\frac{N\pi^2}{s^2}(1-\mu)\Delta_o = -0.6169\ lb.ft.^{-1}\Delta_o \cdot n^2$ (Structure symmetrical about center-line.)

$n=1 \quad \frac{b}{s}=0.5$

$\Delta_{of} = +8210\ ft.$	$\Delta_{og} = +8210\ ft.$
$M_{of} = -887\ lb.$	$M_{og} = -887\ lb.$
$-M_{owf} = -5065\ lb.$	$-M_{owg} = -5065\ lb.$
$M'_{of} = +4178\ lb.$	$M'_{og} = +4178\ lb.$
For $\frac{v_f}{b}=0$:	For $\frac{v_g}{b}=1.0$, (in direction of $-y$):
$C_{xy} = +1.0903$ (Table 7)	$C_{xy} = +0.4345$
$m_{xy} = -0.3969$ (Table 10)	$m_{xy} = +0.1548$

$M_{oxy} = [(+1.0903)(5065\ lb.) + (-0.3969)(4178\ lb.)] - [(+0.4345)(5065\ lb.) + (+0.1548)(4178\ lb.)] = +1015\ lb.$

$n=3$	$M_{oxy} = +219\ lb.$
$n=5$	$M_{oxy} = +\ 58\ lb.$
$n=7$	$M_{oxy} = +\ 23\ lb.$
$n=9$	$M_{oxy} = +\ 11\ lb.$

Due to continuity with the flexible edge beams, the twisting moment on line $y=0$ is

$$M_{xy} = (1015\ lb.)\cos\frac{\pi x}{a} + (219\ lb.)\cos\frac{3\pi x}{a} + (58\ lb.)\cos\frac{5\pi x}{a} + (23\ lb.)\cos\frac{7\pi x}{a} + (11\ lb.)\cos\frac{9\pi x}{a} + \cdots\cdots$$

At the corner $x=0$, the twisting moment is:

$$M_{xy} = 1015\ lb. + 219\ lb. + 58\ lb. + 23\ lb. + 11\ lb. + \cdots\cdots = +1330\ lb.$$

(2)–Due to the loading acting on a simply supported slab, the twisting moment at the corner $x=0$, $y=0$, is:

$$M_{xy} = -0.0656\,pb^2 = -2620\ lb.$$

(3)–Total twisting moment at the corner is:

$$M_{xy} = -2620\ lb. + 1330\ lb. = -1290\ lb.$$

Computation of bending moment in edge beams

n	Δ_o	U	$F_o = U \cdot \Delta_o$ (Equation 35)	Component of $M_{beam} = \frac{s^2}{\pi^2}F = \frac{a^2}{\pi^2 n^2}F_o \sin\frac{n\pi x}{a}$ (Equation 71)
1	8210 ft	0.1142 $lb.ft.^{-2}$	938 $lb.ft.^{-1}$	$+152\,000\ ft.lb.\ \sin\frac{\pi x}{a}$
3	26.0	9.25	241	$+\ 4\,340\ ft.lb.\sin\frac{3\pi x}{a}$
5	1.43	71.4	102	$+\ 660\ ft.lb.\sin\frac{5\pi x}{a}$
7	0.202	274	55	$+\ 180\ ft.lb.\sin\frac{7\pi x}{a}$
9	0.0463	747	35	$+\ 70\ ft.lb.\sin\frac{9\pi x}{a}$

At $x=\frac{a}{2}$, (center of beam), $M_{beam} = +148\,200\ ft.lb.$

The total moment on line $x=\frac{a}{2}$, in both beams and in the slab is obtained from statics:

$$\Sigma M = \tfrac{1}{8}\cdot 100\ lb.ft.^{-2}\cdot 20ft.(40ft.)^2 = 400\,000\ ft.lb.$$

The ratio of the moment in both beams to the total moment is:

$$\frac{2M_{beam}}{\Sigma M} = \frac{2(148\,200)\ ft.lb.}{400\,000\ ft.lb.} = 0.741$$

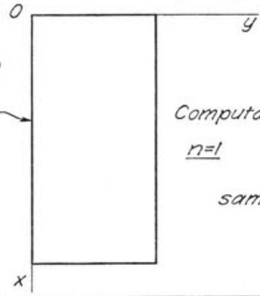
The ratio of the modulus of elasticity times moment of inertia of both beams, to the total of modulus of elasticity times moment of inertia of beams and slab at the cross section is:

$$\frac{2EI}{2EI + 20ft{\cdot}N} = \frac{2\cdot 30}{2\cdot 30 + 20} = 0.750$$

3-2

Illustrative Problem 4

Structure same as in Problem 3, except for loading.



Uniform line load on edge of slab, $F=1000$ lb.ft.$^{-1}$

Computations made for $n=1$, only

$\underline{n=1}$ $F_o=1273$ lb.ft.$^{-1}$

Other constants same as in Problem 3.

"Successive Distribution" Procedure

(1)-Distribute reactions:

$U=0.1142$ lb.ft.$^{-2}$ $T=0.2516$ lb.ft.$^{-2}$ $-t=-0.8499$ $[T(1-t)=0.0378$ lb.ft.$^{-2}]$

	0.312	0.688	$\frac{V}{b}=0$, $C_R=-1.0$	0.688	0.312	
Fixed-end Reaction, R_o^F	0	-1273 lb.ft.$^{-1}$ $=-1.0F_o$		0	0	
R_o^F due to uniform deflection	+343 lb.ft.$^{-1}$	+113 lb.ft.$^{-1}$ $=3000$ ft $[T(1-t)]$		+113 lb.ft.$^{-1}$	+343 lb.ft.$^{-1}$ $=3000$ ft$\cdot U$	
Distribute	+255	+562	carry-over →	−478		
		−13	←	+15	+7	Distribute
R_o^F due to uniform deflection	+3	+1		+1	+3	
Distribute	+3	+6	→	−5		
				+1		
F_o	+604 lb.ft.$^{-1}$	−604 lb.ft.$^{-1}$	R_o	−353 lb.ft.$^{-1}$	+353 lb.ft.$^{-1}$	F_o
Deflections, $\Delta_o=\frac{F_o}{U}$	+5290 ft.			+3090 ft.		

(2)-Distribute Moments:

$L=6.17$ lb. $K=23.34$ lb. $-k=-0.3902$

	0.209	0.791		0.791	0.209
M_o^F, due to loading	0	0		0	0
M_o^F due to deflections in (1)	0	+4090 lb. $=+5290$ ft.$(Q)+3090$ ft.$(-qQ)$		−3110 lb.	0
Distribute	+855	−3235	→	+1260	
		−571	←	+1463	−387
	−119	+452	→	−176	
		−54		+139	−37
	−11	+43		−17	
		−5		+13	−4
	−1	+4		−2	
				+2	0
M_o	+724 lb.			−428 lb.	
Torsional moment, m_o, in beam	+724 lb.				−428 lb.
$\Phi_o=\frac{m_o}{L}$	+117.3	−117.3		+69.4	−69.4

4-1

Illustrative Problem 4 (concluded)

(3)-Distribute reactions:				
R_0^F	0	-309 lb.ft.$^{-1}$ $= -117.3(Q) + 69.4(-qQ)$ (Distribution not shown.)	$+303$ lb.ft.$^{-1}$	0
F_0	$+62.6$ lb.ft.$^{-1}$			-58.0 lb.ft.$^{-1}$
Δ_0	$+548$ ft.		-508 ft.	
(4)-Distribute moments:				
M_0^F	0	$+1731$ lb.	-1726 lb.	0
M_0	$+277$ lb.		-276 lb.	
Φ_0		-44.8	$+44.7$	
(5)-Distribute reactions:				
R_0^F	0	-146.3 lb.ft.$^{-1}$	$+146.3$ lb.ft.$^{-1}$	0
F_0	$+28.9$ lb.ft.$^{-1}$			-28.9 lb.ft.$^{-1}$
Δ_0	$+253$ ft.		-253 ft.	
		etc.		
Final Values:–				
ΣM_0	$+1260$ lb. $=$ $(724$ lb. $+ 277$ lb. $+ 133$ lb. $+ 63$ lb. $+ 30$ lb. $+ \cdots)$		-960 lb.	
$\Sigma \Delta_0$	$+6326$ ft. $=$ $(5290$ ft. $+ 548$ ft. $+ 253$ ft. $+ 121$ ft. $+ 58$ ft. $+ \cdots)$		$+2094$ ft.	

4-2

of the symmetry of the structure the modified flexural stiffness, $K(1 - k)$, of the slab may be used to avoid carry-overs during the distribution. In the computations the beams may be considered to act as panels of a slab adjacent to the loaded panel, but with no carry-over factors. All elastic constants are stated in terms of N for the slab. In the computations N is taken as 100 lb. ft. with no effect on the stresses in the slab. The deflections and slopes are affected by this choice of a value of N, but these quantities are not of interest in this problem.

The intermediate steps in the calculations are handled in the same manner as in Problem 2. The computations are shown in detail for $n = 1$, and the results are given for $n = 3, 5, 7$, and 9.

The magnitude of the twisting moment at the corner $x = 0$, $y = 0$ is computed in two steps. The first computation is for M_{xy} due to edge deflections and moments. For example, for $n = 1$, the center deflection of both edges is 8210 ft. (with the value of $N = 100$ lb. ft.); the center moments at the edges are -887 lb. The fictitious edge moment acting with the deflection, $-\mathsf{M}_{0w}$, is -5065 lb., and consequently the edge moment to be considered acting alone without deflection, to give the total effect of deflection and edge moment, must be 4178 lb. One finds the result, for $n = 1$:

$$M_{0xy} = C_{xy}\,(5065 \text{ lb.}) + m_{xy}\,(4178 \text{ lb.})$$

at $y = 0$, due to $-\mathsf{M}_w$ and M' at $y = 0$, where C_{xy} and m_{xy} are taken for $v/b = 0$; and

$$M_{0xy} = -[C_{xy}\,(5065 \text{ lb.}) + m_{xy}\,(4178 \text{ lb.})]$$

at $y = 0$, due to $-\mathsf{M}_w$ and M' at $y = b$, where C_{xy} and m_{xy} are taken for $v/b = 1$.

For all the cosine waves of twisting moment one finds the total due to edge deflections and moments, $M_{xy} = +1330$ lb.

The twisting moment due to the load acting on a simply-supported rectangular slab is computed from coefficients given by Westergaard.* Since, in the present notation, a positive twisting moment produces compression along the diagonal at a corner such as the one at $x=0$, $y=0$, and since Westergaard's coefficient is for positive moment across the diagonal, one has for $b/a = 0.5$

$$M_{xy} = -0.0656\ pb^2 = -2620 \text{ lb.}$$

Then the total twisting moment at the corner $x = 0$, $y = 0$ is

$$M_{xy} = -2620 \text{ lb.} + 1330 \text{ lb.} = -1290 \text{ lb.}$$

The bending moment at the center of one of the supporting beams is also computed. The procedure is straightforward and easily followed. It is seen that for this case the total moment at the cross-section $x = a/2$ is divided between beams and slab approximately in proportion to the values of EI for the various members.

26. *Illustrative Problem 4.*—The same structure that is considered in Problem 3, a single-span slab with stiffened edges having torsional resistance, is loaded with a uniform line load on one of the edges of the slab directly over a supporting beam. The procedure of successive distribution is used, no advantage being taken of symmetry, in order to show some of the details of the procedure more fully. The computations are given only for the first term in the series, that is, for $n = 1$.

The fixed-end reaction in the slab for a load at one edge is minus the load, of course. The sine wave component of load for $n = 1$ is $F_0 = 1273$ lb. ft.$^{-1}$. The fixed-end reaction is $-F_0$. Before distribu-

*H. M. Westergaard and W. A. Slater, "Moments and Stresses in Slabs," Proc. American Concrete Institute, Vol. 17, 1921, p. 415-538. See Fig. 3, p. 431 for coefficient for moment across diagonal at corner, for $\mu = 0$.

tion of reactions a uniform deflection of both joints is written in to hasten the convergence by making the unbalanced reaction smaller. It is noted that for the uniform deflection the fixed-end reactions in the beams are proportional to U and in the slab, to $T(1 - t)$. The left-hand joint is balanced first, and the carry-over to the right-hand joint is considered with the reactions due to the uniform deflection previously introduced. The joint is then balanced, and a carry-over is made to the first joint. Again fixed-end reactions due to a uniform deflection are written in, proportional to the first set of reactions, and after one further distribution the structure is practically balanced.

Then the deflections are computed, and the fixed-end moments due to the deflection are determined. These are distributed without deflection. The rotations due to distributing moments are determined next, and the changes in fixed-end reaction due to the rotations are computed. These are again distributed as in the first step. The moments due to this second distribution of reactions are computed, and so on until the computations converge.

All the computations are not shown, but enough numerical values are given to indicate the slow convergence of this procedure. The "indirect procedure" leads to results much more rapidly.

VII. Notes on General and Special Cases

27. *Convergence of the Numerical Computations.*—The method of solution described here is based on a resolution of the loading into components in such a way that the loading is expressed as an infinite series. In certain cases, such as concentrated loads, the series representing the load does not converge. In other cases, such as distributed loads and loads distributed along a line in the x direction, the series for the load does converge.

The results obtained from the computations for deflections, moments, reactions, etc., will be expressed also as infinite series, and the computations may be performed for a sufficient number of terms in the series to obtain results of any precision desired.

For ordinary cases only a relatively small number of terms in the series are required to compute the deflections accurately. Successively larger number of terms are required to compute slopes, curvatures or moments, and changes in curvatures or shears and reactions. For some quantities the series will not converge. For example, in the case of a concentrated load, the moments under the load are infinitely large as given by the ordinary theory of flexure of slabs upon which this method is based. As is explained elsewhere in this report, one may obtain the effects of a concentrated load by assuming the load distributed over some area. Also for a concentrated load on a slab over a flexible beam the load carried by the beam will be given by an infinite series that does not converge. These faults of the method of analysis are common to all methods of solution of the fundamental differential equation in terms of infinite trigonometric series.

In most cases the number of terms, or number of values of n required to find effects to a certain degree of accuracy, will be obvious from an examination of the results obtained as the distribution procedure is carried out. Sometimes, however, care is necessary in order to be sure of the convergence of the results. Perhaps the writer's meaning will be clearer if one examines the ordinary series

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots .$$

where the sum of the first two terms is 1.5, the sum of the first three terms is 1.833, etc., and the terms are successively decreasing in value. But this series, of course, diverges, since if one takes enough terms one may make the sum greater than any previously assigned quantity.

In general it may be stated that for distributed or concentrated loads the series for deflection always converges. The series for moments converges except at the point of application of a concentrated load and the series for reactions at a joint converges except for a concentrated load at some point along the joint.

In a continuous slab, for large values of nb/a the effects near a joint approach those in an infinitely long slab continuous over a beam of the same flexibility as that in the joint, and with the same values of N on either side of the joint, as in the panels of the slab adjacent thereto of the actual structure. Formulas for this case of the infinitely long slab are given in Section 28. One may compare the results of the distributions with the formulas given therein to obtain the general form of the terms in the series for high values of n. In many cases one can establish the limits within which the sum of such a series must lie, for values of n greater than some given value, by comparison with an integral.

The effects in the interior of a panel due to the moments and deflections at the edge of the panel converge much more rapidly than the moments and deflections at the edge. Also, for common cases of loading, the series for the moment in the flexible beams converges even when the series for the load on the flexible beam diverges.

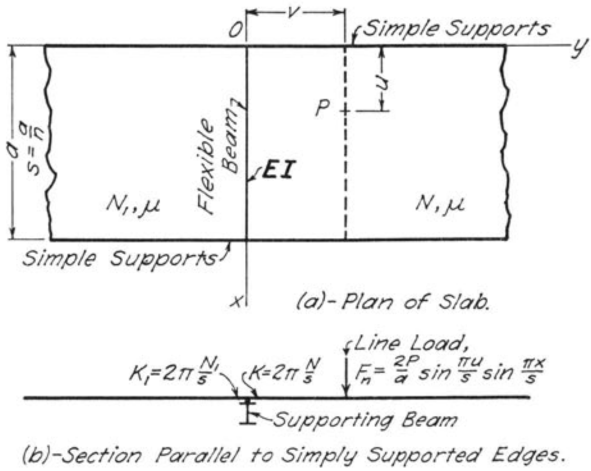


FIG. 9. CONCENTRATED LOAD ON INFINITELY LONG SLAB WITH ONE FLEXIBLE BEAM

28. *The Infinitely Long Slab With One Flexible Beam.*—The problem of the infinitely long slab continuous over a flexible beam is of importance as a limiting case, and serves also to illustrate a possible use of the distribution procedure to find formulas for simple cases. Needless to say, the procedure is much simpler for a numerical case.

The slab shown in Fig. 9 is subjected to a concentrated load P at the point $x = u$, $y = v$, and is supported on the line $y = 0$ by a beam having a modulus of elasticity E and a moment of inertia I. For the infinitely long panel of the slab to the right of the beam the elastic properties of the material are stated as N and μ; and for the infinitely long panel to the left of the beam, as N_1 and μ.

The series for the loading is obtained from Fig. 5 (c) and the various coefficients for the slab are taken from Table 12. The distribution is carried out for only one value of n, since the same results are obtained for other values as only s changes with n. The procedure is developed in three steps, as indicated.

Step 1. Distribution of moments, beam temporarily held against deflection. See Fig. 9(b).

The fixed-end moment in the loaded panel, to the right of the beam, is

$$\mathsf{M}^F = -\frac{2P}{a} v e^{-\frac{\pi v}{s}} \sin \frac{\pi u}{s} \sin \frac{\pi x}{s}$$

This moment is distributed in proportion to the flexural stiffnesses of the two panels meeting at the joint. Since the values of K differ only because N is different for the two panels, the moment may be distributed in proportion to N and N_1. One finds the following result for the final moment at the joint:

$$M = -\frac{N_1}{N + N_1}\frac{2P}{a} v e^{-\frac{\pi v}{s}} \sin\frac{\pi u}{s} \sin\frac{\pi x}{s}$$

The change in edge moment in the left-hand panel is equal to the final moment at the joint. The rotation of the edge of the left-hand panel is equal to the change in moment divided by the quantity K_1. The rotation of the edge of the right-hand panel is equal in magnitude but opposite in sign to the rotation of the left-hand panel, and may be written as

$$\Phi = \frac{Ps}{a\pi}\frac{1}{N + N_1} v e^{-\frac{\pi v}{s}} \sin\frac{\pi u}{s} \sin\frac{\pi x}{s}$$

Then the unbalanced reaction at the joint is as follows:
Unbalanced reaction

$$= R^F + \Sigma Q\Phi$$

$$= -\frac{2P}{a}\left(1 + \frac{\pi v}{s}\right) e^{-\frac{\pi v}{s}} \sin\frac{\pi u}{s} \sin\frac{\pi x}{s}$$

$$+ (1+\mu)\frac{P}{a}\frac{\pi v}{s}\frac{N - N_1}{N + N_1} e^{-\frac{\pi v}{s}} \sin\frac{\pi u}{s} \sin\frac{\pi x}{s}$$

$$= -\left[1 + \frac{\pi v}{s} - \frac{1+\mu}{2}\frac{N - N_1}{N + N_1}\frac{\pi v}{s}\right]\frac{2P}{a} e^{-\frac{\pi v}{s}} \sin\frac{\pi u}{s} \sin\frac{\pi x}{s}$$

Step 2. Deflection of the joint at the supporting beam introduced.

The fixed-end moments, due to a sine wave of deflection of the joint with maximum magnitude of unity, are equal to the quantities $Q_1 \sin \pi x/s$ and $Q \sin \pi x/s$ to the left and to the right of the beam respectively, where Q_1 and Q are given by the equations

$$Q_1 = (1 + \mu) \frac{\pi^2}{s^2} N_1 \quad , \quad Q = (1 + \mu) \frac{\pi^2}{s^2} N$$

The final moment at the joint may be expressed as follows, after simplifying:

$$M = 2 (1 + \mu) \frac{\pi^2}{s^2} \frac{N N_1}{N + N_1} \sin \frac{\pi x}{s}$$

The rotation of the edges of the panels are $+\Phi$ on the left and $-\Phi$ on the right, where Φ is given by the equation

$$\Phi = \frac{1 + \mu}{2} \frac{\pi}{s} \frac{N - N_1}{N + N_1} \sin \frac{\pi x}{s}$$

Then the unbalanced reaction at the joint is as follows:

$$\text{Unbalanced reaction} = \Sigma Q \Phi + (T_1 + T + U) \sin \frac{\pi x}{s}$$

$$= \frac{\pi^4 EI}{s^4} \left[1 + \frac{2s}{\pi} \frac{N + N_1}{EI} - \frac{(1 + \mu)^2}{2} \frac{s}{\pi} \frac{N - N_1}{N + N_1} \frac{N - N_1}{EI} \right] \sin \frac{\pi x}{s}$$

Step 3. Combine Steps 1 and 2 to eliminate the unbalanced reaction at the joint.

The actual deflection of the beam is a sine wave with maximum ordinate equal to minus the unbalanced reaction from Step 1 divided by the unbalanced reaction from Step 2. Then the actual deflection, for the n^{th} term of the series, is:

$$\frac{\pi^4 EI}{s^4} \Delta_0 = \frac{\left(1 + \frac{\pi v}{s} - \frac{1 + \mu}{2} \frac{N - N_1}{N + N_1} \frac{\pi v}{s} \right) \frac{2P}{a} e^{-\frac{\pi v}{s}} \sin \frac{\pi u}{s}}{1 + \frac{2s}{\pi} \frac{N + N_1}{EI} - \frac{(1 + \mu)^2 s}{2\pi} \frac{N - N_1}{N + N_1} \frac{N - N_1}{EI}} \qquad (83)$$

The moment in the beam is obtained from the relation

$$M_{\text{beam}} = \frac{\pi^2 EI}{s^2} \Delta$$

whence it follows that, for the n^{th} term,

$$M_{\text{beam}} = \frac{\left(1+\frac{\pi v}{s}-\frac{1+\mu}{2}\,\frac{N-N_1}{N+N_1}\,\frac{\pi v}{s}\right)\frac{2Pa}{n^2\pi^2}\,e^{-\frac{\pi v}{s}}\sin\frac{\pi u}{s}\sin\frac{\pi x}{s}}{1+\frac{2s}{\pi}\,\frac{N+N_1}{\mathsf{EI}}-\frac{(1+\mu)^2 s}{2\pi}\,\frac{N-N_1}{N+N_1}\,\frac{N-N_1}{\mathsf{EI}}} \tag{84}$$

The moment at the joint is obtained by adding the moment in Step 1 to Δ times the moment in Step 2. One finds the result, for the n^{th} term,

$$\mathsf{M} = -\frac{N_1}{N+N_1}\,\frac{2P}{a}\,ve^{-\frac{\pi v}{s}}\sin\frac{\pi u}{s}\sin\frac{\pi x}{s}+\frac{2(1+\mu)\;N\;N_1}{(N+N_1)\mathsf{EI}}\,M_{\text{beam}} \tag{85}$$

All the values for the interior of a panel due to a deflection Δ of the edge are stated in terms of an edge moment $-\mathsf{M}_w$ defined by the relation

$$-\mathsf{M}_w = -N\,(1-\mu)\,\frac{\pi^2}{s^2}\,\Delta$$

One has the result, then, for the n^{th} term,

$$-\mathsf{M}_w = -(1-\mu)\,\frac{N}{\mathsf{EI}}\,M_{\text{beam}} \tag{86}$$

For the special case of a rigid beam, $\mathsf{EI} = \infty$ and Equations (84) and (85) become

$$M_{\text{beam}} = \left(1+\frac{\pi v}{s}-\frac{1+\mu}{2}\,\frac{N-N_1}{N+N_1}\,\frac{\pi v}{s}\right)\frac{2Pa}{n^2\pi^2}\,e^{-\frac{\pi v}{s}}\sin\frac{\pi u}{s}\sin\frac{\pi x}{s} \tag{87}$$

$$\mathsf{M} = -\frac{2PN_1}{N+N_1}\,\frac{v}{a}\,e^{-\frac{\pi v}{s}}\sin\frac{\pi u}{s}\sin\frac{\pi x}{s} \tag{88}$$

For $N_1 = N$, the series in which Equation (88) is one term becomes the moment M_y at the beam in an infinitely long slab continuous over a rigid beam

$$M_y\Big]_{y=0} = -\frac{Pv}{a} \sum_{n=1,2,3,\cdots} e^{-\frac{n\pi v}{a}} \sin\frac{n\pi u}{a} \sin\frac{n\pi x}{a} \tag{89}$$

The infinite series in Equation (89) for the moment over a rigid beam in an infinitely long slab can be expressed in finite form by comparison with a derivative of the potential function introduced by Nádai* in his study of concentrated loads on an infinitely long slab. This has been done by Jensen.†

For a flexible beam, when $N_1 = N$, Equations (84) and (85) become

$$M_{\text{beam}} = \frac{1 + \dfrac{\pi v}{s}}{1 + \dfrac{4sN}{\pi EI}} \frac{2Pa}{n^2\pi^2} e^{-\frac{\pi v}{s}} \sin\frac{\pi u}{s} \sin\frac{\pi x}{s} \tag{90}$$

and

$$M = -\frac{Pv}{a} e^{-\frac{\pi v}{s}} \sin\frac{\pi u}{s} \sin\frac{\pi x}{s} + \frac{N(1+\mu)}{EI} M_{\text{beam}} \tag{91}$$

For a load directly over the beam, $v = 0$, and Equations (90) and (91) become

$$M_{\text{beam}} = \frac{2Pa}{n^2\pi^2} \frac{1}{1 + \dfrac{4sN}{\pi EI}} \sin\frac{\pi u}{s} \sin\frac{\pi x}{s} \tag{92}$$

$$M = \frac{N(1+\mu)}{EI} M_{\text{beam}} \tag{93}$$

From the form of Equation (93), and from the fact that the slope in the y direction is zero at the beam, one concludes that the curvatures in the slab in the x and y directions, directly over the beam must be

*A. Nádai, "Die elastischen Platten," 1925, Julius Springer, Berlin, p. 95.

†V. P. Jensen, "Solutions for Certain Rectangular Slabs Continuous Over Flexible Supports," Univ. of Ill. Eng. Exp. Sta., Bul. 303, 1938, see Section 6.

equal. Then for this case of loading, over the beam the slab moments are related by the equation

$$M_x\Big]_{y=0} = M_y\Big]_{y=0}$$

This can also be shown by computing M_x from the relations in Table 12.

For $N_1 = 0$, which corresponds to the slab with an edge supported only by a beam, one finds the relations, from Equations (84) and (85),

$$M_{\text{beam}} = \frac{1 + \dfrac{1-\mu}{2}\,\dfrac{\pi v}{s}}{1 + \dfrac{2s\,N}{\pi\,EI} - \dfrac{(1+\mu)^2\,sN}{2\pi\,EI}}\,\frac{2\,Pa}{n^2\pi^2}\,e^{-\frac{\pi v}{s}}\sin\frac{\pi u}{s}\sin\frac{\pi x}{s} \quad (94)$$

and

$$M = 0.$$

For $N_1 = \infty$, the slab is fixed at the line $y = 0$, and one has

$$M = -2P\frac{v}{a}\,e^{-\frac{\pi v}{s}}\sin\frac{\pi u}{s}\sin\frac{\pi x}{s}$$

which is the n^{th} term in the series for the fixed-end moment.

29. *Application of the Procedure to Special Problems.*—The manner of treatment of some special types of construction to obtain a solution by the distribution procedure is described briefly. Consider the case of a slab with integral supporting beams. One of the principal assumptions upon which the distribution procedure is based is that the beams offer only vertical support to the slab or in special cases also offer torsional restraint to the edge. In the case, however, where the beams are integral with the slab but the center of gravity of the beam is not in the same plane as the middle plane of the slab, the slab in the neighborhood of the beam acts somewhat like a T beam and is subjected to direct stress as well as flexure. Consequently the solutions described here, or any solution based on the fundamental equation given in Appendix A, are not applicable since those solutions

depend on the assumption that there are no resultant forces in the direction of the plane of the slab. For most practical purposes, and particularly for the effects in the interior of the panel, it is valid to make the approximation that the supporting beam has some equivalent moment of inertia of magnitude equal to the actual moment of inertia of the beam plus that of the part of the slab acting with it as a T beam, and then to assume that the beam offers only vertical support to the slab. In extreme cases it is likely that this approximation is not valid.

The treatment of beams having resistance to torsion is described in Section 8, and an illustrative problem is given in Section 25 of a single panel of a slab having two stiffened edges. In all treatment of beams having torsional resistance it must be assumed that the ends of the beam are not permitted to rotate about a longitudinal axis through the beam. If this assumption is not fulfilled in the actual structure the treatment of torsion made herein is not valid, but, in any case, it seems reasonable that the stresses in the structure should be between the stresses computed for no torsional resistance of the beam and the stresses computed for the case of torsional resistance with the assumption described in the foregoing.

Cases in which there are point supports on the slab or under the beams can be treated in the following manner:

(1) With the load on the structure, and with the point supports temporarily removed, find the total deflections at the points where the point supports are to be placed.

(2) In succession, at each place where a point support is desired, apply a concentrated load to the structure and find the total deflections at all proposed point supports.

(3) With the values obtained from (1) and (2), write a series of simultaneous equations which express the deflection at each point support in terms of the deflection due to the loading, the deflection due to each of the point reactions, and the flexibility of the supporting structure.

(4) Solve these equations to find the reactions at the point supports.

(5) Find the effects in the slab due to the loading and the reactions of the point supports.

Rigid supports or elastic supports are treated in the manner indicated. If the supports are rigid the resultant deflections at each point of support are set equal to zero. If the supports are elastic the resultant deflections are set equal to the deflections that the unknown reactions produce in the elastic support. The case of a diaphragm between the flexible beams, having no contact with the slab, may be treated in a similar manner, for, in this case, the deflections are the relative deflections of the various points connected by a particular diaphragm, and the flexibility of the various points of connection of the beams with the diaphragm is determined by the elastic properties of the diaphragm.

Other cases of distribution of loading than those described in Section 5 may be treated in various ways. One case of some interest is that of a load distributed uniformly over a rectangular area. Such a case may be treated by assuming temporary beam supports coinciding with two opposite sides of the loaded rectangular area. The temporary beam supports may be considered to have zero modulus of elasticity, and then the analysis can be carried out in the manner indicated for the other problems solved herein. In many other cases one may use to advantage this introduction of an arbitrary joint in a structure where no actual joint exists. For example, one may find the fixed-end moments and reactions due to a line loading by considering a panel of the slab with some deflection along the line of loading. After determining the fixed-end forces and the load on the line, one may establish the coefficients for the fixed-end reactions and moments.

One may use the tabulated coefficients for other purposes than for the distribution procedure. For example, one may derive an expression, in terms of a trigonometric series, for the reaction at an edge of a simply supported slab due to a line or point load. The numerical values of the coefficients in the series may be obtained from values reported in Tables 6 and 8, as is shown by the derivation given in Section 8 of Appendix B.

VIII. Influence Surfaces for Slabs

30. *Reciprocal Relations in Slabs With Two Opposite Edges Simply Supported.*—Certain special laws of reciprocity,* in addition to the usual general reciprocal relations derived from Maxwell's Theorem of reciprocal deflections, apply to slabs with two simply-supported opposite edges.

*Several special cases have been pointed out by others. See: H. M. Westergaard, "Computation of Stresses in Bridge Slabs Due to Wheel Loads," Public Roads, Vol. 11, No. 1, 1930, p. 1-23, especially p. 7.

D. L. Holl, "Analysis of Thin Rectangular Plates Supported on Opposite Edges," Bulletin 129, Iowa Engineering Experiment Station, Iowa State College, 1936, p. 35.

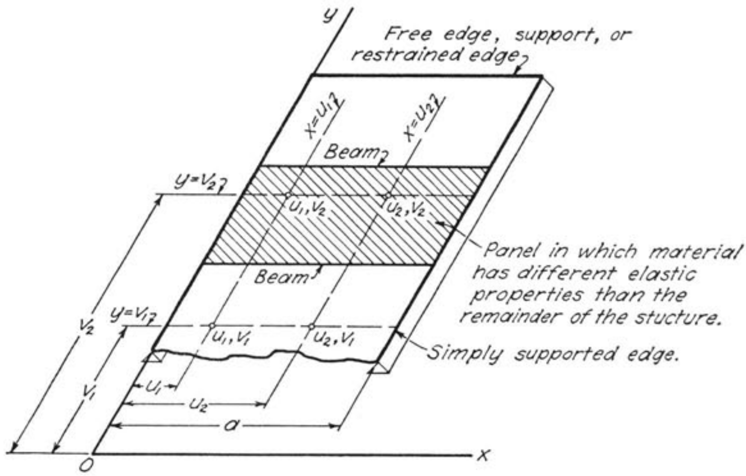


FIG. 10. GENERAL TYPE OF SLAB TO WHICH CERTAIN RECIPROCAL RELATIONS APPLY

Consider a slab with simply-supported edges parallel to the axis of y, continuous over any number and spacing of simply-supported flexible beams of constant section parallel to the axis of x. In each panel formed by lines parallel to the x axis and by the simply-supported sides, the slab is of constant thickness. It is assumed that Hooke's Law applies to the material in the slab and supporting beams. The slab may extend indefinitely far in the y direction, or it may be supported or restrained in some manner on the edges parallel to the x axis. An example is shown in Fig. 10.

A concentrated load P at the point u_1, v_1 may be represented by the line load on the line $y = v_1$ expressed by the trigonometric series

$$F = \frac{2P}{a} \sum_{n=1,2,3,\cdots} \sin \frac{n\pi u_1}{a} \sin \frac{n\pi x}{a} \tag{95}$$

as is shown in Section 5. Each sine wave component of loading

$$F_n = \frac{2P}{a} \sin \frac{n\pi u_1}{a} \sin \frac{n\pi x}{a} \tag{96}$$

produces a sine wave of deflection, or of moment in the x direction, or of moment in the y direction, or other particular effect on the line $y = v_2$, of the form

$$c_n F_n = \frac{2P}{a} c_n \sin \frac{n\pi u_1}{a} \sin \frac{n\pi x}{a} \tag{97}$$

where c_n depends on the properties of the entire structure.

Similarly, a concentrated load P at the point u_2, v_1 may be represented by the line load on the line $y = v_1$,

$$F = \frac{2P}{a} \sum_{n=1,2,3,\ldots} \sin \frac{n\pi u_2}{a} \sin \frac{n\pi x}{a} \tag{98}$$

Each sine wave component of loading,

$$F_n = \frac{2P}{a} \sin \frac{n\pi u_2}{a} \sin \frac{n\pi x}{a} \tag{99}$$

produces certain effects on the line $y = v_2$ of the form

$$c_n F_n = \frac{2P}{a} c_n \sin \frac{n\pi u_2}{a} \sin \frac{n\pi x}{a} \tag{100}$$

where c_n is the same as in Equation (97), since the structure is the same and the loading is of the same type and is proportional to the loading producing (97).

Comparison of (100) and (97) shows that the effects at $x = u_2$ in (97) are exactly the same as the corresponding effects at $x = u_1$ in (100). Since the component effects are the same, the total effect is the same. Consequently, one may state the following reciprocal theorem:

Theorem I. The effects expressible as a sine series (deflection w, curvatures $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$, bending moments M_x, M_y, etc., but not M_{xy}, or $\frac{\partial w}{\partial x}$, etc.) at the point u_2, v_2 due to a load P at the point u_1, v_1 are equal to the corresponding effects at u_1, v_2 due to the same load P at u_2, v_1.

This statement may be readily extended to the case of similar line loads on the lines $x = u_1$ and $x = u_2$.

It is noted that there is a corresponding reciprocal law for simply-supported beams: the simple beam moment at point 1 due to a concentrated load at point 2 is equal to the moment at point 2 due to the same concentrated load at point 1.

For the special case of a rectangular panel simply supported on four sides one may apply Theorem I to obtain a reciprocal relation between effects at the points u_1, v_1 and u_2, v_2 by considering the axes of x and y interchanged temporarily; certain effects at u_2, v_2 due to a load P at u_1, v_1 are equal to corresponding effects at u_1, v_2 due to the same load P at u_2, v_1, which, in turn, from consideration of the other direction, are equal to the corresponding effects at u_1, v_1 due to the same load P at u_2, v_2. In this case, the only effects that may be considered are those expressible as sine series in both directions; namely, deflections w, curvatures $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$, moments M_x and M_y, and certain other quantities such as $\frac{\partial^4 w}{\partial x^4}$, $\frac{\partial^2 M_{xy}}{\partial x \partial y}$, etc. Therefore one may state the following corollary which holds also for one dimension of the slab infinitely long:

Corollary I. In a rectangular slab simply supported on four sides the deflection, curvatures, and bending moments at a point u_2, v_2 due to a load P at the point u_1, v_1 are equal to the corresponding quantities at the point u_1, v_1 due to the same load P at u_2, v_2.

This corollary can be derived also from a consideration of the infinitely long slab, and has been used by Holl.*

A second general reciprocal theorem applies to the deflections along the lines $y = v_1$ and $y = v_2$ due to loads at u_1, v_2 and u_1, v_1, respectively. Let the load P at the point u_1, v_1 be expressed as the line load on the line $y = v_1$ given by Equation (95). Similarly a load P at the point u_1, v_2 may be expressed as the line load on the line $y = v_2$:

$$F = \frac{2P}{a} \sum_{n=1,2,3,\cdots} \sin \frac{n\pi u_1}{a} \sin \frac{n\pi x}{a} \tag{101}$$

where (101) and (95) are identical, term for term.

*See preceding footnote.

The general form of the "Theorem of Reciprocal Deflections" due to Maxwell and others may be stated as follows:* The forces of a given system acting on a structure do the same amount of work during the application of a second system of forces to the structure as the forces of the second system would do during the application of the first system. The theorem is derived by means of the principle of conservation of energy and applies to a structure in which Hooke's Law holds, and where superposition of effects is valid. A special case is the ordinary statement that the vertical deflection of point 1 due to the application of a vertical load at point 2 is the same as the vertical deflection of point 2 due to the application of the same vertical load at point 1. The extension to sine waves of line loading and sine waves of deflection along a line is obvious.

Then, for each value of n, the sine wave of deflection on the line $y = v_2$ due to a sine wave of loading on the line $y = v_1$ is equal to the sine wave of deflection on the line $y = v_1$ due to the same sine wave of loading on the line $y = v_2$. Consequently, the resultant deflections on the line $y = v_1$ due to the load P at u_1, v_2 are equal to the resultant deflections on the line $y = v_2$ due to the load P at u_1, v_1. Then the slopes $\frac{\partial w}{\partial x}$, and the curvatures in the x direction $\frac{\partial^2 w}{\partial x^2}$, as well as higher derivatives with respect to x along these lines, also obey the same reciprocal law. One may state the following theorem:

Theorem II. The deflection w, the slope $\frac{\partial w}{\partial x}$, and the curvature $\frac{\partial^2 w}{\partial x^2}$ at the point u_2, v_2 due to a load P at u_1, v_1 are equal to the corresponding quantities at the point u_2, v_1 due to the same load P at u_1, v_2.

A third law of reciprocity for deflections w, curvatures $\frac{\partial^2 w}{\partial x^2}$, and derivatives with respect to x of even order only, may be derived by combining Theorems II and I.

Theorem III. The deflection w and the curvature in the x direction $\frac{\partial^2 w}{\partial x^2}$ at the point u_1, v_1 due to a load P at u_2, v_2 are equal to the corresponding quantities at the point u_2, v_2 due to the same load P at u_1, v_1.

31. *Use of the Deflected Structure as an Influence Surface for the Slab.*—The principle established by Muller-Breslau, that an influence

*See, for example, R. V. Southwell, "An Introduction to the Theory of Elasticity," 1936, The Clarendon Press, Oxford, p. 11.

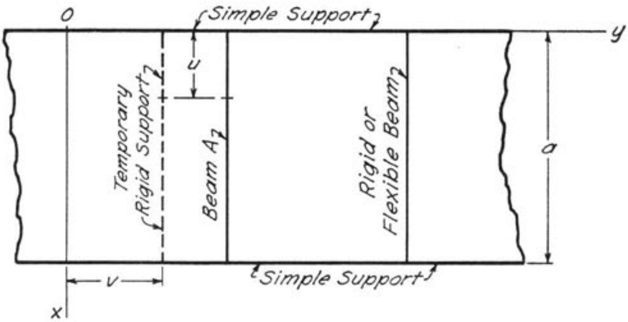


FIG. 11. TYPE OF STRUCTURE FOR WHICH INFLUENCE SURFACES CAN BE OBTAINED BY THE DISTRIBUTION PROCEDURE

diagram for an effect at one point due to a load at a second point may be obtained as the deflection of the structure due to a distortion of a particular sort, may be applied to slabs.* The procedure is discussed here because certain cases are very simply treated with the use of the distribution procedure.

Suppose it is desired to find the influence surface for M_y at the point $x = u$ on the line $y = v$ of the slab shown in Fig. 11. The influence surface is obtained as the deflection of the slab due to a unit angle change or change in slope in the y direction concentrated at the point u, v. If there is no rigid supporting beam on the line $y = v$, let there be a rigid support temporarily placed there, and consider the slab cut transversely over the support. Introduce a negative unit concentrated rotation at the point $x = u$ on the edge of a panel of the slab adjacent to the cut. This is done by rotating the edge according to the equation

$$\Phi = -\frac{2}{a} \sum_{n=1,2,3,\cdots} \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a} \tag{102}$$

That this represents a concentrated negative rotation of unit magnitude is seen by comparison with Equation (20) for the trigonometric series corresponding to a concentrated load.

Consider each term, or each value of n separately, as in the other cases studied. Write in fixed-end moments and fixed-end reactions in the panel due to the rotations of the edge. Then imagine the slab cemented together at the place where it was previously cut, and the

*See for example, H. M. Westergaard, "Graphostatics of Stress Functions," Trans. Am. Soc. Mech. Eng., March 1934, p. 141-150, especially p. 148.

temporary support removed. Distribute the unbalanced moments and reactions throughout the structure. One will find certain edge moments and deflections at every joint. From these, by use of the tabulated values of deflection in the interior of a panel due to moments and deflections of an edge one may compute deflections throughout, and obtain the complete influence surface.

It is noted that this procedure is applicable to continuous beams or frames also.*

One may obtain the influence surface for curvature in the x direction, $\frac{\partial^2 w}{\partial x^2}$, by use of the reciprocal relation given as Theorem III in Section 30. Apply a unit load at the point where the influence is desired, and after analysis of the structure compute $\frac{\partial^2 w}{\partial x^2}$ throughout. Then the curvatures $\frac{\partial^2 w}{\partial x^2}$ at the original point for a load at some other point are equal to the curvatures $\frac{\partial^2 w}{\partial x^2}$ at the other point for a load at the original load point.

The influence surface for moment in a supporting beam is obtained in the following manner. Let it be required to find the influence surface for moment at the point $x = u$ in Beam A of Fig. 11. Consider the beam separated from the slab, and broken at the point considered; then introduce a unit angle change in the direction of positive moment in the beam at the point $x = u$. The beam will deflect in the shape of the moment diagram for a unit load at $x=u$. The deflection will be given by the equation

$$\Delta = \frac{2a}{\pi^2} \sum_{n=1,2,3,\cdots} \frac{1}{n^2} \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a} \tag{103}$$

Now consider the beam cemented together at the discontinuity and compute the load F on the beam required to pull the beam up to a horizontal line. The loading is as follows:

$$F = -\frac{2\pi^2 EI}{a^3} \sum_{n=1,2,3,\cdots} n^2 \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a} \tag{104}$$

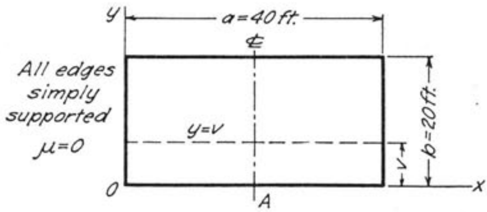
This loading reversed, or applied downward, on the complete structure along the line of the supporting beam, will produce a deflection which is the influence surface desired.

*See, for example, Hardy Cross, "Statically Indeterminate Structures," published in mimeographed form in 1926, Chapter IV.

If the beam is rigid it is necessary to compute the loads required to deflect the slab to fit the beam. In this case, joint or edge deflections in the slab at the panels adjacent to the beam are introduced. These deflections are, of course, those given by Equation (103).

In all the cases discussed, the influence ordinates are the resultant deflections due to all the sine-wave components combined.

32. *Numerical Example Illustrating Computation of Influence Surface for Moment in a Rigid Supporting Beam.*—It is desired to find the influence surface for moment at the center (point A) of one of the rigid beams supporting a long side of the simply-supported slab shown in the sketch below.



The deflection of the rigid beam, and consequently of the edge of the slab, that must be introduced, is a triangle with a maximum ordinate at A of 10.0 ft. This deflection may also be expressed as an infinite series, derived from Equation (103):

$$\Delta = \frac{2a}{\pi^2} \sum_{n=1,2,3,\cdots} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{a}$$

Since the slab is simply supported there will be no edge moments. However, in expressing the effect of the deflection of an edge, one must introduce, for each value of n, a fictitious edge moment, $-\mathsf{M}_w$, where, for $\mu = 0$,

$$-\mathsf{M}_w = -\frac{N\pi^2}{s^2} \Delta^{n=n} = -\frac{N\pi^2 n^2}{a^2} \Delta^{n=n}$$

Then, one must add an edge moment M', to cancel $-\mathsf{M}_w$. Therefore, one has the equation

$$\mathsf{M}' = \mathsf{M}_w = \frac{N\pi^2}{s^2} \Delta^{n=n}$$

The deflection at $y = v$ is determined from the relation, for each value of n,

$$w = C\Delta^{n=n} + C_w \frac{b^2}{N} M'$$

This last equation, written for the sum of all the component deflections, with some simplification, becomes

$$w = \sum_{n=1,3,5,\cdots} \left[\frac{2a}{\pi^2} \frac{C}{n^2} + \frac{2b^2}{a} C_w \right] \sin \frac{n\pi}{2} \sin \frac{n\pi x}{a}$$

or, with numerical values,

$$w = \sum_{n=1,3,5,\cdots} \left[8.10 \frac{C}{n^2} + 20.0\, C_w \right] \text{ft. } \sin \frac{n\pi}{2} \sin \frac{n\pi x}{a}$$

The values of C and C_w are taken from Tables 6 and 11 for the various values of $\frac{b}{s}$ and $\frac{v}{b}$ considered.

The following values of the coefficients of $\sin \frac{n\pi x}{a}$ in the expression for w were computed:

n	Values of Coefficients of $\sin n\pi x/a$		
	$v/b = 0.2$	$v/b = 0.5$	$v/b = 0.8$
1	+6.36 ft.	+3.85 ft.	+1.51 ft.
3	−0.51	−0.18	−0.05
5	+0.12	+0.02	0.00
7	−0.04	0.00	0.00

With these coefficients the influence ordinates for $u = \frac{a}{2}$ and $u = \frac{a}{4}$ or $\frac{3a}{4}$ were computed for particular values of $\frac{v}{b}$. The results are given in the following table:

u/a	Ordinates to Influence Surface for M_{beam}				
	$v/b = 0$	$v/b = 0.2$	$v/b = 0.5$	$v/b = 0.8$	$v/b = 1.0$
¼ or ¾	5.00 ft.	4.08 ft.	2.58 ft.	1.03 ft.	0
½	10.00 ft.	7.03 ft.	4.05 ft.	1.56 ft.	0

These results may be interpreted in the following way: a load of 10 lb. at $u = \dfrac{1}{4}a$, $v = 0.2b$ produces a bending moment in the beam at point A of a magnitude of 40.8 ft. lb.

IX. General Aspects of the Analysis

33. *Summary.*—The procedure given here relates the analysis of slabs continuous in one direction to the analysis of continuous girders. The moment distribution procedure is used for the analogous continuous girder. It is noted that a number of analogous girders must be treated, corresponding to the number of terms in the infinite trigonometric series for the slab which are required to obtain results of the desired precision. The analysis of a slab continuous in one direction is somewhat more tedious than the analysis of a continuous girder, but is based on the same principles. The solution of the problem of a slab continuous over yielding supports, such as flexible beams, is best handled by an indirect procedure similar to that suggested by Cross for the treatment of sidesway of frames.

The flexural stiffnesses of a panel of a slab depend on both span lengths. For panels very short in the direction of continuity the stiffness varies nearly inversely as the panel length, but for long panels the stiffness varies nearly inversely as the transverse span. Consequently, the flexural stiffnesses for slabs continuous over rigid supports do not differ as much as for continuous beams. Moreover, the carry-over factors are always less than 0.50 for slabs, and grow considerably smaller as the number of terms in the series, n, increases. Therefore one may develop the approximate rule that a slab continuous over a rigid support with any series of panels is approximately "50 per cent fixed" at that support for loads in the panel considered.

The tabulated values given in this report are developed for the distribution procedure, but they may be used in other analyses or for other purposes.

The distribution procedure may also be used to compute influence surfaces for a slab. One may estimate very readily, by rough calculations based on the distribution, the form of the influence surface for moment in a beam. This may be used to form some judgment of the way in which the load is "carried" to the simple supports in view of the static control for total bending moment in a cross-section of the structure. On any cross-section parallel to the simply-supported edges through the entire structure consisting of a continuous slab and supporting beams, the magnitude of the total bending moment on the section is determined by statics. Part of the total moment is resisted by flexure in the beam, the remainder by flexure in the slab, but the manner in which the moment in the slab is distributed over the section, and the amount of the moment resisted by each beam, is statically indeterminate. One may, however, in many cases assign reasonable values to the magnitudes of the various moments. For loads great enough to cause cracking of the concrete, a rational analysis founded on good judgment is probably a better basis for design than an "exact" analysis developed on the assumption of a homogeneous, elastic material.

APPENDIX A

Summary of Fundamental Relations of the Ordinary Theory of Flexure for Slabs

The ordinary theory of flexure of slabs is based upon the following assumptions, similar to those upon which the ordinary theory of flexure of beams is based, namely:

(1) The stresses acting on any cross-section have no resultant force in the direction of the plane of the slab, and the slab is loaded only by forces normal to its plane.

(2) The slab is of constant thickness of homogeneous, elastic, isotropic material.

(3) The strains and detrusions in the plane of the slab vary linearly through the depth of the slab.

The derivations of the fundamental equations, based on these or equivalent assumptions, are available in a number of places in the literature.* The relations necessary for the present work are summarized here.

The forces acting on cross-sections of the slab are shown in Fig. 1. With the use of the quantity

$$N = \frac{EI}{1 - \mu^2} = \frac{Eh^3}{12\,(1 - \mu^2)} \tag{A1}$$

as a measure of the stiffness of an element of the slab, the relations between the bending and twisting moments and the deflections are

$$\left.\begin{aligned} M_x &= -N\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] \\ M_y &= -N\left[\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right] \\ M_{xy} &= -N\,(1 - \mu)\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \tag{A2}$$

*For example: H. M. Westergaard, "Computation of Stresses in Bridge Slabs Due to Wheel Loads," Public Roads, Vol. 11, No. 1, 1930, p. 1-23, see p. 2.
H. M. Westergaard and W. A. Slater, "Moments and Stresses in Slabs," Proc. American Concrete Institute, Vol. 17, 1921, p. 415-538, see p. 424.
A. Nádai, "Die elastischen Platten," 1925, Julius Springer, Berlin, p. 18.

These moments are the resultants, per unit of length, of stresses on the cross-section that are linearly distributed through the depth of the slab. At the bottom of the slab the stresses as shown in Fig. 1 are given by the well known relations

$$\left.\begin{aligned} \sigma_x\Big]_{z=\frac{h}{2}} &= \frac{M_x \dfrac{h}{2}}{I} = \frac{6}{h^2} M_x \\ \sigma_y\Big]_{z=\frac{h}{2}} &= \frac{6}{h^2} M_y \\ \tau_{xy}\Big]_{z=\frac{h}{2}} &= \frac{6}{h^2} M_{xy} \end{aligned}\right\} \tag{A3}$$

With the use of Laplace's differential operator for two dimensions

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{A4}$$

the shears are given by the equations

$$\left.\begin{aligned} V_x &= -N \frac{\partial}{\partial x} (\nabla^2 w) \\ V_y &= -N \frac{\partial}{\partial y} (\nabla^2 w) \end{aligned}\right\} \tag{A5}$$

These shears are the resultants, per unit of length, of the shearing stresses τ_{zx} and τ_{zy}, respectively, shown in Fig. 1, which are distributed over the depth of the slab according to the ordinates of a parabola, in a manner similar to the distribution of the shearing stresses in a rectangular beam.

Substitution of the moments and shears expressed in (A2) and (A5) into the equation of statics expressing the equilibrium of elements of the slab gives the result

$$\nabla^2 (\nabla^2 w) = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{N} \tag{A6}$$

This is the fundamental equation for the flexure of slabs.

The reactions on faces such as those shown in Fig. 1 may be stated as follows:

On a face normal to the x axis,

$$R_x = V_x + \frac{\partial M_{xy}}{\partial y} \tag{A7}$$

and on a face normal to the y axis,

$$R_y = V_y + \frac{\partial M_{xy}}{\partial x} \tag{A8}$$

In addition, at each corner formed by the intersection of faces normal to the axes of x and y, there are concentrated reactions R_c. The magnitude of these corner reactions is

$$R_c = 2M_{xy} \tag{A9}$$

For positive M_{xy} the reaction R_c at a corner such as that shown in Fig. 1 is positive upward; at the corner diagonally opposite, R_c is also positive upward; and at the other two corners, R_c is positive downward.

The ordinary theory of flexure is not valid in the neighborhood of a concentrated load or near an edge of the slab where the assumptions upon which the ordinary theory is based are not fulfilled. For the determination of the tensile stresses at the point of application of a concentrated load Westergaard* suggests using, with the ordinary theory, an equivalent circular area of distribution of the load determined by a special theory. He gives values of the diameter of the equivalent circular area of loading in terms of the actual diameter of the loaded area.

*H. M. Westergaard, "Computation of Stresses in Bridge Slabs Due to Wheel Loads," Public Roads, Vol. 11, No. 1, 1930, p. 1-23, see p. 1 and Table 1, p. 9.

APPENDIX B

Derivation of Formulas

1. *General Solution of the Fundamental Equation for Slabs.*—Under certain conditions the fundamental equation of the ordinary theory of flexure for slabs

$$\nabla^2 (\nabla^2 w) = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{N} \tag{B1}$$

may be solved in the form

$$w = \sum_{n=1,2,3,\cdots} Y_n \left(\frac{n\pi y}{a}\right) \sin \frac{n\pi x}{a} \tag{B2}$$

where Y_n is a function of $n\pi y/a$, or of the variable y only.

Consider one term of the infinite series, for a particular value of n. In terms of $s = a/n$, one has

$$w = Y\left(\frac{\pi y}{s}\right) \sin \frac{\pi x}{s} \tag{B3}$$

Equation (B2) or the single term (B3) applies to the rectangular slab shown in Fig. 2, with the sides $x = 0$ and $x = a$ simply supported, and the sides $y=0$ and $y=b$ restrained in some arbitrary manner.

With the notation

$$\eta = \frac{\pi y}{s} \quad , \quad \xi = \frac{\pi x}{s} \tag{B4}$$

the slopes, moments, shears, reactions, and loading corresponding to the deflection given by Equation (B3) are determined from the fundamental equations of Appendix A with the following results:

$$
\left.
\begin{aligned}
&\text{Slopes} \\
&\qquad \frac{\partial w}{\partial y} = \frac{\pi}{s} Y' \sin \xi \\
&\qquad \frac{\partial w}{\partial x} = \frac{\pi}{s} Y \cos \xi \\
&\text{Moments} \\
&\qquad M_x = N \frac{\pi^2}{s^2} (Y - \mu Y'') \sin \xi \\
&\qquad M_y = N \frac{\pi^2}{s^2} (-Y'' + \mu Y) \sin \xi \\
&\qquad M_{xy} = -N \frac{\pi^2}{s^2} (1 - \mu)\, Y' \cos \xi \\
&\text{Shears} \\
&\qquad V_x = N \frac{\pi^3}{s^3} (Y - Y'') \cos \xi \\
&\qquad V_y = N \frac{\pi^3}{s^3} (Y' - Y''') \sin \xi \\
&\text{Reactions} \\
&\qquad R_x = N \frac{\pi^3}{s^3} [Y - (2 - \mu)\, Y''] \cos \xi \\
&\qquad R_y = N \frac{\pi^3}{s^3} [(2 - \mu)\, Y' - Y'''] \sin \xi \\
&\text{Load} \\
&\qquad p = N \frac{\pi^4}{s^4} [Y - 2Y'' + Y''''] \sin \xi
\end{aligned}
\right\} \qquad \text{(B5)}
$$

where

$$
Y' = \frac{dY}{d\eta} = \frac{s}{\pi} \frac{dY}{dy}, \qquad Y'' = \frac{s^2}{\pi^2} \frac{d^2 Y}{dy^2}, \text{ etc.} \qquad \text{(B6)}
$$

It is noted that a sine wave of deflection is accompanied by slopes, moments, shears, reactions, and loading which are all given by sine curves or cosine curves in the x direction multiplied by functions of y only.

In general, the deflection w in Equation (B3) may be stated in two parts,

$$w = w_p + w_c \tag{B7}$$

where w_p accounts for the loading for the particular value of n, and w_c accounts for certain of the boundary conditions. The function w_p satisfies Equation (B1) but may not satisfy the boundary conditions. The function w_c satisfies the differential equation

$$\frac{\partial^4 w_c}{\partial x^4} + 2\frac{\partial^4 w_c}{\partial x^2 \partial y^2} + \frac{\partial^4 w_c}{\partial y^4} = 0 \tag{B8}$$

and the sum of w_c and w_p satisfies all of the boundary conditions for the particular value of n.

Of special interest in so far as the derivations in this work are concerned is the solution of Equation (B8) in the form

$$w_c = Y \sin \xi \tag{B9}$$

where Y is a function of η, or of y only. This solution applies to the case, shown in Fig. 2, of a rectangular slab with two opposite edges simply supported, and the other two restrained in some arbitrary manner, with no load on the slab. The function Y, as stated by Nádai,* is in the present notation

$$Y = c_1 \sinh \eta + c_2 \cosh \eta + c_3\, \eta \sinh \eta + c_4\, \eta \cosh \eta. \tag{B10}$$

The following quantities are necessary in the use of Equations (B5) when Y is given by Equation (B10):

$$\left.\begin{aligned}
Y' &= (c_2 + c_3) \sinh \eta + (c_1 + c_4) \cosh \eta + c_4\, \eta \sinh \eta + c_3\, \eta \cosh \eta \\
Y'' &= (c_1 + 2c_4) \sinh \eta + (c_2 + 2c_3) \cosh \eta + c_3\, \eta \sinh \eta + c_4\, \eta \cosh \eta \\
Y''' &= (c_2 + 3c_3) \sinh \eta + (c_1 + 3c_4) \cosh \eta + c_4\, \eta \sinh \eta + c_3\, \eta \cosh \eta \\
Y'''' &= (c_1 + 4c_4) \sinh \eta + (c_2 + 4c_3) \cosh \eta + c_3\, \eta \sinh \eta + c_4\, \eta \cosh \eta
\end{aligned}\right\} \tag{B11}$$

*A. Nádai, "Die elastischen Platten," 1925, Julius Springer, Berlin, p. 69.

From these equations one may verify the fact that w_c corresponds to no load on the slab and no deflections or moments M_x at $x = 0$ and at $x = s$. The four constants in Equation (B10) are determined by the boundary conditions at the edges of the slab $y=0$ and $y=b$.

2. *Formulas for Elastic Constants.*—In Fig. 3 consider the case of a sine wave of rotation without deflection at the edge $y = b$, the edge $y = 0$ remaining fixed. With the notation for the edge moments and forces used in the text and illustrated in Fig. 3, the boundary conditions may be stated as follows:

at $y = 0$

$$\left.\begin{aligned} \Phi &= \frac{\partial w}{\partial y} = 0 \\ \Delta &= w = 0 \end{aligned}\right\} \tag{B12}$$

and at $y = b$

$$\left.\begin{aligned} \Phi &= -\frac{\partial w}{\partial y} = \Phi_0 \sin \xi \\ \Delta &= w = 0 \end{aligned}\right\} \tag{B13}$$

Since there is no load on the slab $w_p = 0$ and $w = w_c$.

For these boundary conditions one finds the following values of the constants in Equation (B10):

$$\left.\begin{aligned} c_1 &= -c_4 = \frac{\beta \sinh \beta}{\sinh^2 \beta - \beta^2} \cdot \frac{s}{\pi} \Phi_0 \\ c_2 &= 0 \\ c_3 &= \frac{\beta \cosh \beta - \sinh \beta}{\sinh^2 \beta - \beta^2} \cdot \frac{s}{\pi} \Phi_0 \end{aligned}\right\} \tag{B14}$$

where

$$\beta = \frac{\pi b}{s} \tag{B15}$$

The moments **M** and reactions **R** at the edges are found to be as follows:

At $y = b$

$$\mathsf{M} = M_y = K\Phi_0 \sin \xi \tag{B16}$$

where

$$K = 2\pi \frac{\sinh \beta \cosh \beta - \beta}{\sinh^2 \beta - \beta^2} \frac{N}{s} = 2\beta \frac{\sinh \beta \cosh \beta - \beta}{\sinh^2 \beta - \beta^2} \frac{N}{b} \tag{B17}$$

and

$$\mathsf{R} = R_y = Q\Phi_0 \sin \xi \tag{B18}$$

where

$$Q = \pi^2 \left[\frac{\sinh^2 \beta + \beta^2}{\sinh^2 \beta - \beta^2} + \mu \right] \frac{N}{s^2} = \beta^2 \left[\frac{\sinh^2 \beta + \beta^2}{\sinh^2 \beta - \beta^2} + \mu \right] \frac{N}{b^2} \tag{B19}$$

At $y = 0$

$$\mathsf{M} = M_y = -kK\Phi_0 \sin \xi \tag{B20}$$

where

$$k = \frac{\beta \cosh \beta - \sinh \beta}{\sinh \beta \cosh \beta - \beta} \tag{B21}$$

and

$$\mathsf{R} = -R_y = -qQ\Phi_0 \sin \xi \tag{B22}$$

where

$$qQ = \pi^2 \frac{2\beta \sinh \beta}{\sinh^2 \beta - \beta^2} \frac{N}{s^2} = \frac{2\beta^3 \sinh \beta}{\sinh^2 \beta - \beta^2} \frac{N}{b^2} \tag{B23}$$

Now consider the case, in Fig. 2, of a sine wave of deflection without rotation at the edge $y = b$, the edge $y = 0$ remaining fixed. The boundary conditions may be stated as follows:

At $y = 0$

$$\left.\begin{aligned} \Phi &= \frac{\partial w}{\partial y} = 0 \\ \Delta &= w = 0, \end{aligned}\right\} \tag{B24}$$

and at $y = b$

$$\left.\begin{aligned} \Phi &= -\frac{\partial w}{\partial y} = 0 \\ \Delta &= w = \Delta_0 \sin \xi. \end{aligned}\right\} \tag{B25}$$

Since there is no load, $w_p = 0$ and $w = w_c$.

For these boundary conditions one finds the following values of the constants in Equation (B10):

$$\left.\begin{aligned} c_1 &= -c_4 = \frac{\beta \cosh \beta + \sinh \beta}{\sinh^2 \beta - \beta^2} \Delta_0 \\ c_2 &= 0 \\ c_3 &= \frac{\beta \sinh \beta}{\sinh^2 \beta - \beta^2} \Delta_0 \end{aligned}\right\} \tag{B26}$$

The moments M and reactions R at the edges are found to be as follows:

At $y = b$

$$\mathsf{M} = M_y = Q\Delta_0 \sin \xi \tag{B27}$$

where Q is given by Equation (B19), and

$$\mathsf{R} = R_y = T\Delta_0 \sin \xi \tag{B28}$$

where

$$T = 2\pi^3 \frac{\sinh \beta \cosh \beta + \beta}{\sinh^2 \beta - \beta^2} \frac{N}{s^3} = 2\beta^3 \frac{\sinh \beta \cosh \beta + \beta}{\sinh^2 \beta - \beta^2} \frac{N}{b^3} \tag{B29}$$

At $y = 0$

$$\mathsf{M} = M_y = -qQ\Delta_0 \sin \xi \tag{B30}$$

where qQ is given by Equation (B23), and

$$\mathsf{R} = -R_y = -tT\Delta_0 \sin \xi \tag{B31}$$

where

$$t = \frac{\beta \cosh \beta + \sinh \beta}{\sinh \beta \cosh \beta + \beta}\,. \tag{B32}$$

3. *Formulas for Moment at the Fixed Edges of a Panel of a Slab Due to a Line Load.*—It is desired to find the end moments on the lines $y = 0$ and $y = b$ of the slab in Fig. 3, when these ends are fixed against rotation and deflection, due to a loading

$$F = F_0 \sin \xi \tag{B33}$$

on the line $y = y_1 = b - v$.

In view of the fact that a sine wave of loading produces a sine wave of fixed-end moment, and also that a sine wave of end rotation produces a deflection with ordinates varying as a sine wave, one may apply Maxwell's Theorem of reciprocal deflections to obtain the influence line for fixed-end moments in the following manner:

With all quantities given as sine waves in the x direction, the deflection in the slab on the line $y = y_1$ due to a unit negative rotation at $y = b$ when the edge $y = 0$ is fixed, is equal to the end moment at the edge $y = b$ due to a unit line load applied to the slab on the line $y = y_1$. These reciprocal relations may be formulated as follows:

$$\left.\begin{aligned} \Phi &= \Phi_0 \sin \xi = -F = -F_0 \sin \xi \\ w &= w_0 \sin \xi = \mathsf{M}^F = \mathsf{M}_0^F \sin \xi \end{aligned}\right\} \tag{B34}$$

The case of rotation without deflection at the edge $y = b$, the edge $y = 0$ remaining fixed, is treated in Section 2 of this Appendix in deriving formulas for the elastic constants. One may write the deflection of the slab due to the rotation

$$\Phi = \Phi_0 \sin \xi$$

at the edge $y = b$, in terms of the constants c_1, c_2, c_3 and c_4 stated in Equation (B14). With the reciprocal relations of (B34) one obtains the following formula for the fixed-end moments at the edge $y = b$ due to a line loading on the line $y = y_1$:

$$\mathsf{M}_0^F = -\frac{F_0 s}{\pi} \frac{1}{\sinh^2 \beta - \beta^2} [\beta \sinh \beta \sinh \eta_1$$

$$+ (\beta \cosh \beta - \sinh \beta)\, \eta_1 \sinh \eta_1 - (\beta \sinh \beta)\, \eta_1 \cosh \eta_1]$$

This equation may be simplified and stated in the form:

$$\mathsf{M}_0^F = -\frac{F_0 s}{\pi} \frac{\alpha \sinh \beta \sinh \eta_1 - \beta \eta_1 \sinh \alpha}{\sinh^2 \beta - \beta^2} \tag{B35}$$

where

$$\alpha = \beta - \eta_1 = \frac{\pi}{s}(b - y_1) = \frac{\pi v}{s} \tag{B36}$$

If the distances y_1 and v are interchanged in (B35) one obtains the formula for the fixed end moment at the edge $y = 0$ due to a load on the line $y = v$. This formula may be stated as

$$\mathsf{M}_0^F = -C_M F_0 b \tag{B37}$$

where

$$C_M = \frac{\sinh \beta \, (\eta_1 \sinh \alpha) - \beta \alpha \sinh \eta_1}{\beta \, (\sinh^2 \beta - \beta^2)} \tag{B38}$$

4. *Formulas for Reaction at the Fixed Edges of a Panel of a Slab Due to a Line Load.*—In a manner similar to that used in the preceding section the influence line for fixed-end reaction at the edge $y = b$ in Fig. 3 due to a line load on the line $y = y_1$, given by the equation

$$F = F_0 \sin \xi$$

is obtained from the case of a deflection without rotation at the edge $y = b$, the edge $y = 0$ remaining fixed, as treated in Section 2 of this Appendix. The reciprocal relations in this case may be stated as follows:

$$\left.\begin{aligned} \Delta &= \Delta_0 \sin \xi = -F = -F_0 \sin \xi \\ w &= w_0 \sin \xi = \mathsf{R}^F = \mathsf{R}_0^F \sin \xi \end{aligned}\right\} \tag{B39}$$

With these relations and the values of the constants given in Equation (B26) one finds the fixed-end reaction at $y = b$

$$\mathsf{R}_0^F = -F_0 \frac{1}{\sinh^2 \beta - \beta^2} [(\sinh \beta + \beta \cosh \beta) \sinh \eta_1$$

$$+ \beta \eta_1 \sinh \beta \sinh \eta_1 - (\sinh \beta + \beta \cosh \beta) \eta_1 \cosh \eta_1]$$

This formula may be simplified and rearranged in the form

$$\mathsf{R}_0^F = -F_0 \frac{\sinh\beta\,(\sinh\eta_1 + \alpha\cosh\eta_1) - \beta\,(\sinh\alpha + \eta_1\cosh\alpha)}{\sinh^2\beta - \beta^2} \quad \text{(B40)}$$

By interchanging the distances y_1 and v in (B40) one obtains the fixed-end reaction on the edge $y = 0$ due to a line load on the line $y = v$

$$\mathsf{R}_0^F = -C_R F_0 \quad \text{(B41)}$$

where

$$C_R = \frac{\sinh\beta\,(\sinh\alpha + \eta_1\cosh\alpha) - \beta\,(\sinh\eta_1 + \alpha\cosh\eta_1)}{\sinh^2\beta - \beta^2} \quad \text{(B42)}$$

5. *Formulas for Effect of a Moment Applied at One Edge of a Panel of a Slab.*—Consider a moment applied at the edge $y = b$ of the slab shown in Fig. 3. Let the edge at $y = 0$ be simply supported, and the applied moment be given by the equation

$$\mathsf{M} = \mathsf{M}_0 \sin\xi \quad \text{(B43)}$$

The boundary conditions are stated as follows:

at $y = 0$,

$$\Delta = w = 0$$

$$\mathsf{M} = M_y = 0,$$

and at $y = b$,

$$\Delta = w = 0$$

$$\mathsf{M} = M_y = \mathsf{M}_0 \sin\xi$$

For these boundary conditions one finds the following values of the constants in Equation (B10):

$$\left.\begin{aligned} c_1 &= \frac{\mathsf{M}_0 b^2}{2N}\frac{\cosh\beta}{\beta\sinh^2\beta} \\ c_2 &= c_3 = 0 \\ c_4 &= -\frac{\mathsf{M}_0 b^2}{2N}\frac{1}{\beta^2\sinh\beta} \end{aligned}\right\} \quad \text{(B44)}$$

With these constants one may write the bending moments M_x and M_y, and the twisting moment M_{xy} as follows:

$$\left.\begin{aligned} M_x &= (m_x + \mu m_y)\,\mathsf{M}_0 \sin \xi \\ M_y &= (m_y + \mu m_x)\,\mathsf{M}_0 \sin \xi \\ M_{xy} &= (1 - \mu)\, m_{xy}\mathsf{M}_0 \cos \xi \end{aligned}\right\} \tag{B45}$$

in which

$$\left.\begin{aligned} m_x &= \frac{1}{2}\frac{\sinh \eta}{\sinh \beta}(\beta \coth \beta - \eta \coth \eta) \\ m_y &= \frac{\sinh \eta}{\sinh \beta} - m_x \\ m_{xy} &= \frac{1}{2}\frac{\cosh \eta}{\sinh \beta}(1 + \eta \tanh \eta - \beta \coth \beta) \end{aligned}\right\} \tag{B46}$$

The deflection w may be written

$$w = C_w \mathsf{M}_0 \frac{b^2}{N} \sin \xi \tag{B47}$$

where

$$C_w = \frac{1}{\beta^2} m_x \tag{B48}$$

If the moment M in Equation (B43) is applied at the edge $y = 0$ the bending moments M_x and M_y and the deflection w on the line $y = v$ are the same as the quantities in Equations (B45), (B46), (B47) and (B48) for $y = b - v$. However, the twisting moment on the line $y = v$ is of the same magnitude as, but opposite in sign to, the twisting moment given by Equations (B45) and (B46) for the line $y = b - v$. This follows from the manner in which positive twisting moment is defined.

6. *Formulas for Effect of a Deflection of One Edge of a Panel of a Slab Combined With a Moment at That Edge.*—Consider a deflection of the edge $y = b$ of the panel of the slab shown in Fig. 3. Let the edge deflection be given by the equation

$$\Delta = \Delta_0 \sin \xi \tag{B49}$$

Combined with the deflection let there be an edge moment at the edge $y = b$, of magnitude given by the equation

$$-\mathsf{M}_w = -\frac{(1-\mu)\,N\pi^2}{s^2}\,\Delta_0 \sin\xi = -\mathsf{M}_{0w}\sin\xi \qquad \text{(B50)}$$

If the edge $y = 0$ is simply supported, the boundary conditions are stated as follows:

at $y = 0$

$$\Delta = w = 0$$

$$\mathsf{M} = M_y = 0,$$

and at $y = b$

$$\Delta = \Delta_0 \sin\xi$$

$$\mathsf{M} = -\mathsf{M}_w$$

For these boundary conditions one finds the following values of the constants in Equation (B10):

$$\left.\begin{aligned} c_1 &= \frac{\Delta_0}{\sinh\beta} \\ c_2 &= c_3 = c_4 = 0 \end{aligned}\right\} \qquad \text{(B51)}$$

With these constants one may write the bending and twisting moments in the slab as follows:

$$\left.\begin{aligned} M_y &= -\frac{\sinh\eta}{\sinh\beta}\,\mathsf{M}_w = -C\mathsf{M}_w \\ M_x &= -M_y \\ M_{xy} &= -\frac{(1-\mu)\,N\pi^2}{s^2}\,\Delta_0\,\frac{\cosh\eta}{\sinh\beta}\cos\xi = -C_{xy}\mathsf{M}_{0w}\cos\xi \end{aligned}\right\} \text{(B52)}$$

The deflection w may be written as

$$w = \frac{\sinh\eta}{\sinh\beta}\,\Delta = C\Delta \qquad \text{(B53)}$$

If the deflection Δ in (B49) and the moment $-\mathsf{M}_w$ in (B50) are applied at the edge $y = 0$ the bending moments M_x and M_y and the deflection w on the line $y = v$ are the same as the quantities stated in (B52) and (B53) for $y = b - v$. However, the twisting moment on the line $y = v$ is of the same magnitude as, but opposite in sign to, the twisting moment given by (B52) for the line $y = b - v$. This follows from the definition of positive twisting moment.

7. *Formulas for Moment and Reaction at the Fixed Edges of a Panel of a Slab Due to a Distributed Load.*—Let the slab shown in Fig. 3 be subjected to the loading

$$p = p_0 \sin \xi \tag{B54}$$

where p_0 is a constant, and let the deflection of the slab be given by the equation

$$w = w_0 \sin \xi \tag{B55}$$

in which

$$w_0 = \frac{p_0 s^4}{N\pi^4} \tag{B56}$$

Then w will satisfy the fundamental equation for the flexure of slabs. The boundary conditions corresponding to (B55) are

at $y = 0$ or b

$$\left.\begin{aligned} \Delta &= \frac{p_0 s^4}{N\pi^4} \sin \xi \\ \Phi &= 0 \\ \mathsf{M} &= \mu \frac{p_0 s^2}{\pi^2} \sin \xi \\ \mathsf{R} &= 0 \end{aligned}\right\} \tag{B57}$$

It is desired to make the edges at $y = 0$ and $y = b$ fixed. This is done by introducing a deflection

$$\Delta = -\frac{p_0 s^4}{N\pi^4} \sin \xi \tag{B58}$$

at each edge, without rotation. The reactions at the edges due to the deflection will be the fixed-end reactions, and the moments at the edges due to the deflection plus the moment given by (B57) will be the fixed-end moments. Since Poisson's ratio μ does not enter into the moments or reactions at a fixed edge, one may take $\mu = 0$ in the subsequent work, and set $\mathsf{M} = 0$ in (B57).

By use of the elastic constants for the slab one finds the fixed-end moments and reactions as follows:

$$\left.\begin{aligned} \mathsf{M}^F &= -(Q - qQ)\frac{p_0 s^4}{N\pi^4}\sin\xi \\ \mathsf{R}^F &= -(T - tT)\frac{p_0 s^4}{N\pi^4}\sin\xi \end{aligned}\right\} \tag{B59}$$

With the values of Q, q, T, and t given by Equations (B19), (B23), (B29), and (B32), one finds

$$\mathsf{M}^F = -p_0\frac{s^2}{\pi^2}\frac{\sinh\beta - \beta}{\sinh\beta + \beta} = -c_m p_0 b^2 \tag{B60}$$

where

$$c_m = \frac{1}{\beta^2}\frac{\sinh\beta - \beta}{\sinh\beta + \beta} \tag{B61}$$

and

$$\mathsf{R}^F = -\frac{2p_0 s}{\pi}\frac{\cosh\beta - 1}{\sinh\beta + \beta} = -c_r p_0 b \tag{B62}$$

where

$$c_r = \frac{2}{\beta}\frac{\cosh\beta - 1}{\sinh\beta + \beta} \tag{B63}$$

For β approaching zero, M^F approaches the moment for a fixed-end beam, $-\frac{1}{12}p_0b^2$, and R^F approaches the reaction for a fixed-end beam, $-\frac{1}{2}p_0b$.

8. *Formulas for Reaction at the Edge of a Simply-Supported Panel of a Slab Due to a Line Load.*—One may obtain the influence ordinates for reaction at an edge of a simply-supported panel of a slab due to the line loading

$$F = F_0 \sin \xi$$

in a manner similar to that used for the fixed-end reaction. Let the edge $y = b$ of the slab in Fig. 3 have the deflection

$$\Delta = \Delta_0 \sin \xi$$

The boundary conditions are stated as follows:

at $y = b$

$$\Delta = \Delta_0 \sin \xi$$

$$M = 0$$

and at $y = 0$

$$\Delta = 0$$

$$M = 0$$

With these boundary conditions one finds the following values of the constants in Equation (B10):

$$c_1 = \frac{1-\mu}{2}\frac{\beta \cosh \beta}{\sinh^2 \beta} + \frac{1}{\sinh \beta}$$

$$c_2 = c_3 = 0$$

$$c_4 = -\frac{1-\mu}{2}\frac{1}{\sinh \beta}$$

The deflection of the slab may then be stated as

$$w = w_0 \sin \xi \tag{B64}$$

where

$$w_0 = \frac{\sinh \eta}{\sinh \beta} \left[1 + \frac{1 - \mu}{2} (\beta \coth \beta - \eta \coth \eta) \right] \qquad \text{(B65)}$$

This may be restated in terms of the notation used in Equations (B46) and (B52). One obtains the result

$$w_0 = C + (1 - \mu) m_x \qquad \text{(B66)}$$

Then with the reciprocal relations

$$\Delta = -F = -F_0 \sin \xi$$

$$w = w_0 \sin \xi = \mathbf{R} = \mathbf{R}_0 \sin \xi$$

where $\mathbf{R}$ is the reaction at the edge $y = b$ due to the line loading F, one finds the following expression for the reaction at the edge $y = b$ due to a load on the line $y = y_1$ a distance v away from the edge b

$$\mathbf{R} = \mathbf{R}_0 \sin \xi \qquad \text{(B67)}$$

where

$$\mathbf{R}_0 = - [C + (1 - \mu) m_x] F_0 \qquad \text{(B68)}$$

Equation (B68) also gives the reaction on the edge $y = 0$ due to a line load on the line $y = v$, in a slab simply supported on all sides.

RECENT PUBLICATIONS OF THE ENGINEERING EXPERIMENT STATION†

Bulletin No. 271. Determination of Mean Specific Heats at High Temperatures of Some Commercial Glasses, by Cullen W. Parmelee and Alfred E. Badger. 1934. *Thirty cents.*

Bulletin No. 272. The Creep and Fracture of Lead and Lead Alloys, by Herbert F. Moore, Bernard B. Betty, and Curtis W. Dollins. 1934. *Fifty cents.*

Bulletin No. 273. Mechanical-Electrical Stress Studies of Porcelain Insulator Bodies, by Cullen W. Parmelee and John O. Kraehenbuehl. 1935. *Seventy-five cents.*

Bulletin No. 274. A Supplementary Study of the Locomotive Front End by Means of Tests on a Front-End Model, by Everett G. Young. 1935. *Fifty cents.*

Bulletin No. 275. Effect of Time Yield in Concrete upon Deformation Stresses in a Reinforced Concrete Arch Bridge, by Wilbur M. Wilson and Ralph W. Kluge. 1935. *Forty cents.*

Bulletin No. 276. Stress Concentration at Fillets, Holes, and Keyways as Found by the Plaster-Model Method, by Fred B. Seely and Thomas J. Dolan. 1935. *Forty cents.*

Bulletin No. 277. The Strength of Monolithic Concrete Walls, by Frank E. Richart and Nathan M. Newmark. 1935. *Forty cents.*

Bulletin No. 278. Oscillations Due to Corona Discharges on Wires Subjected to Alternating Potentials, by J. Tykocinski Tykociner, Raymond E. Tarpley, and Ellery B. Paine. 1935. *Sixty cents.*

Bulletin No. 279. The Resistance of Mine Timbers to the Flow of Air, as Determined by Models, by Cloyde M. Smith. 1935. *Sixty-five cents.*

Bulletin No. 280. The Effect of Residual Longitudinal Stresses upon the Load-Carrying Capacity of Steel Columns, by Wilbur M. Wilson and Rex L. Brown. 1935. *Forty cents.*

Circular No. 24. Simplified Computation of Vertical Pressures in Elastic Foundations, by Nathan M. Newmark. 1935. *Twenty-five cents.*

Reprint No. 3. Chemical Engineering Problems, by Donald B. Keyes. 1935. *Fifteen cents.*

Reprint No. 4. Progress Report of the Joint Investigation of Fissures in Railroad Rails, by Herbert F. Moore. 1935. *None available.*

Circular No. 25. Papers Presented at the Twenty-Second Annual Conference on Highway Engineering, Held at the University of Illinois, Feb. 21 and 22, 1935. 1936. *Fifty cents.*

Reprint No. 5. Essentials of Air Conditioning, by Maurice K. Fahnestock. 1936. *Fifteen cents.*

Bulletin No. 281. An Investigation of the Durability of Molding Sands, by Carl H. Casberg and Carl E. Schubert. 1936. *Sixty cents.*

Bulletin No. 282. The Cause and Prevention of Steam Turbine Blade Deposits, by Frederick G. Straub. 1936. *Fifty-five cents.*

Bulletin No. 283. A Study of the Reactions of Various Inorganic and Organic Salts in Preventing Scale in Steam Boilers, by Frederick G. Straub. 1936. *One dollar.*

Bulletin No. 284. Oxidation and Loss of Weight of Clay Bodies During Firing, by William R. Morgan. 1936. *Fifty cents.*

Bulletin No. 285. Possible Recovery of Coal from Waste at Illinois Mines, by Cloyde M. Smith and David R. Mitchell. 1936. *Fifty-five cents.*

Bulletin No. 286. Analysis of Flow in Networks of Conduits or Conductors, by Hardy Cross. 1936. *Forty cents.*

Circular No. 26. Papers Presented at the First Annual Conference on Air Conditioning, Held at the University of Illinois, May 4 and 5, 1936. *Fifty cents.*

Reprint No. 6. Electro-Organic Chemical Preparations, by S. Swann, Jr. 1936. *Thirty-five cents.*

Reprint No. 7. Papers Presented at the Second Annual Short Course in Coal Utilization, Held at the University of Illinois, June 11, 12, and 13, 1935. 1936. *None available.*

Bulletin No. 287. The Biologic Digestion of Garbage with Sewage Sludge, by Harold E. Babbitt, Benn J. Leland, and Fenner H. Whitley, Jr. 1936. *One dollar.*

Reprint No. 8. Second Progress Report of the Joint Investigation of Fissures in Railroad Rails, by Herbert F. Moore. 1936. *Fifteen cents.*

†Copies of the complete list of publications can be obtained without charge by addressing the Engineering Experiment Station, Urbana, Ill.

Reprint No. 9. Correlation Between Metallography and Mechanical Testing, by Herbert F. Moore. 1936. *None available.*

Circular No. 27. Papers Presented at the Twenty-third Annual Conference on Highway Engineering, Held at the University of Illinois, Feb. 26-28, 1936. 1936. *Fifty cents.*

Bulletin No. 288. An Investigation of Relative Stresses in Solid Spur Gears by the Photoelastic Method, by Paul H. Black. 1936. *Forty cents.*

Bulletin No. 289. The Use of an Elbow in a Pipe Line for Determining the Rate of Flow in the Pipe, by Wallace M. Lansford. 1936. *Forty cents.*

Bulletin No. 290. Investigation of Summer Cooling in the Warm-Air Heating Research Residence, by Alonzo P. Kratz, Maurice K. Fahnestock, and Seichi Konzo. 1937. *One dollar.*

Bulletin No. 291. Flexural Vibrations of Piezoelectric Quartz Bars and Plates, by J. Tykocinski Tykociner and Marion W. Woodruff. 1937. *Forty-five cents.*

Reprint No. 10. Heat Transfer in Evaporation and Condensation, by Max Jakob. 1937. *Thirty-five cents.*

Circular No. 28. An Investigation of Student Study Lighting, by John O. Kraehenbuehl. 1937. *Forty cents.*

Circular No. 29. Problems in Building Illumination, by John O. Kraehenbuehl. 1937. *Thirty-five cents.*

Bulletin No. 292. Tests of Steel Columns; Thin Cylindrical Shells; Laced Channels; Angles, by Wilbur M. Wilson. 1937. *Fifty cents.*

Bulletin No. 293. The Combined Effect of Corrosion and Stress Concentration at Holes and Fillets in Steel Specimens Subjected to Reversed Torsional Stresses, by Thomas J. Dolan. 1937. *Fifty cents.*

Bulletin No. 294. Tests of Strength Properties of Chilled Car Wheels, by Frank E. Richart, Rex L. Brown, and Paul G. Jones. 1937. *Eighty-five cents.*

Bulletin No. 295. Tests of Thin Hemispherical Shells Subjected to Internal Hydrostatic Pressure, by Wilbur M. Wilson and Joseph Marin. 1937. *Thirty-five cents.*

Circular No. 30. Papers Presented at the Twenty-fourth Annual Conference on Highway Engineering, Held at the University of Illinois, March 3-5, 1937. 1937. *Fifty cents.*

Reprint No. 11. Third Progress Report of the Joint Investigation of Fissures in Railroad Rails, by H. F. Moore. 1937. *Fifteen cents.*

Bulletin No. 296. Magnitude and Frequency of Floods on Illinois Streams, by George W. Pickels. 1937. *Seventy cents.*

**Bulletin No. 297.* Ventilation Characteristics of Some Illinois Mines, by Cloyde M. Smith. 1937. *Seventy cents.*

**Bulletin No. 298.* Resistance to Heat Checking of Chilled Iron Car Wheels, and Strains Developed Under Long-Continued Application of Brake Shoes, by Edward C. Schmidt and Herman J. Schrader. 1937. *Fifty-five cents.*

**Bulletin No. 299.* Solution of Electrical Networks, by Successive Approximations, by Laurence L. Smith. 1937. *Forty-five cents.*

**Circular No. 31.* Papers Presented at the Short Course in Coal Utilization, Held at the University of Illinois, May 25-27, 1937. 1938. *Fifty cents.*

**Bulletin No. 300.* Pressure Losses Resulting from Changes in Cross-Sectional Area in Air Ducts, by Alonzo P. Kratz and Julian R. Fellows. 1938. *Seventy cents.*

**Bulletin No. 301.* The Friction of Railway Brake Shoes at High Speed and High Pressure, by Herman J. Schrader. 1938. *Sixty cents.*

**Bulletin No. 302.* Fatigue Tests of Riveted Joints, by Wilbur M. Wilson and Frank P. Thomas. 1938. *One dollar.*

**Circular No. 32.* Two Investigations on Transit Instruments, by William H. Rayner. 1938. *Twenty-five cents.*

**Circular No. 33.* Papers Presented at the Twenty-fifth Annual Conference on Highway Engineering, Held at the University of Illinois, March 2-4, 1938. 1938. *Fifty cents.*

**Bulletin No. 303.* Solutions for Certain Rectangular Slabs Continuous Over Flexible Supports, by Vernon P. Jensen. 1938. *One dollar.*

**Bulletin No. 304.* A Distribution Procedure for the Analysis of Slabs Continuous Over Flexible Beams, by Nathan M. Newmark. 1938. *One dollar.*

---

*A limited number of copies of bulletins starred are available for free distribution.

UNIVERSITY OF ILLINOIS

Colleges and Schools at Urbana

College of Liberal Arts and Sciences.—General curriculum with majors in the humanities and sciences; specialized curricula in chemistry and chemical engineering; general courses preparatory to the study of law and journalism; pre-professional training in medicine, dentistry, and pharmacy.

College of Commerce and Business Administration.—Curricula in general business, trade and civic secretarial service, banking and finance, insurance, accountancy, transportation, commercial teaching, foreign commerce, industrial administration, public utilities, and commerce and law.

College of Engineering.—Curricula in agricultural engineering, ceramics, ceramic engineering, chemical engineering, civil engineering, electrical engineering, engineering physics, general engineering, mechanical engineering, metallurgical engineering, mining engineering, and railway engineering.

College of Agriculture.—Curricula in agriculture, floriculture, general home economics, and nutrition and dietetics.

College of Education.—Curricula in education, agricultural education, home economics education, and industrial education. The University High School is the practice school of the College of Education.

College of Fine and Applied Arts.—Curricula in architecture, art, landscape architecture, and music.

College of Law.—Professional curriculum in law.

School of Journalism.—General and special curricula in journalism.

School of Physical Education.—Curricula in physical education for men and for women.

Library School.—Curriculum in library science.

Graduate School.—Advanced study and research.

University Extension Division.—For a list of correspondence courses conducted by members of the faculties of the colleges and schools at Urbana and equivalent to courses offered to resident students, address the Director of the Division of University Extension, 354 Administration Building, Urbana, Illinois.

Colleges in Chicago

College of Medicine.—Professional curriculum in medicine.

College of Dentistry.—Professional curriculum in dentistry.

College of Pharmacy.—Professional curriculum in pharmacy.

University Experiment Stations, and Research and Service Bureaus at Urbana

Agricultural Experiment Station
Engineering Experiment Station
Extension Service in Agriculture and Home Economics
Bureau of Business Research
Bureau of Community Planning
Bureau of Educational Research
Bureau of Institutional Research

State Scientific Surveys and Other Divisions at Urbana

State Geological Survey
State Natural History Survey
State Water Survey
State Historical Survey
State Diagnostic Laboratory (Animal Pathology)
U. S. Soybean Products Laboratory
U. S. Weather Bureau Station

For general catalog of the University, special circulars, and other information, address

The Registrar, University of Illinois
Urbana, Illinois