THE GREEN'S FUNCTION FOR A PLANE CONTOUR

BY

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INTRODUCTION

A Green's function is a function of a boundary and two
points. The points, of which one is considered fixed and one var-
iable, both lie inside (or outside) the boundary. In two dimen-
sions the boundary is a contour. A large part of mathematical phys-
ics is devoted to the solution of boundary problems in two dimen-
sions. All such problems can be solved immediately if the Green's
function for the given boundary or contour is known. The character
of the Green's function depends on three things, (1) the differen-
tial equation of which it is a solution, (2) the contour or bound-
ary to which it applies, and (3) the boundary conditions. The class
of Green's functions to which we shall confine ourselves in the fol-
lowing pages is the one of most frequent occurence in math-
ematical physics, namely, the Green's function for Laplace's differ-
ential equation, with what Hilbert calls boundary condition (I), that
is, $G = 0$ on the boundary. Having found this particular solution
G, we can solve the problem of Dirichlet in the plane, that is we
can determine a solution u satisfying any given condition of the
form $u = U(\theta)$ on the boundary. This we can do by making use of the
formula

$$u = 1/2\pi \int_C U \frac{\partial G}{\partial n} \, ds,$$

which in polar co-ordinates becomes

$$u(r,\theta) = 1/2\pi \int_0^{2\pi} U(\theta) \left[- r \frac{\partial G}{\partial r} \, d\theta + \frac{\partial G}{r \partial \theta} \, dr \right]$$

There are two well recognized methods of obtaining the Green's function in two dimensions, both of which however are very limited in their applications. These methods are the following.-

(1) The method of images. This method is easily applicable to the circle, semi-circle, infinite strip, half plane, equilateral triangle, 30 degree right-triangle, and the rectangle: but for other contours it would involve complicated Riemann surfaces and hyper-elliptic functions. Apparently no one has even attempted to apply the method to other polygons.

(2) The method of Schwarz. Schwarz obtained a general formula w = f(z), where f(z) is in the form of a definite integral, by means of which any given polygon can be mapped conformally on to the unit circle. Knowing this mapping function f(z) we can obtain the Green's function for the polygon immediately from the formula

G = - Re log f(z).   (Re = real part of)

By this method we are able to get the Green's function for the equilateral triangle, the 30 degree right-triangle, the 45 degree right-triangle, an infinite strip, a regular five pointed star, and a rectangle. Beyond this it is difficult to go. Other polygons lead to Abelian integrals and hyper-elliptic functions. Moreover, except for regular polygons, it seems to be impossible to determine the unknown constants which appear under the integral sign in the f(z). For regular polygons however we could evaluate Schwarz' integral ,

$$w = \int_0^z \frac{dz}{(1 - z^n)^{\frac{2}{n}}} \; ,$$

in the form of an infinite series, and take the real part of the logarithm.   In other words, the series obtained by expanding

$$G = - \operatorname{Re} \int_0^z \frac{dz}{(1 - z^n)^{\frac{2}{n}}}$$

gives us the value of the Green's function for a regular polygon of n sides, inscribed in a circle of radius

$$r_0 = \int_0^1 \frac{dz}{(1 - z^n)^{\frac{2}{n}}} \, ,$$

where the point of discontinuity is at the center of the polygon. This work of Schwarz' has been considerably discussed and amplified by other writers, particularly in a recent book by Study.[1]   However very little of fundamental importance has been added to Schwarz' original memoirs.

The method of arriving at the Green's function which we shall employ in the following pages is new, although in essence the same as employed by Fourier and Neuman in various physical problems. We select an infinite set of linearly independent functions which are solutions of the given differential equation, and seek to expand the Green's function, or rather the quantity $G - \gamma$, where γ is the principle solution (log 1/r in this case), in a series of these functions, determining the coefficients of the series so that the boundary conditions are satisfied.   The determination of these coefficients depends upon the solution of an infinite set of equations in an infinity of unknowns.   In the problems solved by Fourier the solution of this set of equations was very simple. In the problem before us the solution is much more involv-

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1) <u>Vorlesungen</u> <u>über</u> <u>Ausgewählte</u> <u>Gegenstände</u> <u>der</u> <u>Geometrie,</u> 2tes Heft

ed though entirely manageable in a large class of cases.

We have confined our attention to a single class of Green's functions, but the same procedure might be employed to get the Green's function for any differential equation with any boundary conditions.[2] It might also be extended to problems in three dimensions.

To summarize briefly:- The procedure which we employ in the following pages is much more widely applicable than either of the two ordinary methods described above. These two methods give workable results for, at most, the regular polygons, two other triangles, the five pointed star, and the rectangle. Our method is applicable, not only to all regular polygons, but to a large class of irregular polygons as well, and to many other contours. It gives the Green's function in the form, log 1/r plus an expansion in plane harmonics $r^n \cos n\theta$ and $r^n \sin n\theta$. The coefficients of this expansion are each given in the form of an infinite series of constants which can be determined to any desired degree of accuracy by taking a sufficient number of terms. We are convinced that the method is actually workable in a practical sense in a large number of cases.

We append here a list of a few of the articles and treatises which were consulted in the preparation of this dissertation.-

Bateman, <u>Inversion of the Definite Integral</u>, Proc. Lond. Math.
      Soc., Vol. 4 (1907), pp 90-115, and 461-498.

Bôcher, <u>Region of Convergence of Power Series which Represent
      Two-Dimensional Harmonic Functions</u>, Trans. Am. Math. Soc.,
      Vol. 10 (1909) pp 271-278.

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2) See Hadamard's Columbia lectures, pp47-52.

Carmichael, On Non-Homogeneous Equations with an Infinite Number
    of Variables, Am. J. of Math., Vol. 36 (1914) pp 13-20.

Goursat, Traité d'Analyse, Vol. III.

Hadamard, Kempton Adams Lectures on Mathematics at Columbia
    University, (1911), 51 pages.

Kneser, Integral Gleichungen.

Lery, La Fonction de Green pour une Contour Algébrique, Ann. Sci.
    de l'Ecole Normale, Vol. 51 (1915) pp 49-136.

Osgood, Lehrbuch der Functionen Theorie, 2nd Edition, Part IV.

Pell, Non-Homogeneous Equations in an Infinity of Unknowns, Ann.
    of Math., Vol. 28 (1914-15) pp 32-38.

Riemann, (Jules), Sur le Problème de Dirichlet, Ann. Sci. de
    l'Ecole Normale, Ser. 3, Vol. 5, (1888), pp 327-410.

Riesz, Les Systèmes d'Équations Linéaires à une Infinité d'In-
    connues, 180 pages.

Schwarz, Über einige Abbildungsaufgaben, Math. Abhandlungen,
    Vol. II, pp 65-83.

Study, Vorlesungen über Ausgewählte Gegenstände der Geometrie,
    Zweites Heft.

I. FORMULATION OF THE PROBLEM.

By a plane contour we mean any closed plane curve which is
regular in the sense of Osgood, that is, is composed of a finite
number of analytic arcs or straight segments.   An important case is
that of a polygon.

The Green's function for any such contour is given by the
following formula in polar co-ordinates, -

$$G(r, \theta) = - \log r + u(r, \theta) \quad , \tag{1}$$

where the function u is harmonic, that is, satisfies Laplace's equa-
tion,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad ,$$

and has no singularities in the region.

Consider this function $u(r, \theta)$.   We know that it is every-
where harmonic in the area S enclosed by
the contour C and has no singularities
in S.   Moreover, on the boundary,
for which $r = R(\theta)$ and $u = U(\theta)$,
we have G = 0 and

$$U(R, \theta) = \log R(\theta) \quad . \tag{2}$$

Draw any circle, radius r_0, about O
so that it lies wholly in S.

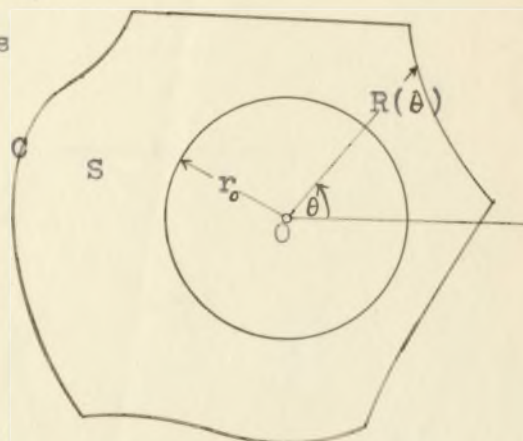By a well known theorem [3]

Figure 1.

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3) Osgood, p 656, Satz 1.

we can expand the harmonic function u uniquely in the series

$$u = A_o/2 + \sum_1^\infty r^n(A_n \cos n\theta + B_n \sin n\theta) \, , \qquad (3)$$

where A_n and B_n are the Fourier constants for the area enclosed by

the circle $0 \leqq \theta \leqq 2\pi, \; 0 \leqq r \leqq r_o$. The functions $r^n \cos n\theta$ and

$r^n \sin n\theta$ form a complete set of orthogonal functions for the circu-

lar boundary. Let us write for convenience

$$u = c_o + c_1 r \cos\theta + c_3 r^2 \cos 2\theta + c_5 r^3 \cos 3\theta + \ldots\ldots$$
$$+ c_2 r \sin\theta + c_4 r^2 \sin 2\theta + c_6 r^3 \sin 3\theta + \ldots\ldots$$
$$= c_o + \sum_{n=1}^\infty (c_{2n-1} r^n \cos n\theta + c_{2n} r^n \sin n\theta), \qquad (4)$$

or

$$u = \sum_{m=0}^\infty c_m \varphi_m(r,\theta),$$

where $\varphi_o = 1, \; \varphi_1 = r \cos\theta, \; \varphi_2 = r \sin\theta, \; \ldots\ldots\ldots\ldots\ldots$

We can continue the function u analytically across the

boundary C (fig. 1.) by means of Schwarz' principle of symmetry.[4]

The extended function thus obtained takes values equal but opposite

in sign at every pair of points which are symmetrical[5] with respect

to an arc or segment of the contour. At the symmetrical points or

images O' of the point O, the function

becomes infinite as log **r**,

but at all other symmetrical

points it is harmonic. Hence

if we construct the images[6]

of the point O in the sev-

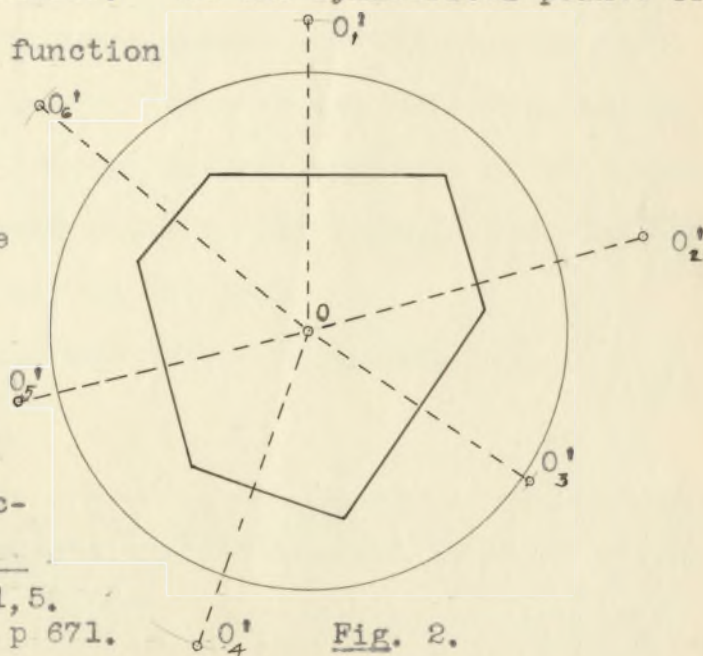eral arcs or segments of the

contour, we know that the func-

Fig. 2.

4) Osgood, pp 666, 672, Satz 1,5.
5) For definition see Osgood, p 671.
6) For the method see Léry's thesis, p 49.

tion u exists and is harmonic at least in a circle extending to the nearest image.   Figure 2. is drawn for a special case where the contour is a polygon.

Now, is the function u represented by the same expansion in the whole circle?   To answer this question we cite the following theorem due to Bôcher.[7]

Theorem: If u is a function harmonic throughout the neighborhood of the point $x_0 y_0$ and if, when this function is continued analytically, the distance from $x_0 y_0$ to the nearest singular point of u, which lies in the same sheet of the Riemann surface generated by the analytic continuation in which $x_0 y_0$ lies, is K, then the development[8] in polar co-ordinates, (the point $x_0 y_0$ is taken as pole),

$$u = A_0/2 + \sum (A_n r^n \cos n\theta + B_n r^n \sin n\theta),$$

converges and represents u throughout the interior of the circle of radius r =K and does not converge throughout any continuum which does not lie in this circle.

If we restrict the problem then to contours of such a nature that a point 0 can be found whose images all lie outside of a circle about 0 enclosing the contour (this we can make the unit circle), we have from the above theorem, for all contours of this type and all such points 0 within each contour, the Green's function for the point 0 and the given contour in the form

$$G = - \log r + A_0/2 + \sum_1^\infty (A_n r^n \cos n\theta + B_n r^n \sin n\theta)$$
$$= - \log r + \sum_0^\infty c_m \varphi_m(r,\theta) \tag{5}$$

7) Trans. Am. Math. Soc., Ser. 2, Vol. 10, p 278.
8) Bôcher gives here also an equivalent development in terms of polynomials in x and y.

This expansion holds for all points inside the polygon and on the boundary.   Moreover it is unique and we must be able to determine the constants A_n, B_n or c_n from the boundary conditions.

For the boundary we have $r = R(\theta)$ and $G = 0$, that is,

$$A_0/2 + \sum_1^\infty \left(A_n R^n(\theta) \cos n\theta + B_n R^n(\theta) \sin n\theta \right) = \log R(\theta)$$

$$\tag{6}$$

or $\qquad\qquad\qquad\qquad\qquad\qquad \sum_m^\infty c_m \phi_m(\theta) = \log R(\theta) ,$

where $\qquad \phi_0 = 1$; $\phi_{2n-1} = R^n \cos n\theta$; $\phi_{2n} = R^n \sin n\theta.$

Now we can apply a method of analytic continuation due to Borel[9] to obtain a new expression for the function u which will hold for a much more extended region, as follows.   Equation (3) may be written

$$u = A_0/2 + \mathrm{Re} \sum_1^\infty (A_n - iB_n) z^n ,$$

where z is a complex variable.   Since u is harmonic the expression $\sum_1^\infty (A_n - iB_n) z^n$ is analytic in a circle extending to the nearest image O_1' of O.   Consider for example the polygon[10] ABCDE (Fig. 3.)
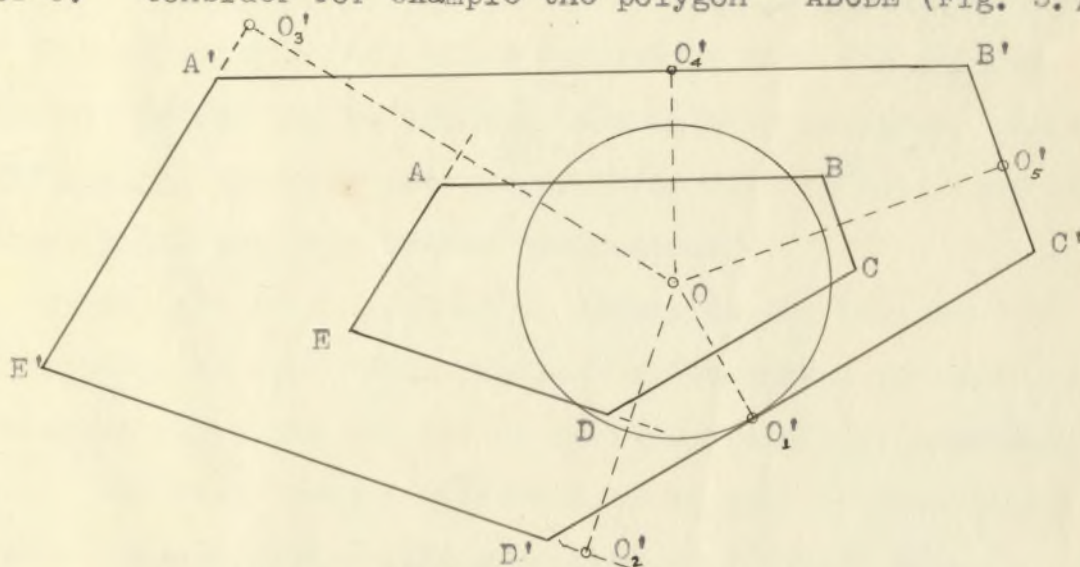

Fig. 3.

with the point of discontinuity at a point O inside it.

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9) Whittaker and Watson, _Modern Analysis_, 2nd Ed., p 141.
10) Some or all of the sides of the polygon could be replaced by analytic arcs.

By means of Borel's integral we can continue the function $\sum_{0}^{\infty} A_n z^n$ analytically to the region enclosed by the polygon A'B'C'D'E' whose sides are lines through the several images O' and perpendicular respectively to the several lines OO'. The expression for the function thus extended is

$$f(z) = \int_0^{\infty} e^{-t} \sum_0^{\infty} \frac{A_n (zt)^n}{n!} \, dt \, .$$

Applying this to the problem in hand we have, in the real plane,

$$u = A_0/2 + \mathrm{Re} \int_0^{\infty} e^{-t} \sum_1^{\infty} \frac{(A_n - i\, B_n)(zt)^n}{n!} \, dt$$

$$= A_0/2 + \int_0^{\infty} e^{-t} \sum_1^{\infty} (A_n r^n \cos n\theta + B_n r^n \sin n\theta) \frac{t^n}{n!} \, dt \, .$$

This function is harmonic inside the polygon A'B'C'D'E' whose sides are respectively parallel to the sides of ABCDE. Hence this gives us a formula,

$$G = -\log r + A_0/2 + \int_0^{\infty} e^{-t} \sum_1^{\infty} (A_n r^n \cos n\theta + B_n r^n \sin n\theta) \frac{t^n}{n!} \, dt, \quad (7)$$

for the Green's function for any convex polygon whatever, with the point of discontinuity at any point whatever inside the polygon. Evidently this method can be applied, not only to polygons, but to any convex contour whatever and the point of discontinuity may be taken at any point whatever inside the contour.

Now we must find a method to determine the infinite set of constants A_n, B_n, or c_n. This problem may be solved in two ways, (1) by integral equations and (2) by an infinite set of algebraic equations. The first method turns out to be only of theoretical interest; the second is a practical method as we shall show.

II. SOLUTION BY INTEGRAL EQUATIONS

The c's are Fourier constants. Hence by a well-known the-
orem of Fischer and Riesz,[11] the sum of their squares, $\sum_{0}^{\infty} c_m^2$, con-
verges. Moreover, if we select any set of orthogonal functions
whatever, linear, planar or solid, for example $h_m(\chi)$, there exists
one and only one function[11] $H(\chi)$ for which c_m are the Fourier con-
stants with respect to this orthogonal set. Whether the series
thus obtained converges or actually represents the function at all
is immaterial, for in any case the formulas for the coefficients
hold,

$$c_m = \int_{\chi_1}^{\chi_2} H(\chi)\, h_m(\chi)\ d\chi, \tag{8}$$

where the definite integral is either single, double or triple,
according as the orthogonal functions chosen are linear, planar
or solid. Now substitute these constants back in equation (6)
and we have, provided the series in the bracket converges,

$$\int_{\chi_1}^{\chi_2} H(\chi) \left[\sum_{0}^{\infty} h_m(\chi)\, \phi_m(\theta) \right] d\chi = \log R(\theta), \tag{9}$$

an integral equation of the first kind with unsymmetric kernel
$K(\chi,\theta) = \sum_{0}^{\infty} h_m(\chi)\, \phi_m(\theta)$. That a set of functions $h_m(\chi)$ for which
the expression for the kernel converges can be found we shall show
farther on by an actual example.

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11) Schmidt, Rendiconti di Palermo, Vol. 25 (1908), p 53;
    Riesz, Comptes Rendus, Vol. 144 (1907), p 615.
    Fischer, Comptes Rendus, Vol. 144 (1907), p 1022.

We have then to solve an integral equation of the first kind of the form

$$\int K(\chi,\theta) \ H(\chi) \ d\chi = f(\theta).$$ (10)

No practical solution of this equation with unsymmetric kernel has appeared in the mathematical literature, although theoretical solutions or existence theorems have been developed by Schmidt, Picard, Bateman, and others. A solution due to Picard and Schmidt[12] may be outlined briefly as follows. -

Picard's solution: We determine by Schmidt's method two "associated" sets of orthogonal functions $\varphi_n(\theta)$ and $\psi_n(\theta)$ which are simultaneous solutions of the two integral equations of the first kind,

$$\varphi(\theta) = \lambda \int_{\chi_1}^{\chi_2} K(\theta,\chi) \ \psi(\chi) \ d\chi$$
$$\psi(\theta) = \lambda \int_{\chi_1}^{\chi_2} K(\chi,\theta) \ \varphi(\chi) \ d\chi .$$

These associated functions are obtained by Fredholm's method from the two homogeneous integral equations of the second kind, -

$$\varphi(\theta) = \lambda^2 \int_{\chi_1}^{\chi_2} \overline{K(\theta,\chi)} \ \varphi(\chi) \ d\chi$$
$$\psi(\theta) = \lambda^2 \int_{\chi_1}^{\chi_2} \underline{K(\theta,\chi)} \ \psi(\chi) \ d\chi ,$$ (11)

where the symmetric kernels $\overline{K(\theta,\chi)}$ and $\underline{K(\theta,\chi)}$ are given by

$$\overline{K(\theta,\chi)} = \int_{\chi_1}^{\chi_2} K(\theta,t) \ K(\chi,t) \ dt$$
$$\underline{K(\theta,\chi)} = \int_{\chi_1}^{\chi_2} K(t,\theta) \ K(t,\chi) \ dt .$$ (12)

Now, referring back to equation (10), let $f(\theta)$ be expanded in terms of the orthogonal functions $\psi_n(\theta)$ and let A_1, A_2, A_3, ... be the Fourier constants of the expansion, that is,

$$A_n = \int_{\theta_1}^{\theta_2} f(\theta) \ \psi_n(\theta) \ d\theta .$$

12) Lalesco, Théorie des Équations Intégrales, pp 96-100; Hobson, Pro. Lond. Math. Soc., Ser. 2, Vol. 12 (1913), p 297.

Substituting from equation (10) we have

$$A_n = \int_{\theta_1}^{\theta_2} \int_{\chi_1}^{\chi_2} K(\chi,\theta) \, H(\chi) \, \psi_n(\theta) \, d\theta d\chi = 1/\lambda_n^2 \int_{\chi_1}^{\chi_2} H(\chi) \, \varphi_n(\chi) \, d\chi$$

or

$$A_n \lambda_n^2 = \int_{\chi_1}^{\chi_2} H(\chi) \, \varphi_n(\chi) \, d\chi .$$

That is to say the quantities $A_n \lambda_n^2$ provided they converge, are the Fourier constants for the expansion of $H(\chi)$ in terms of the orthogonal functions $\varphi_n(\chi)$, and we have

$$H(\chi) = \sum_0^\infty A_n \lambda_n^2 \, \varphi_n(\chi) . \tag{13}$$

The actual carrying out of the above process even for the simplest cases presents insuperable difficulties. To evaluate Fredholm's expansion for only a few terms is very difficult even when the kernel is of the simplest character, but in the form given by equations (12) the task is impossible. In fact the method is given by Picard as an existence proof rather than a practical solution. Probably a much more practical method is the following.

Solution by the author.- Referring to equation (10) again let both $K(\theta,\chi)$ and $f(\theta)$ be expanded as functions of θ in terms of some convenient set of orthogonal functions $k_n(\theta)$, for example in a Fourier series. Then we have

$$\int_{\chi_1}^{\chi_2} \left[\sum_0^\infty D_n \, k_n(\theta) \right] H(\chi) \, d\chi = \sum_0^\infty d_n \, k_n(\theta) ,$$

where the D_n are functions of χ and the d_n are constants. Comparing the coefficients of $k_n(\theta)$ on each side of the equation we have

$$d_n = \int_{\chi_1}^{\chi_2} H(\chi) \, D_n(\chi) \, d\chi . \qquad n = 0, 1, 2, \ldots\ldots\ldots$$

Now, let us take the set $D_n(\chi)$, provided they are linearly independent,[13] together with any other convenient set of linearly independ-

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13) If not, use only those which are linearly independent, expressing the others in terms of them.

ent functions, and form the bi-orthogonal set[14]

$$B_1(\chi) \qquad B_2(\chi) \qquad B_3(\chi) \qquad \cdots\cdots\cdots$$

$$A_1(\chi) \qquad A_2(\chi) \qquad A_3(\chi) \qquad \cdots\cdots\cdots ,$$

where the B's are linear expressions in the D's as follows.-

$$B_0(\chi) = b_{00}D_0(\chi)$$
$$B_1(\chi) = b_{10}D_0(\chi) + b_{11}D_1(\chi)$$
$$B_2(\chi) = b_{20}D_0(\chi) + b_{21}D_1(\chi) + b_{22}D_2(\chi) \qquad\qquad\qquad (14)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$B_n(\chi) = b_{n0}D_0(\chi) + b_{n1}D_1(\chi) + b_{n2}D_2(\chi) + \cdots\cdots\cdots\cdots + b_{nn}D_n(\chi)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Now we may write

$$a_n = b_{n0}d_0 + b_{n1}d_1 + b_{n2}d_2 + \cdots = \int_{\chi_1}^{\chi_2} H(\chi)\, B_n(\chi)\, d\chi. \qquad n = 1, 2, \ldots$$

Since $A_n(\chi)$ and $B_n(\chi)$ are bi-orthogonal this set of equations gives us the coefficients for the expansion of $H(\chi)$ in terms of the functions $A_n(\chi)$, that is,

$$H(\chi) = \sum_0^\infty a_n A_n(\chi) . \qquad\qquad\qquad (15)$$

Having found $H(\chi)$ by this method or some other we could substitute in equations (8) and obtain the values of the required constants c_m. Putting these constants in equation (5) we obtain the desired expansion for the Green's function. However, these constants can not be used in equation (7) unless the polygon ABCDE lies wholly inside the circle of convergence, in which case equation (7) reduces to equation (5). It is important to notice also that the constants c_m will be different for different points O inside the contour and must be re-determined for each different O.

The Integral Equation in a Special Form:- Returning now to equation (9) we select a particular set of functions for $h_n(\chi)$, namely the linear set $\cos n\chi$, $\sin n\chi$. This choice gives some results

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14) Goursat, Vol. III, p 392.

of considerable interest. The integral equation becomes

$$\frac{1}{2} \int_0^{2\pi} H(\chi) \left[1 + 2 \sum_1^\infty (R^n \cos n\theta \, \cos n\chi + R^n \sin n\theta \, \sin n\chi) \right] d\chi = \log R$$

Condensing the kernel in the bracket we have $1 - 2 \sum_1^\infty R^n \cos n(\chi - \theta)$,

the well-known expansion for

$$\frac{1 - R^2}{1 - 2 R \cos(\chi - \theta) + R^2} \ . \qquad\qquad R < 1 \ .$$

Evidently from the way we have now chosen the set $h_m(\chi)$ the unknown

function H is nothing else than the value of u on the unit circle:

$$H(\chi) = u(1, \chi) \ .$$

Thus the integral equation (9) becomes

$$\frac{1}{2} \int_\theta^{2\pi} u(1, \chi) \, \frac{1 - R^2(\theta)}{1 - 2 R(\theta) \cos(\chi - \theta) + R^2(\theta)} \, d\chi = \log R(\theta), \qquad (16)$$

where the left hand member is Poisson's integral. Hence we may

state the solution of the problem before us as follows.-

~~Given~~ *Suppose we have given* any closed contour of such a nature that a circle,

with center at the point O, can be drawn completely enclosing it,

but not enclosing any of the images of O with respect to the contour

or arcs composing the contour. Make this the unit circle. Then

the Green's function for this contour is obtained by using Poisson's

integral backwards, as it were, to determine the value on the unit

circle, namely $H(\chi) = u(1, \chi)$, of a harmonic function $u(r, \theta)$ which

reduces to $\log R(\theta)$ for points of the given contour. Having found

H we determine the Fourier coefficients corresponding to it and sub-

stitute them in formula (5) to get the Green's function. On ac-

count of the complicated character of the kernel in equation (16)

the solution of the equation is practically impossible. We shall

therefore take up in the following pages an entirely different meth-

od for determining the constants for equation (5).

III. SOLUTION BY AN INFINITE SET OF EQUATIONS

Going back to equation (6), we reduce the problem of deter-
mining the unknown constants c_m to the solution of an infinite num-
ber of equations for this infinite set of unknowns. One way to do this
would be to expand the various powers of $R(\theta)$ as well as the func-
tion log $R(\theta)$ in Fourier series, and compare the coefficients of
cos $n\theta$ and sin $n\theta$ on each side of the equation. This leads im-
mediately to an infinite set of equations for the constants c_m
but these equations are unmanageable.

In order to get a manageable set of equations let us con-
struct a normalized, orthogonal set of functions from the set
$\Phi_n(\theta) : \left[1, \; R^n(\theta) \cos n\theta, \; R^n(\theta) \sin n\theta \right]$, functions which will be
orthogonal for the variable θ between 0 and 2π. These functions we
can construct by the method of Goursat [15] provided the set $\Phi_n(\theta)$
are linearly independent. We shall first investigate this matter
of linear independence.

An infinite set of functions $\Phi_n(\theta)$ are said to be linear-
ly independent if there exists no relation of the form

$$ C_0 \, \Phi_{n_0} + C_1 \, \Phi_{n_1} + C_2 \, \Phi_{n_2} + \ldots\ldots\ldots + C_n \, \Phi_{n_n} = 0 \; . \qquad (17) $$

for any finite number of them. For the set of functions we are
considering, there certainly do exist such relations for some func-
tions $R(\theta)$. For example if $R = \sec \theta$ we have

15) Goursat, Vol. III, p. 391-2.

$$\overline{\Phi}_0 - \overline{\Phi}_1 = 0 ; \qquad 2\overline{\Phi}_2 - \overline{\Phi}_4 = 0 .$$

However for no closed finite contour can such a relation exist. For, suppose we consider a contour $R = R(\theta)$ such that a linear relation exists between the first n of the set $\overline{\Phi}_n$. That is, suppose there exists a set of constants C_m such that

$$V(R,\theta) = C_o + C_1 R \cos\theta + C_2 R \sin\theta + C_3 R^2 \cos 2\theta + C_4 R^2 \sin 2\theta +$$

$$\cdots\cdots\cdots\cdots + C_{2n-1} R^n \cos n\theta + C_{2n} R^n \sin n\theta \equiv 0 . \quad (18)$$

The left hand member of this equation is the value of a harmonic function $V(r,\theta)$ on a finite closed contour. Hence by Gauss' mean-value theorem[16],

$$V(r,\theta) \equiv 0$$

for all points inside the contour, which is only possible if all the coefficients vanish. Hence the set of functions 1, $R^n \cos n\theta$, and $R^n \sin n\theta$ are linearly independent for <u>all closed contours</u>, in particular for polygons.

We can now apply Goursat's method and form a set $\psi_1, \psi_2, \psi_3, \cdots\cdots\cdots$ of normal, orthogonal functions, as follows.-

$$
\begin{aligned}
\psi_0 &= a_{oo}\\
\psi_1 &= a_{10} + a_{11} R \cos\theta\\
\psi_2 &= a_{20} + a_{21} R \cos\theta + a_{22} R \sin\theta\\
\psi_3 &= a_{30} + a_{31} R \cos\theta + a_{32} R \sin\theta + a_{33} R^2 \cos 2\theta \qquad (19)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\psi_{2n-1} &= a_{2n-1,0} + a_{2n-1,1} R \cos\theta + a_{2n-1,2} R \sin\theta + a_{2n-1,3} R^2 \cos 2\theta + \cdots\cdots + a_{2n-1,2n-1} R^n \cos n\theta\\
\psi_{2n} &= a_{2n,0} + a_{2n,1} R \cos\theta + a_{2n,2} R \sin\theta + a_{2n,3} R^2 \cos 2\theta + \cdots\cdots + a_{2n,2n-1} R^n \cos n\theta + a_{2n,2n} R^n \sin n\theta
\end{aligned}
$$

Solving for the $\overline{\Phi}$'s we have

$$
\begin{aligned}
1 &= b_{oo}\psi_0\\
R \cos\theta &= b_{10}\psi_0 + b_{11}\psi_1\\
R \sin\theta &= b_{20}\psi_0 + b_{21}\psi_1 + b_{22}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (20)\\
R^n \cos n\theta &= b_{2n-1,0}\psi_0 + b_{2n-1,1}\psi_1 + b_{2n-1,2} + \cdots\cdots + b_{2n-1,2n-1}\psi_{2n-1}\\
R^n \sin n\theta &= b_{2n,0}\psi_0 + b_{2n,1}\psi_1 + b_{2n,2} + \cdots\cdots + b_{2n,2n-1}\psi_{2n-1} + b_{2n,2n}\psi_{2n}
\end{aligned}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

16) Osgood, p 623.

where

$$b_{nm} = (-1)^{n+m} \frac{1}{a_{mm}\, a_{m+1,m+1} \cdots \cdots a_{nn}}
\begin{vmatrix}
a_{m+1,m} & a_{m+1,m+1} & 0 & 0 & \cdots\cdots & 0 \\
a_{m+2,m} & a_{m+2,m+1} & a_{m+2,m+2} & 0 & \cdots\cdots & 0 \\
\multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
 & & & & & 0 \\
a_{n-1,m} & a_{n-1,m+1} & a_{n-1,m+2} & a_{n-1,m+3} & \cdots\cdots & a_{n-1,n-1} \\
a_{n,m} & a_{n,m+1} & a_{n,m+2} & a_{n,m+3} & \cdots\cdots & a_{n,n-1}
\end{vmatrix}
. \qquad (21)$$

Now expand $\log R(\theta)$ in terms of ψ_0, ψ_1, ψ_2, ... We have

$$\log R(\theta) = A_0\psi_0 + A_1\psi_1 + A_2\psi_2 + \cdots\cdots = \sum_{i=0}^{\infty} A_i \psi_i,$$

where the coefficients are given by

$$A_i = \int_0^{2\pi} \log R(\theta)\, \psi_i(\theta)\, d\theta,$$

and, substituting in equation (6),

$$c_0(b_{00}\psi_0) + c_1(b_{10}\psi_0 + b_{11}\psi_1) + c_2(b_{20}\psi_0 + b_{21}\psi_1 + b_{22}\psi_2) + \cdots = \sum_{i=0}^{\infty} A_i\psi_i, \qquad (22)$$

where c_0, c_1, c_2, ... are the unknown constants required. Comparing coefficients of the ψ's on each side of the equation we have

$$
\begin{aligned}
b_{00}c_0 + b_{10}c_1 + b_{20}c_2 + b_{30}c_3 + \cdots\cdots\cdots\cdots\cdots &= A_0 \\
b_{11}c_1 + b_{21}c_2 + b_{31}c_3 + \cdots\cdots\cdots\cdots\cdots &= A_1 \\
b_{22}c_2 + b_{32}c_3 + \cdots\cdots\cdots\cdots\cdots &= A_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\,,
\end{aligned}
\qquad (23)
$$

an infinite set of equations which is not difficult to solve in a formal way, giving each c in the form of an infinite series involving the b's and the A's. Substituting for the b's in terms of the a's from equation (21), we arrive at the ~~surprisingly~~ simple result,-

$$
\begin{aligned}
c_0 &= a_{00}A_0 + a_{10}A_1 + a_{20}A_2 + a_{30}A_3 + a_{40}A_4 + \cdots\cdots\cdots\cdots\cdots \\
c_1 &= \phantom{a_{00}A_0 +} a_{11}A_1 + a_{21}A_2 + a_{31}A_3 + a_{41}A_4 + \cdots\cdots\cdots\cdots\cdots \\
c_2 &= \phantom{a_{00}A_0 + a_{11}A_1 +} a_{22}A_2 + a_{32}A_3 + a_{42}A_4 + \cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
c_m &= \sum_{i=m}^{\infty} a_{im}A_i \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\qquad (24)
$$

Substituting this solution in equation (6) we have,

$$\sum_{i=o}^{\infty} a_{ic} A_i + \sum_{n=1}^{\infty} (\sum_{i=2n-1}^{\infty} a_{i,2n-1} A_i R^n \cos n\theta + \sum_{i=2n}^{\infty} a_{i,2n} A_i R^n \sin n\theta) = \sum_{i=o}^{\infty} A_i \psi_i .$$

It is sufficient to multiply equations (19) respectively by A_o, A_1, A_2, ... and add, to verify formally the correctness of this equation. Since we know the c's exist and are unique, if we can prove that equations (24) give an actual solution of (23), this is the only solution.

To prove this we shall make use of the follwing theorem due to Mrs. Pell.[17)]

Theorem: If the sequences $\{\lambda_k\}$ and $\{\mu_j\}$ are such that the sequence $\{b_{ik}/\lambda_k\}$ is of finite norm for every **k**, and the matrix $(\lambda_i a_{ki}/\mu_i)$ is limited, then for every sequence $\{A_k\}$ such that $\{\mu_k A_k\}$ is of finite norm, the system of equations

$$\sum_{i=k}^{\infty} b_{ik} c_i = A_k \qquad\qquad (\mathbf{k} = 0, 1, 2, 3, ...) \qquad (23)$$

has a solution c_k such that $\{\lambda_k c_k\}$ is of finite norm, and the solution is given by

$$c_k = \sum_{k=L}^{\infty} a_{ik} A_i \qquad\qquad (i = 0, 1, 2, 3, ...) \qquad (24)$$

Making $\lambda_k = \mu_j = 1$, we can show that the three conditions required by this theorem are satisfied, as follows.-

(1) The first condition to be satisfied is that the sequence $\{b_{ik}\}$ shall be of finite norm for every k, that is, the coefficients in equations (20), taken by columns, must give sequences of finite norm. This we can prove as follows. Since the ψ's are normal, orthogonal functions we have for every n,

---

17) Ann. of Math., Vol. 28 (1914-15), p 35. We have changed Mrs.
    Pell's notation to conform to our own.

$$\int_0^{2\pi} R^{2n} \cos n\theta \; d\theta = b^2_{2n-1,0} + b^2_{2n-1,1} + \cdots\cdots\cdots\cdots + b^2_{2n-1,2n-1}$$

$$\int_0^{2\pi} R^{2n} \sin n\theta \; d\theta = b^2_{2n,0} + b^2_{2n,1} + \cdots\cdots\cdots\cdots + b^2_{2n,2n} .$$

$$(25)$$

Hence for every k we have,

$$b^2_{2n-1,k} < \int_0^{2\pi} R^{2n} \cos n\theta \; d\theta < \int_0^{2\pi} R^{2n} d\theta \leqq 2\pi R^{2n}_{max.}$$

$$b^2_{2n,k} < \int_0^{2\pi} R^{2n} \sin n\theta \; d\theta < \int_0^{2\pi} R^{2n} d\theta \leqq 2\pi R^{2n}_{max.},$$

so that we have for the norm of the sequence $\{b_{ik}\}$,

$$\sum_{i=k}^{\infty} b^2_{ik} < 4\pi \sum_{2n=k}^{\infty} R^{2n}_{max.} . \qquad\qquad k = 0, 1, 2, \ldots\ldots \qquad (26)$$

But the maximum value of $R(\theta)$ is always less than unity. Hence the right hand member of equation (26) is an absolutely convergent series. Thus we have proved that $\{b_{ik}\}$ is of finite norm for every k.

(2) The second condition is that the matrix (a_{ki}) of the coefficients in equations (24) shall be limited. This we can show to be true as follows. Consider first the matrix (b_{ik}) of equations (20). From (25) we have,

$$\sum_{i=0}^{\infty} \sum_{k=0}^{i} b^2_{ik} < 4\pi \sum_{2n=k}^{\infty} R^{2n}_{max.} < M,$$

that is, the b's are of finite norm and hence[18] the matrix (b_{ik}) is limited. Now, the matrix (a_{ik}) of equations (19) is the unique reciprocal of the matrix (b_{ik}) and hence is itself limited.[19] Therefore its conjugate, the matrix (a_{ki}) of equations (23) is limited.

(3) The third condition is evidently satisfied, for the sequence $\{A_i\}$ of the right hand members of equation (23) is of finite norm since the A's are the Fourier constants for the expansion of $\log R(\theta)$ in terms of the ψ's.

Now by ~~Mrs.~~ Pell's theorem it follows that equations (24)

---

18) Hellinger and Toeplitz, Math. Ann., Vol. 69, p 307.
19) Hellinger and Toeplitz, 1. c., p 311.

give an actual solution of (23) and hence give us the required expressions for the constants c.

IV. APPLICATION TO A SQUARE CONTOUR

As a simple application of the above let us find the Green's function for a square with discontinuity at the center. If we make the apothem of the square unity and measure θ from it, we have for the function R,

$$
\begin{aligned}
R(\theta) &= \sec\theta & -\pi/4 \leqq \theta &\leqq \pi/4 \\
R(\theta) &= \csc\theta & \pi/4 \leqq \theta &\leqq 3\pi/4 \\
R(\theta) &= -\sec\theta & 3\pi/4 \leqq \theta &\leqq 5\pi/4 \\
R(\theta) &= -\csc\theta & 5\pi/4 \leqq \theta &\leqq 7\pi/4 \quad .
\end{aligned}
\tag{27}
$$

Beginning with the first function $\phi_o = 1$ we shall first orthogonalize all the other functions to it by determining C_{2n-1} and C_{2n} to satisfy the following equations;

$$
\begin{aligned}
\int_0^{2\pi} (\phi_{2n-1} - C_{2n-1}\phi_o)\,\bar{\phi}_o \; d\theta &= 0 \\
\int_0^{2\pi} (\phi_{2n} - C_{2n}\phi_o)\,\bar{\phi}_o \; d\theta &= 0 \quad .
\end{aligned}
\tag{28}
$$

From the first of these equations we have $C_{2n-1} = 0$ for n odd and n/2 odd, that is,

$$
C_1 = C_3 = C_5 = C_9 = C_{11} = C_{13} = C_{17} = \ldots\ldots = 0 \quad .
$$

For n/2 even

$$
C_{2n-1} = 2/\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^n\theta \, \cos n\theta \; d\theta \; .
$$

From the second of equations (28) we have $C_{2n} = 0$ for every n. Thus we get a new set of functions ϕ', all of which are orthogonal to ϕ_o, as follows:

$$
\begin{array}{ll}
\phi_1' = R\cos\theta & \phi_2' = R\sin\theta \\
\phi_3' = R^2\cos 2\theta & \phi_4' = R^2\sin 2\theta \\
\phi_5' = R^3\cos 3\theta & \phi_6' = R^3\sin 3\theta
\end{array}
$$

$$\Phi_7' = R^4 \cos 4\theta + .48826 \qquad \Phi_8' = R^4 \sin 4\theta$$

$$\Phi_9' = R^5 \cos 5\theta \qquad \Phi_{10}' = R^5 \sin 5\theta$$

. . . .

$$\Phi_{15}' = R^8 \cos 8\theta - 1.11259 \qquad \Phi_{16}' = R^8 \sin 8\theta$$

. . . .

$$\Phi_{23}' = R^{12} \cos 12\theta + 3.01549 \qquad \Phi_{24}' = R^{12} \sin 12\theta$$

. . . .

Now we take Φ_1' and orthogonalize Φ_2', Φ_3', Φ_4',...... to it in the same way as before. Continue this process until a sufficient number of orthogonal functions have been obtained. Finally these orthogonal functions must be normalized by dividing each function Φ by $\left[\int_0^{2\pi} \Phi^2 d\theta \right]^{\frac{1}{2}}$. The results, which we obtained with the aid of a "Millionaire" calculating machine, are given in the following table, page 24, which is a tabulation of the matrix (a_{ik}) of equations (19), for the square. Denote by ψ_0, ψ_1, ψ_2, ... , as before, the normal, orthogonal functions whose coefficients are given by the respective rows of the matrix.

Next we determine the A's by means of the formula,

$$A_m = \int_0^{2\pi} \log R(\theta) \, \psi_m(\theta) \, d\theta .$$

The a's are independent of the dimensions of the square chosen but the A's are not. The A's which appear in the table, page 24, in the first column, are calculated for a square with apothem $1/2 \cdot \sqrt{2}$, so that the unit circle passes through the corners.

Substituting in equations (24) we have the required constants for the square, namely,-

A	1	R cos θ	R sin θ	R² cos 2θ	R² sin 2θ	R³ cos 3θ	R³ sin 3θ	R⁴ cos 4θ	R⁴ sin 4θ	R⁵ cos 5θ	R⁵ sin 5θ	R⁶ cos 6θ	R⁶ sin 6θ	R⁷ cos 7θ	R⁷ sin 7θ	R⁸ cos 8θ	R⁸ sin 8θ	R⁹ cos 9θ	R⁹ sin 9θ	R¹⁰cos10θ	R¹⁰sin10θ	R¹¹cos11θ	R¹¹sin11θ	R¹²cos12θ	R¹²sin12θ
-.59294	+.59294																								
0	0	+.50000																							
0	0	0	+.50000																						
0	0	0	0	+.51303																					
0	0	0	0	0	+.38160																				
0	0	+.14622	0	0	0	+.38199																			
0	0	0	-.14622	0	0	0	+.38129																		
-.40681	+.13558	0	0	0	0	0	0	+.27748																	
0	0	0	0	0	0	0	0	0	+.38573																
0	0	+.21689	0	0	0	-.10769	0	0	0	+.27036															
0	0	0	+.22171	0	0	0	+.09149	0	0	0	+.27714														
0	0	0	0	*	0	0	0	0	0	*	0	*													
0	0	0	0	0	*	0	0	0	0	0	0	0	*												
0	0	*	0	0	0	*	0	0	0	*	0	0	0	*											
0	0	0	*	0	0	0	*	0	0	0	*	0	0	0	*										
-.15917	-.05805	0	0	0	0	0	0	+.13799	0	0	0	0	0	0	0	+.02818									
0	0	0	0	0	0	0	0	0	*	0	0	0	0	0	0	0	*								
0	0	*	0	0	0	*	0	0	0	*	0	0	0	*	0	0	0	*							
0	0	0	*	0	0	0	*	0	0	0	*	0	0	0	*	0	0	0	*						
0	0	0	0	*	0	0	0	0	0	0	0	*	0	0	0	0	0	0	0	*					
0	0	0	0	0	*	0	0	0	0	0	0	0	*	0	0	0	0	0	0	0	*				
0	0	0	0	0	0	*	0	0	0	0	0	0	0	*	0	0	0	0	0	0	0	*			
0	0	*	0	0	0	0	0	0	0	*	0	0	0	0	0	*	0	0	0	0	0	0	*		
0	0	0	*	0	0	0	0	0	0	0	*	0	0	0	0	0	*	0	0	0	0	0	0	*	
+.05961	+.07945	0	0	0	0	0	0	-.13967	0	0	0	0	0	0	0	0	0	-.00913	0	0	0	0	0	+.02558	
0	0	0	0	0	0	0	*	0	0	0	0	0	0	0	0	0	*	0	0	0	0	0	0	0	*

NOTE: The asterisks * indicate non-zero figures which we did not compute. The only figures needed are those underlined in red.

$$c_o = \sum_{i=0}^{\infty} a_{i0} A_i = -.23655 -.05516 +.00883 +.00474 + \dots = -.28 \dots$$

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$$

$$c_7 = \sum_{i=7}^{\infty} a_{i7} A_i = \qquad -.11296 -.02100 -.00834 + \dots = -.14 \dots$$

$$c_8 = c_9 = c_{10} = c_{11} = c_{12} = c_{13} = c_{14} = 0$$

$$c_{15} = \sum_{i=15}^{\infty} a_{i,15} A_i = \qquad -.00794 -.00013 + \dots = -.01 \dots$$

$$c_{16} = c_{17} = c_{18} = c_{19} = c_{20} = c_{21} = c_{22} = 0$$

$$c_{23} = \sum_{i=23}^{\infty} a_{i,23} A_i = \qquad +.00153 + \dots = +.002 \dots$$

. . . .

Putting these constants in equation (5) we have the following formula for the Green's function for a square with the point of discontinuity at the center and diagonal of length 2:-

$$G = - \log r -.28 -.14\, r^4 \cos 4\theta -.01\, r^8 \cos 8\theta +.002\, r^{12} \cos 12\theta + \dots$$
$$= \mathrm{Re}\,(-\log z -.28 -.14\, z^4 -.01\, z^8 +.002\, z^{12} + \dots) \qquad (29)$$

approximately.

Thus we see that the expansion for u, that is, $G + \log r$, is the real part of an expansion in fourth powers of z. This we verified by examining the expression in elliptic functions, obtained by the method of images,[20] for the square with the point of discontinuity and the origin of co-ordinates at the center. For if we write

$$u = \mathrm{Re} \big[g(z) - \log z \big] = \mathrm{Re}\, f(z)$$

it is possible to show by a series of transformations that

$$\mathrm{Re}\, f(iz) = \mathrm{Re}\, f(z)$$

This peculiarity in the expansion for the Green's function for the square suggests immediately that for a regular polygon of p sides with the discontinuity at the center (p any positive integer),

20) Léry, p 66.

we have for u the real part of an expansion in pth powers.   We see at once that this is true when p becomes infinite, for in that case we have a circle, for which u is constant.   To show this is true in general we can make use of Schwarz' mapping function for a regular polygon of p sides, namely,

$$w = \int_0^z \frac{dz}{(1 - z^p)^{2/p}} \, .$$

Expanding the denominator by the binomial theorem and integrating term by term we obtain, using the relation $G = -\operatorname{Re} \log w$,

$$G = -\operatorname{Re}\left[\log z + \log(1 + b_1 z^p + b_2 z^{2p} + b_3 z^{3p} + \ldots\ldots\ldots)\right]$$

$$= \operatorname{Re}\left[-\log z - b_1 z^p - (b_2 - \tfrac{1}{2} b_1^2) z^{2p} - (b_3 - b_1 b_2 + \tfrac{1}{3} b_1^3) \, z^{3p} - \ldots\right]$$

$$= -\log r - b r^p \cos p\theta - (b_2 - \tfrac{1}{2} b_1^2) r^{2p} \cos 2p\theta - (b_3 - b_1 b_2 + \tfrac{1}{3} b_1^3) r^{3p} \cos 3p\theta - \ldots$$

where the coefficients b are functions of the number of sides p of the polygon.   To compare Schwarz's formula directly with ours it is necessary to replace θ by $\theta - \pi/p$ in the above, so that the initial line, $\theta = 0$, is perpendicular to a side of the polygon instead of passing through a vertex as Schwarz has taken it.   Then, putting in the actual values of the coefficients b, this gives us, for a regular polygon of p sides whose radius is

$$r_0 = \int_0^1 \frac{dz}{(1 - z^p)^{2/p}} \, ,$$

the Green's function

$$G = -\log r + \left(\frac{2}{p(p+1)}\right) r^p \cos p\theta + \left(\frac{2}{p^2(p+1)^2} - \frac{2(2+p)}{p(2p+1)}\right) r^{2p} \cos 2p\theta +$$

$$\left(\frac{8}{3p^3(p+1)^3} - \frac{4(2+p)}{p^2(p+1)(2p+1)} + \frac{2(2+p)(2+2p)}{p^3(3p+1)}\right) r^{3p} \cos 3p\theta + \, . (30)$$

Hence we see that for a regular polygon of p sides all the coefficients c in equation (4) will vanish except c_{2p-1}, c_{4p-1}, c_{6p-1}, $\ldots\ldots$

For a square, for which $r_0 = 1.29961$ and the length of the apothem is $a = .91896$, we have by Schwarz' formula,

$$G = -\log r + .1 \, r^4 \cos 4\theta - .3233 \, r^8 \cos 8\theta - .2229 \, r^{12} \cos 12\theta + .$$

This is what we would get if we used the apothem a = .91896 in computing the constants for equation (29).

V. CONCLUSION: SUMMARY OF MAIN RESULTS.

Let $R = R(\theta)$ be the polar equation of the contour, the point of discontinuity being taken as the pole. Let

$$\varphi_0 = 1 \; ; \qquad \varphi_{2n-1} = r^n \cos n\theta \; ; \qquad \varphi_{2n} = r^n \sin n\theta :$$

$$\Phi_0 = 1 \; ; \qquad \Phi_{2n-1} = R^n \cos n\theta \; ; \qquad \Phi_{2n} = R^n \sin n\theta .$$

(1) The Green's function for any convex closed contour with the point of discontinuity at any point inside it, is given by,

$$G = - \log r + \int_0^\infty e^{-t} \sum_0^\infty c_m \varphi_m(r, \theta) \frac{t}{n!} \, dt . \tag{7}$$

(2) For all contours and points of discontinuity O inside them, of such a nature that a circle can be drawn with O as center enclosing the polygon but not enclosing any of the images of O with respect to the contour, the above formula (7) reduces to,

$$G = - \log r + \sum_0^\infty c_m \varphi_m(r, \theta) \quad . \tag{5}$$

The constants c in equation (5) can be determined from,

$$c_m = \sum_{i=m}^\infty a_{im} A_i \; , \; \checkmark \tag{24}$$

where the a's are the coefficients of the normalized, orthogonalized functions,

$$\psi_m = \sum_{i=0}^m a_{mi} \Phi_i \; , \tag{19}$$

and the A's are the Fourier coefficients for log R, -

$$A_i = \int_0^{2\pi} \log R(\theta) \, \psi_i(\theta) \, d\theta .$$

This class of contours includes all regular polygons as well as many irregular polygons and other closed contours.

(3) For a regular polygon of p sides all the coefficients of equation (5) vanish except $c_0, \; c_{2p-1}, \; c_{4p-1}, \; c_{6p-1}, \; \ldots\ldots\ldots .$

BIOGRAPHY OF THE AUTHOR

Born at Minneapolis, Minnesota, April 28, 1887.

Educated in the public schools of Minneapolis.

Entered University of Minnesota, September 1904, to study mechanical engineering.

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Assistant in engineering mathematics, University of Minnesota, 1908-10.

Magnetic Observer and Navigating Officer, in the scientific staff of the magnetic survey yacht "Carnegie", Department of Terrestrial Magnetism, Carnegie Institution of Washington, 1910-12.

Student Engineer, Testing Department of the General Electric Company, Schenectady, N. Y., six months, 1913-14.

Two senesters, University of Goettingen, Germany, studying applied mathematics under Professors Runge, Hilbert, Prantl, etc.

Instructor in mathematics, University of Iowa, 1914-15.

Student, University of Chicago, half of summer quarter, 1915.

Student, University of Illinois, September 1915 to February 1918; studying under Professors Shaw, Townsend, Miller, Sisam, Carmichael, and Dr. Kempner in mathematics; Professor Kunz in mathematical physics; Professor Goodenough in Mechanical Engineering.

Assistant in mathematics, University of Illinois, 1915-16 and 1917-18, including the summer sessions, 1916 and 1917.

Fellow in mathematics, 1916-17.

PUBLICATIONS

Smoke Prevention in Steam Boiler Plants, Minnesota Engineer, January and March, 1910.