October 1994

UILU-ENG-94-2239
DC-163

Decision and Control

Robustness of Minimax Controllers to Nonlinear Perturbations

Zigang Pan and Tamer Başar

Coordinated Science Laboratory
College of Engineering
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY 2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
4. PERFORMING ORGANIZATION REPORT NUMBER(S) UILU-ENG- 94-2239 DC-163	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Coordinated Science Laboratory University of Illinois 6b. OFFICE SYMBOL *(If applicable)*	7a. NAME OF MONITORING ORGANIZATION U. S. Department of Energy
6c. ADDRESS *(City, State, and ZIP Code)* 1308 West Main Street Urbana, IL 61801-2307	7b. ADDRESS *(City, State, and ZIP Code)* Germantown, MD 20585
8a. NAME OF FUNDING / SPONSORING ORGANIZATION U. S. Department of Energy 8b. OFFICE SYMBOL *(If applicable)*	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER DEFG_02-88-ER-13939
8c. ADDRESS *(City, State, and ZIP Code)* Germantown, MD 20585	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE *(Include Security Classification)*

ROBUSTNESS OF MINIMAX CONTROLLERS TO NONLINEAR PERTURBATIONS

12. PERSONAL AUTHOR(S)

PAN, ZIGANG and BAŞAR, TAMER

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT *(Year, Month, Day)*	15. PAGE COUNT
Technical	FROM ____ TO ____	October 1994	54

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
FIELD	GROUP	SUB-GROUP	Minimax controllers, robustness, nonlinear systems, H^∞ control differential games.

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

See attached.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE *(Include Area Code)* / 22c. OFFICE SYMBOL

DD Form 1473, JUN 86 *Previous editions are obsolete.*

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

Abstract

—We study the robustness of H^∞ controllers, originally designed for nominal linear or nonlinear systems, to unknown static nonlinear perturbations in the state dynamics, the measurement equation, and the performance index. When the nominal system is linear, we consider both perfect state measurements and general imperfect state measurements, and in the case of nominally nonlinear systems, we consider perfect state measurements only. Using a differential game theoretic approach, we show for the former class that as the perturbation parameter (say, $\epsilon > 0$) approaches zero, the optimal disturbance attenuation level for the overall system converges to the optimal disturbance attenuation level for the *nominal* system if the nonlinear structural uncertainties satisfy certain prescribed growth conditions. We also show that the H^∞-optimal controller designed based on a chosen performance level for the *nominal* linear system achieves the same performance level when the parameter ϵ is smaller than a computable threshold, except for the finite-horizon imperfect state measurements case. For that case, we show that the design of the *nominal* controller must be based on a decreased confidence level of the initial data, and a controller thus designed again achieves a desired performance level in the face of nonlinear perturbations satisfying a computable norm bound. In the case of nominally nonlinear systems, and assuming that the *nominal* system is solvable, we obtain sufficient conditions such that the nominal controller achieves a desired performance in the face of perturbations satisfying computable norm bounds. In this way, we provide a characterization of the class of uncertainties that are tolerable for a controller designed based on the nominal system. The paper also presents two numerical examples; in one of these the nominal system is linear, and in the other one it is nonlinear.

Key Words. Minimax controllers, robustness, nonlinear systems, H^∞ control, differential games.

Robustness of Minimax Controllers to Nonlinear Perturbations *†

Zigang Pan and **Tamer Başar**

Decision and Control Laboratory
Coordinated Science Laboratory and the
Department of Electrical and Computer Engineering
University of Illinois
Urbana, IL 61801/USA

Address for Correspondence: Prof. Tamer Başar
Coordinated Science Laboratory
University of Illinois
1308 West Main Street
Urbana, IL 61801/USA
Tel: 217/333-3607; Fax: 217/244-1653
Email: tbasar@black.csl.uiuc.edu

*Research supported in part by the U.S. Department of Energy under Grant DE-FG-02-88-ER-13939; and in part by the National Science Foundation under Grant ECS 91-13153.

†August 25, 1993. An abridged version has been accepted for presentation at the 32nd IEEE Conference on Decision and Control, to be held in San Antonio, Texas, December 15–17, 1993, and will appear in the conference proceedings.

Abstract

We study the robustness of H^∞ controllers, originally designed for nominal linear or nonlinear systems, to unknown static nonlinear perturbations in the state dynamics, the measurement equation, and the performance index. When the nominal system is linear, we consider both perfect state measurements and general imperfect state measurements, and in the case of nominally nonlinear systems, we consider perfect state measurements only. Using a differential game theoretic approach, we show for the former class that as the perturbation parameter (say, $\epsilon > 0$) approaches zero, the optimal disturbance attenuation level for the overall system converges to the optimal disturbance attenuation level for the *nominal* system if the nonlinear structural uncertainties satisfy certain prescribed growth conditions. We also show that the H^∞-optimal controller designed based on a chosen performance level for the *nominal* linear system achieves the same performance level when the parameter ϵ is smaller than a computable threshold, except for the finite-horizon imperfect state measurements case. For that case, we show that the design of the *nominal* controller must be based on a decreased confidence level of the initial data, and a controller thus designed again achieves a desired performance level in the face of nonlinear perturbations satisfying a computable norm bound. In the case of nominally nonlinear systems, and assuming that the *nominal* system is solvable, we obtain sufficient conditions such that the nominal controller achieves a desired performance in the face of perturbations satisfying computable norm bounds. In this way, we provide a characterization of the class of uncertainties that are tolerable for a controller designed based on the nominal system. The paper also presents two numerical examples; in one of these the nominal system is linear, and in the other one it is nonlinear.

Key Words. Minimax controllers, robustness, nonlinear systems, H^∞ control, differential games.

1 Introduction

After the completion of a satisfactory theory for linear H^∞-optimal control, using either frequency-domain or state-space methods, attention has been focused more recently on more general classes of problems, where the objective is either to develop a counterpart of the linear theory for nonlinear systems (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9]) or to study performance robustness of the linear theory to structural perturbations ([10], [11], [12], [13], [14], [15]) In these extensions the state-space framework, and especially the dynamic game theoretic approach ([16]) has been adopted, as it provides the most natural setting for a study of these problems.

In the present paper, our objective is to advance the current theory on both fronts, (i.e., contribute to nonlinear control as well as to performance robustness) by performing a careful analysis of a special class of nonlinear control problems, in terms of a small (regular perturbation) parameter, say ϵ. More specifically, we first consider a class of nonlinear systems which are (nonlinearly) perturbed versions of linear systems, where the small nonlinear perturbations are present in the system dynamics, measurement equation, and the performance index. The perturbation terms are linear in the control and disturbance, and generally nonlinear in the state, and satisfy some growth conditions. For this class our prime objective is to study the performance robustness of an H^∞ controller designed for the nominal system — which we do in both finite and infinite horizons, and for both perfect and imperfect state measurements. One of the objectives is to show that to a given central H^∞ controller there corresponds a computable norm bound on the perturbations, so that for all perturbations satisfying that bound the attained performance level of the nominal systems is attained also for the perturbed system. Another objective is to show the continuity of the optimal performance level for the perturbed systems when the perturbations asymptotically vanish. Following this, we take the nominal system to be nonlinear, but solvable in state feedback, in the sense that the underlying minimax control problem admits a well-defined Isaacs equation from which the "H^∞-optimal" controller can be constructed. For this class, we again study performance robustness, under the same type of nonlinear perturbations as above.

The results obtained are quite comprehensive and "positive". In the case of nominal linear systems with perfect state measurements, we show that the optimal performance level for the perturbed system converges to the optimal performance level for the nominal linear system as the perturbations asymptotically vanish. Furthermore, the H^∞-optimal controller designed for the nominal linear system achieves the desired performance level for the nonlinear system, when the nonlinearities are "relatively small". In the case of nominal linear systems with imperfect state measurements, we again show that the optimal performance level for the perturbed system converges to the optimal performance level of the nominal linear system, as perturbations vanish. In this case, and when the horizon is finite, the controller designed for a certain achievable performance level for the linear system is not guaranteed to yield the same performance level for the nonlinear system even if the perturbations are small. We show, however, that an appropriate design still exists, which requires a modification on the nominal control law. For the infinite-horizon case, such a discrepancy does not exist, and the controller designed for the nominal system is robust with respect to small perturbations. When the nominal system is nonlinear, we obtain a set of sufficient conditions for the H^∞-optimal controller designed for the nominal system to be robust, and identify the class of nonlinear perturbations that can be tolerated by such a controller.

The balance of the present paper is organized as follows. In the next section (Section 2) we formulate the H^∞-optimal control problem for nominally linear systems subject to small nonlinear static perturbations. In Section 3, we solve the problem formulated in Section 2 under perfect state measurements. The same problem under imperfect state measurements is solved in Section 4, which is followed by the treatment of the nominally nonlinear problem in Section 5. We include two illustrative examples in Section 6, and the paper ends with the concluding remarks of Section 7, and three Appendices, the first of which lists and proves some useful robustness results on generalized algebraic and differential Riccati equations, and the other two present some auxiliary results used in the main body of the paper.

2 Problem Formulation

The nonlinearly perturbed linear system under consideration is described by

$$\left\{ \begin{array}{rcl} \dot{x} &=& A(t)x + B(t)u + D(t)w + \epsilon(a(t,x) + b(t,x)u + d(t,x)w) \\ y &=& C(t)x + E(t)w + \epsilon(c(t,x) + n(t,x)u + e(t,x)w) \end{array} \right. \tag{2.1}$$

where x is the n-dimensional state vector; y is the measured output; u is the control input, and w is the disturbance, each belonging to appropriate ($\mathcal{L}^2$) Hilbert spaces $\mathcal{H}_x$, $\mathcal{H}_y$, $\mathcal{H}_u$ and $\mathcal{H}_w$, respectively, defined on the time interval $[t_0, t_f]$. The nonlinear perturbation terms, a, b, d, c, n, and e are functions of both t and x, satisfying some growth conditions to be specified shortly; ϵ is a small scalar quantity, and the initial state for the system is x_0.

In the perfect state measurement case, the initial state is taken to be *zero*, and the control input is generated by a closed-loop policy μ according to

$$u(t) = \mu(t, x_{[t_0,t]}) \tag{2.2}$$

where $\mu : [t_0, t_f] \times \mathcal{H}_x \to \mathcal{H}_u$ is piecewise continuous in t and Lipschitz continuous in x, further satisfying the given causality condition. Let us denote the class of all these controllers by $\mathcal{M}$. With this system, we associate the finite-horizon performance index:[1]

$$\begin{aligned} L(u,w;\epsilon) &= \int_{t_0}^{t_f} \left(|x(t)|^2_{Q(t)} + |u(t)|^2 + \epsilon(q(t,x(t)) + u(t)'r(t,x(t))u(t)) \right) dt \\ &\quad + |x(t_f)|^2_{Q_f}; \end{aligned} \tag{2.3}$$

where q and r are nonquadratic perturbation terms, which satisfy some growth conditions to be specified shortly. When the control (2.2) is substituted into L, let us denote the resulting function (on $\mathcal{M} \times \mathcal{H}_w$) by $J_\epsilon(\mu, w)$. For each ϵ, by direct analogy with the linear problem, the "H^∞-optimal control problem" is the minimization of the quantity

[1] Here, $|x|^2_Q := x'Qx$, a convention that applies throughout.

$$\sup_{w\in\mathcal{H}_w} \{[J_\epsilon(\mu, w)]^{1/2}/\|w\|\} \tag{2.4}$$

over all permissible controllers μ, and in the case a minimum does not exist, the derivation of a controller μ that will ensure a performance in a given neighborhood of the infimum of (2.4). Let us denote this infimum by $\gamma^*(\epsilon)$, i. e.

$$\inf_{\mu\in\mathcal{M}} \sup_{w\in\mathcal{H}_w} \{[J_\epsilon(\mu, w)]^{1/2}/\|w\|\} = \gamma^*(\epsilon) \tag{2.5}$$

where we explicitly show the dependence of γ^* on the perturbation parameter ϵ.

It has been shown in [16] that the above robust control problem is closely related to a class of zero-sum differential games with game kernel:

$$L_\gamma(u, w; \epsilon) \;\; = \;\; L(u, w; \epsilon) - \gamma^2\|w\|^2 \tag{2.6}$$

In the imperfect state measurement case, the initial state is considered to be unknown and is taken as a part of the disturbance. In this case, the control input u is generated by a control policy μ_I, according to

$$u(t) = \mu_I(t, y_{[t_0,t]}) \tag{2.7}$$

where $\mu_I : [t_0, t_f] \times \mathcal{H}_y \to \mathcal{H}_u$ is piecewise continuous in t and Lipschitz continuous in y, further satisfying the given causality condition. Let us denote the class of all such controllers by $\mathcal{M}_I$. With this system, we associate the performance index $L(u, w, x_0; \epsilon)$, defined exactly as in (2.3). Let $J_\epsilon(\mu_I, w, x_0)$ be the counterpart of $J_\epsilon(\mu, w)$ in this case. For each fixed ϵ, the "H^∞-optimal control problem" is again the minimization of the quantity

$$\sup_{w\in\mathcal{H}_w;\, x_0\in\mathcal{H}_x} \{[J_\epsilon(\mu_I, w, x_0)]^{1/2}/[\|w\|^2 + |x_0|^2_{Q_0}]^{1/2}\} \tag{2.8}$$

over all permissible controllers μ_I, or if a minimum does not exist, the derivation of a controller μ_I that will ensure a performance in a given neighborhood of the infimum of (2.8). Let us denote this infimum by $\gamma_I^*(\epsilon)$, i. e.,

$$\inf_{\mu_I\in\mathcal{M}_I} \sup_{w\in\mathcal{H}_w;\, x_0\in\mathcal{H}_x} \{[J_\epsilon(\mu_I, w, x_0)]^{1/2}/[\|w\|^2 + |x_0|^2_{Q_0}]^{1/2}\} := \gamma_I^*(\epsilon). \tag{2.9}$$

The associated zero-sum differential game problem admits the cost function:

$$L_{I\gamma}(u, w, x_0; \epsilon) \;\; = \;\; L(u, w, x_0; \epsilon) - \gamma^2(|x_0|^2_{Q_0} + \|w\|^2) \tag{2.10}$$

We now impose the following conditions on the parameters of the problem:

Assumption 1 The matrices $A(t)$, $B(t)$, $D(t)$, $C(t)$, $E(t)$ and $Q(t)$ are piecewise continuous in t on $[t_0, t_f]$, and $Q(t) \geq 0$, $Q_0 > 0$, $Q_f \geq 0$, $D(t)E(t)' = 0$ and $N(t) := E(t)E(t)' > 0$. $\diamond$

Assumption 2 The nonlinear matrix-valued functions, of proper dimensions, $a(t,x)$, $b(t,x)$, $d(t,x)$, $c(t,x)$, $n(t,x)$, $e(t,x)$, $q(t,x)$ and $r(t,x)$ are piecewise continuous in t and locally Lipschitz continuous in x, and further satisfy the following growth conditions: There exist positive constants $M_a, M_b, M_d, M_c, M_n, M_e, M_q$ and M_r, such that

$$
\begin{array}{llll}
|a(t,x)| \le M_a|x|, & |b(t,x)| \le M_b, & |d(t,x)| \le M_d, & \\
|c(t,x)| \le M_c|x|, & |n(t,x)| \le M_n, & |e(t,x)| \le M_e, & \forall t \in [t_0,t_f], \quad x \in \mathcal{R}^n \\
|q(t,x)| \le M_q|x|^2, & |r(t,x)| \le M_r. & &
\end{array}
$$

where $|\cdot|$ denotes the Euclidean norm on vectors or its induced norm on matrices, the matrix $r(t,x)$ is symmetric, and $q(t,x) \ge 0$. ◇

For the infinite-horizon case (i. e., as $t_f \to \infty$ and $t_0 = 0$ in the perfect state measurement case, and $t_0 \to -\infty$ in the imperfect state measurement case, as well as when $t_f = \infty$ and $t_0 = 0$ in the perfect state measurement case, and $t_0 = -\infty$ in the imperfect state measurement case), we take A, B, D, C, E, Q to be time-invariant, $Q_f = 0$ and $Q_0 = 0$, and require that $x(t) \to 0$ as $t \to -\infty$ and $t \to \infty$. We furthermore impose the following standard conditions on the nominal linear system:

Assumption 3a The matrix pair (A,B) is controllable, and the pair (A,Q) is observable.

Assumption 3b The matrix pair (A,D) is controllable, and the pair (A,C) is observable. ◇

One of our objectives in this paper is to study (for both the perfect and imperfect state measurement cases) the dependence of the optimal performance level on the perturbation parameter ϵ, as $\epsilon \to 0$, and design controllers, independent of ϵ, that attain desired performance levels for the perturbed system for for all values of ϵ not exceeding a computable bound.

We now first study, in the next section, the perfect state measurement case, in both finite and infinite horizons.

3 Perfect State Measurements

The finite-horizon case

The H^∞-optimal control problem formulated in the previous section is solvable for the nominal linear system, for a fixed $\gamma > 0$, if the GRDE on the time interval $[t_0, t_f]$:

$$
\dot{Z} + A'Z + ZA - Z(BB' - \frac{1}{\gamma^2}DD')Z + Q = 0 \quad Z(t_f) = Q_f \tag{3.1}
$$

admits a nonnegative definite solution $Z_\gamma(t)$, which implies that $\gamma > \gamma^*(0)$. For $\gamma < \gamma^*(0)$, on the other hand, the GRDE (3.1) has at least one conjugate point in the interval $[t_0, t_f]$ and the value of the soft-constrained zero-sum differential game is infinite for the nominal linear system. Fix a $\gamma > \gamma^*(0)$; then, an H^∞ controller for the nominal linear system, guaranteeing this level of performance, is:

$$
\mu^*_\gamma(t,x) = -B(t)'Z_\gamma(t)x \tag{3.2}
$$

To study the robustness of this controller with respect to small nonlinear perturbations, we substitute the controller μ_γ^* into the system dynamics (2.1), as well as the game kernel (2.6), and arrive at the following maximization problem (with respect to $w \in \mathcal{H}_w$), where we suppress the dependence of Z on γ:

$$\dot{x} = (A - BB'Z)x + Dw + \epsilon(a - bB'Zx + dw) \tag{3.3}$$

$$L_\gamma^* = |x(t_f)|^2_{Q_f} + \int_{t_0}^{t_f} (|x|^2_{Q+ZBB'Z} + \epsilon(q(t,x) + x'ZBr(t,x)B'Zx) - \gamma^2|w|^2)\, dt \tag{3.4}$$

Consider the following related partial differential inequality:

$$V_{\gamma t} + V_{\gamma x}((A - BB'Z)x + \epsilon(a - bB'Zx)) + \frac{1}{4\gamma^2}|(D' + \epsilon d')V'_{\gamma x}|^2 + x'Qx + x'ZBB'Zx$$
$$+\epsilon(q + x'ZBrB'Zx) \leq 0; \quad V_\gamma(t_f, x; \epsilon) = x'Q_f x, \tag{3.5}$$

If $V_\gamma(t,x;\epsilon)$ is a solution to (3.5), then, by the standard "completion of squares" argument [17], $V_\gamma(t,x;\epsilon)$ constitutes an upper bound for the value function of the maximization problem (3.3)–(3.4), and hence the controller given by (3.2) attains the performance level γ for the perturbed system.

Extending this bounding technique further, we now consider a related partial differential inequality for the perturbed system without substituting the controller μ_γ^*:

$$W_{\gamma t} + W_{\gamma x}(Ax + \epsilon a) - \frac{1}{4}|(I + \epsilon r)^{-1}(B' + \epsilon b')W'_{\gamma x}|^2_{I+\epsilon r} + \frac{1}{4\gamma^2}|(D' + \epsilon d')W'_{\gamma x}|^2$$
$$+x'Qx + \epsilon q \geq 0; \quad W_\gamma(t_f, x) = x'Q_f x, \tag{3.6}$$

If there exists a solution $W_\gamma(t,x,\epsilon)$ to (3.6), then, again by the "completion of squares" argument, $W_\gamma(t,x,\epsilon)$ this time constitutes a lower bound for the value function of the zero-sum differential game defined by (2.1) and (2.6), for sufficiently small ϵ such that $I + \epsilon r > 0$. As an illustration of the "completion of squares" method, we will carry out the following algebraic manipulations:

$$\begin{aligned} L_\gamma &= L_\gamma(u,w;\epsilon) + \int_{t_0}^{t_f} \frac{d}{dt} W_\gamma\, dt - W_\gamma(t_f, x(t_f);\epsilon) + W_\gamma(t_0, 0;\epsilon) \\ &= W_\gamma(t_0,0;\epsilon) + \int_{t_0}^{t_f} (|x|^2_Q + |u|^2 + \epsilon(q(t,x) + u'r(t,x)u) - \gamma^2|w|^2 + W_{\gamma t} \\ &\quad + W_{\gamma x}(A(t)x + B(t)u + D(t)w + \epsilon(a(t,x) + b(t,x)u + d(t,x)w)))\, dt \\ &\geq W_\gamma(t_0,0;\epsilon) + \int_{t_0}^{t_f} (|u + \frac{1}{2}(I + \epsilon r)^{-1}(B' + \epsilon b')W'_{\gamma x}|^2_{I+\epsilon r} \\ &\quad -\gamma^2|w - \frac{1}{2\gamma^2}(D' + \epsilon d')W'_{\gamma x}|^2)\, dt \end{aligned}$$

from which it follows that

$$\inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{H}_w} L_\gamma \geq \sup_{w \in \mathcal{H}_w} \inf_{\mu \in \mathcal{M}} L_\gamma \geq W_\gamma(t_0, 0; \epsilon).$$

This argument holds even if the initial time t_0 is replaced by any $t \in [t_0, t_f]$ and the intial state $x_0 = 0$ is replaced by some arbitrary $x \in \mathcal{R}^n$. Hence, $W_\gamma(t, x, \epsilon)$ is a lower bound for the value function of the zero-sum differential game.

Now, we can prove the following theorem, which states in precise terms the relationship between the achievable performances for the nominal and perturbed systems.

Theorem 1 *Consider the H^∞-optimal control problem formulated in (2.1)–(2.5), under Assumptions 1 and 2. Then, we have the following:*

1. $\limsup_{\epsilon \to 0} \gamma^*(\epsilon) \le \gamma^*(0)$.
2. $\forall \gamma > \gamma^*(0)$, $\exists \epsilon_\gamma > 0$ *such that the control policy μ^*_γ, defined in (3.2), achieves the performance level γ for the perturbed system,* $\forall |\epsilon| < \epsilon_\gamma$. *Hence,* $\gamma \ge \gamma^*(\epsilon)$, $\forall |\epsilon| < \epsilon_\gamma$.
3. *If, in addition,* $Q_f > 0$, *then,* $\forall \gamma < \gamma^*(0)$, $\exists \epsilon'_\gamma > 0$ *such that* $\forall |\epsilon| < \epsilon'_\gamma$, *no control policy achieves the performance level γ for the perturbed system; hence* $\lim_{\epsilon \to 0} \gamma^*(\epsilon) = \gamma^*(0)$.

Proof It is well-known ([16], Chapter 4, Section 2) that $\gamma^*(0) < \infty$.

To prove part 2), we fix a $\gamma > \gamma^*(0)$, and seek a solution to the inequality (3.5) in the form $x'K(t)x$. Note that we are seeking a solution that is independent of ϵ.

By standard (continuity) results on ordinary differential equations, for some $\delta > 0$, there exists a solution $Z_\delta(t)$ to the following GRDE on the interval $[t_0, t_f]$:

$$\dot{Z}_\delta + (A' - ZBB')Z_\delta + Z_\delta(A - BB'Z) + \frac{1}{\gamma^2} Z_\delta DD' Z_\delta + Q + ZBB'Z + \delta I = 0$$
$$Z_\delta(t_f) = Q_f. \tag{3.7}$$

Since GRDE (3.1) admits a nonnegative definite solution $Z(t)$, and this solution satisfies (3.7) for $\delta = 0$, then the matrix $Z_\delta(t)$ is nonnegative definite for each $t \in [t_0, t_f]$; in fact, $Z_\delta(t) \ge Z(t)$.

Substitute $V_\gamma(t, x; \epsilon) = x' Z_\delta(t) x$ into LHS of (3.5), to obtain:

$$\begin{aligned} \text{LHS} \;&=\; -\delta x'x + \epsilon(2x'Z_\delta(a - bB'Zx) + \frac{2}{\gamma^2} x'Z_\delta Dd' Z_\delta x + \epsilon \frac{1}{\gamma^2} x' Z_\delta dd' Z_\delta x \\ &\quad + q + x'ZBrB'Zx) \le 0 \quad \forall (t, x) \in [t_0, t_f] \times \mathcal{R}^n \end{aligned}$$

Hence, under Assumption 2, the inequality (3.5) holds for sufficiently small ϵ. Note also that $V_\gamma(t_0, 0; \epsilon) = 0$, which implies that the maximization problem (3.3)–(3.4) has a value bounded above by 0. This completes the proof for part 2).

It is obvious that part 2) implies part 1).

For part 3), first fix a $\gamma < \gamma^*(0)$. Note that $\gamma < \gamma^*(0)$ implies that the GRDE (3.1) has at least one conjugate point on the time interval $[t_0, t_f]$. By Lemma 1 of Appendix A, $\exists T \in [t_0, t_f)$, such that the following GRDE:

$$\dot{Z}^\# - AZ^\# - Z^\# A' - Z^\# Q Z^\# + BB' - \frac{1}{\gamma^2} DD' = 0 \quad Z^\#(t_f) = Q_f^{-1} \tag{3.8}$$

admits a symmetric solution $Z^{\#}(t)$ on the time interval $[T, t_f]$ and the matrix $Z^{\#}(T)$ has at least one negative eigenvalue. Let T_1 be the largest conjugate point of $Z(t)$; then, $Z(t)$ exists and is equal to $Z^{\#}(t)^{-1}$ on $(T_1, t_f]$.

By a standard result on ordinary differential equations, for some small positive scalar δ, there exists a solution $Z_\delta^{\#}(t)$ to the following GRDE on the time interval $[T, t_f]$:

$$\dot{Z}_\delta^{\#} - AZ_\delta^{\#} - Z_\delta^{\#}A' - Z_\delta^{\#}(Q - \delta I)Z_\delta^{\#} + BB' - \frac{1}{\gamma^2}DD' + \delta I = 0; \quad Z_\delta^{\#}(t_f) = Q_f^{-1} \tag{3.9}$$

such that $Z_\delta^{\#}(T)$ has at least one negative eigenvalue. It should be noted here that $Z_\delta^{\#}(t) \geq Z^{\#}(t)$ on $[T, t_f]$. Define T_δ as

$$T_\delta := \sup\{t \in [T, t_f] : Z_\delta^{\#}(t) \text{ has at least one nonpositive eigenvalue}\}$$

Thus, the following GRDE:

$$\dot{\tilde{Z}}_\delta + A'\tilde{Z}_\delta + \tilde{Z}_\delta A - \tilde{Z}_\delta(BB' - \frac{1}{\gamma^2}DD' + \delta I)\tilde{Z}_\delta + Q - \delta I = 0; \quad \tilde{Z}_\delta(t_f) = Q_f \tag{3.10}$$

admits a solution $\tilde{Z}_\delta(t)$ on the time interval $(T_\delta, t_f]$, which is equal to $Z_\delta^{\#}(t)^{-1}$, and has a conjugate point at T_δ. Now, substitute $x'\tilde{Z}_\delta(t)x$ for $W_\gamma(t, x, \epsilon)$ in (3.6). After some straightforward algebraic manipulations, we obtain

$$\begin{aligned}
\text{LHS} &= x'(\dot{\tilde{Z}}_\delta + A'\tilde{Z}_\delta + \tilde{Z}_\delta A - \tilde{Z}_\delta(BB' - \frac{1}{\gamma^2}DD')\tilde{Z}_\delta + Q)x \\
&\quad + \epsilon(2x'\tilde{Z}_\delta a - 2x'\tilde{Z}_\delta(Bb' - \frac{1}{\gamma^2}Dd')\tilde{Z}_\delta x \\
&\quad - \epsilon x'\tilde{Z}_\delta(bb' - \frac{1}{\gamma^2}dd')\tilde{Z}_\delta x + x'\tilde{Z}_\delta(B + \epsilon b)r(I + \epsilon r)^{-1}(B' + \epsilon b')\tilde{Z}_\delta x + q) \\
&= \delta x'x + \delta x'\tilde{Z}_\delta\tilde{Z}_\delta x + \epsilon(2x'\tilde{Z}_\delta a - 2x'\tilde{Z}_\delta(Bb' - \frac{1}{\gamma^2}Dd')\tilde{Z}_\delta x \\
&\quad - \epsilon x'\tilde{Z}_\delta(bb' - \frac{1}{\gamma^2}dd')\tilde{Z}_\delta x + x'\tilde{Z}_\delta(B + \epsilon b)r(I + \epsilon r)^{-1}(B' + \epsilon b')\tilde{Z}_\delta x + q) \\
&= x'\tilde{Z}_\delta(\frac{\delta}{2}I - 2\epsilon(Bb' - \frac{1}{\gamma^2}Dd') + \epsilon(B + \epsilon b)r(I + \epsilon r)^{-1}(B' + \epsilon b') \\
&\quad - 2\epsilon^2(bb' - \frac{1}{\gamma^2}dd'))\tilde{Z}_\delta x + \frac{\delta}{2}x'\tilde{Z}_\delta\tilde{Z}_\delta x + 2\epsilon x'\tilde{Z}_\delta a + \delta x'x + q \\
&\geq 0 \quad \forall (t, x) \in (T_\delta, t_f] \times \mathcal{R}^n
\end{aligned}$$

where, under Assumption 2, the inequality holds for sufficiently small values of ϵ. Hence, the function $x'\tilde{Z}_\delta(t)x$ provides a lower bound for the value function of the soft-constrained zero-sum differential game on the interval $(T_\delta, t_f]$. Since the matrix $\tilde{Z}_\delta(t)$ has a conjugate point at T_δ, it follows that, the value of the game is infinite. Hence, $\gamma \leq \gamma^*(\epsilon)$.

Note that the above implies that $\liminf_{\epsilon \to 0} \gamma^*(\epsilon) \geq \gamma^*(0)$. Coupled with the result of part 1), this completes the proof of part 3). □

Remark 1 *The statement 3) of the above theorem holds under the condition $Q_f > 0$. In the case of Q_f is instead just nonnegative definite, the case is much more difficult to show. For the special case of time-invariant system, however, validity of statement 3) can be established following derivations very similar to those that will be used later to prove statement 3) of Theorem 2 in the infinite-horizon case. It can be shown to hold also if the $Z(t) >$ for all $t < t_f$. In the most general case, the validity of the statement is not known.* ◇

We now obtain a lower bound on the maximal allowable ϵ_γ defined in Theorem 1. Let $P(t)$ be the solution to the following Lyapunov equation:

$$\dot{P} + (A' - Z(BB' - \frac{1}{\gamma^2}DD'))P + P(A - (BB' - \frac{1}{\gamma^2}DD')Z) + I = 0; \quad P(t_f) = 0 \quad (3.11)$$

It is then straightforward to verify that $K_\delta := Z + \delta P$ satisfies the GRDE:

$$\dot{K}_\delta + (A' - ZBB')K_\delta + K_\delta(A - BB'Z) + \frac{1}{\gamma^2}K_\delta DD'K_\delta$$
$$+Q + ZBB'Z + \delta I - \frac{\delta^2}{\gamma^2}PDD'P = 0; \qquad K_\delta(t_f) = Q_f.$$

In view of this, the inequality (3.5) is satisfied by $x'K_\delta x$ if the following scalar inequality holds:

$$\begin{aligned}\delta \;\geq\; & |\epsilon|(2(|Z| + \delta|P|)(M_a + M_b|B||Z|) + \frac{2}{\gamma^2}(|Z| + \delta|P|)^2|D|M_d + M_r|Z|^2|B|^2 + M_q) \\ & + \frac{\epsilon^2}{\gamma^2}M_d^2(|Z| + \delta|P|)^2 + \frac{\delta^2}{\gamma^2}|P|^2|D|^2 \end{aligned} \quad (3.12)$$

Hence, we have the following result:

Corollary 1 *A lower bound for the maximal allowable value of the quantity ϵ_γ used in Theorem 1 is*

$$\epsilon_\gamma^* = \sup\{\epsilon > 0 : \quad \exists\delta > 0 \textit{ such that inequality (3.12) is satisfied } \}$$

The infinite-horizon case

For the nominal linear system, i.e., with $\epsilon = 0$ in the perturbed system, the optimal H^∞-performance level is $\gamma_\infty^*(0) < \infty$. For $\gamma > \gamma_\infty^*(0)$, the following GARE:

$$A'\bar{Z} + \bar{Z}A - \bar{Z}(BB' - \frac{1}{\gamma^2}DD')\bar{Z} + Q = 0 \quad (3.13)$$

admits a minimal nonnegative definite solution $\bar{Z}_\gamma$, and a control policy that achieves the disturbance attenuation level γ for the nominal linear system is:

$$\mu_{\gamma\infty}^*(x) = -B'\bar{Z}_\gamma x \quad (3.14)$$

Again, the robustness of this controller with respect to the nonlinear perturbations can be established by the existence of a nonnegative solution to the following partial differential inequality:

$$V_{\gamma t} + V_{\gamma x}((A - BB'Z)x + \epsilon(a - bB'Zx)) + \frac{1}{4\gamma^2}|(D' + \epsilon d')V'_{\gamma x}|^2 + x'Qx$$
$$+x'ZBB'Zx + \epsilon(q + x'ZBrB'Zx) \leq 0 \tag{3.15}$$

We have the following counterpart of Theorem 1, which states again in precise terms the relationship between the achievable performances of the nominal and perturbed systems.

Theorem 2 *Consider the H^∞-optimal control problem formulated in (2.1)–(2.5) with $t_0 = 0$, $t_f = \infty$ (as well as $t_f \to \infty$), and the matrices A, B, D, Q being time-invariant, $Q_f = 0$. Let Assumptions 1, 2 and 3a hold. Then,*

1. $\lim_{\epsilon \to 0} \gamma^*_\infty(\epsilon) = \gamma^*_\infty(0)$.

2. $\forall \gamma > \gamma^*_\infty(0)$, $\exists \epsilon_\gamma > 0$ *such that the optimal controller for the nominal linear system, $\mu^*_{\gamma\infty}$, achieves the performance level γ for the perturbed system, $\forall |\epsilon| < \epsilon_\gamma$, and internally stabilizes it. Hence, $\gamma \geq \gamma^*_\infty(\epsilon)$, $\forall |\epsilon| < \epsilon_\gamma$.*

3. $\forall \gamma < \gamma^*_\infty(0)$, $\exists \epsilon'_\gamma > 0$ *such that $\forall |\epsilon| < \epsilon'_\gamma$, no control policy achieves the performance level γ for the perturbed system, and hence, $\gamma \leq \gamma^*_\infty(\epsilon)$.*

Proof We first note the known fact that, under Assumption 3, $\gamma^*_\infty(0) < \infty$.

To prove part 2) of this theorem, we fix a $\gamma > \gamma^*_\infty(0)$ and seek a solution to inequality (3.15) in the form $x'Kx$, where K is some nonnegative definite matrix.

By the observability of the pair (A, Q), we have $\bar{Z} > 0$. By Corollary 7 of Appendix A, the matrix $A - (BB' - \frac{1}{\gamma^2}DD')\bar{Z}$ is Hurwitz. Now, consider the following GARE:

$$(A - BB'\bar{Z})'\bar{Z}_\delta + \bar{Z}_\delta(A - BB'\bar{Z}) + \frac{1}{\gamma^2}\bar{Z}_\delta DD'\bar{Z}_\delta + Q + \bar{Z}BB'\bar{Z} + \delta I = 0 \tag{3.16}$$

The GARE (3.16) admits a solution $\bar{Z}$ for $\delta = 0$. Let $\vec{p}$ be the vector form of $\bar{Z}_\delta$, $\vec{p_0}$ be the vector form of $\bar{Z}$ and view GARE (3.16) as a vector equation $\chi(\vec{p}, \delta) = 0$. Then,

$$\frac{\partial \chi}{\partial \vec{p}}|_{(\vec{p_0},0)} = I \otimes F + F' \otimes I$$

where $F := A - (BB' - \frac{1}{\gamma^2}DD')\bar{Z}$ is Hurwitz. Thus, by the Implicit Function Theorem [18], we can find a $\delta > 0$ such that GARE (3.16) admits a positive definite solution $\bar{Z}_\delta$.

Substitute $V_\gamma(t, x; \epsilon) = x'\bar{Z}_\delta x$ into the LHS of (3.15), to obtain:

$$\text{LHS} = -\delta x'x + \epsilon(2x'\bar{Z}_\delta(a - bB'\bar{Z}x) + \frac{2}{\gamma^2}x'\bar{Z}_\delta Dd'\bar{Z}_\delta x + \epsilon\frac{1}{\gamma^2}x'\bar{Z}_\delta dd'\bar{Z}_\delta x + q$$
$$+x'\bar{Z}BrB'\bar{Z}x) \leq 0, \qquad \forall (t, x) \in [0, \infty) \times \mathcal{R}^n$$

which shows, under Assumption 2, that the inequality holds for sufficiently small ϵ. Hence, the performance level γ can be achieved by using the nominal controller (3.14) for the perturbed system.

To prove part 3) of the theorem, we use a finite horizon argument. Fix a $\gamma < \gamma_\infty^*(0)$. Let $\gamma_1 > \gamma_\infty^*(0)$, and $\bar{Z}_{\gamma_1}$ be the minimal positive definite solution to the GARE:

$$A'\bar{Z}_{\gamma_1} + \bar{Z}_{\gamma_1}A - \bar{Z}_{\gamma_1}(BB' - \frac{1}{\gamma_1^2}DD')\bar{Z}_{\gamma_1} + Q = 0$$

Then, the matrix $A - (BB' - \frac{1}{\gamma_1^2}DD')\bar{Z}_{\gamma_1}$ is Hurwitz, by Corollary 7 of Appendix A. By Corollary 9 of Appendix A, there exists $T_1 > 0$ such that the GRDE:

$$\dot{Z}_{\gamma_1} + A'Z_{\gamma_1} + Z_{\gamma_1}A - Z_{\gamma_1}(BB' - \frac{1}{\gamma_1^2}DD')Z_{\gamma_1} + Q = 0; \quad Z_{\gamma_1}(T_1) = 0$$

admits a nonnegative definite solution $Z_{\gamma_1}(t)$ on $[0, T_1]$, and further satisfies $Z_{\gamma_1}(0) > 0$. By a standard result on ordinary differential equations, the GRDE:

$$\dot{Z}_{\gamma_1\delta} + A'Z_{\gamma_1\delta} + Z_{\gamma_1\delta}A - Z_{\gamma_1\delta}(BB' - \frac{1}{\gamma_1^2}DD')Z_{\gamma_1\delta} + Q - \delta I = 0; \quad Z_{\gamma_1\delta}(T_1) = 0 \quad (3.17)$$

admits a solution $Z_{\gamma_1\delta}(t)$ on $[0, T_1]$, and $Z_{\gamma_1\delta}(0) > 0$, for some $\delta > 0$. We note here that $\bar{Z}_{\gamma_1} \geq Z_{\gamma_1}(t) \geq Z_{\gamma_1\delta}(t), \quad \forall t \in [0, T_1]$.

Substitute $W_\gamma(t, x; \epsilon) = x'Z_{\gamma_1\delta}(t)x$ in inequality (3.6), to obtain

$$\begin{aligned} \text{LHS} &= \delta x'x + \epsilon(2x'Z_{\gamma_1\delta}a - 2x'Z_{\gamma_1\delta}(Bb' - \frac{1}{\gamma^2}Dd')Z_{\gamma_1\delta}x \\ &\quad -\epsilon x'Z_{\gamma_1\delta}(bb' - \frac{1}{\gamma^2}dd')Z_{\gamma_1\delta}x + x'Z_{\gamma_1\delta}(B + \epsilon b)r(I + \epsilon r)^{-1}(B' + \epsilon b')Z_{\gamma_1\delta}x + q) \\ &\geq 0 \qquad \forall (t, x) \in (T_\delta, t_f] \times \mathcal{R}^n \end{aligned}$$

where, under Assumption 2, the inequality holds for sufficiently small values of ϵ. Hence, the function $x'Z_{\gamma_1\delta}(t)x$ provides a lower bound for the value function of the soft-constrained zero-sum differential game with state equation (2.1) and game kernel

$$L_{\gamma T_1} = \int_0^{T_1} (|x|_Q^2 + |u|^2 + \epsilon(q(t, x) + u'r(t, x)u) - \gamma_1^2|w|^2)\, dt$$

for sufficiently small ϵ. Note that the preceding argument holds even if the time interval $[0, T_1]$ is translated to any $[t, t + T_1]$.

For the infinite-horizon soft-constrained zero-sum differential game, we have the following simple relationships:

$$\begin{aligned} \inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{H}_w} L_\gamma &\geq \inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{H}_w} \{ \int_0^T (|x|_Q^2 + |u|^2 + \epsilon(q(t, x) + u'r(t, x)u) - \gamma^2|w|^2)\, dt\} \\ &\geq \inf_{\mu \in \mathcal{M}} \sup_{w \in \mathcal{H}_w} \{\{ \int_0^{T-T_1} (|x|_Q^2 + |u|^2 + \epsilon(q(t, x) + u'r(t, x)u) - \gamma^2|w|^2)\, dt\} \end{aligned}$$

$$+ \inf_{\mu\in\mathcal{M}} \sup_{w\in\mathcal{H}_w} \{ \int_{T-T_1}^{T} (|x|_Q^2 + |u|^2 + \epsilon(q + u'ru) - \gamma_1^2|w|^2)\, dt \} \}$$

$$\geq \inf_{\mu\in\mathcal{M}} \sup_{w\in\mathcal{H}_w} \{ \int_0^{T_2} (|x|_Q^2 + |u|^2 + \epsilon(q + u'ru) - \gamma^2|w|^2)\, dt + x'(T_2) Z_{\gamma_1\delta}(0) x(T_2) \}$$

where $T_2 := T - T_1$, and the last inequality holds for sufficiently small ϵ.

By Lemma 4 of Appendix A, $\forall\gamma_2$ such that $\gamma < \gamma_2 < \gamma_\infty^*(0)$, and a fixed large enough T_2, the GRDE:

$$\dot{Z} + A'Z + ZA - Z(BB' - \frac{1}{\gamma_2^2} DD')Z + Q = 0; \quad Z(T_1) = Z_{\gamma_1\delta}(0)$$

admits at least one conjugate point in $[0, T_2)$. Thus, by part 2) of Theorem 1, we have

$$\inf_{\mu\in\mathcal{M}} \sup_{w\in\mathcal{H}_w} \{ \int_0^{T_2} (|x|_Q^2 + |u|^2 + \epsilon(q + u'ru) - \gamma^2|w|^2)\, dt + x'(T_2) \tilde{Z}_\delta(0) x(T_2) \} = \infty$$

Hence, part 3) follows.

Part 2) implies that $\limsup_{\epsilon\to 0} \gamma_\infty^*(\epsilon) \leq \gamma_\infty^*(0)$, and part 3) implies that $\liminf_{\epsilon\to 0} \gamma_\infty^*(\epsilon) \geq \gamma_\infty^*(0)$. Hence, this verifies part 1). □

To obtain a lower bound for the maximal allowable ϵ_γ of Theorem 2, let $\bar{P}$ be the positive definite solution to the following Lyapunov equation:

$$(A' - \bar{Z}(BB' - \frac{1}{\gamma^2} DD'))\bar{P} + \bar{P}(A - (BB' - \frac{1}{\gamma^2} DD')\bar{Z}) + I = 0 \tag{3.18}$$

It is then straightforward to verify that $\bar{K}_\delta := \bar{Z} + \delta\bar{P}$ provides a solution to the following GARE:

$$(A' - \bar{Z}BB')\bar{K}_\delta + \bar{K}_\delta(A - BB'\bar{Z}) + \frac{1}{\gamma^2} \bar{K}_\delta DD' \bar{K}_\delta$$
$$+ Q + \bar{Z}BB'\bar{Z} + \delta I - \frac{\delta^2}{\gamma^2} \bar{P}DD'\bar{P} = 0$$

In view of this, the inequality (3.15) is satisfied by $x'\bar{K}_\delta x$ if the following scalar inequality is satisfied:

$$\delta \;\geq\; |\epsilon|(2(|\bar{Z}| + \delta|\bar{P}|)(M_a + M_b|B||\bar{Z}|) + \frac{2}{\gamma^2}(|\bar{Z}| + \delta|\bar{P}|)^2|D|M_d + M_r|\bar{Z}|^2|B|^2 + M_q)$$
$$+ \frac{\epsilon^2}{\gamma^2} M_d^2 (|\bar{Z}| + \delta|\bar{P}|)^2 + \frac{\delta^2}{\gamma^2} |\bar{P}|^2|D|^2 \tag{3.19}$$

Hence, we have the following result:

Corollary 2 *A lower bound for the maximal allowable value of the perturbation constant ϵ_γ of Theorem 2 is*

$$\epsilon_\gamma^* = \sup\{\epsilon > 0 : \quad \exists\delta > 0 \textit{ such that inequality (3.19) is satisfied } \}$$

4 Imperfect State Measurements

In this section, we turn to the study of the imperfect state measurement problem formulated in Section 2. Again we first consider the finite-horizon case, and subsequently the infinite-horizon problem.

The finite-horizon case

We consider the H^∞-optimal control problem formulated by (2.1), (2.7)–(2.9).

When $\epsilon = 0$, this problem has been solved completely (see, for example, Chapter 5 of [16]). For a fixed γ, any controller that renders a value 0 for the nominal soft-constrained zero-sum differential game also guarantees the disturbance attenuation level γ for the nominal linear system in the H^∞ sense. For any $\gamma > \gamma_I^*(0)$, the optimal control policy achieving the desired performance level is

$$\mu_{I\gamma}^*(t, y_{[t_0,t]}) = -B(t)'Z(t)x_a \tag{4.1}$$

$$\dot{x}_a = (A - (BB' - \frac{1}{\gamma^2}DD')Z)x_a + (I - \frac{1}{\gamma^2}\Sigma Z)^{-1}\Sigma C'N^{-1}(y - Cx_a); \quad x_a(t_0) = 0 \tag{4.2}$$

where $Z(t)$ is the nonnegative definite solution to GRDE (3.1) on the time interval $[t_0, t_f]$, and $\Sigma(t)$ is the positive definite solution to the following GRDE:

$$\dot{\Sigma} = A\Sigma + \Sigma A' - \Sigma(C'N^{-1}C - \frac{1}{\gamma^2}Q)\Sigma + DD'; \quad \Sigma(t_0) = Q_0^{-1} \tag{4.3}$$

on the time interval $[t_0, t_f]$, which further satisfies the spectral radius condition

$$\gamma^2\Sigma(t)^{-1} > Z(t); \quad \forall t \in [t_0, t_f]. \tag{4.4}$$

For any $\gamma < \gamma_I^*(0)$, either at least one of the GRDE's (3.1) or (4.3) has a conjugate point in the interval $[t_0, t_f]$, or the spectral radius condition (4.4) is violated for some $t \in [t_0, t_f]$.

In fact, the nominal controller $\mu_{I\gamma}^*$ cannot always achieve the desired performance level γ for the perturbed system even if ϵ is taken to be arbitrarily small. To achieve the desired performance level $\gamma > \gamma_I^*(0)$ for the perturbed system, we introduce the following GRDE to replace (4.3) (for reasons that will become clear in the proof of Theorem 3):

$$\dot{\tilde{\Sigma}} = A\tilde{\Sigma} + \tilde{\Sigma}A' - \tilde{\Sigma}(C'N^{-1}C - \frac{1}{\gamma^2}Q)\tilde{\Sigma} + DD'; \quad \tilde{\Sigma}(t_0) = (Q_0 - \eta I)^{-1} \tag{4.5}$$

where $\eta > 0$ is a scalar parameter. Since $Q_0 > 0$, it is obvious that we can choose η sufficiently small such that $Q_0 - \eta I > 0$ and the GRDE (4.5) admits a positive definite solution $\tilde{\Sigma}(t)$ on the time interval $[t_0, t_f]$, which further satisfies inequality (4.4). Then, the control law that will be shown to achieve the performance level γ for the perturbed system for sufficiently small ϵ is

$$\tilde{\mu}^*_{I\gamma}(t, y_{[t_0,t]}) = -B(t)'Z(t)\hat{x} \tag{4.6}$$

$$\dot{\hat{x}} = (A - (BB' - \frac{1}{\gamma^2}DD')Z)\hat{x} + (I - \frac{1}{\gamma^2}\tilde{\Sigma}Z)^{-1}\tilde{\Sigma}C'N^{-1}(y - C\hat{x}); \quad \hat{x}(0) = 0 \tag{4.7}$$

By applying this controller to system (2.1), and also using it in the soft-constrained game kernel, we arrive at the following maximization problem (with respect to $w \in \mathcal{H}_w$):

$$\begin{aligned}
\dot{x}^e \;&=\; \begin{bmatrix} A & -BB'Z \\ (I - \frac{1}{\gamma^2}\tilde{\Sigma}Z)^{-1}\tilde{\Sigma}C'N^{-1}C & \Pi \end{bmatrix} x^e + \begin{bmatrix} D \\ (I - \frac{1}{\gamma^2}\tilde{\Sigma}Z)^{-1}\tilde{\Sigma}C'N^{-1}E \end{bmatrix} w \\
&\quad + \epsilon \begin{bmatrix} a - bB'Z\hat{x} \\ (I - \frac{1}{\gamma^2}\tilde{\Sigma}Z)^{-1}\tilde{\Sigma}C'N^{-1}(c - nB'Z\hat{x}) \end{bmatrix} + \epsilon \begin{bmatrix} d \\ (I - \frac{1}{\gamma^2}\tilde{\Sigma}Z)^{-1}\tilde{\Sigma}C'N^{-1}e \end{bmatrix} w \\
&:= \; F(t)x^e + G(t)w + \epsilon(f(t, x^e) + g(t, x^e)w); \quad x^e(t_0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
L^*_{I\gamma} &= -\gamma^2 x^e(t_0)' \begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix} x^e(t_0) + x^e(t_f)' \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix} x^e(t_f) \\
&\quad + \int_{t_0}^{t_f} (x^e \begin{bmatrix} Q & 0 \\ 0 & ZBB'Z \end{bmatrix} x^e - \gamma^2 w'w + \epsilon(q + \hat{x}'ZBrB'Z\hat{x}))\, dt \\
&:= \; -\gamma^2 x_0^{e\prime} Q_0^e x_0^e + x^e(t_f)' Q_f^e x^e(t_f) + \int_{t_0}^{t_f} (x^{e\prime} H(t) x^e - \gamma^2 w'w + \epsilon h(t, x^e))\, dt
\end{aligned} \tag{4.9}$$

where $x^e := [x' \; \hat{x}']'$ and

$$\Pi := A - (BB' - \frac{1}{\gamma^2}DD')Z - (I - \frac{1}{\gamma^2}\tilde{\Sigma}Z)^{-1}\tilde{\Sigma}C'N^{-1}C,$$

and $x_0^e := x^e(t_0)$. Note that there exist $M_f > 0$, $M_g > 0$ and $M_h > 0$[2] such that

$$\begin{matrix} |f(t, x^e)| \leq M_f |x^e|; \quad |g(t, x^e)| \leq M_g; \\ |h(t, x^e)| \leq M_h |x^e|^2; \end{matrix} \qquad \forall (t, x^e) \in (t_0, t_f) \times \mathcal{R}^{2n}$$

Now, we are in a position to state the following theorem which establishes the robustness of the control policy $\tilde{\mu}^*_{I\gamma}$.

Theorem 3 *Consider the H^∞-optimal control problem formulated in (2.1), (2.7)–(2.9), under Assumptions 1 and 2. Then,*

1. $\lim_{\epsilon \to 0} \gamma_I^*(\epsilon) \leq \gamma_I^*(0)$.
2. *$\forall \gamma > \gamma_I^*(0)$, $\exists \epsilon_\gamma > 0$ such that the controller $\tilde{\mu}^*_{I\gamma}$ achieves the performance level γ for the perturbed system $\forall |\epsilon| < \epsilon_\gamma$, and hence $\gamma \geq \gamma_I^*(\epsilon)$.*

[2]See Appendix B for expressions for these constants.

3. *If, in addition,* $Q_f > 0$, *then,* $\forall\gamma < \gamma_I^*(0)$, $\exists\bar{\epsilon}_\gamma > 0$ *such that no controller can achieve the performance level* γ *for the perturbed system* $\forall|\epsilon| < \epsilon_\gamma$, *and hence* $\gamma \leq \gamma_I^*(\epsilon)$.

Proof It is well-known that the quantity $\gamma_I^*(0)$ is finite.

For a fixed $\gamma > \gamma_I^*(0)$, we now seek an upper bound for the value function of the maximization problem (4.8)–(4.9). Note that

$$\begin{aligned} Q_f^e &\leq Q_f^e + \begin{bmatrix} \gamma^2\tilde{\Sigma}^{-1}(t_f) - Z(t_f) & -\gamma^2\tilde{\Sigma}^{-1}(t_f) + Z(t_f) \\ -\gamma^2\tilde{\Sigma}^{-1}(t_f) + Z(t_f) & \gamma^2\tilde{\Sigma}^{-1}(t_f) - Z(t_f) \end{bmatrix} \\ &= \begin{bmatrix} \gamma^2\tilde{\Sigma}^{-1}(t_f) & -\gamma^2\tilde{\Sigma}^{-1}(t_f) + Z(t_f) \\ -\gamma^2\tilde{\Sigma}^{-1}(t_f) + Z(t_f) & \gamma^2\tilde{\Sigma}^{-1}(t_f) - Z(t_f) \end{bmatrix} := \tilde{Q}_f^e \end{aligned}$$

Then,

$$\begin{aligned} &\sup_{x_0\in\mathcal{R}^n}\sup_{w\in\mathcal{H}_w} L_{I\gamma}^*(w,x_0^e) \leq \sup_{x_0\in\mathcal{R}^n}\{-\gamma^2x^e(t_0)'Q_0^ex^e(t_0) + \sup_{w\in\mathcal{H}_w}\{x^e(t_f)'\tilde{Q}_f^ex^e(t_f) \\ &\quad + \int_{t_0}^{t_f}(x^{e\prime}H(t)x^e - \gamma^2w'w + \epsilon h(t,x^e))\,dt\}\} \end{aligned}$$

Now, we first find an upper bound for the maximization of

$$\hat{L}_\gamma(w,x_0^e) := \sup_{w\in\mathcal{H}_w}\{x^e(t_f)'\tilde{Q}_f^ex^e(t_f) + \int_{t_0}^{t_f}(x^{e\prime}H(t)x^e - \gamma^2w'w + \epsilon h(t,x^e))\,dt\} \tag{4.10}$$

For $\epsilon = 0$, this problem admits a maximum $x_0^{e\prime}\Xi(t_0)x_0^e$, where $\Xi(t)$ is the nonnegative definite solution to the following GRDE:

$$\dot{\Xi} + F'\Xi + \Xi F + \frac{1}{\gamma^2}\Xi GG'\Xi + H = 0; \quad \Xi(t_f) = \tilde{Q}_f^e$$

on the interval $[t_0, t_f]$. It is straightforward to check that

$$\Xi(t) = \begin{bmatrix} \gamma^2\tilde{\Sigma}^{-1}(t) & -\gamma^2\tilde{\Sigma}^{-1}(t) + Z(t) \\ -\gamma^2\tilde{\Sigma}^{-1}(t) + Z(t) & \gamma^2\tilde{\Sigma}^{-1}(t) - Z(t) \end{bmatrix}$$

is the solution to the above GRDE. The proof for this is standard and is therefore omitted here; interested reader may refer to [14] for a detailed proof in a more general context.

In the case of a nonzero ϵ, we consider the following GRDE:

$$\dot{\Xi}_\delta + F'\Xi_\delta + \Xi_\delta F + \frac{1}{\gamma^2}\Xi_\delta GG'\Xi_\delta + H + \delta I = 0; \quad \Xi_\delta(t_f) = \tilde{Q}_f^e.$$

By a standard (continuity) result on ordinary differential equations, for sufficiently small scalar $\delta > 0$, there exists a solution Ξ_δ. Let $\Xi_\delta(t)$ be partitioned as

$$\Xi_\delta(t) = \begin{bmatrix} \Xi_{\delta 11}(t) & \Xi_{\delta 12}(t) \\ \Xi_{\delta 21}(t) & \Xi_{\delta 22}(t) \end{bmatrix}$$

Note that $\Xi_\delta(t) \geq \Xi(t)$, $\forall t \in [t_0, t_f]$; in fact, $\Xi_\delta(t) > \Xi(t)$, $\forall t \in [t_0, t_f)$, since $\delta I > 0$. Thus, $\Xi_{\delta 11}(t) \geq \gamma^2 \tilde{\Sigma}^{-1}(t)$. But $\Xi_\delta(t)$ converges to $\Xi(t)$ as $\delta \downarrow 0$. Hence, for some sufficiently small $\delta > 0$, we have

$$\Xi_{\delta 11}(t_0) \leq \gamma^2 \tilde{\Sigma}^{-1}(t_0) + \gamma^2 \eta I = \gamma^2 Q_0$$

Fixing such a $\delta > 0$, and using the upper bounding technique introduced, the function $x^{e\prime}\Xi_\delta(t)x^e$ provides an upper bound for the value function of the maximization problem described by (4.8) and (4.10) for sufficiently small ϵ. Hence,

$$\sup_{w \in \mathcal{H}_w} \{x^e(t_f)' \tilde{Q}^e_f x^e(t_f) + \int_{t_0}^{t_f} (x^{e\prime} H(t) x^e - \gamma^2 w' w + \epsilon h(t, x^e))\, dt\} \leq x_0^{e\prime} \Xi_\delta(t_0) x_0^e$$

for sufficiently small ϵ. Then, we have

$$\begin{aligned} &\sup_{x_0 \in \mathcal{R}^n} \sup_{w \in \mathcal{H}_w} L^*_{I\gamma} \leq \sup_{x_0 \in \mathcal{R}^n} \{-\gamma^2 x^e(t_0)' Q_0^e x^e(t_0) + x_0^{e\prime} \Xi_\delta(t_0) x_0^e\} \\ &= \sup_{x_0 \in \mathcal{R}^n} \{-\gamma^2 x_0' Q_0 x_0 + x_0' \Xi_{\delta 11}(t_0) x_0\} \leq 0 \end{aligned}$$

Hence the soft-constrained zero-sum differential game has an upper value less than or equal to 0, which implies that the controller $\tilde{\mu}^*_{I\gamma}$ achieves the disturbance attenuation level γ for the perturbed system for sufficiently small ϵ. This proves part 2).

Obviously, part 2) further implies part 1).

For part 3), introduce the following two quantities:

$$\begin{aligned} \gamma_{Ic} &:= \inf\{\gamma > 0 : \text{The GRDE (3.1) admits a positive definite solution on } [t_0, t_f].\} \\ \gamma_{If} &:= \inf\{\gamma > 0 : \text{The GRDE (4.3) admits a positive definite solution on } [t_0, t_f].\} \end{aligned}$$

Obviously, these quantities are less than or equal to $\gamma_I^*(0)$. There are three possible relationship between $\gamma_I^*(0)$, γ_{Ic} and γ_{If}:

(a) $\gamma_I^*(0) = \gamma_{Ic}$.
(b) $\gamma_I^*(0) = \gamma_{If} > \gamma_{Ic}$.
(c) $\gamma_I^*(0) > \gamma_{If}$ and $\gamma_I^*(0) > \gamma_{Ic}$.

We will show that part 3) holds in all three cases.

Note that the differential game with index γ_1 has value function that is larger or equal to that of the differential game with index γ_2 if $0 < \gamma_1 \leq \gamma_2$. Hence, we only need to show part 3) for $\gamma < \gamma_I^*(0)$ that is sufficiently close to $\gamma_I^*(0)$.

In case (a), $\forall \gamma < \gamma_I^*(0)$ implies $\gamma < \gamma_{Ic}$. Then, part 3) follows from part 3) of Theorem 1.

In case (b), let us consider $\gamma \in (\gamma_{Ic}, \gamma_{If})$. The GRDE (4.3) has at least one conjugate point in $(t_0, t_f]$. By the dual result of Lemma 1 in Appendix A, $\exists T \in (t_0, t_f]$, such that the following GRDE:

$$\dot{\Sigma}^\# + A'\Sigma^\# + \Sigma^\# A + \Sigma^\# DD'\Sigma^\# - C'N^{-1}C + \frac{1}{\gamma^2} Q = 0 \quad \Sigma^\#(t_0) = Q_0 \tag{4.11}$$

admits a symmetric solution $\Sigma^{\#}(t)$ on the time interval $[t_0, T]$ and the matrix $\Sigma^{\#}(T)$ has at least one negative eigenvalue.

By a standard result on ordinary differential equations, for some small positive scalar δ_1, there exists a solution $\Sigma^{\#}_{\delta_1}(t)$ to the following GRDE on the time interval $[t_0, T]$:

$$\dot{\Sigma}^{\#}_{\delta_1} + A'\Sigma^{\#}_{\delta_1} + \Sigma^{\#}_{\delta_1}A + \Sigma^{\#}_{\delta_1}(DD' - \delta_1 I)\Sigma^{\#}_{\delta_1} - C'N^{-1}C + \frac{1}{\gamma^2}Q - \delta_1 I = 0; \quad \Sigma^{\#}_{\delta_1}(t_0) = Q_0 \tag{4.12}$$

such that $\Sigma^{\#}_{\delta_1}(T)$ has at least one negative eigenvalue. It should be noted here that $\Sigma^{\#}_{\delta_1}(t) \geq \Sigma^{\#}(t)$ on $[t_0, T]$. Define T_{δ_1} as

$$T_{\delta_1} := \inf\{t \in [t_0, T] : \Sigma^{\#}_{\delta_1}(t) \text{ has at least one nonpositive eigenvalue.}\}$$

It is clear that $Z(t) > 0$ for all $t \in [t_0, t_f]$, since $Q_f > 0$. Hence, the matrix $\gamma^2\Sigma^{\#}_{\delta_1}(T_{\delta_1}) - Z(T_{\delta_1})$ has at least one negative eigenvalue. Then, there exists a $\delta_2 > 0$ such that the GRDE (3.10), with $\delta = \delta_2$, admits a positive definite solution, $\tilde{Z}_{\delta_2}(t)$, on $[t_0, t_f]$ and the matrix $\gamma^2\Sigma^{\#}_{\delta_1}(T_{\delta_1}) - \tilde{Z}_{\delta_2}(T_{\delta_1})$ has at least one negative eigenvalue.

The following property must hold for any controller μ_I that achieves a finite H^∞ performance level for the perturbed system when $\epsilon < 1/M_r$:

$$\mu_I(t, y_{[t_0,t]}) = 0 \quad \text{if } y(s) = 0 \ \forall s \in [t_0, t].$$

This property will be referred to as *zero-control property.* The infimum over $\mathcal{M}_I$ in (2.9) can be replaced by the infimum over controllers in $\mathcal{M}_I$ that further satisfy the zero-control property.

Define a function $W(\xi; \epsilon)$ as follows:

$$\begin{aligned} W(\xi;\epsilon) \ :=\ & \max_{x_0, w_{[t_0,T_{\delta_1}]}|x(T_{\delta_1})=\xi, y(t)=0 \ t_0 \leq t \leq T_{\delta_1}} \int_{t_0}^{T_{\delta_1}} (|x(t)|^2_Q - \gamma^2 w(t)'w(t) \\ & \qquad + \epsilon q(t, x(t)))\, dt - \gamma^2 |x_0|^2_{Q_0} && (4.13) \\ \text{s.t.} \quad & \dot{x} = Ax + Dw + \epsilon(a + dw); \quad x(t_0) = x_0 && (4.14) \\ & 0 = Cx + Ew + \epsilon(c + ew); \quad \forall t \in [t_0, T_{\delta_1}], && (4.15) \end{aligned}$$

where the operater before the integral in (4.13) denotes the maximization over all x_0, and $w_{[t_0,T_{\delta_1}]}$ that satisfy the constraint $x(T_{\delta_1}) = \xi$ and $y(t) = 0 \ \forall t \in [t_0, T_{\delta_1}]$ in addition to constraints (4.14) and (4.15). Note that (4.14) is the state equation under any controller satisfying the zero-control property, and (4.15) is the constaint as $y(t) = 0, \ t_0 \leq t \leq T_{\delta_1}$.

Clearly, there exists an $\epsilon_0 > 0$, such that the matrix $(E + \epsilon e)(E + \epsilon e)'$ is positive definite for all $(t, x) \in [t_0, t_f] \times \mathcal{R}^n$ and $\epsilon \in [-\epsilon_0, \epsilon_0]$ under Assumption 2. Take $\epsilon \in [-\epsilon_0/2, \epsilon_0/2]$. Define

$$N_\epsilon(t, x) := [(E + \epsilon e)(E + \epsilon e)']^{-1}; \quad \Psi(t, x; \epsilon) := (E + \epsilon e)'N_\epsilon(t, x).$$

Let

$$w^*(t, x) = -\Psi(Cx + \epsilon c) + (D' - \epsilon\Psi e D')\Sigma^{\#}_{\delta_1} x; \quad t \in [t_0, T_{\delta_1}],$$

and choose x_0^* such that the solution to the differential equation (4.14) with initial condition x_0^* under the disturbance $w^*(t,x)$ is $x^*(t)$, and $x^*(T_{\delta_1}) = \xi$. [3] It is straightforward to check that

$$\Psi'(D' - \epsilon\Psi e D') = 0.$$

Hence, this choices of w^* and x_0^* are admissible for the maximization of (4.13). Then,

$$W(\xi;\epsilon) \geq \int_{t_0}^{T_{\delta_1}} (|x^*|_Q^2 - \gamma^2 w^{*\prime} w^* + \epsilon q(t,x^*))\,dt - \gamma^2 |x_0^*|_{Q_0}^2 \tag{4.16}$$

where x^* is generated by the following differential equation:

$$\dot{x}^* = (A + DD'\Sigma_{\delta_1}^{\#})x^* + \epsilon(a - De'N_\epsilon(Cx^* + \epsilon c) - \epsilon De'N_\epsilon eD'\Sigma_{\delta_1}^{\#} x^*); \quad x^*(t_0) = x_0^*$$

Define

$$\bar{W}(t,x) := -\gamma^2 x'\Sigma_{\delta_1}^{\#}(t)x, \quad t \in [t_0, T_{\delta_1}].$$

Adding the identically zero function $\bar{W}(T_{\delta_1},\xi) - \bar{W}(t_0,x_0^*) - \int_{t_0}^{T_{\delta_1}} \frac{d\bar{W}}{dt}\,dt$ onto the RHS of (4.16), we arrive at the following inequality, by also utilizing GRDE (4.12):

$$\begin{aligned} W(\xi;\epsilon) \geq \bar{W}(T_{\delta_1},\xi) &+ \int_{t_0}^{T_{\delta_1}} (\gamma^2\delta_1|x^*|^2 + \gamma^2\delta_1|x^*|^2_{\Sigma_{\delta_1}^{\#}\Sigma_{\delta_1}^{\#}} + \epsilon q + 2\gamma^2\epsilon x^{*\prime}\Sigma_{\delta_1}^{\#}(a - De'N_\epsilon \\ &\cdot (Cx^* + \epsilon c) - \epsilon De'N_\epsilon eD'\Sigma_{\delta_1}^{\#}x^*) + \gamma^2\epsilon x^{*\prime}C'N^{-1}(I + (2\epsilon Ee' + \epsilon^2 ee')N^{-1})^{-1}(2Ee' \\ &+ \epsilon ee')N^{-1}Cx^* + 2\gamma^2\epsilon x^{*\prime}\Sigma_{\delta_1}^{\#}De'N_\epsilon^{-1}Cx^* + 2\gamma^2\epsilon x^{*\prime}(\Sigma_{\delta_1}^{\#}D - C'\Psi') \\ &\cdot (\Psi c + \Psi eD'\Sigma_{\delta_1}^{\#}x^*) - \gamma^2\epsilon^2|\Psi c + \Psi eD'\Sigma_{\delta_1}^{\#}x^*|^2)\,dt \end{aligned}$$

Then, there exists $\epsilon_1 > 0$ such that $W(\xi;\epsilon) \geq \bar{W}(T_{\delta_1},\xi)$ for $|\epsilon| < \epsilon_1$.

Introduce the following notation:

$$L_{I\gamma}^{T_{\delta_1}} := \int_{t_0}^{T_{\delta_1}} (|x|_Q^2 + u'u - \gamma^2 w'w + \epsilon(q + u'ru))\,dt - \gamma^2|x_0|_{Q_0}^2$$

$$L_{I\gamma T_{\delta_1}} := \int_{T_{\delta_1}}^{t_f} (|x|_Q^2 + u'u - \gamma^2 w'w + \epsilon(q + u'ru))\,dt + |x(t_f)|_{Q_f}^2$$

Clearly, we have $L_{I\gamma} = L_{I\gamma}^{T_{\delta_1}} + L_{I\gamma T_{\delta_1}}$. Then,

$$\begin{aligned} \min_{u[t_0,t_f]} \max_{x_0, w[t_0,t_f]} L_{I\gamma} &= \min_{u[t_0,T_{\delta_1}]} \max_{y[t_0,T_{\delta_1}]} \min_{u[T_{\delta_1},t_f]} \max_{x(T_{\delta_1})=\xi, w[T_{\delta_1},t_f]} \max_{x_0, w[t_0,T_{\delta_1}] | y[t_0,T_{\delta_1}], x(T_{\delta_1})=\xi} L_{I\gamma} \\ &\geq \min_{u[T_{\delta_1},t_f]} \max_{x(T_{\delta_1})=\xi, w[T_{\delta_1},t_f]} \max_{x_0, w[t_0,T_{\delta_1}] | x(T_{\delta_1})=\xi, y(t)=0 \ t_0 \leq t \leq T_{\delta_1}} (L_{I\gamma}^{T_{\delta_1}} + L_{I\gamma T_{\delta_1}}) \\ &\geq \min_{u[T_{\delta_1},t_f]} \max_{x(T_{\delta_1})=\xi, w[T_{\delta_1},t_f]} (\bar{W}(T_{\delta_1},\xi) + L_{I\gamma T_{\delta_1}}) \\ &\geq \max_{x(T_{\delta_1})=\xi} (\bar{W}(T_{\delta_1},\xi) + \max_{w[T_{\delta_1},t_f]} \min_{u[T_{\delta_1},t_f]} L_{I\gamma T_{\delta_1}}) \end{aligned}$$

By the part 3) of Theorem 1 and its proof, there exists $\epsilon_2 > 0$ such that

[3] Such an initial condition x_0^* can be found as the solution to the reverse differential equation of (4.14) with initial state ξ. Since right-hand-side of (4.14) is sectorly bounded and Lipschitz continuous, the reverse differential equation admits a solution on the entire time interval.

$$\max_{w[T_{\delta_1},t_f]} \min_{u[T_{\delta_1},t_f]} L_{I\gamma T_{\delta_1}} \geq \xi' \tilde{Z}_{\delta_2}(T_{\delta_1})\xi \quad \forall \epsilon \in [-\epsilon_2, \epsilon_2]$$

Let $\epsilon'_\gamma = \min\{\epsilon_0, \epsilon_1, \epsilon_2\}$; then, for $|\epsilon| < \epsilon'_\gamma$, we have

$$\min_{u[t_0,t_f]} \max_{x_0, w[t_0,t_f]} L_{I\gamma} \geq \max_{x(T_{\delta_1})=\xi} \xi'(-\gamma^2 \Sigma^{\#}_{\delta_1}(T_{\delta_1}) + \tilde{Z}_{\delta_2}(T_{\delta_1}))\xi = +\infty$$

Hence, part 3) is established.

In case (c), consider $\gamma < \gamma_I^*(0)$ that is larger than γ_{Ic} and γ_{If}. Let $\Sigma^*(t)$ and $Z^*(t)$ be the positive definite solutions to GRDEs (4.8) and (4.3), respectively, with $\gamma = \gamma_I^*(0)$. Then, there exists $T \in [t_0, t_f]$ and $\xi^* \in \mathcal{R}^n$, $\xi^* \neq 0$ such that

$$\xi^{*\prime}(\gamma_I^*(0)^2 \Sigma^*(T)^{-1} - Z^*(T))\xi^* = 0.$$

Since $\Sigma^* \leq \Sigma$ and $Z^* \leq Z$ for all $t \in [t_0, t_f]$, we have

$$\begin{aligned}\xi^{*\prime}(\gamma^2\Sigma(T)^{-1} - Z(T))\xi^* &= \xi^{*\prime}(Z^*(T)\Sigma(T)^{-1} - Z(T))\xi^* + \xi^{*\prime}(\gamma^2\Sigma(T)^{-1} - \gamma^2\Sigma^*(T)^{-1})\xi^* \\ &\quad + (\gamma^2 - \gamma_I^*(0)^2)\xi^{*\prime}\Sigma^*(T)^{-1}\xi^* < 0\end{aligned}$$

This implies that the matrix $\gamma^2\Sigma(T)^{-1} - Z(T)$ has at least one negative eigenvalue. By continuity of $\Sigma(t)$ and $Z(t)$, we may assume that $T \in (t_0, t_f)$. Then, for sufficiently small positive scalars δ_1 and δ_2, GRDEs (4.12) and (3.10) admits positive definite solutions $\Sigma^{\#}_{\delta_1}$ and $\tilde{Z}_{\delta_2}$ on $[t_0, t_f]$, respectively, and furthermore the matrix $\gamma^2\Sigma^{\#}_{\delta_1}(T) - \tilde{Z}_{\delta_2}(T)$ has at least one negative eigenvalue. The rest of the proof follows exactly as in case (b) by replacing T_{δ_1} with T. Hence, part 3) holds in case (c).

Hence, part 3) is proven.

This completes the proof of the Theorem. □

Remark 2 *The matrices $\Sigma(t)$ and $\tilde{\Sigma}(t)$ correspond to the worst "covariance" of the error in the estimation of the state x. The matrix Q_0 corresponds to the level of "confidence" of the designer on the initial state estimate. Thus, Theorem 3 indicates that: in case of small structural uncertainty, one must appear to be somewhat less confident about the initial data, for the "nominal" controller to work well on the perturbed system.* ◇

To obtain a lower bound for the maximal allowable ϵ_γ of Theorem 3.2), let $\Theta(t)$ be the solution to the following Lyapunov equation:

$$\dot{\Theta} + (F' + \frac{1}{\gamma^2}\Xi GG')\Theta + \Theta(F + \frac{1}{\gamma^2}GG'\Xi) + I = 0; \quad \Theta(t_0) = 0$$

Then, we have the following result:

Corollary 3 *A lower bound for the maximal allowable ϵ_γ of Theorem 3.2) is*

$$\epsilon^*_\gamma = \sup\{\epsilon > 0 : \quad \exists\delta > 0 \text{ such that inequality (4.17) is satisfied } \}$$

where

$$\begin{aligned}\delta \geq\ & |\epsilon|(2(|\Xi| + \delta|\Theta|)M_f + \frac{2}{\gamma^2}(|\Xi| + \delta|\Theta|)^2|G|M_g + M_h) \\ & + \frac{\epsilon^2}{\gamma^2}M_g^2(|\Xi| + \delta|\Theta|)^2 + \frac{\delta^2}{\gamma^2}|\Theta|^2|G|^2 \qquad (4.17)\end{aligned}$$

◇

The infinite-horizon case

Next, we consider the infinite-horizon case.

When $\epsilon = 0$, for any $\gamma > \gamma^*_{I\infty}(0)$, the optimal control policy achieving the desired performance level is

$$\mu^*_{I\gamma\infty}(y_{[-\infty,t]}) = -B'\bar{Z}\hat{x} \tag{4.18}$$

$$\dot{\hat{x}} = (A - (BB' - \frac{1}{\gamma^2}DD')\bar{Z})\hat{x} + (I - \frac{1}{\gamma^2}\bar{\Sigma}\bar{Z})^{-1}\bar{\Sigma}C'N^{-1}(y - C\hat{x}); \quad \hat{x}(-\infty) = 0 \tag{4.19}$$

where $\bar{Z}$ is the minimal nonnegative definite solution to GARE (3.13), and $\bar{\Sigma}$ is the minimal nonnegative definite solution to the following GARE:

$$A\bar{\Sigma} + \bar{\Sigma}A' - \bar{\Sigma}(C'N^{-1}C - \frac{1}{\gamma^2}Q)\bar{\Sigma} + DD' = 0 \tag{4.20}$$

which further satisfies the spectral radius condition

$$\gamma^2\bar{\Sigma}^{-1} > \bar{Z} \tag{4.21}$$

On the other hand, for any $\gamma < \gamma^*_{I\infty}(0)$, either at least one of the GAREs (3.13) and (4.20) does not admit a nonnegative definite solution, or the spectral radius condition (4.21) is violated.

Unlike the finite-horizon case, we require here that $x(t) \to 0$ as $t \to -\infty$, which says that we know the initial condition of the system. Hence, we can expect the "nominal" controller $\mu^*_{I\gamma\infty}$ to achieve the desired performance bound for sufficiently small ϵ without any correction term as in the finite-horizon case. Applying the control law $\mu^*_{I\gamma\infty}$ to the perturbed differential game, we arrive at the following infinite-horizon maximization problem:

$$\begin{aligned} \dot{x}^e &= \begin{bmatrix} A & -BB'\bar{Z} \\ (I - \frac{1}{\gamma^2}\bar{\Sigma}\bar{Z})^{-1}\bar{\Sigma}C'N^{-1}C & \bar{\Pi} \end{bmatrix} x^e + \begin{bmatrix} D \\ (I - \frac{1}{\gamma^2}\bar{\Sigma}\bar{Z})^{-1}\bar{\Sigma}C'N^{-1}E \end{bmatrix} w \\ &\quad + \epsilon \begin{bmatrix} a - bB'\bar{Z}\hat{x} \\ (I - \frac{1}{\gamma^2}\bar{\Sigma}\bar{Z})^{-1}\bar{\Sigma}C'N^{-1}(c - nB'\bar{Z}\hat{x}) \end{bmatrix} + \epsilon \begin{bmatrix} d \\ (I - \frac{1}{\gamma^2}\bar{\Sigma}\bar{Z})^{-1}\bar{\Sigma}C'N^{-1}e \end{bmatrix} w \\ &:= \bar{F}x^e + \bar{G}w + \epsilon(\bar{f}(t, x^e) + \bar{g}(t, x^e)w); \quad x^e(-\infty) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{4.22}$$

$$\begin{aligned} L^*_{I\gamma} &= \int_{-\infty}^{\infty} (x^e \begin{bmatrix} Q & 0 \\ 0 & \bar{Z}BB'\bar{Z} \end{bmatrix} x^e - \gamma^2 w'w + \epsilon(q + \hat{x}'\bar{Z}BrB'\bar{Z}\hat{x}))\, dt \\ &:= \int_{-\infty}^{\infty} (x^{e\prime}\bar{H}x^e - \gamma^2 w'w + \epsilon\bar{h}(t, x^e))\, dt \end{aligned} \tag{4.23}$$

where $x^e := [x' \ \hat{x}']'$ and

$$\bar{\Pi} := A - (BB' - \frac{1}{\gamma^2}DD')\bar{Z} - (I - \frac{1}{\gamma^2}\bar{\Sigma}\bar{Z})^{-1}\bar{\Sigma}C'N^{-1}C.$$

Note that there exist $\bar{M}_f > 0$, $\bar{M}_g > 0$ and $\bar{M}_h > 0$[4] such that

$$\begin{array}{l} |\bar{f}(t,x^e)| \le \bar{M}_f|x^e|; \quad |\bar{g}(t,x^e)| \le \bar{M}_g; \\ |\bar{h}(t,x^e)| \le \bar{M}_h|x^e|^2; \end{array} \qquad \forall (t,x^e) \in (-\infty,\infty) \times \mathcal{R}^{2n}$$

We now have the following Theorem, which establishes the robustness of the controller $\mu^*_{I\gamma\infty}$.

Theorem 4 *Consider the H^∞-optimal control problem formulated in (2.1), (2.7)–(2.9) with $t_f \to \infty$ and $t_0 \to -\infty$, as well as when $t_f = \infty$ and $t_0 = -\infty$, and A, B, D, C, E, Q being time-invariant, and $Q_f = 0$ and $Q_0 = 0$, and require that $x(t) \to 0$ as $t \to -\infty$ and $t \to \infty$. Let Assumptions 1, 2, 3a and 3b hold. Then,*

1. $\lim_{\epsilon\to 0} \gamma^*_{I\infty}(\epsilon) = \gamma^*_{I\infty}(0)$.
2. $\forall \gamma > \gamma^*_{I\infty}(0)$, $\exists \epsilon_\gamma > 0$ *such that the controller $\mu^*_{I\gamma\infty}$ achieves the disturbance attenuation level γ for the perturbed system, $\forall |\epsilon| < \epsilon_\gamma$, and internally stabilizes it. Hence, $\gamma \ge \gamma^*_{I\infty}(\epsilon)$, $\forall |\epsilon| < \epsilon_\gamma$.*
3. $\forall \gamma < \gamma^*_{I\infty}(0)$, $\exists \epsilon'_\gamma > 0$ *such that $\forall |\epsilon| < \epsilon'_\gamma$, no control policy achieves the performance level γ for the perturbed system, and hence, $\gamma \le \gamma^*_{I\infty}(\epsilon)$.*

Proof We first note the well-known fact that, under the Assumption 3, the quantity $\gamma^*_{I\infty}(0)$ is finite.

Fix a $\gamma > \gamma^*_{I\infty}(0)$; for $\epsilon = 0$, the maximization problem (4.22)–(4.23) admits a value function $x^{e\prime}\bar{\Xi}x^e$, where $\bar{\Xi}$ is the minimal nonnegative definite solution to the following GARE:

$$\bar{F}'\Xi + \Xi\bar{F} + \frac{1}{\gamma^2}\Xi\bar{G}\bar{G}'\Xi + \bar{H} = 0$$

It is straightforward to check that the matrix

$$\begin{bmatrix} \gamma^2\bar{\Sigma}^{-1} & -\gamma^2\bar{\Sigma}^{-1} + \bar{Z} \\ -\gamma^2\bar{\Sigma}^{-1} + \bar{Z} & \gamma^2\bar{\Sigma}^{-1} - \bar{Z} \end{bmatrix}$$

constitutes a positive definite solution to the above GARE. The proof for this can again be found in [14], in a more general context. Thus, the matrix $\bar{\Xi}$ exists. Furthermore, the matrix $\bar{F} + \frac{1}{\gamma^2}\bar{G}\bar{G}'\bar{\Xi}$ is Hurwitz by Theorem 7 in [19].

Thus, by the same Implicit Function Theorem argument as in the proof for part 2) of Theorem 2, there exists a minimal positive definite solution $\bar{\Xi}_\delta$ to the following GARE, for some scalar $\delta > 0$:

$$\bar{F}'\bar{\Xi}_\delta + \bar{\Xi}_\delta\bar{F} + \frac{1}{\gamma^2}\bar{\Xi}_\delta\bar{G}\bar{G}'\bar{\Xi}_\delta + \bar{H} + \delta I = 0. \qquad (4.24)$$

[4]See Appendix B for expressions for these constants.

It is straightforward to show that the smooth function $x^{e\prime}\bar{\Xi}_\delta x^e$ provides an upper bound for the value function of the maximization problem (4.22)–(4.23) for sufficiently small ϵ, which implies that the soft-constrained zero-sum differential game has an upper value bounded above by 0. This proves part 2).

For part 3), we first choose a γ_1 that is larger than $\gamma^*_{I\infty}(0)$. By Lemma 2, there exists a sufficiently small $\delta_1 > 0$ such that the following GARE:

$$A'\bar{\Sigma}^{\#}_{\gamma_1\delta_1} + \bar{\Sigma}^{\#}_{\gamma_1\delta_1}A + \bar{\Sigma}^{\#}_{\gamma_1\delta_1}(DD' - \delta_1 I)\bar{\Sigma}^{\#}_{\gamma_1\delta_1} - C'N^{-1}C + \frac{1}{\gamma_1^2}Q - \delta_1 I = 0$$

admits a positive definite solution $\bar{\Sigma}^{\#}_{\gamma_1\delta_1}$; furthermore, the matrix $A + DD'\bar{\Sigma}^{\#}_{\gamma_1\delta_1}$ is anti-stable.

Fix a $\gamma < \gamma^*_{I\infty}(0)$. Define a function $W(\xi;\epsilon)$ by

$$W(\xi;\epsilon) := \max_{w_{(-\infty,0]}|x(0)=\xi, y(t)=0\ -\infty<t\le 0} \int_{-\infty}^{0} (|x(t)|_Q^2 - \gamma^2 w(t)'w(t) + \epsilon q(t,x(t)))\,dt \quad (4.25)$$

$$\text{s.t.} \quad \lim_{t\to-\infty} x(t) = 0 \quad (4.26)$$

$$\dot{x} = Ax + Dw + \epsilon(a + dw) \quad (4.27)$$

$$0 = Cx + Ew + \epsilon(c + ew) \quad \forall t \in (-\infty, 0] \quad (4.28)$$

Choose ϵ sufficiently small such that $(E + \epsilon e)(E + \epsilon e)'$ is positive definite for all t and x, and is uniformly bounded from below. Introduce the matrix functions $N_\epsilon(t,x)$ and $\Psi(t,x;\epsilon)$ as in the finite-horizon case. Let

$$w^*(t,x) = -\Psi(Cx + \epsilon c) + (D' - \epsilon\Psi\epsilon D')\bar{\Sigma}^{\#}_{\gamma_1\delta_1}x; \quad t \in (-\infty, 0].$$

It is straightforward to show that w^* satisfies the constraint (4.28). Let x^* be the solution to differential equation (4.27) with terminal condition $x^*(0) = \xi$ under the disturbance $w^*(t,x)$. Then, x^* is generated by the following differential equation:

$$\dot{x}^* = (A + DD'\bar{\Sigma}^{\#}_{\gamma_1\delta_1})x^* + \epsilon(a - De'N_\epsilon(Cx^* + \epsilon c) - \epsilon De'N_\epsilon eD'\bar{\Sigma}^{\#}_{\gamma_1\delta_1}x^*); \quad x^*(0) = \xi$$

Since the matrix $A + DD'\bar{\Sigma}^{\#}_{\gamma_1\delta_1}$ is anti-stable and the nonlinear perturbation terms are sectorly bounded, the above differential equation is globally asymptotically stable in reverse time for sufficiently small ϵ. This implies that $\lim_{t\to-\infty} x^*(t) = 0$. Hence, w^* is an admissible choice for the maximization of (4.25). Then, we have

$$\begin{aligned} W(\xi;\epsilon) &\ge \int_{-\infty}^{0} (|x^*(t)|_Q^2 - \gamma^2 w^*(t)'w^*(t) + \epsilon q(t,x^*(t)))\,dt \\ &\ge \int_{-\infty}^{0} (|x^*(t)|_Q^2 - \gamma_1^2 w^*(t)'w^*(t) + \epsilon q(t,x^*(t)))\,dt \end{aligned}$$

Following the derivations in the finite-horizon case, the function $W(\xi;\epsilon)$ is further lower bounded by $-\gamma_1^2\xi'\bar{\Sigma}^{\#}_{\gamma_1\delta_1}\xi$ for sufficiently small ϵ. This leads to:

$$W(\xi;\epsilon) \ge -\gamma_1^2\xi'\bar{\Sigma}^{\#}_{\gamma_1\delta_1}\xi$$

Introduce the following notation:

$$L^0_{I\gamma} := \int_{-\infty}^{0} (|x|^2_Q + u'u - \gamma^2 w'w + \epsilon(q + u'ru))\,dt$$
$$L^T_{I\gamma 0} := \int_{0}^{T} (|x|^2_Q + u'u - \gamma^2 w'w + \epsilon(q + u'ru))\,dt$$
$$L_{I\gamma T} := \int_{T}^{\infty} (|x|^2_Q + u'u - \gamma^2 w'w + \epsilon(q + u'ru))\,dt$$

where T is some large positive scalar to be chosen shortly. Then, for controllers that satisfy the zero-control property,

$$\begin{aligned}\min_{u(-\infty,\infty)} \max_{w(-\infty,\infty)} L_{I\gamma} &= \min_{u(-\infty,0]} \max_{y(-\infty,0]} \min_{u[0,\infty)} \max_{x(0)=\xi, w[0,\infty)} \max_{w(-\infty,0]|y(-\infty,0],x(0)=\xi} L_{I\gamma} \\ &\geq \min_{u[0,\infty)} \max_{x(0)=\xi, w[0,\infty)} \max_{w(-\infty,0]|x(0)=\xi, y(t)=0 \ -\infty<t\leq 0} (L^0_{I\gamma} + L^T_{I\gamma 0} + L_{I\gamma T}) \\ &\geq \min_{u[0,\infty)} \max_{x(0), w[0,\infty)} (-\gamma_1^2 x(0)'\bar{\Sigma}^{\#}_{\gamma_1\delta_1} x(0) + L^T_{I\gamma 0} + L_{I\gamma T})\end{aligned}$$

The last inequality holds for sufficiently small ϵ.

There exists some sufficiently small $\delta_2 > 0$ such that, by Corollaries 9 and 7, the GRDE (3.17) with $\delta = \delta_2$ admits a nonnegative definite solution $Z_{\gamma_1\delta_2}(t)$ on $[0, T_1]$, for some large enough T_1, with the property $Z_{\gamma_1\delta_2}(0) > 0$. Using the lower bounding technique for the perfect state measurements case, we have the following inequalities for sufficiently small ϵ:

$$\begin{aligned}\max_{w[T,\infty)} \min_{u[T,\infty)} L_{I\gamma T} &\geq \max_{w[T,\infty)} \min_{u[T,\infty)} \int_T^{T+T_1} (|x|^2_Q + u'u - \gamma^2 w'w + \epsilon(q + u'ru))\,dt \\ &\geq x(T)'Z_{\gamma_1\delta_2}(0)x(T)\end{aligned}$$

Then,

$$\begin{aligned}\min_{u(-\infty,\infty)} \max_{w(-\infty,\infty)} L_{I\gamma} &\geq \min_{u[0,T]} \max_{x(0), w[0,T]} \max_{w[T,\infty)} \min_{u[T,\infty)} (-\gamma_1^2 x(0)'\bar{\Sigma}^{\#}_{\gamma_1\delta_1} x(0) + L^T_{I\gamma 0} + L_{I\gamma T}) \\ &\geq \min_{u[0,T]} \max_{x(0), w[0,T]} (-\gamma_1^2 x(0)'\bar{\Sigma}^{\#}_{\gamma_1\delta_1} x(0) + L^T_{I\gamma 0} + x(T)'Z_{\gamma_1\delta_2}(0)x(T))\end{aligned}$$

The RHS of the above inequality defines a finite-horizon zero-sum differential game on $[0, T]$ under imperfect state measurements. Let the optimal performance level for the finite-horizon H^∞-optimal control problem corresponding to this differential game be $\tilde{\gamma}^*_T(\epsilon)$. Then, part 3) follows from part 3) of Theorem 3, if we can show that $\gamma < \tilde{\gamma}^*_T(0)$ for sufficiently large T.

Define

$$\begin{aligned}\gamma_{Ic\infty} &:= \inf\{\gamma > 0 : \text{ The GARE (3.13) admits a positive definite solution.}\} \\ \gamma_{If\infty} &:= \inf\{\gamma > 0 : \text{ The GARE (4.20) admits a positive definite solution.}\}\end{aligned}$$

For any $\gamma_2 \in (\gamma, \gamma^*_{I\infty}(0))$, let $\bar{Z}_{\gamma_2}$ be the minimal positive definite solution to GARE (3.13) with $\gamma = \gamma_2$ if $\gamma_2 > \gamma_{Ic\infty}$; and let $\bar{\Sigma}_{\gamma_2}$ be the minimal positive definite solution to GARE (4.20) with $\gamma = \gamma_2$ if $\gamma_2 > \gamma_{If\infty}$. Introduce the following two GRDEs:

$$\dot{\Sigma}_{\gamma_2} = A\Sigma_{\gamma_2} + \Sigma_{\gamma_2}A' - \Sigma_{\gamma_2}(C'N^{-1}C - \frac{1}{\gamma^2}Q)\Sigma_{\gamma_2} + DD'; \quad \Sigma_{\gamma_2}(0) = \frac{\gamma^2}{\gamma_1^2}\bar{\Sigma}^{\#}_{\gamma_1\delta_1}{}^{-1} \tag{4.29}$$

and

$$\dot{Z}_{\gamma_2} + A'Z_{\gamma_2} + Z_{\gamma_2}A - Z_{\gamma_2}(BB' - \frac{1}{\gamma^2}DD')Z_{\gamma_2} + Q = 0; \quad Z_{\gamma_2}(T) = Z_{\gamma_1\delta_2}(0) \tag{4.30}$$

When $\gamma_2 < \gamma_{Ic\infty}$, by Lemma 4 of Appendix A, the GRDE (4.30) admits at least one conjugate point on $[0, T]$ for sufficiently large T, which implies that $\gamma_2 \leq \tilde{\gamma}_T^*(0)$. When $\gamma_2 < \gamma_{If\infty}$, by duality, the GRDE (4.29) admits at least one conjugate point on $[0, T]$ for sufficiently large T, which again implies that $\gamma_2 \leq \tilde{\gamma}_T^*(0)$. When γ_2 is larger than both $\gamma_{Ic\infty}$ and $\gamma_{If\infty}$, both $\bar{Z}_{\gamma_2}$ and $\bar{\Sigma}_{\gamma_2}$ are well defined. Then, it is easy to see that the matrix $\gamma_2^2\bar{\Sigma}_{\gamma_2}^{-1} - \bar{Z}_{\gamma_2}$ has at least one negative eigenvalue. By Corollary 9, we have $\lim_{t\to\infty} Z_{\gamma_2}(t) = \bar{Z}_{\gamma_2}$ and $\lim_{T\to\infty} \Sigma_{\gamma_2}(t) = \bar{\Sigma}_{\gamma_2}$. Therefore, for sufficiently large T, the matrix $\gamma_2^2\Sigma_{\gamma_2}(t)^{-1} - Z_{\gamma_2}(t)$ has at least one negative eigenvalue for some $t \in [0, T]$. Hence, $\gamma_2 \leq \tilde{\gamma}_T^*(0)$ in this case too. Thus, $\gamma < \tilde{\gamma}_T^*(0)$, and this completes the proof of part 3).

Part 1) follows from part 2) and part 3). □

To obtain a lower bound for the maximal allowable ϵ_γ of Theorem 4, let $\bar{\Theta}$ be the positive definite solution to the following Lyapunov equation:

$$(\bar{F}' + \frac{1}{\gamma^2}\bar{\Xi}\bar{G}\bar{G}')\bar{\Theta} + \bar{\Theta}(\bar{F} + \frac{1}{\gamma^2}\bar{G}\bar{G}'\bar{\Xi}) + I = 0$$

Then, following the proof of Theorem 4 and applying Corollary 2, we can obtain the following result:

Corollary 4 *A lower bound for the maximal allowable* ϵ_γ *of Theorem 4 is*

$$\epsilon_\gamma^* = \sup\{\epsilon > 0 : \quad \exists\delta > 0 \textit{ such that inequality (4.31) is satisfied } \}$$

where

$$\delta \geq |\epsilon|(2(|\bar{\Xi}| + \delta|\bar{\Theta}|)\bar{M}_f + \frac{2}{\gamma^2}(|\bar{\Xi}| + \delta|\bar{\Theta}|)^2|\bar{G}|\bar{M}_g + \bar{M}_h)$$
$$+\frac{\epsilon^2}{\gamma^2}\bar{M}_g^2(|\bar{\Xi}| + \delta|\bar{\Theta}|)^2 + \frac{\delta^2}{\gamma^2}|\bar{\Theta}|^2|\bar{G}|^2 \tag{4.31}$$

◇

5 An Extension to Nominal Nonlinear Systems

In this section, we consider the problem when the "nominal" system is also nonlinear. We shall solve the problem for the perfect state measurement case, in both finite- and infinite-horizon formulations. We will characterize the class of disturbances that the controlled system can tolerate, while guaranteeing a certain level of performance.

The system under consideration is described by

$$\dot{x} = f(t,x) + g(t,x)u + h(t,x)w + \epsilon(a(t,x) + b(t,x)u + d(t,x)w) \tag{5.1}$$

where we take the initial state to be zero.

The performance index, as the counterpart of (2.3), is

$$\begin{aligned} L(u,w) &= l_f(x(t_f)) + \int_{t_0}^{t_f} (l(t,x(t)) + |u(t)|^2 + \epsilon(q(t,x(t)) \\ &\quad + u(t)'r(t,x(t))u(t)))\,dt \end{aligned} \tag{5.2}$$

with the various nonlinear terms satisfying the following regularity conditions:

Assumption 4 The nonlinear vector-valued and matrix-valued functions $f(t,x)$, $g(t,x)$, $h(t,x)$, $a(t,x)$, $b(t,x)$, $d(t,x)$, $l(t,x)$, $q(t,x)$, $r(t,x)$ and $l_f(x)$ are piecewise continuous in t and locally Lipschitz continuous in x, and further satisfy:

$$\begin{aligned} &f(t,0) = 0; \quad a(t,0) = 0; \quad q(t,0) = 0; l(t,0) = 0; l_f(0) = 0; \quad t \in [t_0, t_f]; \\ &q(t,x) \geq 0; \quad l(t,x) \geq 0; \quad l_f(x) \geq 0; \quad \forall (t,x) \in [t_0,t_f] \times \mathcal{R}^n \\ &|r(t,x)| \leq M_r; \quad \forall (t,x) \in [t_0,t_f] \times \mathcal{R}^n \text{ and some } M_r > 0 \end{aligned}$$

□

The kernel of the associated zero-sum differential game is:

$$\begin{aligned} L_\gamma(u,w) &= l_f(x(t_f)) + \int_{t_0}^{t_f} (l(t,x(t)) + |u(t)|^2 - \gamma^2|w(t)|^2 + \epsilon(q(t,x(t)) \\ &\quad + u(t)'r(t,x(t))u(t)))\,dt \end{aligned} \tag{5.3}$$

Clearly, (see [16]), for any $\gamma > 0$, any controller that attains a value 0 for the soft-constrained zero-sum differential game also guarantees the disturbance attenuation level γ for the H^∞-optimal control problem.

Now, consider the following partial differential inequality:

$$\begin{aligned} &V_{\gamma t} + V_{\gamma x}(f + \epsilon a) - \frac{1}{4}|(I + \epsilon r)^{-1}(g' + \epsilon b')V'_{\gamma x}|^2_{I+\epsilon r} + \frac{1}{4\gamma^2}|(h' + \epsilon d')V'_{\gamma x}|^2 \\ &\quad + l + \epsilon q \leq 0; \quad V_\gamma(t_f, x) = l_f(x). \end{aligned} \tag{5.4}$$

If $V_\gamma(t,x;\epsilon)$ is a solution to it, then, this solution constitutes an upper bound for the value function of the soft-constrained zero-sum differential game by a standard "completion of squares" argument. Hence, $V_\gamma(t_0,0;\epsilon) = 0$ implies that the control policy:

$$u^*_{\epsilon\gamma}(t) = \mu^*_{\epsilon\gamma}(t,x(t)) = -\frac{1}{2}(g' + \epsilon b')V'_{\gamma x} \tag{5.5}$$

achieves the performance level γ for the system.

In view of this, let us introduce the set:

$$\begin{aligned} \Gamma_\epsilon \;:=\; &\{\gamma \in \mathcal{R} : \text{ There exists a solution } V_\gamma(t,x;\epsilon) \text{ to inequality (5.4),} \\ &\quad \text{such that } V_\gamma(t_0, 0; \epsilon) = 0\ \} \end{aligned} \tag{5.6}$$

and define

$$\gamma^*(\epsilon) := \inf \Gamma_\epsilon \tag{5.7}$$

Now, we have the following two facts; we provide a proof for only the second one, as the first one is a simple consequence of (5.4).

Fact 1 *Let $\gamma \in \Gamma_\epsilon$; then, $\forall \gamma' > \gamma$, we have $\gamma' \in \Gamma_\epsilon$.* ◇

Fact 2 *Let Assumption 4 hold, and take $\gamma \in \Gamma_\epsilon$. Then, for sufficiently small ϵ such that $I + \epsilon r(t,x) > 0$,*

1. *$V_\gamma(t, 0; \epsilon) = 0$, $\forall t \in [t_0, t_f]$.*
2. *$V_\gamma(t, x; \epsilon) \geq 0$, $\forall t \in [t_0, t_f] \times \mathcal{R}^n$.*
3. *$V_{\gamma x}(t, 0; \epsilon) = 0$, $\forall t \in [t_0, t_f]$.*

Proof Under Assumption 4, and with $w(t) \equiv 0$, we have $L_\gamma \geq 0$. Then the value function for the soft-constrained zero-sum differential game is nonnegative, provided that it exists. The function $V_\gamma(t, x; \epsilon)$ is an upper bound for the value function, and hence 2) follows.

Suppose $(g' + \epsilon b')V'_{\gamma x}(t, 0; \epsilon) \neq 0$ for some $t \in [t_0, t_f]$. Then, by taking $w(t) \equiv 0$, the control law $\mu^*_{\epsilon\gamma}$ leads to a positive value for the game. This contradicts with $V_\gamma(t_0, 0; \epsilon) = 0$, and hence $(g' + \epsilon b')V'_{\gamma x}(t, 0; \epsilon) \equiv 0$.

Setting $x = 0$ in the inequality (5.4), we obtain

$$V_{\gamma t} \leq 0, \quad \forall t \in [t_0, t_f]$$

Since $l_f(0) = 0$ and $V_\gamma(t_0, 0; \epsilon) = 0$, part 1) follows.

Parts 1) and 2) imply part 3), since $V_\gamma(t, x; \epsilon)$ is differentiable. □

Now, set $\epsilon = 0$, and fix a $\gamma > \gamma^*(0)$. Let $W_\gamma(t, x)$ be a solution to (5.4) such that $W_\gamma(t_0, 0) = 0$. Then the control law:

$$u^*_\gamma(t) = \mu^*_\gamma(t, x(t)) = -\frac{1}{2} g' W'_{\gamma x} \tag{5.8}$$

achieves the performance level γ for the "nominal" system. The corresponding disturbance is:

$$w^*(t) = \nu^*(t, x(t)) = \frac{1}{2\gamma^2} h' W'_{\gamma x} \tag{5.9}$$

To study the robustness of the controller μ^*_γ, we apply the control law μ^*_γ to the perturbed system, to arrive at the following maximization problem with respect to w:

$$\dot{x} = f - \frac{1}{2} g g' W'_{\gamma x} + h w + \epsilon(a - \frac{1}{2} b g' W'_{\gamma x} + d w) \tag{5.10}$$

$$L_\gamma = l_f(x(t_f)) + \int_{t_0}^{t_f} (l + \frac{1}{4} W_{\gamma x} g g' W'_{\gamma x} + \epsilon(q + \frac{1}{4} W_{\gamma x} g r g' W'_{\gamma x}) - \gamma^2 w' w)\, dt \tag{5.11}$$

Consider the following partial differential inequality:

$$\tilde{W}_{\gamma t} + \tilde{W}_{\gamma x}(f - \frac{1}{2}gg'W'_{\gamma x} + \epsilon(a - \frac{1}{2}bg'W'_{\gamma x})) + \frac{1}{4\gamma^2}|(h' + \epsilon d')\tilde{W}'_{\gamma x}|^2$$
$$+ l + \frac{1}{4}W_{\gamma x}gg'W'_{\gamma x} + \epsilon(q + \frac{1}{4}W_{\gamma x}grg'W'_{\gamma x}) \leq 0; \quad \tilde{W}_\gamma(t_f, x) = l_f(x), \qquad (5.12)$$

Any solution to the above inequality is an upper bound for the value function of the maximization problem. Note that W_γ is a solution to (5.12) for $\epsilon = 0$. Hence, it is conceivable that the solution to (5.12) is of the form $\tilde{W}_\gamma(t,x) = W_\gamma(t,x) + \delta\bar{W}_\gamma(t,x)$, for some function $\bar{W}_\gamma$ and sufficiently small scalar $\delta > 0$. Now, let us make the following assumption:

Assumption 5 There exists a nonnegative definite function $\Delta_\gamma(t,x)$ with $\Delta_\gamma(t,0) = 0$, such that:

(i) the following partial differential inequality:

$$\bar{W}_{\gamma t} + \bar{W}_{\gamma x}(f - \frac{1}{2}gg'W'_{\gamma x} + \frac{1}{2\gamma^2}hh'W'_{\gamma x}) + \Delta_\gamma \leq 0; \quad \bar{W}_\gamma(t_f, x) = 0, \qquad (5.13)$$

admits a nonnegative definite solution $\bar{W}_\gamma(t,x)$ on the time interval $[t_0, t_f]$; and

(ii) $\bar{W}_\gamma(t,x)$ satisfies the inequality

$$\bar{W}_{\gamma x}hh'\bar{W}'_{\gamma x} \leq M\Delta_\gamma, \quad \forall (t,x) \in [t_0, t_f] \times \mathcal{R}^n \text{ and for some } M > 0. \qquad (5.14)$$

□

Remark 3 *The function $\bar{W}_\gamma(t,x)$ provides an upper bound for the value function of the following cost:*

$$\int_{t_0}^{t_f} \Delta_\gamma(t,x)\,dt\,,$$

where $x(t)$ is generated by the state dynamics:

$$\dot{x} = f - \frac{1}{2}gg'W'_{\gamma x} + \frac{1}{2\gamma^2}hh'W'_{\gamma x}$$

Hence, $\bar{W}_\gamma(t,x) \geq 0 \quad \forall (t,x) \in [t_0, t_f] \times \mathcal{R}^n$, since Δ_γ is nonnegative. Set $x = 0$, and by Fact 2, we have $\bar{W}_{\gamma t}(t,0) \leq 0$. Thus, $\bar{W}_\gamma(t,0) \equiv 0$. ◇

Now, we invoke the following growth conditions on the perturbation terms:

Assumption 6 The perturbation terms a, b, d, q and r satisfy:

$$\begin{array}{ccc} \bar{W}_{\gamma x}dd'\bar{W}'_{\gamma x} \leq M\Delta_\gamma & |\bar{W}_{\gamma x}(a - \frac{1}{2}bg'W'_{\gamma x})| \leq M\Delta_\gamma & \\ W_{\gamma x}dd'W'_{\gamma x} \leq M\Delta_\gamma & |W_{\gamma x}(a - \frac{1}{2}bg'W'_{\gamma x})| \leq M\Delta_\gamma & \forall (t,x) \in [t_0, t_f] \times \mathcal{R}^n \\ |W_{\gamma x}hd'W'_{\gamma x}| \leq M\Delta_\gamma & |W_{\gamma x}hd'\bar{W}'_{\gamma x}| \leq M\Delta_\gamma & \\ |W_{\gamma x}grg'W'_{\gamma x}| \leq M\Delta_\gamma & q \leq M\Delta_\gamma & \end{array}$$

where M and Δ_γ were introduced in Assumption 5. □

Theorem 5 *Consider the nonlinear H^∞-optimal control problem formulated in (5.1)–(5.2) under Assumption 4. For a fixed $\gamma > \gamma^*(0)$, if Assumptions 5 and 6 hold, then there exists an $\epsilon_\gamma > 0$ such that the control law μ^*_γ, defined by (5.8), achieves the disturbance attenuation level γ for the perturbed system for all $|\epsilon| \leq \epsilon_\gamma$.*

Proof Fix $\gamma > \gamma^*(0)$, and follow the discussion that preceded the Theorem. Substitution of $\tilde{W}_\gamma(t,x) = W_\gamma(t,x) + \delta\bar{W}_\gamma(t,x)$ into the LHS of partial differential inequality (5.12), where $\delta > 0$ is a free parameter, leads to:

$$\begin{aligned}
\text{LHS} &= W_{\gamma t} + W_{\gamma x}f - \frac{1}{4}W_{\gamma x}gg'W'_{\gamma x} + \frac{1}{4\gamma^2}W_{\gamma x}hh'W'_{\gamma x} + l + \delta(\bar{W}_{\gamma t} + \bar{W}_{\gamma x}(f \\
&\quad -\frac{1}{2}gg'W'_{\gamma x} + \frac{1}{2\gamma^2}hh'W'_{\gamma x})) + \epsilon\tilde{W}_{\gamma x}(a - \frac{1}{2}bg'W'_{\gamma x}) + \frac{\delta^2}{4\gamma^2}\bar{W}_{\gamma x}hh'\bar{W}'_{\gamma x} \\
&\quad +\frac{\epsilon}{2\gamma^2}(W_{\gamma x}hd'W'_{\gamma x} + \delta W_{\gamma x}hd'\bar{W}'_{\gamma x} + \delta\bar{W}_{\gamma x}hd'W'_{\gamma x} + \delta^2\bar{W}_{\gamma x}hd'\bar{W}'_{\gamma x}) \\
&\quad +\frac{\epsilon^2}{4\gamma^2}(W_{\gamma x}dd'W'_{\gamma x} + 2\delta W_{\gamma x}dd'\bar{W}'_{\gamma x} + \delta^2\bar{W}_{\gamma x}dd'\bar{W}'_{\gamma x}) \\
&\quad +\epsilon(q + \frac{1}{4}W_{\gamma x}grg'W'_{\gamma x}) \\
&\leq -\delta\Delta_\gamma + \epsilon(1+\delta)M\Delta_\gamma + \frac{\delta^2}{4\gamma^2}M\Delta_\gamma + \frac{\epsilon}{2\gamma^2}(M\Delta_\gamma + \delta M\Delta_\gamma + \delta M\Delta_\gamma \\
&\quad +\delta^2 M\Delta_\gamma) + \frac{\epsilon^2}{4\gamma^2}(M\Delta_\gamma + 2\delta M\Delta_\gamma + \delta^2 M\Delta_\gamma) + \epsilon\frac{5}{4}M\Delta_\gamma
\end{aligned}$$

Hence, we can fix a sufficiently small $\delta > 0$ and find an $\epsilon_\gamma > 0$ such that the above is not positive for any $|\epsilon| \leq \epsilon_\gamma$.

Then, $\tilde{W}_\gamma(t,x)$ constitutes an upper bound for the value function of the single player maximization problem (5.10)–(5.11). Note also that $\tilde{W}_\gamma(t_0,0) = 0$, which implies that the soft-constrained zero-sum differential game with state dynamics (5.1) and cost function (5.3) has a value that is less than or equal to 0. Hence, the controller achieves the disturbance attenuation level γ for the perturbed system for $|\epsilon| \leq \epsilon_\gamma$. □

Corollary 5 *A lower bound for the maximal allowable ϵ_γ of Theorem 5 is*

$$\epsilon^*_\gamma = \sup\{\epsilon > 0 : \quad \exists\delta > 0 \textit{ such that the inequality (5.15) is satisfied } \}$$

where

$$\delta \;\geq\; \epsilon(9/4 + \delta + \frac{1}{2\gamma^2}(1 + 2\delta + \delta^2))M + \frac{\delta^2}{4\gamma^2}M + \frac{\epsilon^2}{4\gamma^2}(M + 2\delta M + \delta^2 M) \qquad (5.15)$$

◇

The infinite-horizon case

Now, we turn to the infinite-horizon case of the nonlinear H^∞-optimal control problem formulated at the beginning of the section. We will take f, g, h and l to be time-invariant, $l_f(x) \equiv 0$, and let $t_0 = 0$ and $t_f \to \infty$, as well as $t_f = \infty$. We will let Assumption 4 hold. Then the class of associated soft-constrained zero-sum differential games has the kernel:

$$\begin{aligned} L_\gamma(u,w) &= \int_0^\infty (l(x(t)) + |u(t)|^2 - \gamma^2 |w(t)|^2 + \epsilon(q(t,x(t)) \\ &\quad + u(t)' r(t,x(t)) u(t)))\, dt \end{aligned} \tag{5.16}$$

As in the finite-horizon case, we will seek an upper bound, $V_\gamma(t,x;\epsilon)$, for the value function such that $V_\gamma(0,0;\epsilon) = 0$, where $V_\gamma(t,x;\epsilon)$ is a nonnegative function that satisfies the following partial differential inequality on the time interval $[0,\infty)$:

$$\begin{aligned} & V_{\gamma t} + V_{\gamma x}(f + \epsilon a) - \frac{1}{4}|(I + \epsilon r)^{-1}(g' + \epsilon b')V'_{\gamma x}|^2_{I+\epsilon r} + \frac{1}{4\gamma^2}|(h' + \epsilon d')V'_{\gamma x}|^2 \\ & \quad + l + \epsilon q \le 0 \end{aligned} \tag{5.17}$$

Then, the control policy:

$$u^*_{\epsilon\gamma\infty}(t) = \mu^*_{\epsilon\gamma\infty}(t,x) = -\frac{1}{2}(g' + \epsilon b')V_{\gamma x} \tag{5.18}$$

achieves the performance level γ for the perturbed system.

Let us introduce the set $\Gamma_{\epsilon\infty}$, as the counterpart of (5.6):

$$\begin{aligned} \Gamma_{\epsilon\infty} := \{ & \gamma \in \mathcal{R} : \text{There exists a nonnegative solution } V_\gamma(t,x,\epsilon) \text{ to} \\ & \text{inequality (5.17), such that } V_\gamma(0,0;\epsilon) = 0. \} \end{aligned} \tag{5.19}$$

and let

$$\gamma^*_\infty(\epsilon) := \inf \Gamma_{\epsilon\infty} \tag{5.20}$$

Now, we have the following two facts, as counterparts of F1 and F2 in the finite-horizon case.

Fact 3 *Let $\gamma \in \Gamma_{\epsilon\infty}$; then, $\forall \gamma' > \gamma$, we have $\gamma' \in \Gamma_{\epsilon\infty}$.* ◇

Fact 4 *Let Assumption 4 hold, and $\gamma \in \Gamma_{\epsilon\infty}$. Then, for sufficiently small ϵ such that $I + \epsilon r(t,x) > 0$,*

1. $V_\gamma(t,0;\epsilon) = 0$, $\forall t \in [0,\infty)$.
2. $V_\gamma(t,x;\epsilon) \ge 0$, $\forall t \in [0,\infty) \times \mathcal{R}^n$.
3. $V_{\gamma x}(t,0;\epsilon) = 0$, $\forall t \in [0,\infty)$. ◇

Now, set $\epsilon = 0$, and fix a $\gamma > \gamma^*_\infty(0)$. Let $W_\gamma(x)$ be a solution to (5.17) such that $W_\gamma(0) = 0$. Note that here we assume that W_γ is time-invariant since the system is time-invariant for $\epsilon = 0$. Then the control law:

$$u^*_{\gamma\infty}(t) = \mu^*_{\gamma\infty}(x(t)) = -\frac{1}{2}g'W'_{\gamma x} \tag{5.21}$$

achieves the performance level γ for the "nominal" system. The corresponding (maximizing) disturbance is:

$$w^*(t) = \nu^*(x(t)) = \frac{1}{2\gamma^2}h'W'_{\gamma x} \tag{5.22}$$

To show the robustness of the controller $\mu^*_{\gamma\infty}$, we apply it to the perturbed system, to arrive at the following maximization problem with respect to w:

$$\dot{x} = f - \frac{1}{2}gg'W'_{\gamma x} + hw + \epsilon(a - \frac{1}{2}bg'W'_{\gamma x} + dw) \tag{5.23}$$

$$L_\gamma = \int_0^\infty (l + \frac{1}{4}W_{\gamma x}gg'W'_{\gamma x} + \epsilon(q + \frac{1}{4}W_{\gamma x}grg'W'_{\gamma x}) - \gamma^2 w'w)\, dt \tag{5.24}$$

Consider the following partial differential inequality:

$$\tilde{W}_{\gamma t} + \tilde{W}_{\gamma x}(f - \frac{1}{2}gg'W'_{\gamma x} + \epsilon(a - \frac{1}{2}bg'W'_{\gamma x})) + \frac{1}{4\gamma^2}|(h' + \epsilon d')\tilde{W}'_{\gamma x}|^2$$
$$+ l + \frac{1}{4}W_{\gamma x}gg'W'_{\gamma x} + \epsilon(q + \frac{1}{4}W_{\gamma x}grg'W'_{\gamma x}) \le 0 \tag{5.25}$$

Any nonnegative solution to the above inequality provides an upper bound for the value function of the maximization problem. We again seek a solution in the form $\tilde{W}_\gamma(x) = W_\gamma(x) + \delta\bar{W}_\gamma(x)$, for some function $\bar{W}_\gamma$ and sufficiently small scalar $\delta > 0$. The following assumption is the counterpart of Assumption 5 here:

Assumption 7 There exists a nonnegative definite function $\Delta_\gamma(x)$ with $\Delta_\gamma(0) = 0$, such that:

(i) the following partial differential inequality:

$$\bar{W}_{\gamma x}(f - \frac{1}{2}gg'W'_{\gamma x} + \frac{1}{2\gamma^2}hh'W'_{\gamma x}) + \Delta_\gamma \le 0 \tag{5.26}$$

admits a nonnegative solution $\bar{W}_\gamma(x)$ with $\bar{W}_\gamma(0) = 0$; and

(ii) $\bar{W}_\gamma(x)$ satisfies the inequality

$$\bar{W}_{\gamma x}hh'\bar{W}'_{\gamma x} \le M\Delta_\gamma, \forall x \in \mathcal{R}^n \text{ and for some } M > 0. \tag{5.27}$$

□

Remark 4 *The function $\bar{W}_\gamma(x)$ provides an upper bound for the value function of the following cost:*

$$\int_0^\infty \Delta_\gamma(x)\, dt$$

where $x(t)$ is generated by the state dynamics:

$$\dot{x} = f - \frac{1}{2}gg'W'_{\gamma x} + \frac{1}{2\gamma^2}hh'W'_{\gamma x}$$

Note also that $\bar{W}_\gamma(x)$ forms a Lyapunov function for the above system if $\bar{W}_\gamma(x)$ and $\Delta_\gamma(x)$ are positive on $\mathcal{R}^n \setminus \{0\}$. We also note that Assumption 7 is ensured under the further assumption that $f - \frac{1}{2}gg'W'_{\gamma x} + \frac{1}{2\gamma^2}hh'W'_{\gamma x}$ is globally asymptotically stable (see Appendix C for detailed explanations). ◇

We now assume that the perturbation terms satisfy the following growth conditions:

Assumption 8 The perturbation terms a, b, d, q and r satisfy:

$$\begin{array}{ccc} \bar{W}_{\gamma x}dd'\bar{W}'_{\gamma x} \le M\Delta_\gamma & |\bar{W}_{\gamma x}(a - \frac{1}{2}bg'W'_{\gamma x})| \le M\Delta_\gamma & \\ W_{\gamma x}dd'W'_{\gamma x} \le M\Delta_\gamma & |W_{\gamma x}(a - \frac{1}{2}bg'W'_{\gamma x})| \le M\Delta_\gamma & \forall (t,x) \in [0,\infty) \times \mathcal{R}^n \\ |W_{\gamma x}hd'W'_{\gamma x}| \le M\Delta_\gamma & |W_{\gamma x}hd'\bar{W}'_{\gamma x}| \le M\Delta_\gamma & \\ |W_{\gamma x}grg'W'_{\gamma x}| \le M\Delta_\gamma & q \le M\Delta_\gamma & \end{array}$$

where M and Δ_γ are from Assumption 7. □

Theorem 6 *Consider the nonlinear H^∞-optimal control problem formulated in (5.1)–(5.2) with f, g, h and l time-invariant, $l_f(x) \equiv 0$, and let $t_0 = 0$ and $t_f \to \infty$, or simply $t_f = \infty$, under Assumption 4. For a fixed $\gamma > \gamma^*_\infty(0)$, if Assumptions 7 and 8 hold, then, there exists an $\epsilon_\gamma > 0$ such that the control law $\mu^*_{\gamma\infty}$, defined in (5.21), achieves the disturbance attenuation level γ for the perturbed system for all $|\epsilon| \le \epsilon_\gamma$.*

Proof Fix a $\gamma > \gamma^*_\infty(0)$, and follow the discussion that preceded the Theorem. By substituting $\tilde{W}_\gamma(x) = W_\gamma(x) + \delta\bar{W}_\gamma(x)$ into the left hand side of the partial differential inequality (5.25), where $\delta > 0$ is a free parameter, some straightforward algebraic manipulations yield:

$$\begin{aligned} \text{LHS} \;=\; & W_{\gamma x}f - \frac{1}{4}W_{\gamma x}gg'W'_{\gamma x} + \frac{1}{4\gamma^2}W_{\gamma x}hh'W'_{\gamma x} + l + \delta(\bar{W}_{\gamma x}(f \\ & -\frac{1}{2}gg'W'_{\gamma x} + \frac{1}{2\gamma^2}hh'W'_{\gamma x})) + \epsilon\tilde{W}_{\gamma x}(a - \frac{1}{2}bg'W'_{\gamma x}) + \frac{\delta^2}{4\gamma^2}\bar{W}_{\gamma x}hh'\bar{W}'_{\gamma x} \\ & +\frac{\epsilon}{2\gamma^2}(W_{\gamma x}hd'W'_{\gamma x} + \delta W_{\gamma x}hd'\bar{W}'_{\gamma x} + \delta\bar{W}_{\gamma x}hd'W'_{\gamma x} + \delta^2\bar{W}_{\gamma x}hd'\bar{W}'_{\gamma x}) \\ & +\frac{\epsilon^2}{4\gamma^2}(W_{\gamma x}dd'W'_{\gamma x} + 2\delta W_{\gamma x}dd'\bar{W}'_{\gamma x} + \delta^2\bar{W}_{\gamma x}dd'\bar{W}'_{\gamma x}) \\ & +\epsilon(q + \frac{1}{4}W_{\gamma x}grg'W'_{\gamma x}) \\ \le\;\; & -\delta\Delta_\gamma + \epsilon(1+\delta)M\Delta_\gamma + \frac{\delta^2}{4\gamma^2}M\Delta_\gamma + \frac{\epsilon}{2\gamma^2}(M\Delta_\gamma + \delta M\Delta_\gamma + \delta M\Delta_\gamma \\ & +\delta^2 M\Delta_\gamma) + \frac{\epsilon^2}{4\gamma^2}(M\Delta_\gamma + 2\delta M\Delta_\gamma + \delta^2 M\Delta_\gamma) + \epsilon\frac{5}{4}M\Delta_\gamma \end{aligned}$$

Hence, we can fix a sufficiently small $\delta > 0$ and find an $\epsilon_\gamma > 0$ such that the above is not positive for any $|\epsilon| \le \epsilon_\gamma$.

Then, $\tilde{W}_\gamma(x)$ constitutes an upper bound for the value function of the maximization problem described by (5.23) and (5.24). Note also that $\tilde{W}_\gamma(0) = 0$, which implies that the soft-constrained zero-sum differential game with state dynamics (5.1) and game kernel (5.16) has a value that is less than or equal to 0. Hence, the controller achieves the disturbance attenuation level γ for the perturbed system for all $|\epsilon| \le \epsilon_\gamma$. □

Corollary 6 *A lower bound for the maximal allowable ϵ_γ of Theorem 6 is*

$$\epsilon_\gamma^* = \sup\{\epsilon > 0 : \quad \exists \delta > 0 \text{ such that the inequality (5.28) is satisfied } \}$$

where

$$\delta \ \ge \ \epsilon(9/4 + \delta + \frac{1}{2\gamma^2}(1 + 2\delta + \delta^2))M + \frac{\delta^2}{4\gamma^2}M + \frac{\epsilon^2}{4\gamma^2}(M + 2\delta M + \delta^2 M) \tag{5.28}$$

◇

6 Examples

In this section, we present two numerical examples on prototype first order systems to illustrate the theoretical results obtained. As indicated in the previous sections, we are interested in finding the level of tolerable uncertainties with respect to an H^∞ controller.

Example 1 Consider the scalar system below, with the given scalar measurement,

$$\dot{x} \ = \ x + u + \begin{bmatrix} 1 & 0 \end{bmatrix} w + \epsilon(a(t,x) + b(t,x)u + d(t,x)w) \tag{6.1}$$

$$y \ = \ 2x + \begin{bmatrix} 0 & 1 \end{bmatrix} w + \epsilon(c(t,x) + n(t,x)u + e(t,x)w) \tag{6.2}$$

$$L_\gamma \ = \ \int_{-\infty}^{\infty} (2x^2 + |u|^2 - \gamma^2|w|^2 + \epsilon(q(t,x) + u'r(t,x)u)\, dt \tag{6.3}$$

where the perturbation terms a, b, d, c, n, e, q and r are chosen to satisfy the conditions of Assumption 2 with $M_a = M_b = M_d = M_c = M_n = M_e = M_q = M_r = 1$. First, we compute the optimal performance level of the nominal linear system to be:

$$\gamma_{I\infty}^*(0) = 1.8229$$

Next, we compute the level of allowable ϵ, which is to be denoted by ϵ_γ^*, for different levels of γ, which are tabulated in Table 1 (where δ^* is the level of δ chosen for GARE (4.24) that leads to ϵ_γ^*).

It is important to note that the ϵ_γ^* obtained here is not the maximum allowable level of uncertainty. However, we observe from Table 1 that the level of allowable uncertainty increases very fast as γ increases, and ϵ_γ^* is very small when γ is close to the optimal value. Hence, the larger γ is, the more robust is the design. ◇

Table 1: The level of tolerable uncertainties for nominal H^∞ controller for Example 1

γ	1.83	1.9	2	2.5	3
δ^*	$7.9726E-5$	$8.7087E-3$	0.041931	0.41813	0.93546
ϵ^*_γ	$1.9650E-10$	$1.8680E-6$	$3.1072E-5$	$1.0741E-3$	$3.2281E-3$

Example 2 Consider the following nonlinear system, under perfect state measurements,

$$\dot{x} = x(u-w) + \epsilon(a(t,x) + b(t,x)u + d(t,x)w) \quad (6.4)$$

$$L_\gamma = \int_0^\infty (x^4 + |u|^2 - \gamma^2|w|^2 + \epsilon(q(t,x) + u'r(t,x)u)\,dt \quad (6.5)$$

where a, b, d, q and r are nonlinear perturbation terms.

For the nominal system (obtained by setting $\epsilon = 0$) the value function for the associated soft-constrained zero-sum differential game is

$$W_\gamma(x) = \frac{\gamma}{\sqrt{\gamma^2-1}}x^2.$$

Hence, the optimal performance level is $\gamma^*_\infty(0) = 1$.

For each $\gamma > 1$, the H^∞ controller guaranteeing this performance level for the nominal system is given by:

$$\mu^*_{\gamma\infty}(x) = -\frac{\gamma}{\sqrt{\gamma^2-1}}x^2$$

and the worst-case disturbance is

$$\nu^*_\gamma(x) = -\frac{1}{\gamma\sqrt{\gamma^2-1}}x^2$$

It is easy to check that conditions of Assumption 7 are satisfied by choosing $\Delta_\gamma(x) = 2x^4$, which leads to $\bar{W}_\gamma(x) = W_\gamma(x)$. This leads to the following growth conditions to be imposed on the perturbation terms:

$$\begin{array}{ll} |a(t,x)| \le M_a|x|^3; \quad |b(t,x)| \le M_b|x|; \quad |d(t,x)| \le M_d|x|; & \\ |r(t,x)| \le M_r; \qquad |q(t,x)| \le M_q x^4; & \forall (t,x) \in [0,\infty) \times \mathcal{R}. \end{array}$$

for some nonnegative constants M_a, M_b, M_d, M_q and M_r. In this example, we assume that these constants are equal to 1.

Now, substitution of $\tilde{W}_\gamma(t,x) = (1+\delta)W_\gamma(x)$ into the partial differential inequality (5.25), yields the following algebraic inequality for δ and ϵ:

$$\epsilon^2(1+\delta)^2 + \epsilon(2(1+\delta)^2 + 2(1+\delta)\gamma\sqrt{\gamma^2-1} + 2(1+\delta)\gamma^2 + 2\gamma^2 - 1)$$
$$-2\delta\gamma^2 + 2\delta + \delta^2 \le 0$$

Table 2: The level of tolerable uncertainties for the nominal H^∞ controller for Example 2

γ	1.01	1.1	1.5	2	3
δ^*	0.019864	0.18853	0.82057	1.5031	2.7585
ϵ_γ^*	$7.3536E-5$	$5.2406E-3$	0.055989	0.11728	0.20342

Now, we again compute the level of allowable ϵ, to be denoted by ϵ_γ^*, for different levels of γ, which are tabulated in Table 2 (where δ^* is the level of δ chosen that leads to ϵ_γ^*).

If we fix $\gamma = 2$, the H^∞ controller for the nominal system becomes:

$$\mu^* = -\frac{2}{\sqrt{3}}x^2$$

and the worst-case disturbance is:

$$\nu^* = -\frac{1}{2\sqrt{3}}x^2$$

To obtain some simulation results, we assume that the perturbation terms are given by

$$a(t,x) = \tfrac{x^5}{1+x^2}\cos(xt); \quad b(t,x) = \tfrac{x^3}{1+x^2}\cos(xt); \quad d(t,x) = \tfrac{x^3}{1+x^2};$$

$$r(t,x) = \sin(x); \qquad q(t,x) = x^4\sin^2(x).$$

We now compare the response of the nominal system with that of the perturbed system under various disturbance inputs. First set $\epsilon = 0.1$; then by Table 2, the nominal controller guarantees the disturbance attenuation level of 2 for the perturbed system. We simulate the dynamic system for three sets of disturbance inputs: 1) the worst-case disturbance for the nominal system, ν^*; 2) constant input; 3) exponentially decaying sinusoid. For each disturbance input, we present four plots, each one comparing the disturbance input, state trajectory, control action, and the game cost incurred, respectively, for the nominal system and the perturbed system. Then, we set $\epsilon = 0.2$ and repeat the same procedure.

From Figures 1, 2, 3, 4, 5 and 6, we see that the H^∞ controller performs very well for the perturbed system, even for the case $\epsilon = 0.2$. ◇

7 Conclusions

This paper has addressed the issue of robustness of H^∞-controllers to structural perturbations in the system dynamics, measurement equation and the performance index. We have considered two types of nominal systems, linear and nonlinear, and for the latter class we have assumed the existence of a well-defined state feedback controller achieving a desired

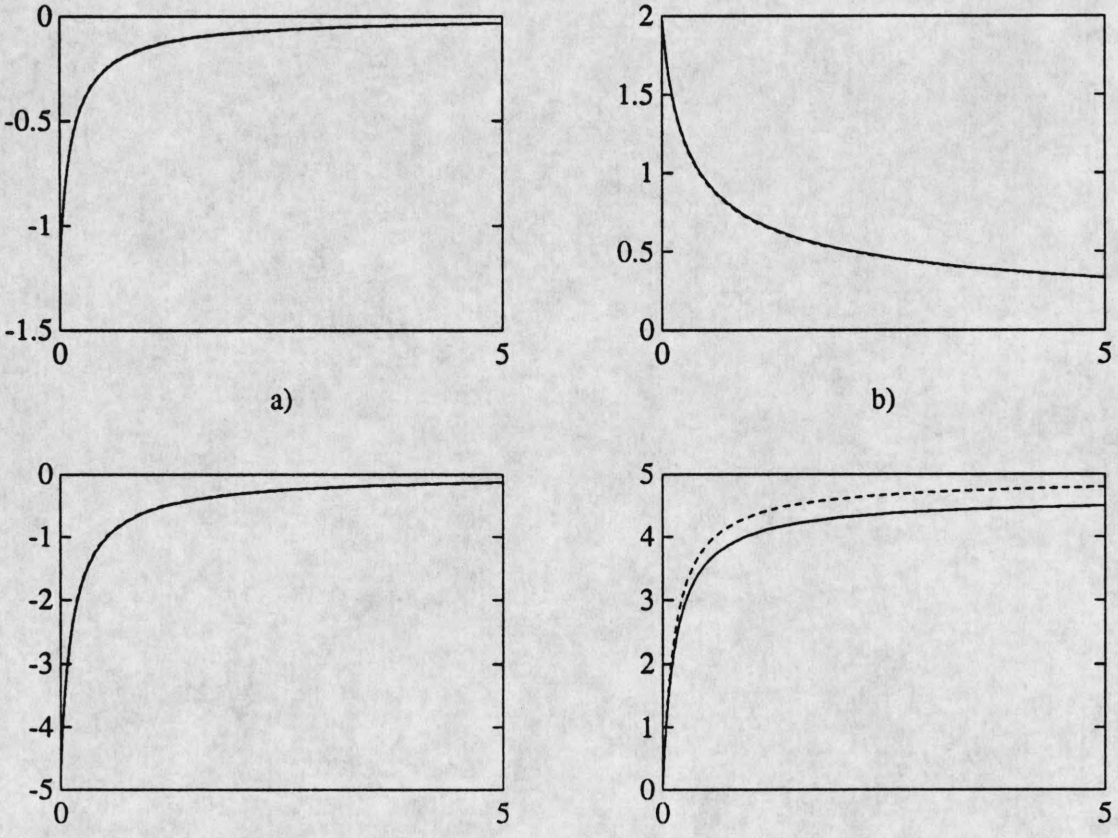

Figure 1: Input/output signals for Example 2: Worst disturbance ν^*; $\epsilon = 0.1$
(a) disturbance input; (b) state trajectory; (c) control input; (d) game cost
——— for the nominal system; - - - - for the perturbed system.

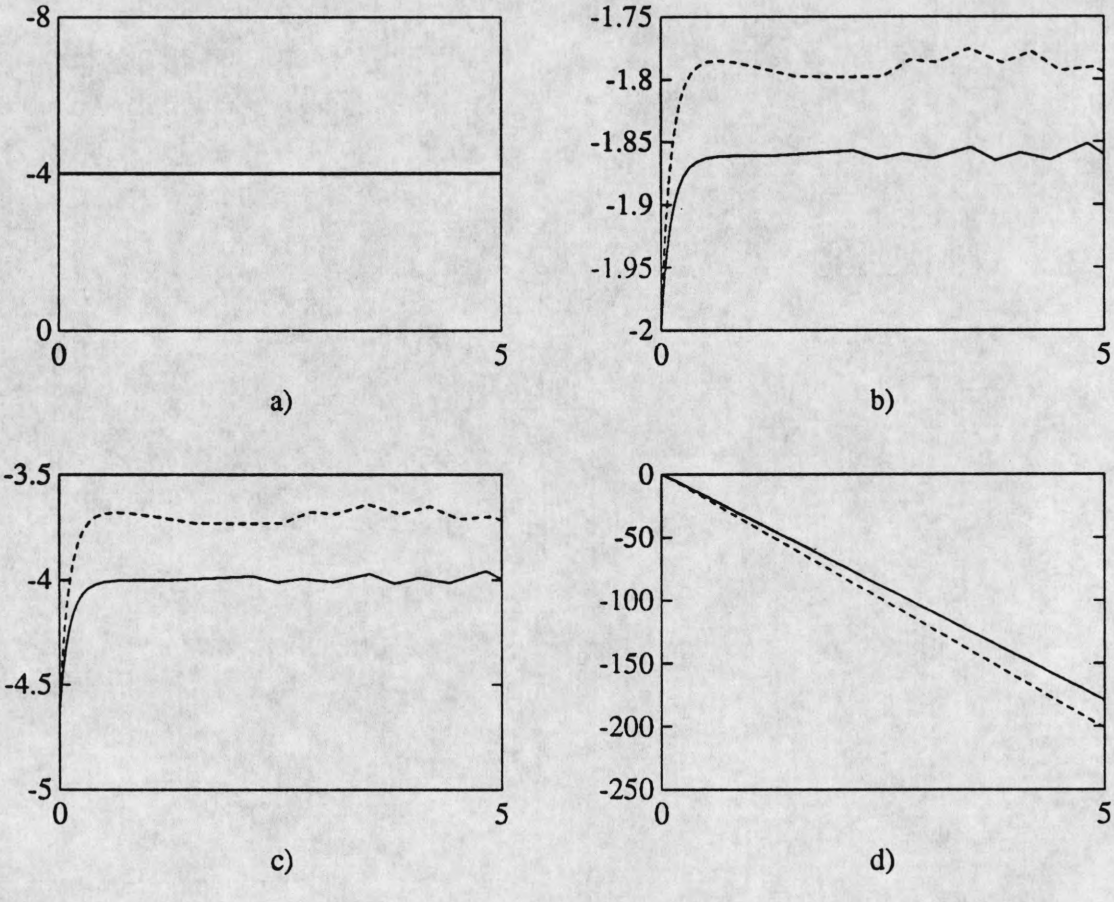

Figure 2: Input/output signals for Example 2: Constant disturbance; $\epsilon = 0.1$
(a) disturbance input; (b) state trajectory; (c) control input; (d) game cost
——— for the nominal system; - - - - for the perturbed system.

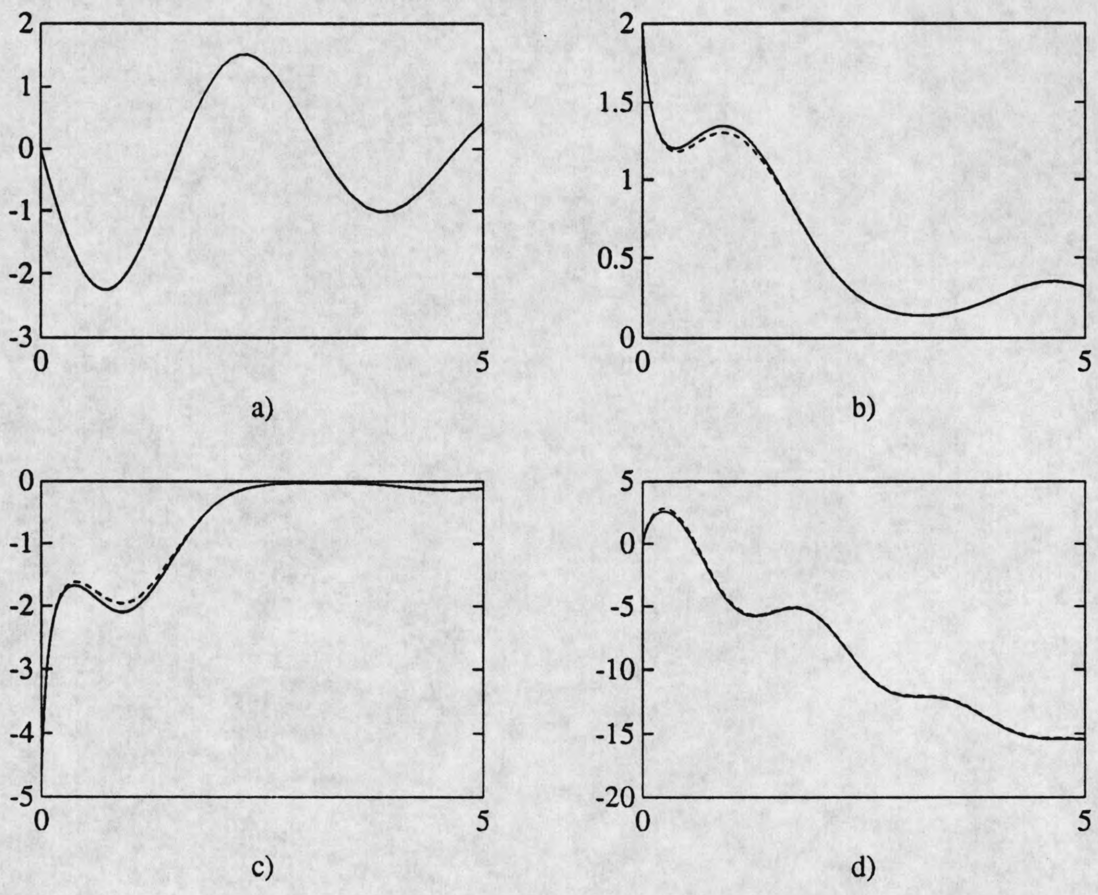

Figure 3: Input/output signals for Example 2: Exponentially decaying sinusoidal disturbance; $\epsilon = 0.1$
(a) disturbance input; (b) state trajectory; (c) control input; (d) game cost
——— for the nominal system; - - - - for the perturbed system.

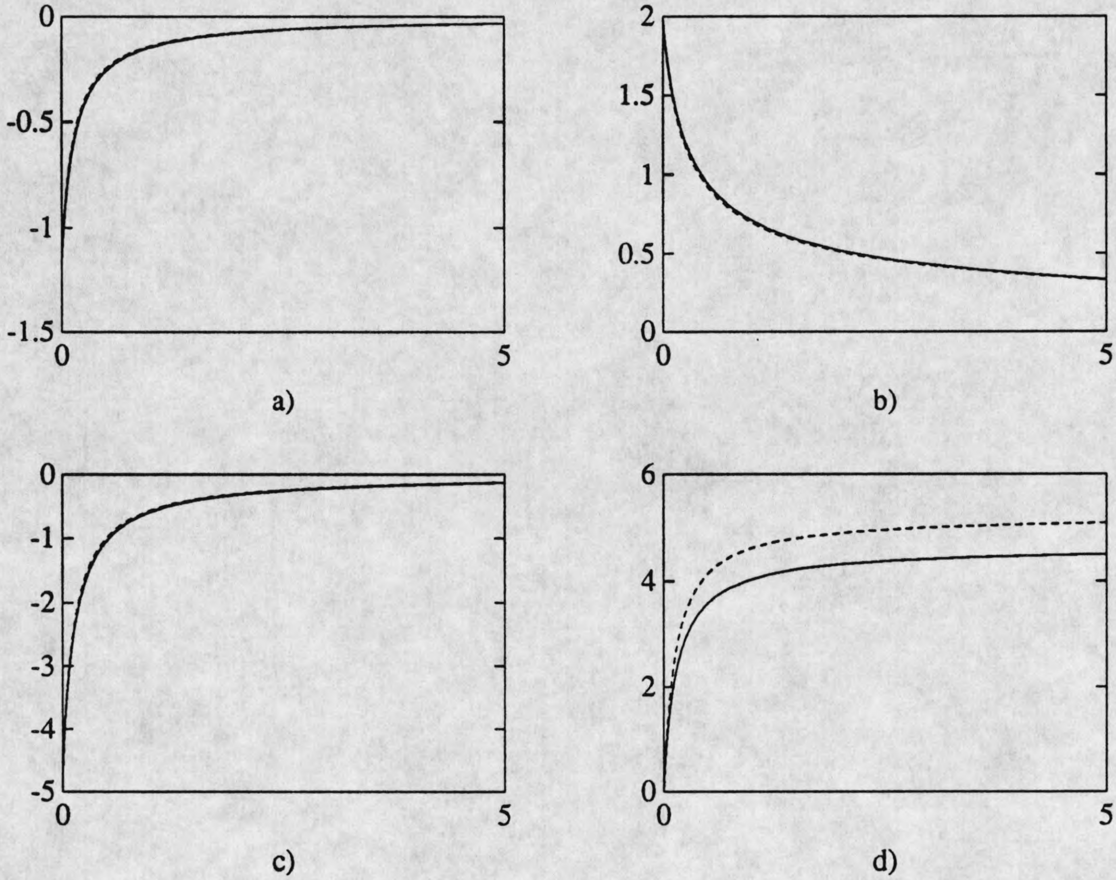

Figure 4: Input/output signals for Example 2: Worst disturbance ν^*; $\epsilon = 0.2$
(a) disturbance input; (b) state trajectory; (c) control input; (d) game cost
—— for the nominal system; – – – – for the perturbed system.

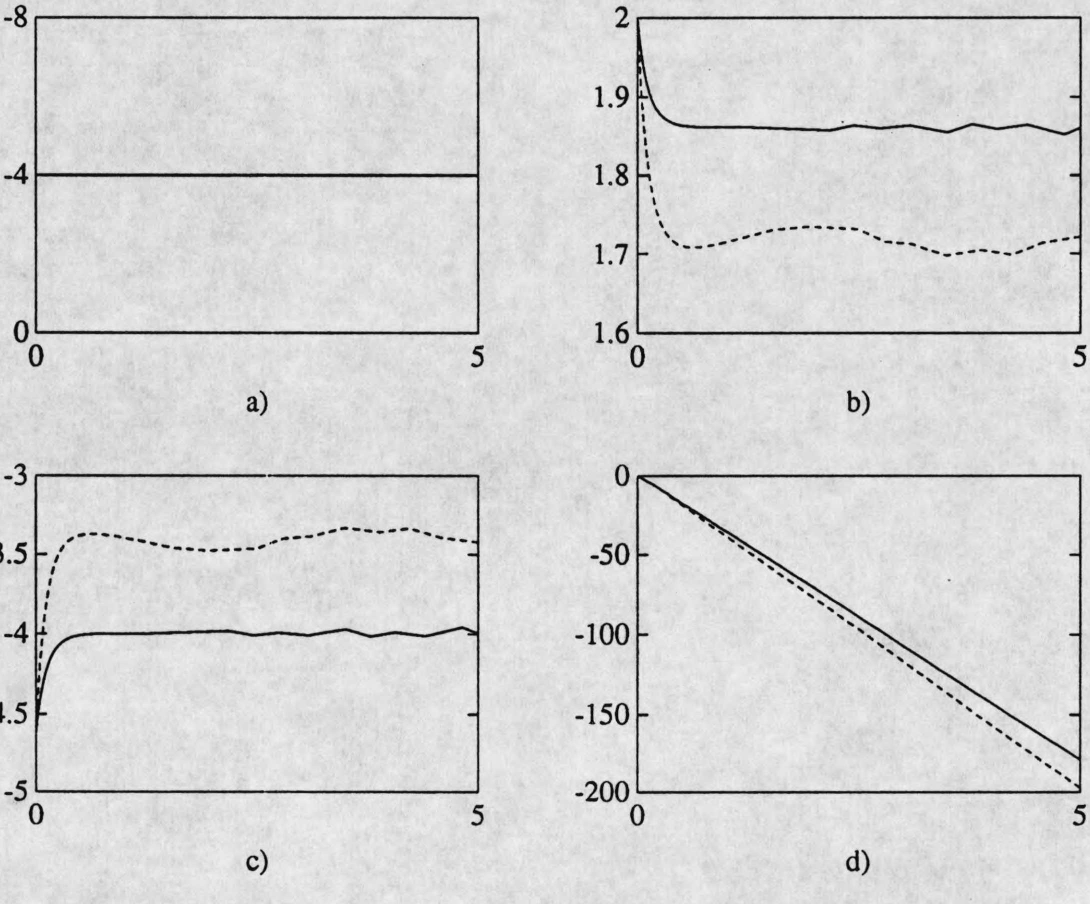

Figure 5: Input/output signals for Example 2: Constant disturbance; $\epsilon = 0.2$
(a) disturbance input; (b) state trajectory; (c) control input; (d) game cost
——— for the nominal system; - - - - for the perturbed system.

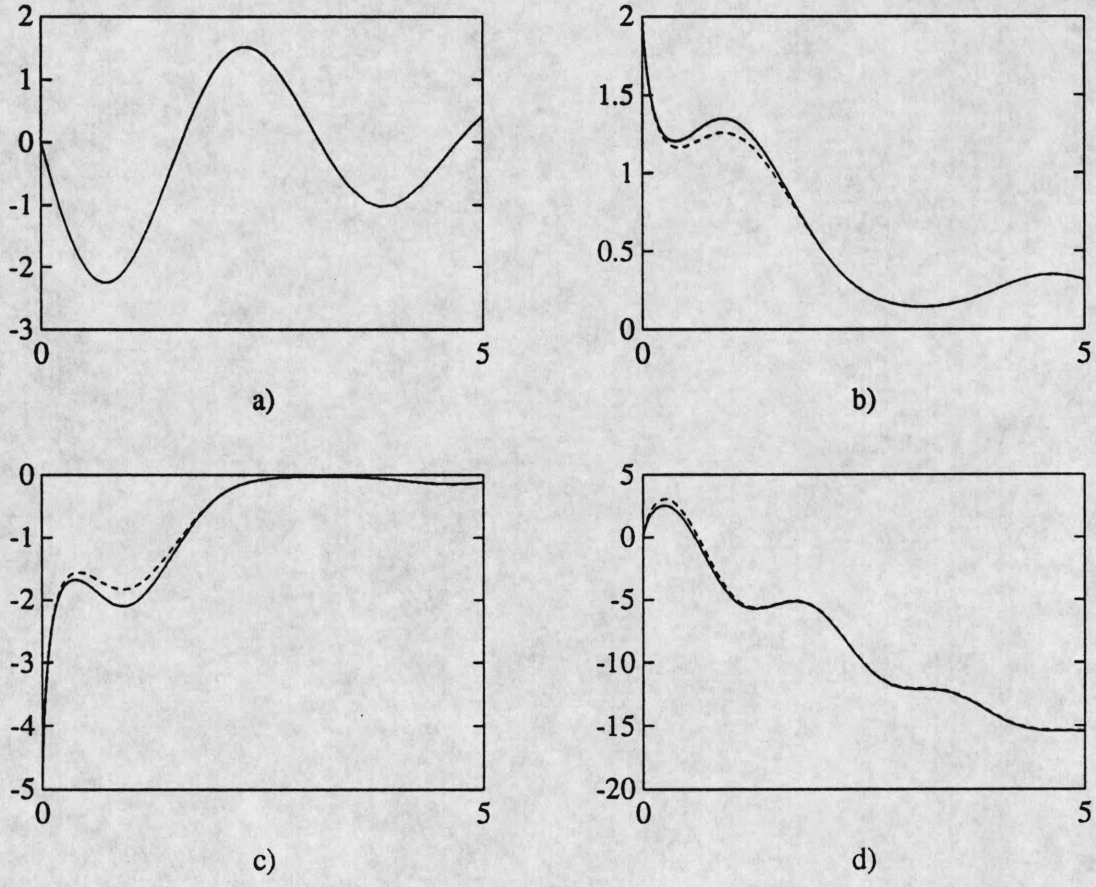

Figure 6: Input/output signals for Example 2: Exponentially decaying sinusoidal disturbance; $\epsilon = 0.2$
(a) disturbance input; (b) state trajectory; (c) control input; (d) game cost
——— for the nominal system; – – – – for the perturbed system.

performance level. In both cases we have studied both finite and infinite-horizon problems, and have taken the perturbations to be nonlinear and satisfying some growth conditions. When the nominal system is linear, we showed that the nominal H^∞ controller designed based on a certain level of disturbance attenuation is robust to small nonlinear structural uncertainties, i. e., the controller guarantees that level of performance for the perturbed system when the perturbation parameter is sufficiently small — this being true for the perfect and imperfect state measurements cases in the infinite horizon, and only for the perfect state measurement case when the horizon is finite. For the remaining case of finite-horizon imperfect state measurements problem, we have developed a modified controller design scheme that also yields controllers that are robust with respect to small structural uncertainties. The design philosophy for such a scheme is that the designer should appear to be less confident on the initial data when he is not perfectly certain about the plant model.

For the case of nominally nonlinear systems, and under perfect state measurements, we have obtained a set of sufficient conditions for an H^∞-controller designed for the original nonlinear system to be robust against small structural uncertainties. These conditions lead to a characterization of the class of uncertainties that the controller can tolerate, which involves the specification of some growth conditions on the uncertainties. The paper has also presented two numerical examples, one being linear and the other one nonlinear.

The results obtained in this paper concern the performance robustness of a given H^∞ controller. The controller is chosen to be the central controller for the nominal linear system, for each fixed achievable performance level γ, and then the size of the sustainable ball in the uncertainty set is determined such that the controller designed for the nominal system achieves the desired performance bound robustly. These results do not follow from the existing small gain type of results. The existing results on performance robustness [20] [21] [22], obtained from the small gain theorem and its variants, require an H^∞ controller design for the nominal system based on some performance level $\gamma - \delta(\epsilon_0)$, for some $\delta(\epsilon_0) > 0$, in order to robustly achieve the desired performance level γ for the perturbed system in face of the ϵ_0- sized ball of uncertainties. This paper shows that the central H^∞ controller designed for the nominal system based on a performance level γ can robustly achieve the same performance level γ for sufficiently small uncertainties. It is a known fact that a central H^∞ controller, designed based on a desired performance γ for a linear system, can acturally achieve a better performance for the system than γ. A byproduct of the analysis of this paper is a lower bound for the difference between the desired performance level and the achieved performance level for the central H^∞ controller.

Robustness (to nonlinear perturbations) is one reason why we have formulated and studied the class of problems addressed in this paper. In this interpretation, there is a nominal, known system, and one wishes to design a controller that yields good performance for a family of systems which are perturbed versions of the original one. Yet another motivation for our formulation has been the desire to obtain simpler controllers for complex nonlinear systems. In this interpretation the perturbed system could be considered as the complex system, and the nominal one is some approximation to it. What has been shown here is that, for the class of systems considered, nonlinear systems can be "approximated" well by either linear or simpler nonlinear systems.

If the solution to the original problem is viewed as a function of ϵ, then what we have obtained is in fact only the *zeroth* order approximation to the true ("optimal") controller. Even though from a robustness point of view this is adequate, from the second view-point mentioned above this may not be totally satisfactory, as higher order terms in ϵ (assuming knowledge of the true value of ϵ) could lead to improved performance. Here, in fact, it turns out that within the framework of our analysis, the higher-order terms (in ϵ) could also be derived without much difficulty — we have not done it here in order not to divert attention from the "robustness" interpretation of the results.

Other extensions are possible. One future topic (though not a trivial one) would be to study the relationship of these results with those obtainable from the regularly perturbed stochastic risk-sensitive optimal control problem ([23]) so as to obtain the counterparts of the results in [24] for regularly perturbed nonlinear systems. Another extension would be to remove the restriction that the nonlinear perturbations necessarily lead to systems that are affine in the control and disturbance variables. Yet another extension would be to allow for dynamic structural perturbations, which then would require (as one possibility) a study of the H^∞-optimal control of nonlinear singularly perturbed systems, so as to obtain the counterparts of the results obtained in [14] [25] in the nonlinear domain. This is a topic currently under study.

Appendices

A

In this appendix, we present a number of fundamental robustness results on algebraic and differential generalized (game) Riccati equations, which were used in the main body of the paper. Some of these results were first derived in the unpublished report [26], which we provide here using the set-up of the current application.

We first let $\bar{\gamma}^*$ denote the quantity $\gamma^*(0)$, $\bar{\gamma}^*_\infty$ denote the quantity $\gamma^*_\infty(0)$, and $Z_\gamma(t, t_f; Q_f)$ denote the solution $Z_\gamma(t)$ to GRDE (3.1), for sake of simplicity and clarity in the presentation to follow.

We introduce the following auxiliary optimal control problem:

$$\dot{z} = -A(t) - \begin{bmatrix} Q_s(t) & 0 \end{bmatrix} u; \quad z(t_1) = z_0 \tag{A.1}$$

with the cost function:

$$\begin{aligned} J_\gamma(z_0, t_1, t_f, u, Q_{if}) \;&=\; |z(t_f)|^2_{Q_{if}} + \int_{t_1}^{t_f} (|z(t)|^2_{B(t)B(t)'} + 2u' \begin{bmatrix} 0 \\ -D(t)' \end{bmatrix} z \\ &\quad + u' \begin{bmatrix} I & 0 \\ 0 & \gamma^2 \end{bmatrix} u)\, dt \end{aligned} \tag{A.2}$$

where $t_1 \in [t_0, t_f]$, $Q_s(t) = Q(t)^{1/2}$ and $Q_{if} \geq 0$. This optimal control problem admits a solution if there exists a solution $Z^{\#}_\gamma(t, t_f; Q_{if})$ to the following GRDE:

$$\begin{array}{c}\dot{Z}^{\#}_\gamma(t) - Z^{\#}_\gamma(t)A(t)' - A(t)Z^{\#}_\gamma(t) - Z^{\#}_\gamma(t)Q(t)Z^{\#}_\gamma(t) + B(t)B(t)' \\ -\frac{1}{\gamma^2}D(t)D(t)' = 0; \quad Z^{\#}_\gamma(t_f) = Q_{if} \quad t \in [t_1, t_f]\end{array} \tag{A.3}$$

Then, the minimum cost, to be denoted by $J^*_\gamma(z_0, t_1, t_f, Q_{if})$, is $z_0' Z^{\#}_\gamma(t_1, t_f; Q_{if})z_0$, and the minimizing controller is

$$u^*_\gamma(t) = \mu^*_\gamma(z, t) = \begin{bmatrix} Q_s' Z^{\#}_\gamma(t, t_f; Q_{if}) \\ -\frac{1}{\gamma^2}D(t)' \end{bmatrix} z(t) \tag{A.4}$$

We note here that the GRDE (A.3) is the "inverse" GRDE of (3.1), i.e., $Z^{\#}_\gamma(t, t_f; Q_{if}) \equiv Z_\gamma(t, t_f; Q_f)^{-1}$, when any one of them exists, provided that $Q_{if} = Q_f^{-1}$. We define the critical level of γ as follows:

$$\begin{array}{rcl}\tilde{\gamma}^* & := & \inf\{\gamma' > 0 : \forall \gamma > \gamma', \text{ the GRDE (A.3) admits a symmetric solution} \\ & & Z^{\#}_\gamma(t, t_f; Q_{if}) \text{ on the interval } [t_0, t_f] \ \}\end{array} \tag{A.5}$$

It is clear that $\tilde{\gamma}^* \leq \bar{\gamma}^*$, if $Q_{if} = Q_f^{-1}$.

To bridge the results obtained by studying the "inverse" GRDE (A.3) and properties for the GARE (3.13), we introduce the "inverse" GARE of (3.13):

$$-A\bar{Z}^{\#}_\gamma - \bar{Z}^{\#}_\gamma A' - \bar{Z}^{\#}_\gamma Q \bar{Z}^{\#}_\gamma + BB' - \frac{1}{\gamma^2}DD' = 0 \tag{A.6}$$

We will denote the maximal symmetric solution to GARE (A.6) by $\bar{Z}^{\#}_\gamma$. Thus, we have $\bar{Z}^{\#}_\gamma = \bar{Z}^{-1}_\gamma$ when any one of them exists and is positive definite. Similar to (A.5), we define the following critical level:

$$\begin{array}{rcl}\tilde{\gamma}^*_\infty & := & \inf\{\gamma' > 0 : \forall \gamma > \gamma', \text{ the GARE (A.6) admits a maximal symmetric} \\ & & \text{solution } \bar{Z}^{\#}_\gamma \ \}\end{array} \tag{A.7}$$

Clearly, the matrix function $\bar{Z}^{\#}_\gamma(t, t_f; Q_{if})$ is nondecreasing in γ.

Now we present a lemma which has been used in the proof of Theorem 1.

Lemma 1 *Under Assumption 1, if $Q_f > 0$, take $Q_{if} = Q_f^{-1}$; then, $\forall \gamma < \bar{\gamma}^*$, $\exists T \in [t_0, t_f)$ such that the GRDE (A.3) admits a solution, $Z^{\#}_\gamma(t, t_f; Q_{if})$, on $[T, t_f]$ and the matrix $Z^{\#}_\gamma(T, t_f; Q_{if})$ has at least one negative eigenvalue.*

Proof Fix a γ, such that $\gamma < \bar{\gamma}^*$. Fix a γ_1, such that $\gamma < \gamma_1 \leq \bar{\gamma}^*$. Then, by the definition of $\bar{\gamma}^*$, there exists $T_1 \in [t_0, t_f]$ such that the GRDE (3.1) admits a positive definite solution $Z_{\gamma_1}(t, t_f; Q_f)$ on $(T_1, t_f]$, but has a conjugate point at $t = T_1$. Since $Q_f > 0$, $Z_{\gamma_1}(t, t_f; Q_f) > 0$ on $[T_1, t_f]$. Hence, the GRDE (A.3) admits solution $Z^{\#}_{\gamma_1}(t, t_f; Q_f^{-1}) = Z_{\gamma_1}(t, t_f; Q_f)^{-1}$ on $(T_1, t_f]$. Note that $\bar{\gamma}^* < \infty$, the matrix $Z^{\#}_{\gamma_1}(t, t_f; Q_f^{-1})$ is always bounded from above uniformly in t, and $Z^{\#}_{\gamma_1}(t, t_f; Q_f^{-1}) > 0$ on $(T_1, t_f]$; hence $Z^{\#}_{\gamma_1}(t, t_f; Q_f^{-1})$ exists on $[T_1, t_f]$, by a standard (continuity) result on differential equations. Again by continuity, we know that $Z^{\#}_{\gamma_1}(T_1, t_f; Q_f^{-1}) \geq 0$. Since GRDE (3.1) has a conjugate point at T_1, $Z^{\#}_{\gamma_1}(T_1, t_f; Q_f^{-1})$ cannot be positive definite. Then, $\exists z_0 \neq 0 \in \mathcal{R}^n$ such that $z_0' Z^{\#}_{\gamma_1}(T_1, t_f; Q_f^{-1})z_0 = 0$. Thus, we have

$$0 = z_0' Z^{\#}_{\gamma_1}(T_1, t_f; Q_f^{-1}) z_0 = J_{\gamma_1}(z_0, T_1, t_f, u^*_{\gamma_1}, Q_f^{-1})$$

where

$$u^*_{\gamma_1}(t) = \begin{bmatrix} Q_s' Z^{\#}_{\gamma_1}(T_1, t_f; Q_f^{-1}) \\ -\frac{1}{\gamma^2} D' \end{bmatrix} z(t); \quad \forall t \in [T_1, t_f]$$

Suppose that $-\frac{1}{\gamma^2} D' z(t) \equiv 0$ on $[T_1, t_f]$. By using this in the expression for J_{γ_1}, we have $|z(t_f)|^2_{Q_{f11}^{-1}} = 0$, and hence $z(t_f) = 0$, which is impossible. This contradiction means that $\exists t \in [T_1, t_f]$ such that $-\frac{1}{\gamma^2} D' z(t) \neq 0$. If $Z^{\#}_{\gamma}(t, t_f; Q_f^{-1})$ exists on $[T_1, t_f]$, we have

$$0 = J_{\gamma_1}(z_0, T_1, t_f, u^*_{\gamma_1}, Q_f^{-1}) > J_{\gamma}(z_0, T_1, t_f, u^*_{\gamma_1}, Q_f^{-1}) \geq z_0' Z^{\#}_{\gamma}(T_1, t_f, Q_f^{-1}) z_0$$

which proves the result with T being T_1.

If $Z^{\#}_{\gamma}(t, t_f; Q_f^{-1})$ does not exist on $[T_1, t_f]$, it must go to $-\infty$ at some point in $[T_1, t_f)$ since it is always uniformly bounded from above. Then, we can find a $T \in (T_1, t_f)$, and the result is proved. This completes the proof. □

This result is critical for our proof of the upper semi-continuity result. If we know that γ chosen is smaller than the nominal $\bar{\gamma}^*$, this result tell us that, under small perturbations, the GRDE (3.1) will still admit at least one conjugate point in the time interval (t_0, t_f), since its inverse will have an eigenvalue crossing over the zero point.

Now, we turn to study the property of the solution to the GARE (3.13), as well as the solution to the GARE (A.6). We will replace the time-varying parameters $A(t)$, $B(t)$, $D(t)$ and $Q(t)$ by their time-invariant counterparts A, B, D and Q in GRDEs (3.1) and (A.3), and in the auxiliary optimal control problem. It is clear that the results obtained in Lemma 1 are readily applicable here.

In the study of the solution to GARE (3.13), we are generally interested also in the stability of the matrix $A - (BB' - \frac{1}{\gamma^2} DD')\bar{Z}_{\gamma}$. For the GARE (A.6), the stability of the matrix $-A' - Q\bar{Z}^{\#}_{\gamma}$ is also useful in many problems. The classical paper on the topic [19], explores this problem in great detail. But [19] has only proven that the matrix $-A' - Q\bar{Z}^{\#}_{\gamma}$ is at least boundarily stable when $\bar{Z}^{\#}_{\gamma}$ exists. There are many occasions where we need strict stability of the matrix $-A' - Q\bar{Z}^{\#}_{\gamma}$, such as the case of Theorem 2. Next, we present a result which provides a complete answer to this question in terms of parameterization on γ.

Lemma 2 *Consider the GARE (A.6), under Assumption 1. $\forall \gamma > \tilde{\gamma}^*_{\infty}$, let $\bar{Z}^{\#}_{\gamma}$ be the maximal symmetric solution to (A.6). If the pair (A, Q) is observable and either the pair (A, B) or the pair (A, D) or the pair $(A', Q_s\bar{Z}^{\#}_{\gamma})$ is controllable, then, the matrix $-A' - Q\bar{Z}^{\#}_{\gamma}$ is Hurwitz.*

Proof First, note that the observability of the pair (A, Q) implies the controllability of the pair $(-A', Q_s)$. This further implies that the GARE (A.6) admits a nonnegative definite solution for $\gamma = \infty$. Hence, $\tilde{\gamma}^*_{\infty} < \infty$. Fix $\gamma > \tilde{\gamma}^*_{\infty}$; since the pair (A, Q) is observable, by Theorem 5 in [19], the matrix $-A' - Q\bar{Z}^{\#}_{\gamma}$ has all its eigenvalues in $\mathcal{C}^-$ (with $\mathcal{C}^-$, we mean the closed left-half plane of the complex plane).

Suppose that the matrix $-A' - Q\bar{Z}^{\#}_{\gamma}$ is not Hurwitz. Then, $\exists z_0 \in \mathcal{R}^n$, $z_0 \neq 0$, and $T > 0$ such that

$$\dot{z} = (-A' - Q\bar{Z}_\gamma^{\#})z; \quad z(0) = z_0$$

leads to $z(T) = z_0$. It is not difficult to see that $Z_\gamma^{\#}(t, T; \bar{Z}_\gamma^{\#}) \equiv \bar{Z}_\gamma^{\#}$ on $[0, T]$. Thus, we have $J_\gamma^*(z_0, 0, T, \bar{Z}_\gamma^{\#}) = z_0'\bar{Z}_\gamma^{\#} Z_0$, and the optimal control for the auxiliary problem is

$$u_\gamma^*(t) = \begin{bmatrix} Q_s'\bar{Z}_\gamma^{\#} \\ -\frac{1}{\gamma^2}D' \end{bmatrix} z(t)$$

The system dynamics associated with this controller is precisely

$$\dot{z} = (-A' - Q\bar{Z}_\gamma^{\#})z; \quad z(0) = z_0.$$

Hence, we have the following:

$$\begin{aligned} z_0'\bar{Z}_\gamma^{\#} z_0 &= J_\gamma^*(z_0, 0, T, \bar{Z}_\gamma^{\#}) \\ &= z_0'\bar{Z}_\gamma^{\#} z_0 + \int_0^T (|z|_{BB'}^2 + 2u_\gamma^{*\prime} \begin{bmatrix} 0 \\ -D' \end{bmatrix} z + u_\gamma^{*\prime} \begin{bmatrix} I & 0 \\ 0 & \gamma^2 \end{bmatrix} u_\gamma^*)\, dt \end{aligned}$$

Thus,

$$\int_0^T (|z|_{BB'}^2 + 2u_\gamma^{*\prime} \begin{bmatrix} 0 \\ -D' \end{bmatrix} z + u_\gamma^{*\prime} \begin{bmatrix} I & 0 \\ 0 & \gamma^2 \end{bmatrix} u_\gamma^*)\, dt = 0 \tag{A.8}$$

This leads to two cases: either $-\frac{1}{\gamma^2}D'z(t) \equiv 0$ on $[0, T]$, or it is not.

First, we consider the case $-\frac{1}{\gamma^2}D'z(t) \equiv 0$ on $[0, T]$. By using this in (A.8), we obtain

$$B'z(t) \equiv 0; \quad Q_s\bar{Z}_\gamma^{\#} z(t) \equiv 0; \quad t \in [0, T]$$

from which it follows that $u_\gamma^* \equiv 0$ on $[0, T]$, and, hence, $z(t)$ is generated by the following differential equation:

$$\dot{z} = -A'z; \quad z(0) = z_0.$$

This means that none of the pairs $(-A', B')$, $(-A', D')$ and $(-A', Q_s\bar{Z}_\gamma^{\#})$ is observable. This contradicts the hypothesis of the Lemma.

Next, we consider the case $-\frac{1}{\gamma^2}D'z(t) \neq 0$ for some $t \in [0, T]$. Let γ_1 be such that $\tilde{\gamma}_\infty^* < \gamma_1 < \gamma$. Then, we have the following strict inequality:

$$0 = J_\gamma(z_0, 0, T, u_\gamma^*, 0) > J_{\gamma_1}(z_0, 0, T, u_\gamma^*, 0)$$

where the first equality is a short-hand notation for (A.8). Note that

$$J_{\gamma_1}(z_0, 0, T, u_\gamma^*, \bar{Z}_{\gamma_1}^{\#}) = z_0'\bar{Z}_{\gamma_1}^{\#} z_0 + J_{\gamma_1}(z_0, 0, T, u_\gamma^*, 0)$$

This leads to the contradiction:

$$z_0'\bar{Z}^{\#}_{s\gamma_1}z_0 = J^*_{\gamma_1}(z_0,0,T,\bar{Z}^{\#}_{s\gamma_1}) \leq J_{\gamma_1}(z_0,0,T,u^*_{\gamma},\bar{Z}^{\#}_{s\gamma_1}) < z_0'\bar{Z}^{\#}_{s\gamma_1}z_0$$

In both cases, we have arrived at a contradiction. Hence, the matrix $-A' - Q\bar{Z}^{\#}_{\gamma}$ must be Hurwitz. □

Because of the close relationship between the two GAREs (3.13) and (A.6), we can deduce a similar result on the matrix $A - (BB' - \frac{1}{\gamma^2}DD')\bar{Z}_{\gamma}$, which is summarized in the following corollary:

Corollary 7 *Consider the GARE (3.13), under Assumption 1, and assume further that $\bar{\gamma}^*_{\infty} < \infty$, and the pair (A, Q) is observable. Then, $\forall\gamma > \bar{\gamma}^*_{\infty}$, the matrix $A-(BB'-\frac{1}{\gamma^2}DD')\bar{Z}_{\gamma}$ is Hurwitz.*

Proof Fix a $\gamma' > \bar{\gamma}^*_{\infty}$, and $\forall\gamma > \gamma'$, let $\bar{Z}_{\gamma}$ be the minimal nonnegative definite solution to GARE (3.13). The observability of the pair (A, Q) implies, by Theorem 4.8 of [16], that the matrix $\bar{Z}_{\gamma} > 0$. Hence, GARE (A.6) admits a maximal symmetric solution $\bar{Z}^{-1}_{\gamma}$. Thus, it follows that $\gamma > \tilde{\gamma}^*_{\infty}$ and the pair $(A', Q_s\bar{Z}^{-1}_{\gamma})$ is observable. Then by Lemma 2, the matrix $-A' - Q\bar{Z}^{-1}_{\gamma}$ is Hurwitz. Since

$$A - (BB' - \frac{1}{\gamma^2}DD')\bar{Z}_{\gamma} = \bar{Z}^{-1}_{\gamma}(-A' - Q\bar{Z}^{-1}_{\gamma})\bar{Z}_{\gamma}$$

it follows that the matrix $A - (BB' - \frac{1}{\gamma^2}DD')\bar{Z}_{\gamma}$ is also Hurwitz, and this completes the proof. □

As a byproduct of the above corollary, we have the following relationship between the two quantities $\bar{\gamma}^*_{\infty}$ and $\tilde{\gamma}^*_{\infty}$:

Corollary 8 *Consider the GARE (3.13), under Assumption 1. Assume that the pair (A, Q) is observable. Then, $\bar{\gamma}^*_{\infty} \geq \tilde{\gamma}^*_{\infty}$.*

Proof The result is true if $\bar{\gamma}^*_{\infty} = +\infty$. In case that $\bar{\gamma}^*_{\infty} < +\infty$, the proof of Corollary 7 shows that if $\forall\gamma > \bar{\gamma}^*_{\infty}$, then $\gamma > \tilde{\gamma}^*_{\infty}$. This completes the proof. □

The following lemmas now give results which are in fact stronger than what we would need in the proof of Theorem 2, but they are included here for the sake of completeness and since they might be of independent interest.

Lemma 3 *Consider GARE (A.6), and assume validity of the same conditions as in Lemma 2. For fixed $\gamma > \tilde{\gamma}^*_{\infty}$, let $Z^{\#}_{\gamma}(t, t_f; Q_{if})$ be the solution to the GRDE (A.6), where $Q_{if} \geq \bar{Z}^{\#}_{\gamma}$. Then*

$$\lim_{t_f\to\infty} Z^{\#}_{\gamma}(t, t_f; Q_{if}) = \bar{Z}^{\#}_{\gamma} \quad \textit{exponentially} \tag{A.9}$$

Proof By the time-invariant nature of the problem, we can take $t_f = 0$ and prove, equivalently,

$$\lim_{t\to-\infty} Z_\gamma^{\#}(t,t_f;Q_{if}) = \bar{Z}_\gamma^{\#}$$

By Lemma 2, the matrix $-A' - Q\bar{Z}_\gamma^{\#}$ is Hurwitz. Consider the case $Q_{if} > \bar{Z}$. Let $\tilde{\Delta} := Z_\gamma^{\#}(t,t_f,Q_{if}) - \bar{Z}_\gamma^{\#}$. Subtract (A.6) from (A.3) to arrive at

$$\dot{\tilde{\Delta}} + \tilde{\Delta}(-A' - Q\bar{Z}_\gamma^{\#}) + (-A - \bar{Z}_\gamma^{\#}Q)\tilde{\Delta} - \tilde{\Delta}Q\tilde{\Delta} = 0; \quad \tilde{\Delta}(0) > 0$$

In this equation, set $\Delta(-\tau) = \tilde{\Delta}^{-1}(t)$, where $\tau = -t$, to get

$$\frac{d}{d\tau}\Delta + (-A' - Q\bar{Z}_\gamma^{\#})\Delta + \Delta(-A - \bar{Z}_\gamma^{\#}Q) - Q = 0; \quad \Delta(0) > 0; \quad \tau \in [0,\infty)$$

Then, we have

$$\Delta(\tau) = e^{-A_f\tau}(\Delta(0) + \int_0^\tau e^{A_f s}Qe^{A_f' s}\,ds\,)e^{-A_f'\tau}; \quad \forall\tau \leq 0$$

where $A_f := -A - \bar{Z}_\gamma^{\#}Q$. Then, the result follows. When $Q_{if} = \bar{Z}$, the result is trivially true. Since the matrix $Z_\gamma^{\#}(t,t_f;Q_{if})$ is nondecreasing in Q_{if}, the result holds also for $Q_{if} \geq \bar{Z}$. □

Now, we state a corollary which relates the solution to the GARE (3.13) to the solution of the GRDE (3.1).

Corollary 9 *Consider the GARE (3.13), and invoke the same conditions as in Corollary 7. For fixed γ, let $Z_\gamma(t,t_f,Q_f)$ be the nonnegative definite solution to the GRDE (3.1), where $Q_f \leq \bar{Z}_\gamma$. Then*

$$\lim_{t_f\to\infty} Z_\gamma(t,t_f;Q_f) = \bar{Z}_\gamma \quad \textit{exponentially} \tag{A.10}$$

Proof If $Q_f > 0$, then $Z_\gamma(t,t_f;Q_f)^{-1}$ exists and is the solution to GRDE (A.3) with $Q_{if} = Q_f^{-1}$, and $\bar{Z}_\gamma^{-1}$ is the maximal symmetric solution to the GARE (A.6). Thus, Lemma 3 applies and completes the proof.

For the case $Q_f \geq 0$, but is not strictly positive definite, $Z_\gamma(t,t_f;Q_f)$ is bounded from above by $\bar{Z}_\gamma$ at all $t \leq t_f$, and bounded from below by the solution $Z_\gamma(t,t_f;0)$. It is well-known that $\lim_{t\to-\infty} Z_\gamma(t,t_f;0) = \bar{Z}_\gamma$, which means that $Z_\gamma(T,t_f;0) > 0$ for some $T < t_f$, since $\bar{Z}_\gamma > 0$. Then, we have that $Z_\gamma(t,T;Z_\gamma(T,t_f;0))$ converges to $\bar{Z}_\gamma$ exponentially as $t \to -\infty$, by the first part of the proof. This completes the proof. □

Lemma 4 *Consider the GARE (3.13), under Assumption 1. Assume that the pair (A,Q) is observable and $\bar{\gamma}_\infty^* < \infty$ and either the pair (A,B) or the pair (A,D) is controllable. Let Q_f be a positive definite matrix such that $Q_f < \bar{Z}_{\gamma_b}$ for some $\gamma_b > \bar{\gamma}_\infty^*$. Then, $\forall\gamma < \gamma_\infty^*$, $\exists T > 0$, such that GRDE (A.3) admits a solution $Z_\gamma^{\#}(t,T,Q_f^{-1})$ on $[0,T]$, and $Z_\gamma^{\#}(0,T;Q_f^{-1})$ has at least one negative eigenvalue.*

Proof By a reasoning similar to that used in Corollary 7, we know that $\bar{Z}_{\gamma_b} > 0$, hence, the lemma makes sense. Fix a $\gamma < \gamma_\infty^*$. Now, we can reduce the problem to two cases: either $\gamma < \tilde{\gamma}_\infty^*$ or $\tilde{\gamma}_\infty^* \leq \gamma < \gamma_\infty^*$, in view of Corollary 8.

First, we consider the case $\gamma < \tilde{\gamma}_\infty^*$. Then, $\exists\gamma_1$, $\gamma < \gamma_1 < \tilde{\gamma}_\infty^*$, such that GARE (A.6) does not admit any symmetric solution. By Theorem 7 in [19], $\exists z_0 \in \mathcal{R}^n$ such that

$$J^*_{\gamma_1}(z_0, 0, \infty) = -\infty; \qquad \text{subject to } \lim_{t\to\infty} z(t) = 0$$

Here, we abuse the notation, and let $J^*_{\gamma_1}(z_0, 0, \infty)$ denote the optimal cost for the infinite-horizon version of the auxiliary optimal control problem, where all coefficient matrices are substituted by the time-invariant ones from GARE (3.13) except that Q_f is set to 0, and $t_0 = 0$ and $t_f = \infty$.

Then, it is not difficult to see that $\exists T_1 > 0$ such that the GRDE (A.3) admits a solution $Z^\#_{\gamma_1}(t, T_1; Q_f^{-1})$ on $[0, T_1]$ and

$$z_0' Z^\#_{\gamma_1}(0, T_1; Q_f^{-1}) z_0 = J^*_{\gamma_1}(z_0, 0, T_1, Q_f^{-1}) < 0,$$

which means that $Z^\#_{\gamma_1}(0, T_1; Q_f^{-1})$ has at least one negative eigenvalue. If $Z^\#_{\gamma}(t, T_1; Q_f^{-1})$ exists on $[0, T_1]$, then the result follows. If $Z^\#_{\gamma}(t, T_1; Q_f^{-1})$ does not exist on $[0, T_1]$, it must exist on $[T_2, T_1]$, for some $0 < T_2 < T_1$, and $Z^\#_{\gamma}(T_2, T_1; Q_f^{-1})$ has at least one negative eigenvalue, since it is always bounded from above by $Z^\#_{\gamma_1}(t, T_1; Q_f^{-1})$. Let $T = T_1 - T_2$, and the result follows.

Next, we consider the case $\tilde{\gamma}^*_\infty \le \gamma < \gamma^*_\infty$. Thus, we can find a γ_1 such that $\gamma < \gamma_1 \le \gamma^*_\infty$. Then, by Corollary 7, the GARE (A.6) admits a maximal symmetric solution $\bar{Z}^\#_{\gamma_1}$. The solution $\bar{Z}^\#_{\gamma_1}$ must not be positive definite, since otherwise the GARE (3.13) would admit a minimal positive definite solution $\bar{Z}^{\#\,-1}_{\gamma_1}$, which implies $\gamma_1 > \bar{\gamma}^*_\infty$. Fix a $\gamma_2 > \gamma$ such that $\gamma_2 < \gamma_1$. Using Lemma 3, we can deduce that $\bar{Z}^\#_{\gamma_2} \le \bar{Z}^\#_{\gamma_1}$.

We now claim that $\bar{Z}^\#_{s\gamma_2}$ has at least one negative eigenvalue. Suppose not; then we have $\bar{Z}^\#_{\gamma_1} \ge 0$. This implies that $\exists z_0 \ne 0$ such that $z_0' \bar{Z}^\#_{\gamma_1} z_0 = 0$. For any $t_F > 0$, $Z^\#_{\gamma_1}(t, t_F; \bar{Z}^\#_{\gamma_1}) \equiv \bar{Z}^\#_{\gamma_1}$, hence,

$$0 = z_0' \bar{Z}^\#_{\gamma_1} z_0 = J_{\gamma_1}(z_0, 0, t_F, u^*_{\gamma_1}, \bar{Z}^\#_{\gamma_1})$$

where

$$u^*_{\gamma_1}(t) = \begin{bmatrix} Q_s' \bar{Z}^\#_{\gamma_1} \\ -\frac{1}{\gamma^2} D' \end{bmatrix} z(t)$$

This again gives us two cases: either the equality $-\frac{1}{\gamma^2} D' z(t) \equiv 0$ holds on $[0, t_F]$, or it does not.

Case a): $-\frac{1}{\gamma^2} D' z(t) \equiv 0$ on $[0, t_F]$. By using this in the expression for $J_{\gamma_1}(z_0, 0, t_F, u^*_{\gamma_1}, \bar{Z}^\#_{\gamma_1})$, we have

$$z(t_F)' \bar{Z}^\#_{\gamma_1} z(t_F) = 0; \quad B' z(t) \equiv 0; \quad Q_s \bar{Z}^\#_{\gamma} z(t) \equiv 0; \quad t \in [0, t_F]$$

which then shows that $u^*_{\gamma_1} \equiv 0$ on $[0, t_F]$, and, hence, $z(t)$ is generated by the following differential equation:

$$\dot{z} = -A' z; \quad z(0) = z_0$$

which contradicts the observability of the pair $(-A', B')$ or the pair $(-A', D')$.

Case b): $-\frac{1}{\gamma^2}D'z(t) \neq 0$ for some $t \in [0, t_F]$. Then, we have the following strict inequality:

$$0 = J_{\gamma_1}(z_0, 0, t_F, u^*_{\gamma_1}, \bar{Z}^\#_{\gamma_1}) > J_{\gamma_2}(z_0, 0, t_F, u^*_{\gamma_1}, \bar{Z}^\#_{\gamma_1})$$

Note that

$$J_{\gamma_2}(z_0, 0, t_F, u^*_{\gamma_1}, \bar{Z}^\#_{\gamma_1}) \geq J_{\gamma_2}(z_0, 0, t_F, u^*_{\gamma_1}, \bar{Z}^\#_{\gamma_2}) \geq J^*_{\gamma_2}(z_0, 0, t_F, \bar{Z}^\#_{\gamma_2}) = z_0' \bar{Z}^\#_{\gamma_2} z_0$$

which leads to the contradiction:

$$0 > z_0' \bar{Z}^\#_{\gamma_2} z_0 \geq 0$$

Hence, our claim that $\bar{Z}^\#_{\gamma_2}$ has at least one negative eigenvalue is verified. Note that

$$Q_f^{-1} > \bar{Z}^{-1}_{\gamma_b} = \bar{Z}^\#_{\gamma_b} \geq \bar{Z}^\#_{\gamma_2}$$

By Lemmas 3 and 2, $Z^\#_{\gamma_2}(t, t_F; Q_f^{-1}) \rightarrow \bar{Z}^\#_{\gamma_2}$ as $t_F \rightarrow \infty$. Then, $\exists T_1 > 0$ such that $Z^\#_{\gamma_2}(0, T_1; Q_f^{-1})$ has at least one negative eigenvalue. Then, we can find $T > 0$ such that $Z^\#_\gamma(0, T; Q_f^{-1})$ has at least one negative eigenvalue.

This completes the proof. □

B

In this Appendix, we provide expressions for the constants M_f, M_g, M_h, $\bar{M}_f$, $\bar{M}_g$ and $\bar{M}_h$, which were introduced in Section 4. The following Fact is needed.

Fact 5 *For scalars $k_1 > 0$, k_2 and $k_3 > 0$,*

$$k_1|x|^2 + k_2|x||\hat{x}| + k_3|\hat{x}|^2 \leq (\frac{k_1 + k_3}{2} + \frac{1}{2}\sqrt{k_2^2 + (k_1 - k_3)^2})(|x|^2 + |\hat{x}|^2)$$

Proof Note that

$$|k_2||x||\hat{x}| \leq \frac{1}{2}(k_4|x|^2 + \frac{k_2^2}{k_4}|\hat{x}|^2)$$

where

$$k_4 = k_3 - k_1 + \sqrt{k_2^2 + (k_1 - k_3)^2})$$

This completes the proof. □

Hence, we have

$$
\begin{aligned}
M_f &= \sqrt{\frac{\lambda_1 + \lambda_3}{2} + \frac{1}{2}\sqrt{\lambda_2^2 + (\lambda_1 - \lambda_3)^2}} \\
\lambda_1 &= M_a^2 + \gamma^4(|(\gamma^2\tilde{\Sigma}^{-1} - Z)^{-1}||C||N^{-1}|M_c)^2 \\
\lambda_2 &= 2M_aM_b|B||Z| + 2\gamma^4(|(\gamma^2\tilde{\Sigma}^{-1} - Z)^{-1}||C||N^{-1}|)^2 M_cM_n|B|Z| \\
\lambda_3 &= (M_b|B||Z|)^2 + \gamma^4(|(\gamma^2\tilde{\Sigma}^{-1} - Z)^{-1}||C||N^{-1}|M_n|B||Z|)^2 \\
M_g &= \sqrt{M_d^2 + \gamma^4(|(\gamma^2\tilde{\Sigma}^{-1} - Z)^{-1}||C||N^{-1}|M_e)^2} \\
M_h &= \max\{M_q\ ,\ M_r|B|^2|Z|^2\} \\
\bar{M}_f &= \sqrt{\frac{\bar{\lambda}_1 + \bar{\lambda}_3}{2} + \frac{1}{2}\sqrt{\bar{\lambda}_2^2 + (\bar{\lambda}_1 - \bar{\lambda}_3)^2}} \\
\bar{\lambda}_1 &= M_a^2 + \gamma^4(|(\gamma^2\ddot{\Sigma}^{-1} - \bar{Z})^{-1}||C||N^{-1}|M_c)^2 \\
\bar{\lambda}_2 &= 2M_aM_b|B||\bar{Z}| + 2\gamma^4(|(\gamma^2\bar{\Sigma}^{-1} - \bar{Z})^{-1}||C||N^{-1}|)^2 M_cM_n|B|\bar{Z}| \\
\bar{\lambda}_3 &= (M_b|B||\bar{Z}|)^2 + \gamma^4(|(\gamma^2\bar{\Sigma}^{-1} - \bar{Z})^{-1}||C||N^{-1}|M_n|B||\bar{Z}|)^2 \\
\bar{M}_g &= \sqrt{M_d^2 + \gamma^4(|(\gamma^2\bar{\Sigma}^{-1} - \bar{Z})^{-1}||C||N^{-1}|M_e)^2} \\
\bar{M}_h &= \max\{M_q\ ,\ M_r|B|^2|\bar{Z}|^2\}
\end{aligned}
$$

C

In this Appendix, we show that, in the infinite horizon case, Assumption 7 can be satisfied if the system

$$
\dot{x} = f - \frac{1}{2}gg'W'_{\gamma x} + \frac{1}{2\gamma^2}hh'W'_{\gamma x} := \chi(x) \tag{C.1}
$$

is globally asymptotically stable.

We first recall the definitions of class $\mathcal{K}$ and class $\mathcal{K}_\infty$ functions ([27], pp. 167):

Definition 1 *A continuous function* $\alpha : [0, a) \to [0, \infty)$ *is said to belong to class* $\mathcal{K}$ *if it is strictly increasing and* $\alpha(0) = 0$. *It is further said to belong to class* $\mathcal{K}_\infty$ *if* $a = \infty$ *and* $\alpha(r) \to \infty$ *as* $r \to \infty$.

Assume that system (C.1) is globally asymptotically stable, i. e., there exists a continuously differentiable function $V : \mathcal{R}^n \to [0, \infty)$ that satisfies:

$$
\begin{aligned}
&\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \\
&V_x\chi(x) := -\xi(x) \leq -\alpha_3(|x|)
\end{aligned}
$$

for all $x \in \mathcal{R}^n$ and some class $\mathcal{K}_\infty$ functions α_1 and α_2 and a class $\mathcal{K}$ function α_3.

Define a class $\mathcal{K}$ function $\alpha(s)$ by:

$$\alpha_s(s) := \min\{1, \min_{\{x:\ \alpha_2^{-1}(s)\leq|x|\leq\alpha_1^{-1}(s)\}} \frac{\alpha_3(|x|)}{V_x hh' V_x'}\}$$

$$\alpha(s) = \int_0^s \alpha_s(\sigma)\, d\sigma$$

for all $s \in [0,\infty)$. Note that α_s is always positive on $(0,\infty)$.

Let $\bar{W}_\gamma(x) := \alpha(V(x))$ and $\Delta_\gamma := \alpha_s(V(x))\xi(x)$. It is easy to see that (5.26) is then satisfied with equality. Also,

$$\frac{\Delta_\gamma}{\bar{W}_{\gamma x} hh' \bar{W}_{\gamma x}'} = \frac{\xi}{\alpha_s(V(x))V_x hh' V_x'} \geq \frac{\alpha_3(|x|)}{\alpha_s(V(x))V_x hh' V_x'} \geq 1$$

Thus, (5.27) is satisfied with $M = 1$. Hence, Assumption 7 is satisfied.

References

[1] A. Isidori and A. Astolfi, "Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems," *IEEE Transactions on Automatic Control*, vol. AC-37, pp. 1283–1293, September 1992.

[2] A. J. Van der Schaft, "On a state-space approach to nonlinear H_∞ control," *Systems & Control Letters*, vol. 16, pp. 1–8, 1991.

[3] A. J. Van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear H_∞ control," *IEEE Transactions on Automatic Control*, vol. AC-37, pp. 770–784, 1992.

[4] A. Isidori, "New results on nonlinear H_∞-control via measurement feedback," in *Advances in Dynamic Games and Applications* (T. Başar and A. Haurie, eds.), Boston, MA: Birkhäuser, 1993.

[5] A. Isidori, "H_∞-control via measurement feedback for affine nonlinear systems," tech. rep., Washington University, St. Louis, Missouri, 1992.

[6] A. Isidori, "Feedback control of nonlinear systems," in *Proceedings of the 1st European Control Conference*, (Grenoble, France), July 1991.

[7] G. Didinsky, T. Başar, and P. Bernhard, "Structural properties of minimax controllers for a class of differential games arising in nonlinear H^∞-control," *Systems & Control Letters*, vol. 21, pp. 433–441, December 1993.

[8] J. A. Ball and J. W. Helton, "H^∞ control for nonlinear plants: Connections with differential games," in *Proceedings of the IEEE 28th Conference on Decision and Control*, (Tampa, FL), pp. 956–962, December 13-15, 1989.

[9] J. A. Ball, J. W. Helton, and M. Walker, "H^∞ control for nonlinear systems with output feedback," *IEEE Transactions on Automatic Control*, vol. AC-38, pp. 546–559, April 1993.

[10] L. Xie, C. E. de Souza, and M. D. Fragoso, "H_∞ filtering for linear periodic systems with parameter uncertainty," *Systems & Control Letters*, vol. 17, pp. 343–350, 1991.

[11] P. P. Khargonekar, I. R. Petersen, and K. Zhou, "Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ control theory," *IEEE Transactions on Automatic Control*, vol. AC-35, pp. 356–361, 1990.

[12] L. Xie, C. E. de Souza, and M. Fu, "H_∞ estimation for linear discrete-time uncertain systems," *Int. J. Robust & Nonlinear Control*, vol. 2, pp. 111–123, 1991.

[13] P. Gahinet and A. A. Stoorvogel, "Continuity Properties of the H_∞-Optimal Gain," in *Proceedings of the Eurpoean Control Conference*, (Groningen, Holland), pp. 1171–1175, 1993.

[14] Z. Pan and T. Başar, "H^∞-optimal control for singularly perturbed systems. Part I: Perfect state measurements," *IEEE Transactions on Automatic Control*, vol. AC-29, pp. 401–423, March 1993.

[15] Z. Pan and T. Başar, "H^∞-optimal control for singularly perturbed systems. Part II: Imperfect state measurements," *IEEE Transactions on Automatic Control*, vol. AC-39, pp. 280–299, February 1994.

[16] T. Başar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Boston, MA: Birkhäuser, 1991.

[17] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory.* London/New York: Academic Press, 1982. Second printing 1989.

[18] W. Rudin, *Principles of Mathematical Analysis.* New York, NY: McGraw-Hill, 1976.

[19] J. C. Willems, "Least squares stationary optimal control and the algebraic Riccati equation," *IEEE Transactions on Automatic Control*, vol. AC-16, pp. 621–634, December 1971.

[20] M. H. Khammash, "Necessary and sufficient conditions for the robustness of time-varying systems with applications to sampled-data systems," *IEEE Transactions on Automatic Control*, vol. AC-38, pp. 49–57, January 1993.

[21] M. Khammash and J. B. Pearson, "Performance robustness of discrete-time systems with structured uncertainty," *IEEE Transactions on Automatic Control*, vol. AC-36, pp. 398–412, April 1991.

[22] M. Khammash and J. B. Pearson, "Analysis and design for robust performance with structured uncertainty." To appear in *Systems & Control Letters.*

[23] P. Whittle, "Likelihood and cost as path integrals," *J. R. Statist. Soc. B*, vol. 53, no. 3, pp. 505–538, 1991.

[24] Z. Pan and T. Başar, "Optimal control of stochastic singularly perturbed systems with exponentiated quadratic cost," in *Proceedings of the 12nd IFAC World Congress*, vol. IX, (Sydney, Australia), pp. 293–298, July 18-23, 1993.

[25] Z. Pan and T. Başar, "H^∞-optimal control for singularly perturbed systems. Part II: Imperfect state measurements," in *Proceedings of the 31st IEEE Conference on Decision and Control*, (Tucson, AZ), pp. 943–948, December 1992.

[26] Z. Pan and T. Başar, "A tight bound for the H^∞ performance of singularly perturbed systems," CSL Report, University of Illinois, Urbana, IL, March 1992.

[27] H. K. Khalil, *Nonlinear Systems.* Macmillan, 1992.