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INFLUENCE CHARTS FOR COMPUTATION OF STRESSES IN ELASTIC FOUNDATIONS

BY

NATHAN M. NEWMARK

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INFLUENCE CHARTS FOR COMPUTATION OF STRESSES IN ELASTIC FOUNDATIONS

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INFLUENCE CHARTS FOR COMPUTATION OF STRESSES IN ELASTIC FOUNDATIONS

I. Introduction

1. *Scope of Investigation and Summary.*—This bulletin describes a simple graphical procedure for computing stresses in the interior of an elastic, homogeneous, isotropic solid bounded by a plane horizontal surface and loaded by distributed vertical loads at the surface. The stresses are computed from charts given herein merely by counting on a chart the number of elements of area, or blocks, covered by a plan of the loaded area drawn to proper scale and laid upon the chart.

Influence charts of a size convenient for practical use are given in Plates 1, 2, and 3 for computing, respectively, vertical stress on horizontal planes, the sum of the principal stresses, and horizontal stress on vertical planes, the latter two charts being constructed for a value of Poisson's ratio of 0.5.

Influence charts are also given to a smaller scale for computing the components of shearing stress on horizontal and vertical planes, and for computing the corrections to the various stresses when Poisson's ratio is different from 0.5. Numerical data and instructions are given in the Appendix which permit drawing any of the charts to whatever scale is desired in particular applications. For all the charts, the influence value of the individual blocks is 0.001.

The calculation of maximum shearing stress at a point is discussed in Chapter IV. It is shown that the so-called octahedral shear, which in most materials is a more significant quantity than the maximum shear, and which lies between the limits of 81.6 and 94.3 per cent of the maximum shear, may be more easily computed than the maximum shear, from the six components of the stresses on horizontal and vertical planes.

The use of the influence charts is simple and rapid, and the accuracy of the calculations is sufficient for all practical purposes. After a few trials one can almost guess at accurate enough values for the stresses from a rough sketch of the loaded area.

This bulletin is not concerned with such questions as why stresses should be computed or what should be done with the stresses after the calculations are made. It is hoped that use of the procedure described herein will enable time to be saved in making calculations that are at present made by more laborious means.

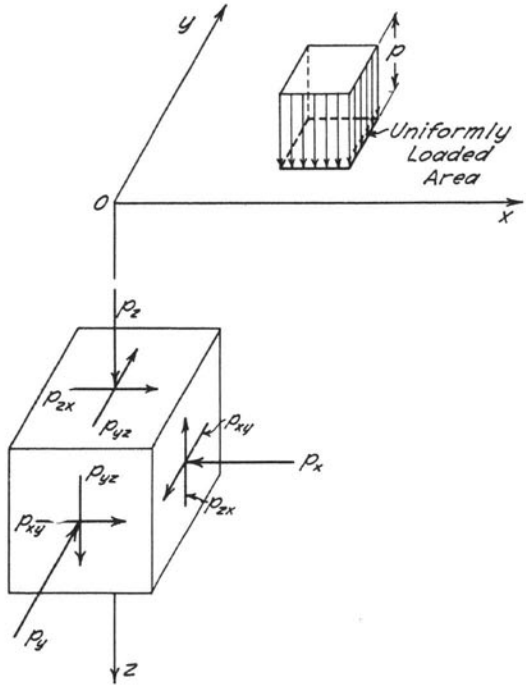

FIG. 1. SIGN CONVENTION AND NOTATION FOR AXES AND STRESSES

2. *Notation and Sign Convention.*—A homogeneous, elastic, isotropic solid of infinite extent, bounded by a horizontal plane surface, and loaded vertically at the surface, is considered. The x and y axes are in the plane of the surface and the z axis is positive downward.

The notation for the stresses acting on vertical and horizontal planes, parallel to the coördinate planes, is indicated in Fig. 1. It will be noted that the ordinary convention for positive stresses used in the theory of elasticity is reversed here in order that pressures or compressions may be positive.

The intensity of load, or the load per unit of area, is denoted by p. The stresses are denoted as follows:

p_z is the normal stress on horizontal planes positive for compression,

p_x and p_y are the normal stresses on vertical planes parallel to the yz and the zx planes, respectively, positive for compression.

p_{yz} and p_{zx} are horizontal shearing stresses on horizontal planes and also vertical shearing stresses on vertical planes, positive as shown in Fig. 1.

p_{xy} is the horizontal shearing stress on vertical planes, positive as shown in Fig. 1.

The following additional notation is used:

$p_{vol} = p_x + p_y + p_z$ = sum of principal stresses.

μ = Poisson's ratio of lateral contraction for the material.

r, α, β = quantities defined in Fig. 8.

$\left.\begin{matrix} a, b, c, d, e, f \\ A, B, C, D, E, F \end{matrix}\right\}$ = quantities defined in Equations (16) and (17).

3. *Acknowledgment.*—The investigation reported herein was conducted as part of the work of the Engineering Experiment Station of the University of Illinois, of which DEAN M. L. ENGER is the head. The calculations were performed by Mr. HAROLD CRATE, student in Civil Engineering.

A chart for computing vertical pressures, similar to that contained herein, was described by the writer in *Transactions*, Am. Soc. C. E., Vol. 103, 1938, p. 321-324, and in *Engineering News-Record,* Vol. 120, 1938, p. 23-24. A condensed description of the process of constructing such charts was also given by the writer in a paper entitled "Stress Distribution in Soils," Proceedings of Conference on Soil Mechanics and Its Applications, Purdue University, September, 1940, p. 295-303. However, the material contained in this bulletin is more complete, and the charts more convenient to use than those in the previous publications.

II. COMPUTATION OF VERTICAL STRESS ON HORIZONTAL PLANES

4. *Use of Influence Chart.*—In Fig. 2 is shown a rough chart for computing vertical stress. The chart represents a plan of the surface of the elastic body drawn to such a scale that the length marked OQ is the depth z at which the pressure is to be computed. The chart is constructed of arcs of concentric circle and radial lines drawn in such a way that each element of area bounded by two adjacent radii and two adjacent arcs contributes the same influence to the stress.

The influence of each element of area is 0.02 times the load on the element. Since there are 10 sectors in the diagram the influence values of the successive circles must be 0.20, 0.40, 0.60, and 0.80. The relative radii of the circles, in terms of the depth z from O to Q, are respectively, 0.401, 0.637, 0.918, and 1.387. The arc for an influence of 1.0 will have an infinite radius.

A more accurate chart in which the value of each elementary area is 0.001 is given in Plate 1. This chart is constructed in much the same

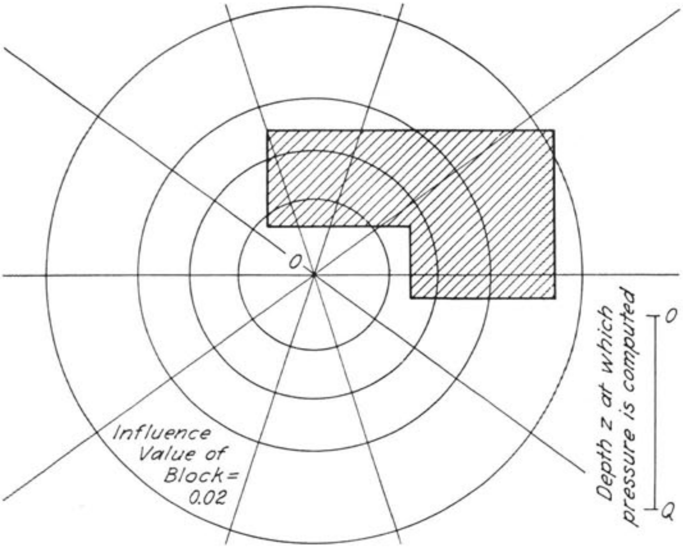

FIG. 2. SKETCH OF INFLUENCE CHART FOR VERTICAL STRESSES ON HORIZONTAL PLANES

manner as the chart in Fig. 2, but the areas are so chosen that they are of convenient shape with the x and y axes both axes of symmetry. The chart can be duplicated or drawn to other scales with the numerical data given in Tables 1 and 2 in the Appendix. The manner of using the chart is best illustrated by reference to Figs. 2 and 3.

Suppose the stress is desired at point Q' at a depth $z' = 80$ ft. under point O' in Fig. 3, where the area outlined is uniformly loaded with a load of 5 000 lb. per sq. ft. The plan of the loaded area is redrawn to such a scale that the distance $O'Q'$ in Fig. 3 becomes the same as the distance OQ in Fig. 2. Then the drawing is placed on Fig. 2 in such a way that point O' falls on point O. The number of blocks covered by the loaded area, multiplied by the magnitude of the load per unit of area and by the influence value of each block, gives the stress required. In the illustration, approximately 8 blocks are loaded, giving a stress of 0.16 times the intensity of load, or 800 lb. per sq. ft. Plate 1 is used in the same way, except that the influence value of each block is 0.001.

It may be noted that the stresses under similar loaded areas, at depths proportional to the sizes of the areas, are equal. For a given loading, different drawings of the plan of the loaded area are required to compute stresses at various depths.

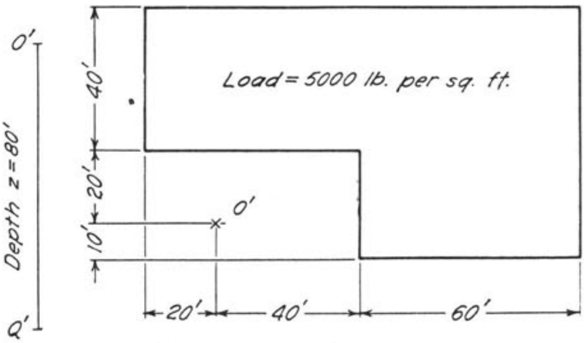

FIG. 3. PLAN OF LOADED AREA

The chart for vertical stress on horizontal planes is radially symmetrical. Therefore the loading plan may be rotated through any angle about a vertical axis through the point where the vertical stress is computed without changing the magnitude of the stress. Consequently the loading plan may be placed on the chart in the manner most convenient for the particular problem.

If the area is not uniformly loaded the chart may still be used, provided that each influence area is considered to be loaded by the average intensity of load on the particular block.

In using Plate 1, parts of blocks may be estimated when counting the blocks, with sufficient accuracy for all practical purposes. In counting the number of blocks when the number is large, the total amount of influence within a given region may be recorded on the chart where it can be taken into account conveniently. In general, the loaded area will be drawn on tracing paper and laid upon the chart so that blocks may be counted through the tracing.

III. COMPUTATION OF VARIOUS COMPONENTS OF STRESS AT A POINT

5. *General Considerations.*—The stresses on any plane at a particular point can be stated in terms of the six components of stress on an elementary cube, such as is indicated in Fig. 1. Of the six components of stress, three are normal stresses and three are shearing stresses. The normal and shearing stresses on any oblique plane can be determined from a consideration of the equilibrium of an infinitesimal pyramid of material. The largest and smallest normal stresses at a point occur on mutually perpendicular planes, called principal planes, on which there is no shearing stress, and these stresses are

called principal stresses. There is a third principal stress of intermediate magnitude on a third principal plane perpendicular to the other principal planes, on which there is no shearing stress either. The greatest shearing stress on any plane is found on the plane that makes angles of 45 degrees with the directions of greatest and least principal stresses, and that is perpendicular to the plane on which the intermediate principal stress acts. The magnitude of the maximum shearing stress is numerically equal to one-half the algebraic difference between the largest and the smallest principal stress.

The sum of three mutually perpendicular stresses at a point is a constant, and is therefore equal to the sum of the principal stresses at the point. In this bulletin this sum is denoted by the symbol p_{vol} since the sum of the principal stresses is associated with the change in volume of an elastic material.

6. *Sum of Principal Stresses.*—An influence chart for computing the sum of the principal stresses, p_{vol}, is given in Plate 2 for a value of Poisson's ratio of one-half. This chart is similar to that for the computation of vertical stress on horizontal planes, and is also radially symmetrical. Each elementary area or block on the chart has an influence value of 0.001. To compute p_{vol} for other values of Poisson's ratio than 0.5, one merely subtracts a correction equal to $\frac{1 - 2\mu}{3}$ times the value computed from Plate 2.

Since

$$p_{vol} = p_x + p_y + p_z, \tag{1}$$

and since p_z can be computed from Plate 1, one can determine the sum of p_x and p_y, the sum of horizontal stresses on two mutually perpendicular vertical planes, from the relation

$$p_x + p_y = p_{vol} - p_z. \tag{2}$$

In certain calculations this sum can be used without it being necessary to compute p_x and p_y individually. This would be the case if one wished to compute vertical strain at a particular point, taking into account the effect of Poisson's ratio. The vertical strain is given by the relation

$$\frac{1}{E_m}\left[p_z - \mu\left(p_x + p_y\right)\right] \tag{3}$$

or, in terms of p_z and p_{vol},

$$\frac{1}{E_m}\left[(1+\mu)\,p_z - \mu p_{vol}\right] \qquad (4)$$

where E_m is the modulus of elasticity of the material.

7. *Horizontal Stress on Vertical Planes.*—The horizontal stress p_x on the vertical plane containing the y axis can be computed for a value of Poisson's ratio of one-half from the influence chart given in Plate 3. It will be noted that the elementary influence areas are not of the same size in a given circle arc; hence, the position of the load must be determined relative to the x and y axes. Each block has an influence value of 0.001. Except for the fact that the chart is not radially symmetrical, Plate 3 is used in the same way as are Plates 1 and 2.

To determine p_x for a value of Poisson's ratio different from 0.5, one subtracts two corrections from the value of p_x for $\mu = 0.5$. One correction is $\dfrac{1-2\mu}{6}$ times the value of p_{vol} for $\mu = 0.5$ determined from Plate 2 and the other correction is $(1 - 2\mu)$ times the quantity obtained from a chart shown reduced in size in Fig. 4. For convenience in actual use such a chart should be drawn to the same scale as Plates 1, 2, and 3. This can be done by those who wish to do so with the numerical values of E and e given in Tables 1 and 2 of the Appendix.

In Fig. 4 each elementary block has an influence value numerically equal to 0.001, but the sign of the influence is positive for points closer to the x axis than to the y axis, and is negative for points closer to the y axis than to the x axis. Thus, for a quadrant of the surface bounded by the positive axes, the net amount of this correction is zero since there is as much negative influence as positive influence.

For certain loaded areas, extending to infinity in a sector, the correction becomes infinitely large. This means that p_x is infinite under certain conditions when μ is different from 0.5. This phenomenon has been noted by Love.* Of course, this means that the material will not remain elastic.

The same charts that are used to compute p_x can also be used to compute p_y by interchanging the x and y axes. More conveniently, p_y

*A. E. H. Love, "The Stress Produced in a Semi-Infinite Solid by Pressure on Part of the Boundary," Phil. Trans. Royal Society, London, Series A, Vol. 228, 1929, p. 377-420.

can be computed from the values of p_{vol}, p_z and p_x when these are known.

It should be noted that the vertical plane on which the stresses p_x act is part of the material and undergoes deformations. It is evident that when loads are applied symmetrically with respect to the y axis, there will be no deflection in the x direction of the vertical plane through the y axis. However there will be deformations in the vertical direction, and in the direction of y axis. Consequently, the stress acting on a perfectly smooth, or frictionless, rigid vertical plane can be computed for a given load by considering a fictitious additional load placed in a symmetrical position. However, this merely amounts to doubling the stress p_x due to the given load.

8. *Shearing Stresses on Horizontal and Vertical Planes.*—Influence charts for computing shearing stresses are shown reduced in size in Figs. 5, 6, and 7. These can be drawn to the same scale as Plates 1, 2, and 3 by those who will have opportunity to use them, with the data given in Tables 1 and 2 of the Appendix.

An influence chart for computing p_{zx} is shown in Fig. 5, constructed from values of B and b from Tables 1 and 2. Each elementary block has an influence value numerically equal to 0.001, but the sign of the influence is negative when x is positive, and positive when x is negative.

The value of p_{yz} can be computed from the same chart if the loaded area is rotated through an angle of 90 degrees clockwise about a vertical axis through O.

An influence chart for computing p_{xy} for a Poisson's ratio of 0.5 is shown in Fig. 6, constructed with values of F and f from Tables 1 and 2. An influence chart for computing the correction to be subtracted from the value of p_{xy} obtained from Fig. 6, when Poisson's ratio is different from 0.5, is shown in Fig. 7, constructed with values of E and f from Tables 1 and 2. The correction is $(1 - 2\,\mu)$ times the quantity obtained from Fig. 7.

In Figs. 6 and 7, each elementary block has an influence value numerically equal to 0.001, but the sign of the influence is positive when x and y have the same sign, and negative when x and y are of opposite sign.

It may be noted that Fig. 7 is identical with Fig. 4 rotated through an angle of 45 degrees counterclockwise.

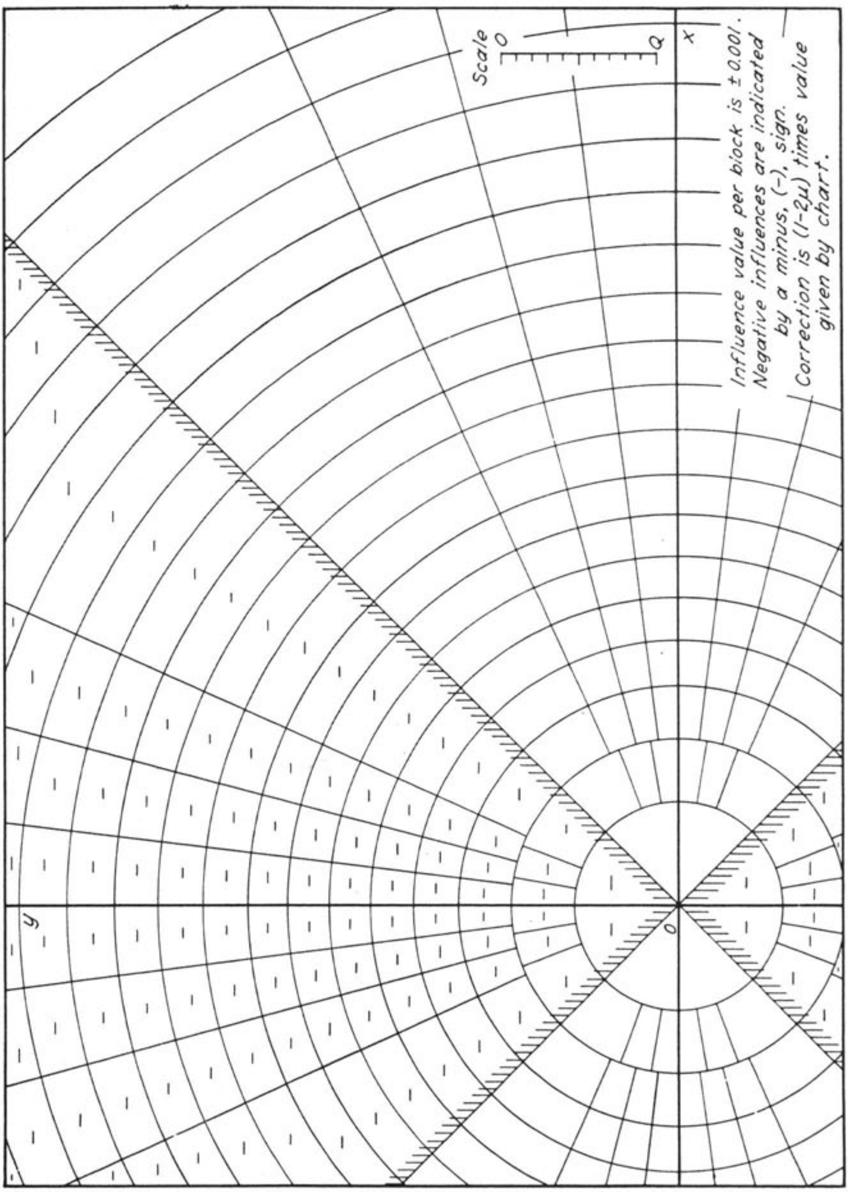

FIG. 4. INFLUENCE CHART FOR PART OF CORRECTION TO p_z WHEN POISSON'S RATIO IS DIFFERENT FROM 0.5

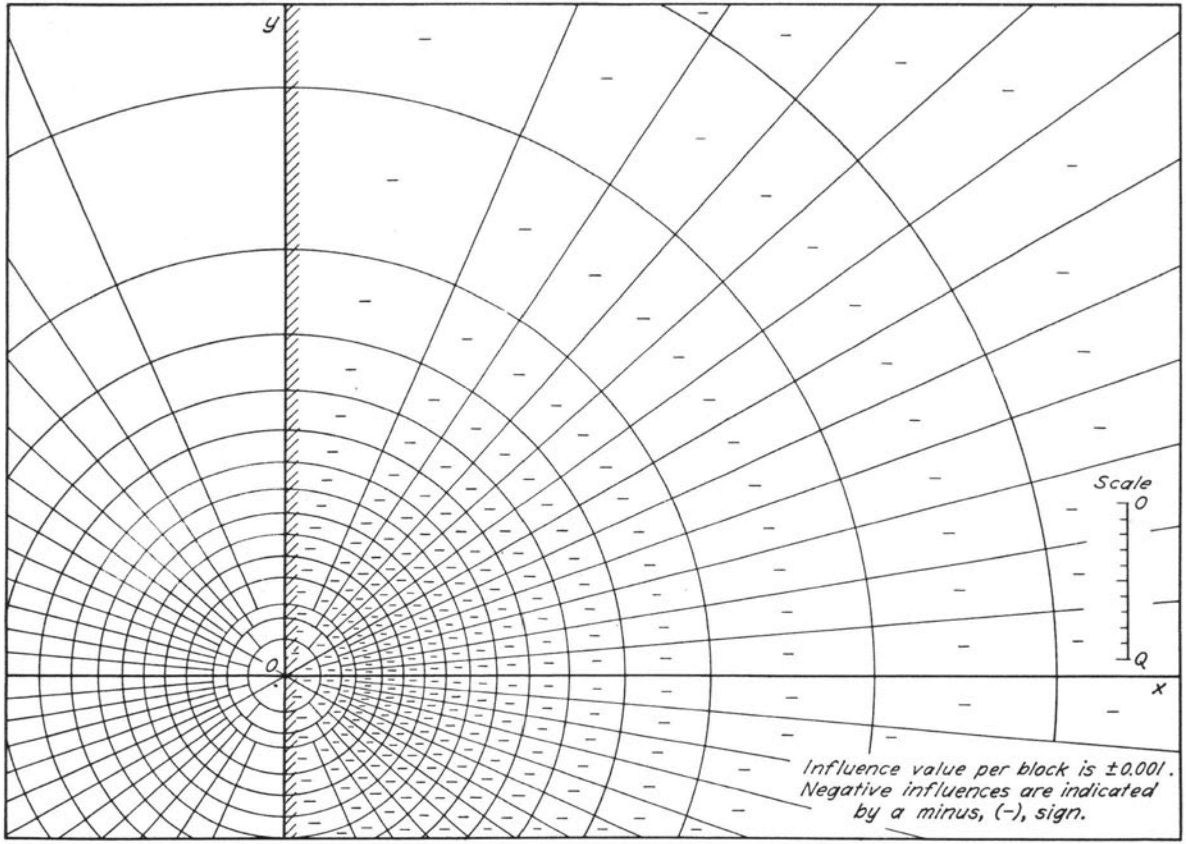

FIG. 5. INFLUENCE CHART FOR p_{zx}, HORIZONTAL SHEARING STRESS ON HORIZONTAL PLANES OR VERTICAL SHEARING STRESS ON VERTICAL PLANES

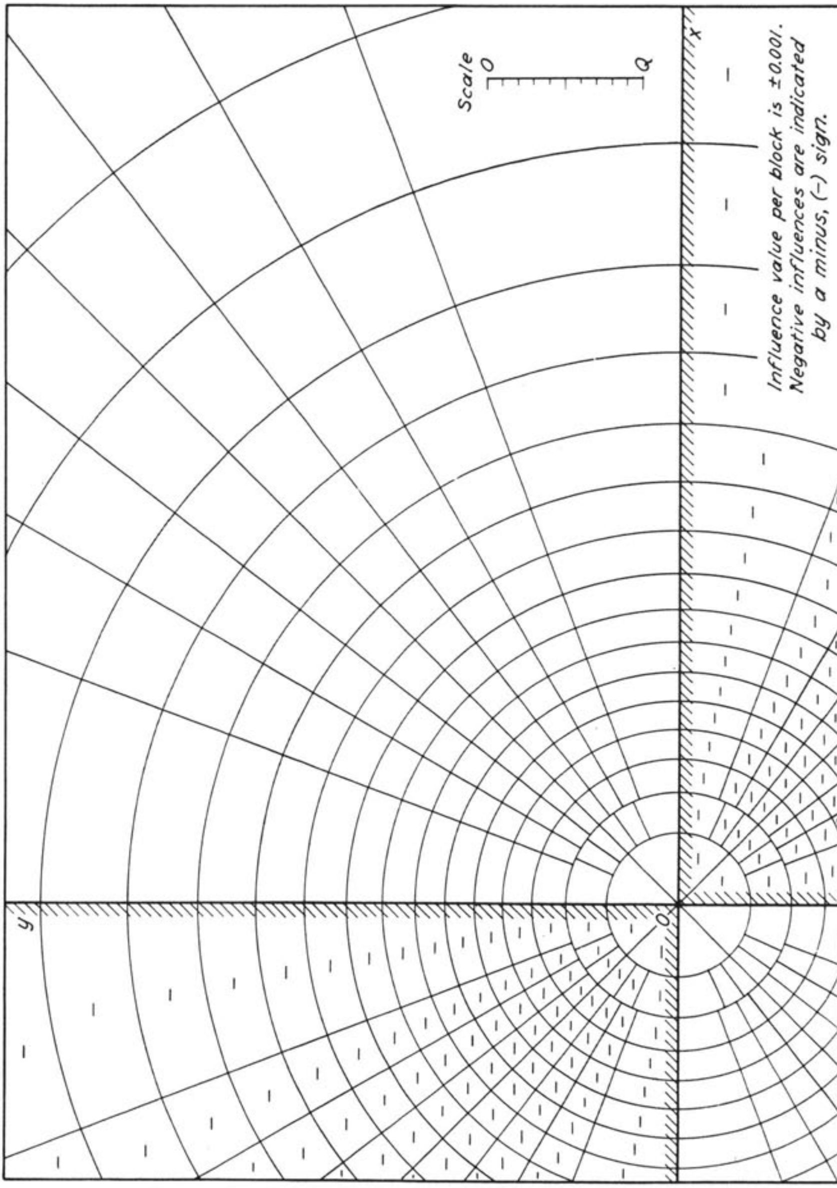

FIG. 6. INFLUENCE CHART FOR p_{zy}, HORIZONTAL SHEARING STRESS ON VERTICAL PLANES, FOR POISSON'S RATIO = 0.5

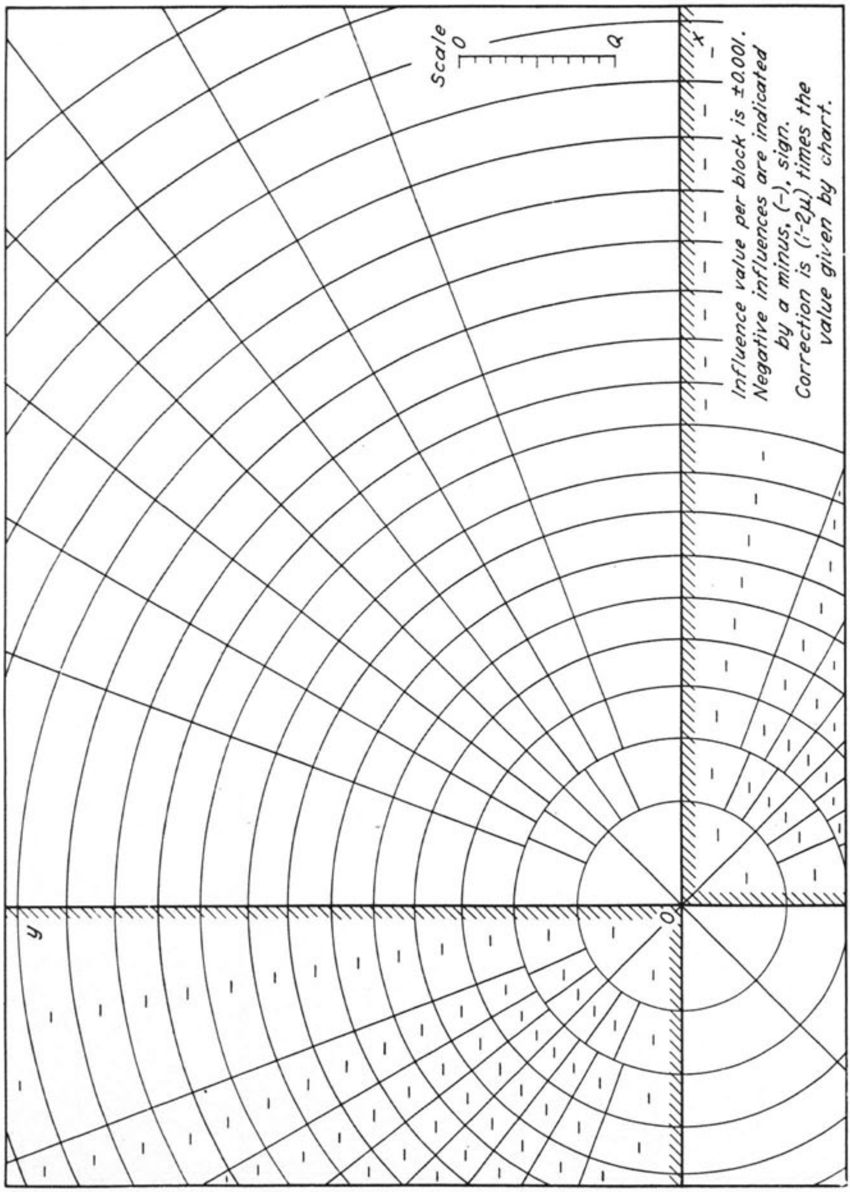

FIG. 7. INFLUENCE CHART FOR CORRECTION TO p_{xy} WHEN POISSON'S RATIO IS DIFFERENT FROM 0.5

IV. Determination of Maximum Shear

9. *Principal Stresses, Maximum Shear, and Octahedral Shear.*— With the components of stress p_x, p_y, p_z, p_{xy}, p_{yz}, and p_{zx}, known at a point the stresses on any plane through the point can be determined. Let l, m, and n be the direction cosines, with respect to the coordinate axes, x, y, and z, respectively, of the normal to the plane on which stresses are to be computed. Then the normal stress p_u on the plane is as follows:*

$$p_u = l^2 p_x + m^2 p_y + n^2 p_z + 2lmp_{xy} + 2mnp_{yz} + 2nlp_{zx}. \tag{5}$$

The maximum shearing stress p_{uv} on the oblique plane is obtained from p_u and the resultant stress S_u on the oblique plane by the relation:

$$\left.\begin{aligned} p_u^2 + p_{uv}^2 = S_u^2 = {} & (lp_x + mp_{xy} + np_{zx})^2 \\ & + (lp_{xy} + mp_y + np_{yz})^2 \\ & + (lp_{zx} + mp_{yz} + np_z)^2 \end{aligned}\right\}. \tag{6}$$

In order that p_u be a principal stress S, p_{uv} must be zero. It can be shown that this condition requires the following relations:

$$\left.\begin{aligned} lS &= lp_x + mp_{xy} + np_{zx} \\ mS &= lp_{xy} + mp_y + np_{yz} \\ nS &= lp_{zx} + mp_{yz} + np_z \end{aligned}\right\}. \tag{7}$$

To find the principal planes these equations must be solved for l, m, and n, or rather for their ratios, since the equations are homogeneous. Then with the relation between the direction cosines of any line,

$$l^2 + m^2 + n^2 = 1, \tag{8}$$

one can obtain the direction cosines of the principal planes, when S is known.

In order that Equations (7) may have a solution for l, m, and n

*For a reference to derivation of the following equations, see, for example, S. Timoshenko, "Theory of Elasticity," McGraw-Hill, New York, 1934, p. 182-188.

different from zero the determinant of the coefficients must vanish. This condition leads to a cubic equation in S, which is:

$$S^3 - (p_x + p_y + p_z)S^2 + (p_x p_y + p_y p_z + p_z p_x - p^2_{xy} - p^2_{yz} - p^2_{zx})S$$
$$- (p_x p_y p_z + 2p_{xy} p_{yz} p_{zx} - p_x p^2_{yz} - p_y p^2_{zx} - p_z p^2_{xy}) = 0 \quad (9)$$

where each of the terms in parentheses is an invariant with respect to directions of the axes at a point.

Equation (9) will always have three real roots which will be the three principal stresses. Let S_{max} and S_{min} denote the algebraically largest and smallest principal stresses. Then the maximum shear τ_{max} is numerically equal to the quantity

$$\tau_{max} = \frac{1}{2}(S_{max} - S_{min}). \quad (10)$$

It will be seen that the determination of the maximum shear requires the determination of the principal stresses, which is a tedious matter involving the solution of a cubic equation.

Part of the difficulty may be avoided by computing a quantity called the octahedral shear,* $\tau_{oct.}$, which differs only slightly from the maximum shear, and is generally a more significant quantity in determining the behavior of materials as they approach plastic action.

The octahedral shear is defined as the shearing stress on an oblique plane the normal to which makes equal angles with the three directions of principal stress at a point. The value of $\tau_{oct.}$ is proportional to the value of the so-called "shear-strain energy" or to the "energy of distortion" of a unit volume of the material. In terms of the principal stresses S_1, S_2, and S_3, the value of $\tau_{oct.}$ is as follows:

$$\tau_{oct.} = \frac{1}{3}\sqrt{(S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2}. \quad (11)$$

However, the octahedral shear can also be stated in terms of the 6 components of stress on planes parallel to the coordinate planes, as follows:

$$\tau_{oct.} = \frac{\sqrt{2}}{3}\sqrt{(S_1 + S_2 + S_3)^2 - 3(S_1 S_2 + S_2 S_3 + S_3 S_1)} \quad (12)$$

*See A. Nadai, "Theories of Strength," Journal of Applied Mechanics, Vol. 1, No. 3, 1933, p. 111-129.

or

$$\tau_{\text{oct.}}=\frac{\sqrt{2}}{3}\sqrt{(p_x+p_y+p_z)^2-3(p_xp_y+p_yp_z+p_zp_x-p^2_{xy}-p^2_{yz}-p^2_{zx})}. \qquad (13)$$

Therefore, the octahedral shear can be computed from the stresses obtained from the influence charts without first determining the principal stresses.

It is possible to show from Equation (11) that the value of $\tau_{\text{oct.}}$ is related to the maximum shear, $\tau_{\max}$, as follows:

$$0.8165\tau_{\max} \leq \tau_{\text{oct.}} \leq 0.9428\tau_{\max}. \qquad (14)$$

That is, the octahedral shear always lies between the limits of 81.6 and 94.3 per cent of the maximum shear, and it is much simpler to compute.

Illustrative Problem

To illustrate the computation of shearing stress, consider the stresses at a depth of 20 ft. beneath the vertex of a uniformly loaded sector of a circle of radius 100 ft., where the central angle of the sector is 45 degrees. One side of the sector lies along the x axis, and the other side along the line $x = y$. Let the load on the area be 1000 lb. per sq. ft. The value of Poisson's ratio is 0.5.

The following stresses are obtained from the formulas in the Appendix, and may be checked by use of the influence charts.

$p_z =$	124	lb. per sq. ft.
$p_{\text{vol}} =$	301	" " " "
$p_x =$	145	" " " "
$p_y =$	32	" " " "
$p_{zx} =$	-106	" " " "
$p_{yz} =$	-44	" " " "
$p_{xy} =$	56	" " " "

The principal stresses are determined from the following equation obtained by substituting the foregoing numerical values in Equation (9).

$$S^3 - 301\,S^2 + 10\,280\,S - 68\,592 = 0.$$

The magnitude of the principal stresses can be obtained by solving this equation by trial, or by any of the methods available for the solution of a cubic equation. The results are, to the nearest unit,

$$S_1 = 263 \text{ lb. per sq. ft.}$$
$$S_2 = 29 \text{ `` `` `` ``}$$
$$S_3 = 9 \text{ `` `` `` ``}$$

Therefore the maximum shear is

$$\tau_{\max} = \tfrac{1}{2}(263 - 9) = 127 \text{ lb. per sq. ft.}$$

The octahedral shear is determined from Equation (13) and is as follows:

$$\tau_{\text{oct.}} = 115 \text{ lb. per sq. ft.}$$

Appendix

Construction of Influence Charts

1. *Stresses Under Vertex of Uniformly Loaded Sector of a Circle.*—Formulas have been given by Boussinesq from which one can obtain the stresses at a point in the interior of a semi-infinite solid bounded by a plane surface, due to a concentrated load applied normal to the surface.* From Boussinesq's formulas one can readily obtain by integration the stresses at a depth z under the vertex of a sector of a circle located as in Fig. 8, uniformly loaded with an intensity of load p.

With an intensity of loading of unity, the following stresses are obtained at the point Q in Fig. 8:

$$\left.\begin{aligned}
p_z &= aA \\
p_{zx} &= -bB \\
p_{yz} &= -cB \\
p_{\text{vol}} &= p_x + p_y + p_z = aC - \frac{1-2\mu}{3}aC \\
p_x &= dD - \frac{1-2\mu}{6}aC - (1-2\mu)\,eE \\
p_{xy} &= fF - (1-2\mu)\,fE
\end{aligned}\right\} \qquad (15)$$

*See, for example, N. M. Newmark, "Stress Distribution in Soils," Proceedings of Conference on Soil Mechanics and its Applications; Purdue University, September 1940, p. 295-303, especially Equation (1).

where

$$\left.\begin{aligned}
a &= \frac{\beta}{2\pi} \\
b &= \sin\beta \\
c &= 1 - \cos\beta \\
d &= \frac{2\beta + \sin 2\beta}{4\pi} \\
e &= \frac{\sin 2\beta}{2} \\
f &= \frac{1 - \cos 2\beta}{2}
\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned}
A &= 1 - \cos^3\alpha \\
B &= \frac{1}{2\pi}\sin^3\alpha \\
C &= 3\,(1 - \cos\alpha) \\
D &= 1 - \frac{3}{2}\cos\alpha + \frac{1}{2}\cos^3\alpha \\
E &= \frac{1}{4\pi}\left(\log_e \frac{1 + \cos\alpha}{2\cos\alpha} + \cos\alpha - 1\right) \\
F &= \frac{1}{4\pi}\,(2 - 3\cos\alpha + \cos^3\alpha)
\end{aligned}\right\}. \tag{17}$$

It will be noted that for $\mu = \frac{1}{2}$, the value of $(1 - 2\mu)$ is zero. For any value of μ one has the results:

$$\left.\begin{aligned}
p_{\mathrm{vol}} &= [p_{\mathrm{vol}}]_{\mu=1/2} - \frac{1 - 2\mu}{3}\,[p_{\mathrm{vol}}]_{\mu=1/2} \\
p_x &= [p_x]_{\mu=1/2} - \frac{1 - 2\mu}{6}\,[p_{\mathrm{vol}}]_{\mu=1/2} - (1 - 2\mu)\,eE \\
p_{xy} &= [p_{xy}]_{\mu=1/2} - (1 - 2\mu)\,fE
\end{aligned}\right\}. \tag{18}$$

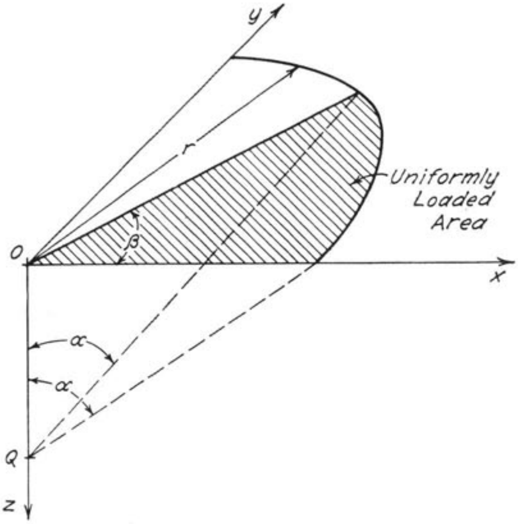

FIG. 8. UNIFORMLY LOADED SECTOR OF A CIRCLE

2. *Numerical Data for Construction of Charts.*—The formulas in the preceding section give stresses at a depth z under the vertex of a uniformly loaded sector of a circle of radius r and central angle β. By combining numerical values of stresses for different sectors one can obtain the stress at Q due to uniform load on an element of area bounded by two radial lines and by two concentric circle arcs. The stress at Q due to such a loading may be interpreted as an influence coefficient. Different areas will in general produce different influences, but values of β and r can be so chosen that elements of area are defined which produce the same influence.

By proper choice of the values of $a, b, \ldots. A, B, \ldots. F$ one can find the values of β and α, or preferably, of β and $\tan \alpha = r/z$, which give, when combined, influences for stresses in steps of uniform amount. In Tables 1 and 2 values of β and r/z are reported, which, when combined, correspond to elementary influence areas of value 0.001. The numerical values are so chosen that the values of β may be laid out with the x and y axes always axes of symmetry for the influence diagrams. By referring to the charts in this bulletin, the manner of using the values of β and r/z to construct influence charts can be seen.

The chart for p_z involves a and A and is independent of μ. Since the values of a correspond to equal divisions of a circle the chart will

TABLE 1

VALUES OF r/z CORRESPONDING TO GIVEN VALUES OF A, B, C, D, E, AND F IN FORMULAS FOR STRESSES, EQUATION (15)

A	Increment in A	r/z
0	0	0
0.008	0.008	0.073
0.024	0.016	0.128
0.048	0.024	0.183
0.072	0.024	0.226
0.096	0.024	0.264
0.144	0.048	0.330
0.192	0.048	0.391
0.240	0.048	0.448
0.288	0.048	0.504
0.336	0.048	0.560
0.384	0.048	0.617
0.432	0.048	0.677
0.480	0.048	0.739
0.528	0.048	0.806
0.576	0.048	0.879
0.624	0.048	0.959
0.672	0.048	1.050
0.720	0.048	1.156
0.768	0.048	1.284
0.816	0.048	1.446
0.864	0.048	1.668
0.912	0.048	2.014
0.944	0.032	2.415
0.976	0.032	3.315
0.992	0.016	4.899
1.000	0.008	inf.

B	Increment in B	r/z
0	0	0
0.002	0.002	0.239
0.006	0.004	0.379
0.012	0.006	0.466
0.024	0.012	0.629
0.036	0.012	0.768
0.048	0.012	0.904
0.060	0.012	1.045
0.072	0.012	1.198
0.084	0.012	1.372
0.096	0.012	1.580
0.108	0.012	1.841
0.120	0.012	2.197
0.132	0.012	2.744
0.144	0.012	3.808
0.150	0.006	4.983
0.156	0.006	8.630
0.159 ($\frac{1}{2}\pi$)	0.003+	inf.

C	Increment in C	r/z
0	0	0
0.008	0.008	0.073
0.024	0.016	0.127
0.048	0.024	0.181
0.080	0.032	0.236
0.112	0.032	0.281
0.144	0.032	0.322
0.192	0.048	0.376
0.240	0.048	0.426
0.288	0.048	0.473
0.336	0.048	0.521
0.384	0.048	0.561
0.480	0.096	0.646
0.576	0.096	0.729
0.672	0.096	0.813
0.768	0.096	0.898
0.864	0.096	0.986
0.960	0.096	1.078
1.056	0.096	1.175
1.152	0.096	1.279
1.248	0.096	1.390
1.344	0.096	1.511
1.440	0.096	1.643
1.536	0.096	1.789
1.632	0.096	1.952
1.728	0.096	2.136
1.824	0.096	2.347
1.920	0.096	2.592
2.016	0.096	2.880
2.112	0.096	3.227
2.208	0.096	3.653
2.304	0.096	4.193
2.352	0.048	4.520
2.400	0.048	4.899
2.448	0.048	5.342
2.496	0.048	5.868
2.592	0.096	7.285
2.688	0.096	9.563
2.784	0.096	13.853
2.880	0.096	24.971
2.976	0.096	125.012
3.000	0.024	inf.

D	Increment in D	r/z
0	0	0
0.008	0.008	0.408
0.024	0.016	0.565
0.048	0.024	0.710
0.096	0.048	0.922
0.144	0.048	1.103
0.192	0.048	1.276
0.240	0.048	1.450
0.288	0.048	1.634
0.336	0.048	1.831
0.384	0.048	2.048
0.432	0.048	2.291
0.480	0.048	2.576
0.528	0.048	2.898
0.576	0.048	3.289
0.624	0.048	3.772
0.672	0.048	4.385
0.720	0.048	5.198
0.768	0.048	6.335
0.816	0.048	8.049
0.864	0.048	10.954
0.912	0.048	16.996
0.960	0.048	37.397
1.000	0.040	inf.

E	Increment in E	r/z
0	0	0
0.002	0.002	0.673
0.008	0.006	1.070
0.016	0.008	1.414
0.024	0.008	1.706
0.032	0.008	1.980
0.040	0.008	2.247
0.048	0.008	2.516
0.056	0.008	2.785
0.064	0.008	3.062
0.072	0.008	3.347
0.080	0.008	3.640
0.088	0.008	3.945
0.096	0.008	4.261
0.104	0.008	4.591
0.112	0.008	4.935
0.120	0.008	5.294
0.128	0.008	5.679
0.136	0.008	6.062
0.144	0.008	6.474
0.152	0.008	6.905
0.160	0.008	7.357
0.168	0.008	7.831
0.176	0.008	8.328
0.184	0.008	8.850
0.192	0.008	9.397
0.200	0.008	9.972
inf.	inf.	inf.

F	Increment in F	r/z
0	0	0
0.002	0.002	0.464
0.008	0.006	0.722
0.016	0.008	0.940
0.024	0.008	1.128
0.032	0.008	1.308
0.040	0.008	1.493
0.048	0.008	1.688
0.056	0.008	1.900
0.064	0.008	2.136
0.072	0.008	2.405
0.080	0.008	2.718
0.088	0.008	3.091
0.096	0.008	3.549
0.104	0.008	4.129
0.112	0.008	4.894
0.120	0.008	5.957
0.128	0.008	7.553
0.136	0.008	10.230
0.144	0.008	15.700
0.152	0.008	33.344
0.159 ($\frac{1}{2}\pi$)	0.007+	inf.

TABLE 2

VALUES OF β FOR GIVEN INCREMENTS IN VALUES OF a, b, c, d, e, AND f IN FORMULAS FOR STRESSES, EQUATION (15)

Increment in β deg.	Increment in a	β deg.	Increment in b	β deg.	Increment in c*	β deg.	Increment in d	β deg.	Increment in e	β deg.	Increment in e	β deg.	Increment in f†	β deg.	Increment in f†
3.75	1/96	0		0		0		0		0		0		0	
7.5	1/48	±4.78	1/12	23.56	1/12	±3.76	1/48	±7.24	1/18	±9.74	1/6	20.70	1/18	29.04	1/6
45.0	1/8	±9.59	1/12	33.56	1/12	±7.54	1/48	±15.00	1/18	±20.91	1/6	30.00	1/18	35.26	1/6
etc., uniform division		±14.48	1/12	41.41	1/12	±11.40	1/48	±24.30	1/18	±45.00	1/6	37.76	1/18	45.00	1/6
		±19.47	1/12	48.19	1/12	±15.36	1/48	±45.00	1/18	±69.09	−1/6	45.00	1/18	54.74	1/6
		±24.62	1/12	54.32	1/12	±19.48	1/48	±65.70	−1/18	±80.26	−1/6	52.24	1/18	65.91	1/6
		±30.00	1/12	60.00	1/12	±23.83	1/48	±75.00	−1/18	±90.00	−1/6	60.00	1/18	90.00	1/6
		±35.68	1/12	65.38	1/12	±28.48	1/48	±82.76	−1/18	±99.74	−1/6	69.30	1/18	114.09	−1/6
		±41.81	1/12	70.53	1/12	±33.59	1/48	±90.00	−1/18	±110.91	−1/6	90.00	1/18	125.26	−1/6
		±48.59	1/12	75.52	1/12	±39.40	1/48	±97.24	−1/18	±135.00	−1/6	110.70	−1/18	135.00	−1/6
		±56.44	1/12	80.41	1/12	±46.39	1/48	±105.00	−1/18	±159.09	1/6	120.00	−1/18	144.74	−1/6
		±66.44	1/12	85.22	1/12	±55.90	1/48	±114.30	−1/18	±170.26	1/6	127.76	−1/18	155.91	−1/6
		±90.00	1/12	90.00	1/12	±90.00	1/48	±135.00	−1/18	±180.00	1/6	135.00	−1/18	180.00	−1/6
		±113.56	−1/12	94.78	1/12	±124.10	1/48	±155.70	1/18			142.24	−1/18		
		±123.56	−1/12	99.59	1/12	±133.61	1/48	±165.00	1/18			150.00	−1/18		
		±131.41	−1/12	104.48	1/12	±140.60	1/48	±172.76	1/18			159.30	−1/18		
		±138.19	−1/12	109.47	1/12	±146.41	1/48	±180.00	1/18			180.00	−1/18		
		±144.32	−1/12	114.62	1/12	±151.52	1/48								
		±150.00	−1/12	120.00	1/12	±156.17	1/48								
		±155.38	−1/12	125.68	1/12	±160.52	1/48								
		±160.53	−1/12	131.81	1/12	±164.64	1/48								
		±165.52	−1/12	138.59	1/12	±168.60	1/48								
		±170.41	−1/12	146.44	1/12	±172.46	1/48								
		±175.22	−1/12	156.44	1/12	±176.24	1/48								
		±180.00	−1/12	180.00	1/12	±180.00	1/48								

*For corresponding values of β between 0 and −180, the increment in c is negative in sign.
†For corresponding values of β between 0 and −180, the increment in f is opposite in sign to that given.

be made up of circles and uniformly spaced radial lines. For convenience in counting it is desirable to make the elementary areas as nearly square as possible. For this reason different numbers of radial divisions are used in different parts of the diagram.

The same general comments apply also to p_{vol} which involves a and C. The value of p_{vol} for $\mu = \frac{1}{2}$ may be used also to obtain p_{vol} for other values of μ merely by subtracting a correction as indicated in the previous section.

The chart for p_{zx} involves b and B, and is independent of μ. The elementary influence areas are negative where x is positive and positive where x is negative. It may be observed that the same chart can be used for p_{zx} and p_{yz} provided that the x and y axes are interchanged; or also, to prevent redrawing of the loading diagram, if the negative y axis is taken as a new x axis and the x axis is taken as a new y axis.

Since p_x involves μ it is convenient to construct a chart for p_x when $\mu = \frac{1}{2}$, which involves d and D, and correction charts for the effect of μ. One correction is proportional to p_{vol} as indicated by Equation (18), and the other requires a separate chart involving e and E. In this chart the elementary influence areas are positive when y is numerically less than x, and negative otherwise.

The same charts can be used for p_x and p_y with proper change of axes, but since p_z and p_{vol} will usually be easier to determine, p_y can be obtained from the relation

$$p_y = p_{\text{vol}} - p_z - p_x.$$

The value of p_{xy} also depends on μ, and it is expedient to construct a chart for p_{xy} when $\mu = \frac{1}{2}$, which involves f and F, and a correction chart for the effect of μ. The correction involves f and E. For both charts the elementary influence areas are positive when x and y have the same sign, and are negative otherwise.

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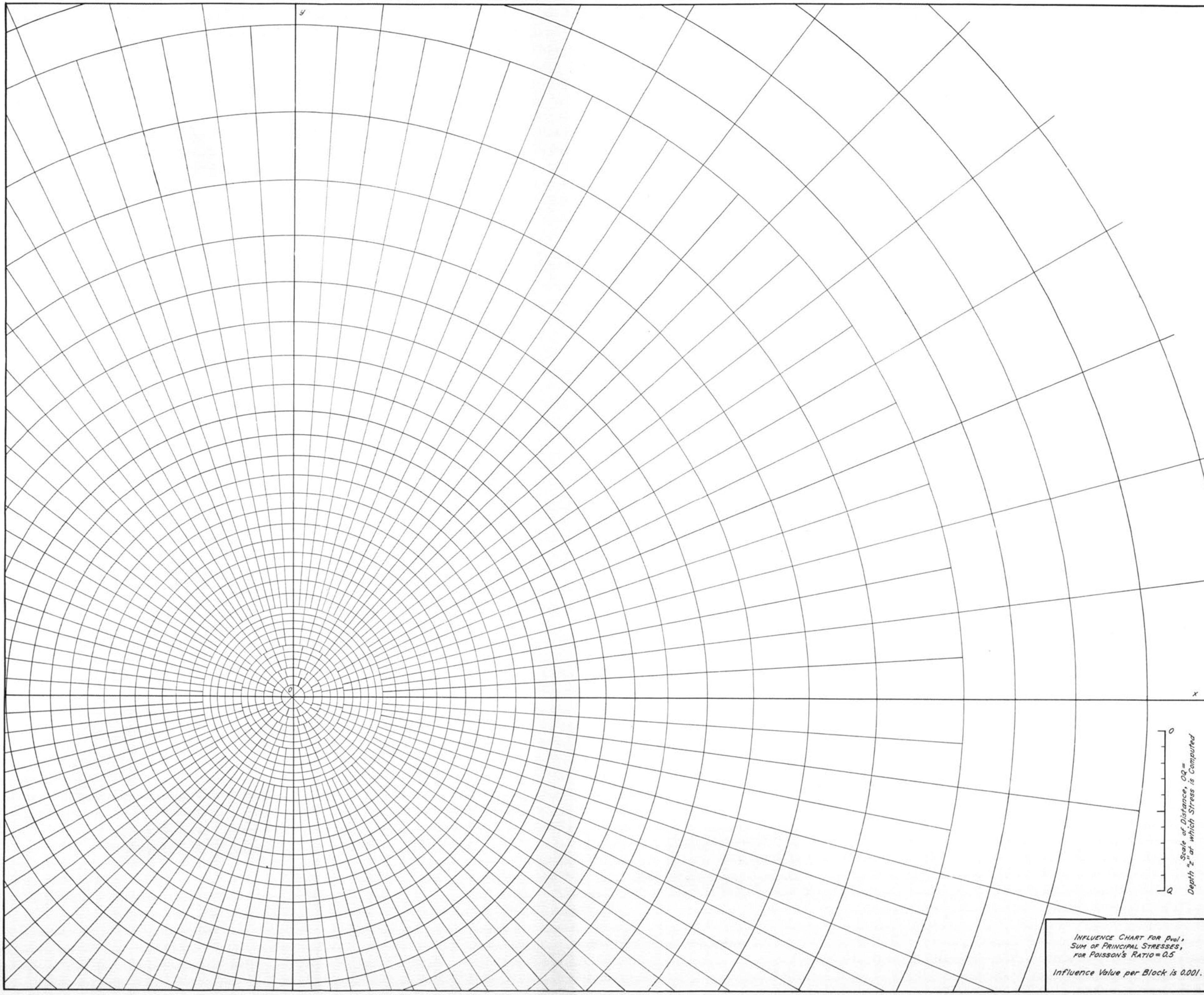

PLATE 2. INFLUENCE CHART FOR p_{vol}, SUM OF PRINCIPAL STRESSES, FOR POISSON'S RATIO = 0.5

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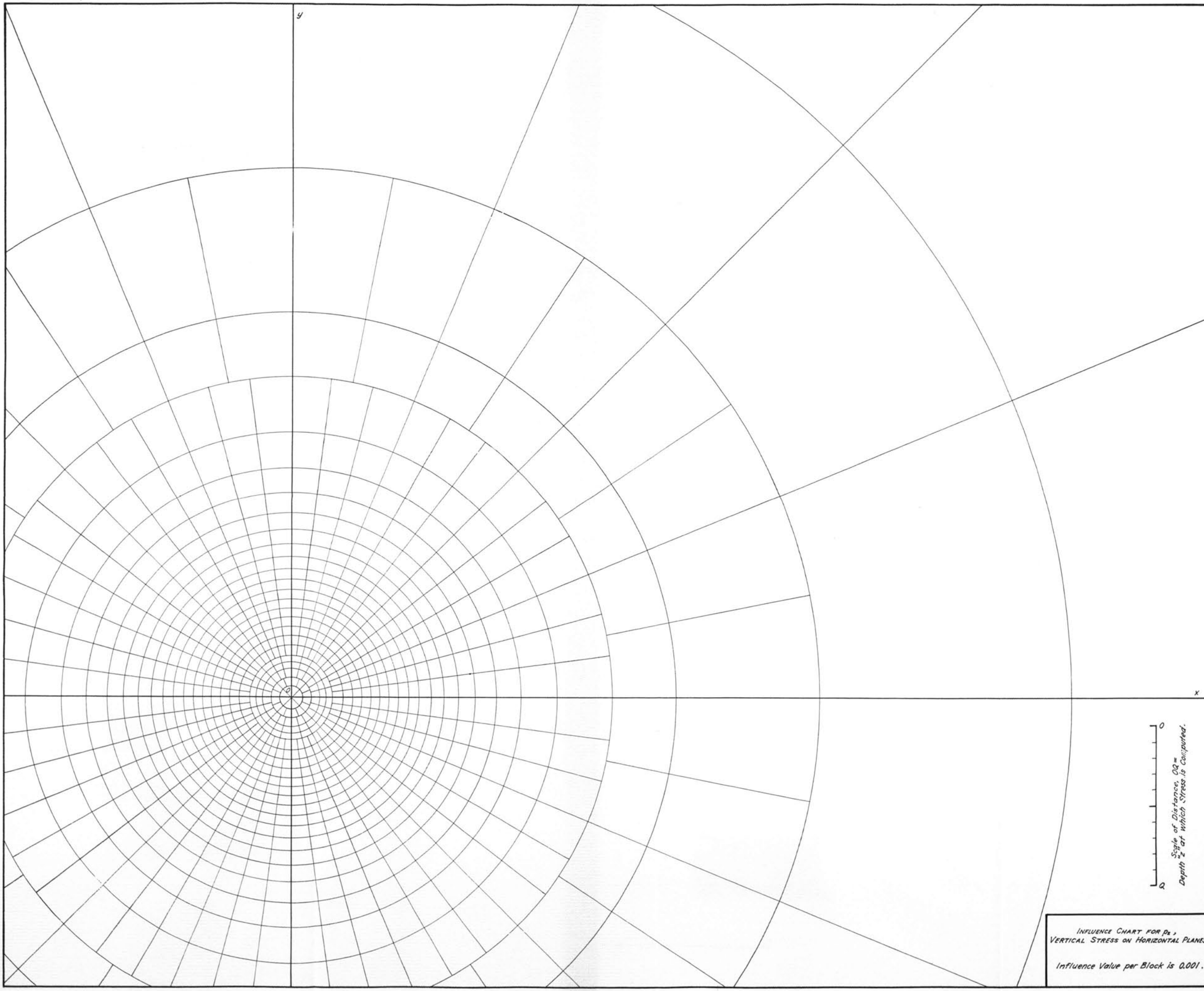

PLATE 1. INFLUENCE CHART FOR p_z, VERTICAL STRESS ON HORIZONTAL PLANES

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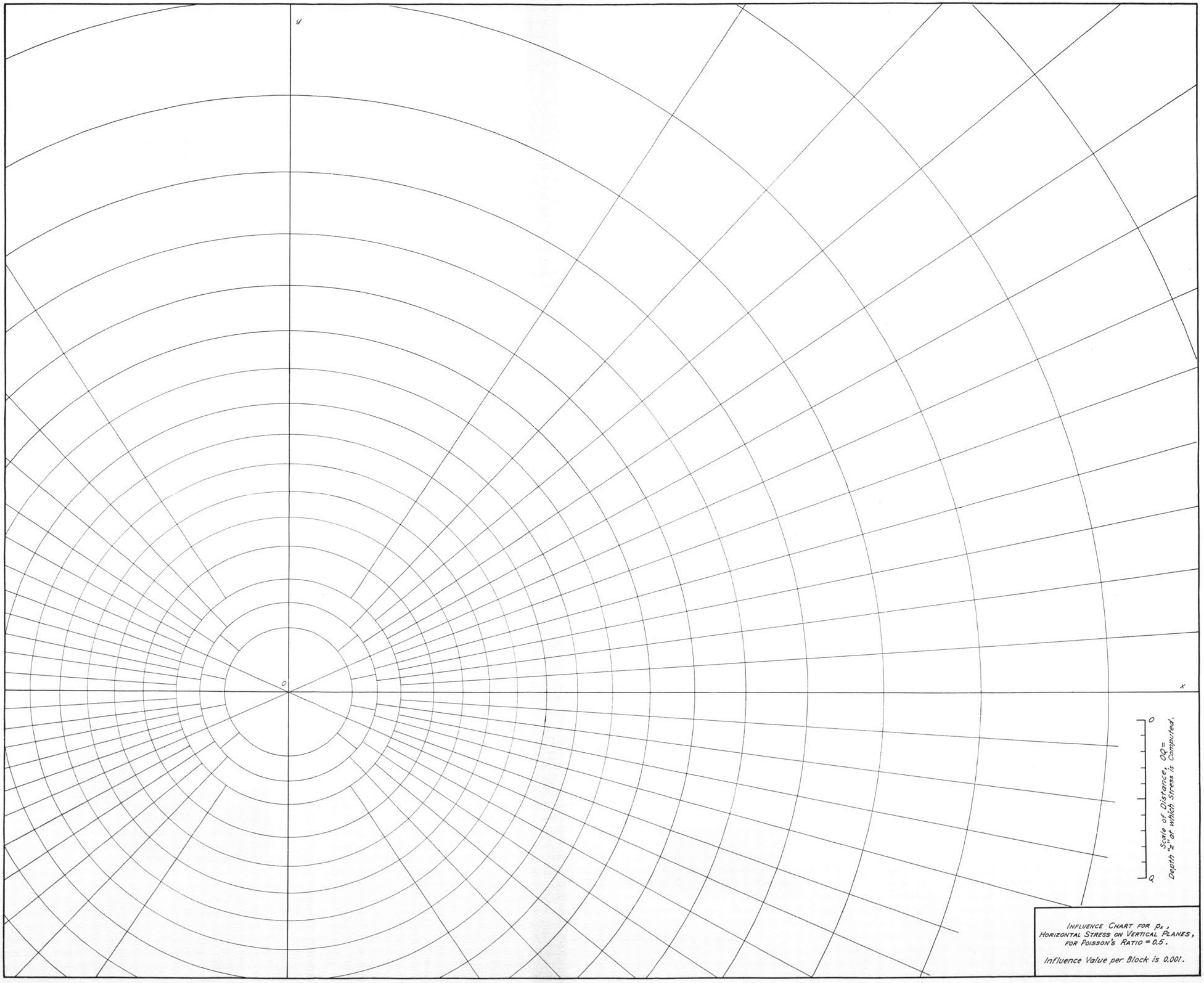

PLATE 3. INFLUENCE CHART FOR p_x, HORIZONTAL STRESS ON VERTICAL PLANES, FOR POISSON'S RATIO = 0.5

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