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ESTIMATING THREE DIMENSIONAL MOTION PARAMETERS OF A RIGID PLANAR PATCH

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In Part I, we show that given two image frames one can determine uniquely (by solving linear equations) eight "pure parameters" which are nonlinear functions of the actual motion parameters. In Part II, a method is presented for determining the motion parameters from the eight pure parameters. The method requires the

singular value decomposition of a 3 x 3 matrix. It also shown that generally there are two distinct solutions for the motion parameters. Two results are given in Part III. First, four point correspondences between two image frames are necessary and sufficient to determine the eight pure parameters. Second, with three image frames, the motion parameters are unique.

Preface

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ESTIMATING THREE-DIMENSIONAL MOTION PARAMETERS OF A RIGID PLANAR PATCH
PART I

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ABSTRACT

We present a new direct method of estimating the three-dimensional motion parameters of a rigid planar patch from two time-sequential perspective views (image frames). First, a set of 8 pure parameters are defined. These parameters can be determined uniquely from the two given image frames by solving a set of linear equations. Then, the actual motion parameters are determined from these pure parameters by a method which requires the solution of a 6th-order polynomial of one variable only, and there exists a certain efficient algorithm for solving a 6th-order polynomial. Aside from a scaling factor for the translation parameters the number of real solutions never exceeds two. In the special case of three-dimensional translation, the motion parameters can be expressed directly as some simple functions of the 8 pure parameters. Thus only a few arithmetic operations are needed.

I. INTRODUCTION

In the past, most work on motion estimation has been restricted to two-dimensional translation. Recently, Roach and Aggarwal [1] and Huang and Tsai [2,3,4] presented methods of estimating three-dimensional motion parameters of rigid bodies based on image-space shifts. The method of Roach and Aggarwal requires the solution of a set of 18 simultaneous non-linear equations; that of Huang and Tsai 5 simultaneous nonlinear equations. Huang and Tsai [4,5] also described a direct method of estimating three-dimensional motion parameters of rigid planar patches based on the relationship between temporal and spatial differentials of image intensity. This method results in the solution of 8 simultaneous nonlinear equations. In none of the above works was the question of the uniqueness of the solution to the nonlinear equations investigated.

In this paper, we present a new direct method of estimating three-dimensional motion parameters of rigid planar patches. We define a set of eight pure parameters and demonstrate using the theory of Lie Transformation Group that given two pictures, these pure parameters are unique. As for the estimation procedure, we first show using the converse of the 2nd Lie theorem [10-13] that these 8 pure parameters can serve as the coordinate system of a certain Lie Transformation Group. Then, we use the result in [10-15] to show that these 8 pure parameters must satisfy a set of linear equations. Furthermore, the real motion parameters can be computed from these pure parameters by solving a six-order polynomial.

Our new direct method has several advantages. First, it requires the solution of a single sixth-order polynomial of one variable only. Second, it demonstrates that more than one solution may exist and therefore answers the uniqueness question. Third, in the special case of three-dimensional translation, the motion parameters can be expressed directly as some simple functions of the eight pure parameters. Therefore, only a few arithmetic operations are needed.

II. THE BASIC MOTION EQUATIONS

We are interested in estimating three-dimensional motion parameters of rigid and deformable bodies from time-sequential perspective views (frames). Throughout this paper, we shall assume that we work with only two frames at times t_1 and t_2 ($t_1 < t_2$).

The basic geometry of the problem is sketched in Fig. 1. Consider a particular point P on an object. Let

(x,y,z) = object-space coordinates of a point P at time t_1.

(x',y',z') = object-space coordinates of P at time t_2.

(X,Y) = image-space coordinates of P at t_1.

(X',Y') = image-space coordinates of P at t_2.

It is obvious from Fig. 1 that

$$X = F\frac{x}{z} \qquad X' = F\frac{x'}{z'}$$
$$Y = F\frac{y}{z} \qquad Y' = F\frac{y'}{z'} \tag{1}$$

Assume that from time t_1 to t_2 the three-dimensional object has undergone translation, rotation, and linear deformation [8]. Then, we have

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = S\begin{bmatrix} x \\ y \\ z \end{bmatrix} + R\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{2}$$

where

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \quad (3)$$

$$R = \begin{bmatrix} 0 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 0 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 0 \end{bmatrix} \quad (4)$$

$$\varphi_1 = n_1\theta, \quad \varphi_2 = n_2\theta, \quad \varphi_3 = n_3\theta \quad (5)$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \quad (6)$$

Note that $(\Delta x, \Delta y, \Delta z)$ is the amount of translation S in the linear deformation matrix, and $(R+I)$, where I is a 3X3 unit matrix, is the rotation matrix. The rotation is around an axis through the origin and with directional cosines (n_1, n_2, n_3). The amount of rotation is θ. Therefore, the φ_1, φ_2, φ_3 defined in (5) are the x, y and z components of the rotation vector with length θ and directional cosines (n_1, n_2, n_3).

Clearly, Eq. (2) represents an afine transformation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \quad (7)$$

Conversely, any afine transformation can be decomposed as in Eq. (2).

III. MOTION OF PLANAR PATCHES

We now restrict ourselves to points on a planar patch with equation

$$ax + by + cz = 1 \tag{8}$$

at time t_1. Then, it is readily shown from Eqs. (1) and (8) that

$$z = \frac{F}{aX + bY + cF} \tag{9}$$

and from Eqs. (1), (9), and (7) that

$$\begin{aligned} X' &= \frac{a_1 X + a_2 Y + a_3}{a_7 X + a_8 Y + 1} \triangleq T_1(X,Y) \\ Y' &= \frac{a_4 X + a_5 Y + a_6}{a_7 X + a_8 Y + 1} \triangleq T_2(X,Y) \end{aligned} \tag{10}$$

where

$$\begin{aligned} a_1 &= \frac{b_{11} + a\Delta x}{b_{33} + c\Delta z} & a_4 &= \frac{b_{21} + a\Delta y}{b_{33} + c\Delta z} \\ a_2 &= \frac{b_{12} + b\Delta x}{b_{33} + c\Delta z} & a_5 &= \frac{b_{22} + b\Delta y}{b_{33} + c\Delta z} \\ a_3 &= \frac{(b_{13} + c\Delta x)F}{b_{33} + c\Delta z} & a_6 &= \frac{(b_{23} + c\Delta y)F}{b_{33} + c\Delta z} \\ a_7 &= \frac{b_{31} + a\Delta z}{(b_{33} + c\Delta z)F} & & \\ a_8 &= \frac{b_{32} + b\Delta z}{(b_{33} + c\Delta z)F} & & \end{aligned} \tag{11}$$

We now specialize to the case of a rigid planar patch. Then Eqs. (11) become

$$a_1 = \frac{1 + a\Delta x}{1 + c\Delta z} \qquad a_4 = \frac{\varphi_3 + a\Delta y}{1 + c\Delta z}$$

$$a_2 = \frac{-\varphi_3 + b\Delta x}{1 + c\Delta z} \qquad a_5 = \frac{1 + b\Delta y}{1 + c\Delta z}$$

$$a_3 = \frac{(\varphi_2 + c\Delta x)F}{1 + c\Delta z} \qquad a_6 = \frac{(-\varphi_1 + c\Delta y)F}{1 + c\Delta z} \tag{12}$$

$$a_7 = \frac{-\varphi_2 + a\Delta z}{(1 + c\Delta z)F}$$

$$a_8 = \frac{\varphi_1 + b\Delta z}{(1 + c\Delta z)F}$$

Eq. (10) defines a mapping from the 2-space (X,Y) onto the 2-space (X',Y'). It will be shown in Section III that corresponding to any specified mapping between the two 2-spaces, there can be only one set of values for the parameters $a_1,\ldots,a_8$. We call them the 8 <u>pure parameters</u>. In Section III, we shall also describe a method of determining these pure parameters from the two given image frames.

Once we have determined the pure parameters $a_1,a_2,\ldots,a_8$, we can attempt to find the actual motion parameters: $\Delta x,\Delta y,\Delta z,\varphi_1,\varphi_2,\varphi_3,a,b$, and c by using Eqs. (12). It is obvious first of all from looking at the right-hand sides of Eqs. (12) that Δz is a scale factor which cannot be determined. We therefore let

$$a'' \triangleq a\Delta z \qquad b'' = b\Delta z \qquad \bar{c} \triangleq c\Delta z$$
$$\Delta x'' \triangleq \frac{\Delta x}{\Delta z} \qquad \Delta y'' \triangleq \frac{\Delta y}{\Delta z} \tag{13}$$

The unknown motion parameters are now:

$$\varphi_1,\ \varphi_2,\ \varphi_3,\ \Delta x'',\ \Delta y'',\ a'',\ b'', \text{ and } \bar{c}.$$

Thus we have 8 nonlinear equations with 8 unknowns. This, however, does not mean that the solution is necessarily unique. In fact, it turns out it is not.

After some tedious manipulations, we get from Eqs. (12)

$$d_6 \Delta x''^6 + d_5 \Delta x''^5 + d_4 \Delta x''^4 + d_3 \Delta x''^3 + d_2 \Delta x''^2 + d_1 \Delta x'' + d_0 = 0 \qquad (14)$$

where

$$
\begin{aligned}
d_6 &= a_{50} h_3^2 - h_6 (h_3 h_2 - h_6 a_{10}) \\
d_5 &= h_3 (h_2^2 + h_6^2 + h_3^2 - 4a_{10} a_{50} \\
d_4 &= -h_3^2 (a_{50} + 5a_{10}) + a_{10} (2h_6^2 - h_2^2) \\
&\qquad - 3h_2 h_3 h_6 + 4a_{10} a_{50} \\
d_3 &= 2h_3 (h_2^2 + h_6^2 - h_3^2 + 4a_{10}^2) \\
d_2 &= (-h_3^2 + 4a_{10}^2) a_{50} + (6h_3^2 - 4a_{10}^2 - 2h_2^2 + h_6^2) a_{10} \\
&\qquad -3h_2 h_3 h_6 \\
d_1 &= h_3 (h_2^2 + h_6^2 + h_3^2 - 4a_{10}^2 + 4a_{10} a_{50}) \\
d_0 &= (a_5 - a_1) h_3^2 - h_2 (h_2 a_{10} + h_3 h_6)
\end{aligned}
\qquad (15)
$$

and

$$
\begin{aligned}
h_2 &= a_2 + a_4 \\
h_3 &= \frac{a_3}{F} + a_7 F \\
h_6 &= \frac{a_6}{F} + a_8 F \\
a_{10} &= a_1 - 1 \\
a_{50} &= a_5 - 1
\end{aligned}
\qquad (16)
$$

And furthermore

$$\Delta y'' = \frac{-h_6\Delta x''^3 + h_2\Delta x''^2 - h_6\Delta x'' + h_2}{-h_3\Delta x''^2 + 2a_{10}\Delta x'' + h_3}$$

$$\bar{c} = \frac{h_3\Delta x'' - a_{10}}{\Delta x''^2 - h_3\Delta x'' + a_1} = \frac{h_6\Delta y'' - a_{50}}{\Delta y''^2 - h_6\Delta y'' + a_5}$$

$$b'' = \bar{c}(h_6 - \Delta y'') + h_6 \qquad (17)$$

$$a'' = \bar{c}(h_3 - \Delta x'') + h_3$$

$$\varphi_1 = b'' - (\bar{c}+1)a_8F = (\bar{c}+1)\frac{a_6}{F} - \Delta y''\bar{c}$$

$$\varphi_2 = -a'' + (\bar{c}+1)a_7F = -(\bar{c}+1)\frac{a_3}{F} + \Delta x''\bar{c}$$

$$\varphi_3 = \Delta y''a'' - a_4(\bar{c}+1) = -\Delta x''b'' + a_2(\bar{c}+1)$$

To find the motion parameters, we first solve Eq. (14) for $\Delta x''$. Then the others are obtained from Eqs. (17). Since Eq. (14) is a sixth-order polynomial equation, we can have potentially 6 real roots which give us 6 solutions for the motion parameters. For all the numerical examples we have tried, only two real roots are found for Eq. (14). One such numerical example follows:

$a_1 = .976$	$a_2 = .058$	$a_3 = .059$
$a_4 = .027$	$a_5 = .976$	$a_6 = .059$
$a_7 = .047$	$a_8 = .047$	

Solution 1 — $\Delta x = .9$ $\quad\Delta y = .9$ $\quad\Delta z = 1$

$\theta = 1°$ $\quad n_1 = \cos 90°$ $\quad n_2 = \cos 90°$ $\quad n_3 = \cos 0°$

$a/A = \cos 60°$ $\quad b/A = \cos 60°$ $\quad c/A = \cos 45°$

$A = \sqrt{a^2 + b^2 + c^2} = 1/10$

Solution 2 — $\Delta x = .707$ $\quad\Delta y = .707$ $\quad\Delta z = 1$

$\theta = 1.59°$ $\quad n_1 = \cos 58.4°$ $\quad n_2 = \cos 121.6°$ $\quad n_3 = \cos 47$

$a/A = \cos 56.2°$ $\quad b/A = \cos 56.2°$ $\quad c/A = \cos 51.8°$

$A = \sqrt{a^2 + b^2 + c^2} = 1/8.74$

We mention in passing that an efficient iterative method for finding the real roots of a sixth-order polynomial equation is given in Ref. 7.

For the special case of three-dimensional translation, the results are considerably simpler. From Eqs. (12), we get

$$\Delta x'' = \frac{a_1(a_1a_8F - a_1 + 1) + (a_1a_8 - a_2a_7)F}{a_7F(a_1a_8F - a_1 + 1)}$$

$$\Delta y'' = \frac{a_5(a_5a_7F - a_5 + 1) + (a_5a_7 - a_4a_8)F}{a_8F(a_5a_7F - a_5 + 1)}$$

$$a'' = \frac{(a_5a_7F - a_5 + 1)a_7}{a_5a_7 - a_4a_8} = \frac{(a_1a_8F - a_1 + 1)a_7}{a_1a_8 - a_2a_7} \tag{18}$$

$$b'' = \frac{(a_5a_7F - a_5 + 1)a_8}{a_5a_7 - a_4a_8} = \frac{(a_1a_8F - a_1 + 1)a_8}{a_1a_8 - a_2a_7}$$

$$\bar{c} = \frac{a_2a_7F - a_1 + 1}{(a_1a_8 - a_7a_2)F} = \frac{a_4a_8F - a_5 + 1}{(a_5a_7 - a_8a_4)F}$$

Therefore, only a few simple arithmetic operations are needed.

IV. DETERMINING THE 8 PURE PARAMETERS

We now go back and examine Eqs. (10). For a particular set of values for the parameters $(a_1, a_2, \ldots, a_8)$, the equations represent a transformation which maps the 2-space (X,Y) (the coordinate space of our image frame at time t_1) onto the 2-space (X',Y') (the coordinate space of our image frame at time t_2). Let us consider the collection G of transformation corresponding to all $(a_1, a_2, \ldots, a_8) \in R^8$. We shall show that it is a continuous (Lie) group of dimension eight and that to any given mapping from the (X,Y)-space onto (X',Y')-space corresponds only one set of values for $(a_1, a_2, \ldots, a_8)$. Furthermore, we shall describe a method of determining the pure parameters $(a_1, a_2, \ldots, a_8)$ from a given pair of image frames at times t_1 and t_2.

In classical continuous group theory, it is known [13] that G satisfies the four group axioms, namely, closure, existence of inverse and identity, and associativity. Furthermore, the composition function for the group parameters a_i's are continuous. It is also known [13] that the a_i's in (10) are essential parameters in the sense that the a_i's are functionally independent. However, it is not known whether the a_i's in (10) are unique, i.e., whether there can be two different sets of values of a_i's such that (10) gives the same mapping $(X,Y) \rightarrow (X',Y')$. Because of this reason, it is not easy to verify whether G is a Lie group according to the modern definition since in modern definition, in addition to the properties satisfied by the classical continuous group according to the classical definition [12,13], several topological properties have to be satisfied, and these properties can not be easily verified unless we are certain that the group parameters a_i's are unique. In the following, we prove that G is strictly a Lie group and that the a_i's are unique.

Before we give the proof, we motivate it by the following considerations. Let us assume that G is indeed a Lie group and that $(a_1,a_2,\ldots,a_8)$ is a coordinate representation for the group G. The identity element of the group is obviously e = (1,0,0,0,1,0,0,0). Then the operators of the Lie algebra associated with the group G are given by:

$$X_j = \frac{\partial T_1}{\partial a_j}\bigg|_{g=e} \frac{\partial}{\partial X} + \frac{\partial T_2}{\partial a_j}\bigg|_{g=e} \frac{\partial}{\partial Y} \qquad (19)$$

where g is used to represent a member of G. From Eqs. (10) we get readily

$$\begin{aligned} X_1 &= X \frac{\partial}{\partial X} \\ X_2 &= Y \frac{\partial}{\partial X} \\ X_3 &= \frac{\partial}{\partial X} \\ X_4 &= X \frac{\partial}{\partial Y} \\ X_5 &= Y \frac{\partial}{\partial Y} \\ X_6 &= \frac{\partial}{\partial Y} \\ X_7 &= -X^2 \frac{\partial}{\partial X} - XY \frac{\partial}{\partial Y} \\ X_8 &= -XY \frac{\partial}{\partial X} - Y^2 \frac{\partial}{\partial Y} \end{aligned} \qquad (20)$$

Now we start our proof. Consider the set of vector fields on the differentiable manifold (X,Y) as given by Eqs. (20). It can be easily verified that none of the X_j can be expressed as a linear combination of the others, i.e., $\{X_j: j=1,2,\ldots,8\}$ are linearly independent; and furthermore for any i,j

$$[X_i,X_j] \triangleq X_iX_j - X_jX_i = \sum_k c_{ij}^k X_k \qquad (21)$$

where c_{ij}^k are constants. From these two properties of $\{X_i\}$ we conclude from the converse of Lie's second fundamental theorem [10-13] that there is a unique Lie group of transformation of order 8 which has $\{X_i\}$ as its Lie algebra. We proceed now to show that G is that group.

From [15,10,11], one can generate the finite equations of a Lie group of transformation with the λ_i's as the canonical coordinates of the second kind as follows:

$$X' = \exp(\lambda_8 X_8)\exp(\lambda_7 X_7)\ \ldots\ \exp(\lambda_1 X_1)X$$
$$Y' = \exp(\lambda_8 X_8)\exp(\lambda_7 X_7)\ \ldots\ \exp(\lambda_1 X_1)Y \tag{22}$$

It is proved in [6] that (22) is equivalent to the following:

$$X' = \frac{a_1 X + a_2 Y + a_3}{a_7 X + a_8 Y + 1} \qquad Y' = \frac{a_4 X + a_5 Y + a_6}{a_7 X + a_8 Y + 1} \tag{23}$$

where

$$a_1 = e^{\lambda_1}\lambda \qquad a_2 = \lambda_2\lambda$$

$$a_3 = (\lambda_3 e^{\lambda_1} + \lambda_6\lambda_2)\lambda \qquad a_4 = e^{\lambda_1}\lambda_4 e^{\lambda_5}\lambda$$

$$a_5 = e^{\lambda_5}(1 + \lambda_2\lambda_4)\lambda \qquad a_6 = [\lambda_6 e^{\lambda_5} + \lambda_4 e^{\lambda_5}(\lambda_3 e^{\lambda_1} + \lambda_6\lambda_2)]\lambda \tag{24}$$

$$a_7 = -(\lambda_7 e^{\lambda_1} + \lambda_8 e^{\lambda_1}\lambda_4 e^{\lambda_5})\lambda \qquad a_8 = -[\lambda_7\lambda_2 + e^{\lambda_5}\lambda_8(1 + \lambda_2\lambda_4)]\lambda$$

$$\lambda = [1 - \lambda_7(\lambda_3 e^{\lambda_1} + \lambda_6\lambda_2) - \lambda_6 e^{\lambda_5}\lambda_8 - \lambda_4 e^{\lambda_5}\lambda_8(\lambda_3 e^{\lambda_1} + \lambda_6\lambda_2)]^{-1}$$

Comparing (10) and (23) shows that G is indeed a Lie Group of transformation, and that since the λ_i's are the canonical coordinates, they are unique, and therefore from (24), the pure parameters $a_1 \ldots a_8$ are also unique.

Let f be any function defined on the 2-space (X,Y) (in our case, f will be the intensity of the picture elements), then from Lie Group Theory [10-15], we have

$$\Delta f = \sum_{i=1}^{8} \beta_i X_i f \qquad (25)$$

where $\beta_i = a_i - e_i$

e_i = ith component of the group parameters at the identity

$\Delta f = f(X',Y') - f(X,Y)$ = frame difference

(The implicit assumption here is that the intensities of the picture elements in the image frames corresponding to the same physical object point are the same.) Clearly,

$$a_1 = \beta_1 + 1$$

$$a_5 = \beta_5 + 1$$

$$a_i = \beta_i, \ i = 2,3,4,6,7,8$$

Eq. (25) is used to determine the β_i's and therefore the 8 pure parameters a_i's. We pick 8 or more points (X,Y), calculate at each point Δf and $X_i f$ ($i = 1,2,\ldots,8$), and substitute into Eq. (25) to obtain 8 or more equations which are linear in the 8 unknowns β_i's. Then we find the least-square solution.

V. DISCUSSIONS

In this paper we have investigated the problem of estimating three-dimensional motion parameters of a rigid planar patch from two image frames. The following results have been established:

1) The fact that we can define 8 pure parameters $a_1, a_2, \ldots, a_8$ which are unique for any given mapping from the 2-space (X,Y) (the coordinate space of the image frame at t_1) onto the 2-space (X',Y') (the coordinate space of the image frame at t_2).

2) A method of determining the actual motion parameters φ_1, φ_2, φ_3, $\Delta x''$, $\Delta y''$, a'', b'', and $\bar{c}$ from the pure parameters a_1, a_2, ..., a_8 which requires the solution of a 6th-order polynomial of one variable only. Aside from a scaling factor in the translation parameters, the number of real solutions never exceeds two.

3) A method of determining the 8 pure parameters a_1, a_2, ..., a_8 from the two given image frames. This requires the solution of a set of linear equations only.

It is to be noted that 1) and 2) are independent of 3). The pure parameters can be determined by other methods. For example, if one can identify 8 or more corresponding point pairs in the two image frames [2], then the a_i's can be determined from Eqs. (10) by solving a set of linear equations.

Recently, an alternative way of analyzing the uniqueness problem and estimating the three-dimensional motion parameters has been developed which stems from the results contained in this paper, and requires computing the singular value decomposition (SVD) of a certain 3X3 matrix only. The eight pure parameters defined by the authors in this work will again be used. It is briefly mentioned in [6], and the detailed paper will be submitted soon.

Furthermore, the conclusion that the motion parameters are generally not unique is of course independent of the method of determining these parameters. The question arises: Are the motion parameters unique, aside from the scaling factor, if the rigid patch is nonplanar? We have solved this problem recently [7], and will publish it in the near future.

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ESTIMATING THREE-DIMENSIONAL MOTION PARAMETERS OF A RIGID PLANAR PATCH, II: SINGULAR VALUE DECOMPOSITION

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Abstract

We show that the three-dimensional motion parameters of a rigid planar patch can be determined by computing the singular value decomposition (SVD) of a 3 x 3 matrix containing the eight so called "pure parameters". Furthermore, aside from a scale factor for the translation parameters, the number of solutions is either one or two, depending on the multiplicity of the singular values of the matrix.

I. INTRODUCTION

The processing of image sequences involving motion has become increasingly important. Because of the key role motion estimation plays in image sequence processing, a considerable amount of effort has been devoted to this topic, for example, see Ref. [1-17]. However, except for [1-8][16][17], most past work considers only 2-D motion, especially translation. Ref. [1-8][16][17] were among the first to consider 3-D motion and [1][3][17] were among the first to consider the problem of uniqueness of solutions. In [1][3], the eight "pure parameters" were introduced for the case of a rigid planar patch undergoing general 3-D motion, and proved to be unique given two successive (in time) perspective views. The proof makes use of the theory of Lie Group of transformations. It was also shown that these 8 pure parameters can be computed by solving a set of linear equations. Furthermore, once the pure parameters are determined, the actual motion parameters can be computed by solving a 6-th order polynomial equation of one variable if the motion is small. Theoretically, the number of solutions cannot exceed six aside from a common scale factor for the translation parameters; experimentally, the maximum number of solutions has been found to be two. In this paper, we show that whether the motion is small or not, once the eight pure parameters are computed using the method described in [1][3], the actual motion parameters can be estimated by computing the SVD of a 3 x 3 matrix consisting of the eight pure parameters. Also, by using the rigidity constraint and the fact that a plane in 3-space can be oriented in at most two possible ways in order to intercept an ellipsoid at a circular cross-section, we prove that the number of solutions is either one or two, depending on the multiplicity of the singular values. Physical description of the motion is stated and justified.

II. THE EIGHT PURE PARAMETERS AND THE MOTION OF PLANAR PATCHES

The basic geometry of the problem considered in [1][3] is repeated here in Fig. 1. Throughout this paper, we shall assume that we work with only two frames at time t_1 and t_2 ($t_1 < t_2$). Consider a particular point P on the object. Let

(x,y,z) = object-space coordinates of a point P at time t_1.

(x',y',z') = object-space coordinates of P at time t_2.

(X,Y) = image-space coordinates of P at t_1.

(X',Y') = image-space coordinates of P at t_2.

It was shown in [1][3] that for a rigid planar patch undergoing 3-D motion (a rotation with a small angle θ around an axis through the origin with directional consines n_1, n_2, n_3, followed by a translation with translation vector $(\Delta x, \Delta y, \Delta z)$), the image-space coordinates before and after the motion are related by the following equations:

$$X' = \frac{a_1X + a_2Y + a_3}{a_7X + a_8Y + 1}$$
$$Y' = \frac{a_4X + a_5Y + a_6}{a_7X + a_8Y + 1} \qquad (1)$$

where

$$a_1 = \frac{1 + a\cdot\Delta x}{1 + c\cdot\Delta z} \qquad a_5 = \frac{1 + b\cdot\Delta y}{1 + c\cdot\Delta z}$$
$$a_2 = \frac{-n_3\theta + b\cdot\Delta x}{1 + c\cdot\Delta z} \qquad a_6 = \frac{(-n_1\theta + c\cdot\Delta y)F}{1 + c\cdot\Delta z}$$
$$a_3 = \frac{(n_2\theta + c\cdot\Delta x)F}{1 + c\cdot\Delta z} \qquad a_7 = \frac{-n_2\theta + a\cdot\Delta z}{(1 + c\cdot\Delta z)F} \qquad (2)$$
$$a_4 = \frac{n_3\theta + a\cdot\Delta y}{1 + c\cdot\Delta z} \qquad a_8 = \frac{n_1\theta + b\cdot\Delta z}{(1 + c\cdot\Delta z)F}$$

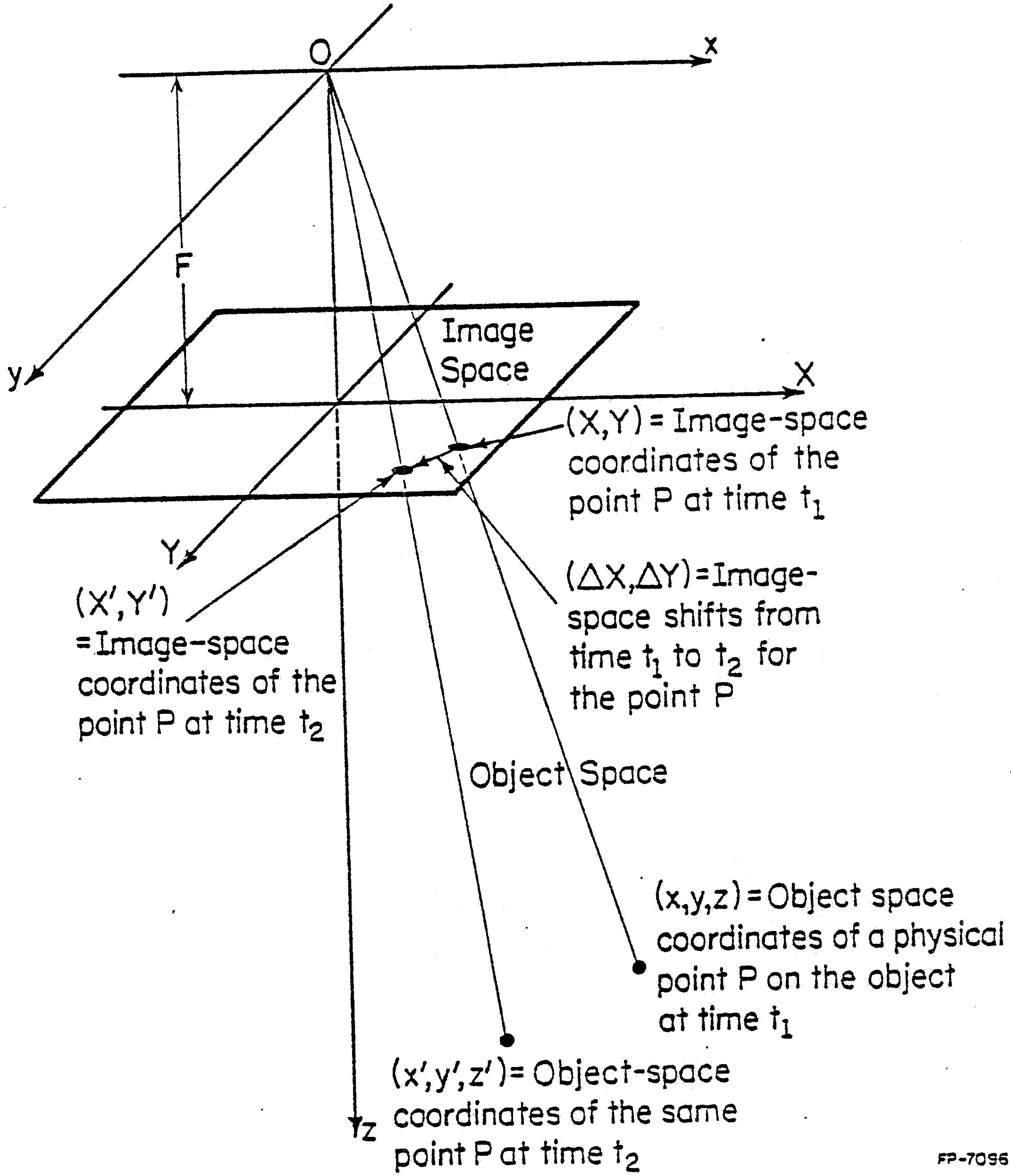

Fig. 1 Basic geometry for three-dimensional motion estimation.

where a, b, c are the parameters that appear in the following equation:

$$ax + by + cz = 1 \tag{3}$$

which describes the surface of the object in the object space coordinate system at time t_1.

Eqs. (2) are applicable when the rotation angle is small. For general 3-D motion, it is well known in mechanics and computer graphics [18] that any 3-D rigid body motion is equivalent to a rotation followed by a translation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{4}$$

where R is a 3 x 3 orthonormal matrix

$$R = \begin{bmatrix} n_1^2+(1-n_1^2)\cos\theta & n_1n_2(1-\cos\theta)-n_3\sin\theta & n_1n_3(1-\cos\theta)+n_2\sin\theta \\ n_1n_2(1-\cos\theta)+n_3\sin\theta & n_2^2+(1-n_2^2)\cos\theta & n_2n_3(1-\cos\theta)-n_1\sin\theta \\ n_1n_3(1-\cos\theta)-n_2\sin\theta & n_2n_3(1-\cos\theta)+n_1\sin\theta & n_3^2+(1-n_3^2)\cos\theta \end{bmatrix} \tag{5}$$

Following exactly the same procedure as in [1][3], one can show that (1) is again valid if (2) is replaced by

$$a_1 = \frac{n_1^2 + (1-n_1^2)\cos\theta + a\cdot\Delta x}{n_3^2 + (1-n_3^2)\cos\theta + c\cdot\Delta z} \qquad a_2 = \frac{n_1n_2(1-\cos\theta) - n_3\sin\theta + b\cdot\Delta x}{n_3^2 + (1-n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_3 = \frac{[n_1n_3(1-\cos\theta)+n_2\sin\theta+c\cdot\Delta x]F}{n_3^2 + (1-n_3^2)\cos\theta + c\cdot\Delta z} \qquad a_4 = \frac{n_1n_2(1-\cos\theta)+n_3\sin\theta + a\cdot\Delta y}{n_3^2 + (1-n_3^2)\cos\theta + c\cdot\Delta z}$$

$$a_5 = \frac{n_2^2 + (1-n_2^2)\cos\theta + b\cdot\Delta y}{n_3^2 + (1-n_3^2)\cos\theta + c\cdot\Delta z} \qquad a_6 = \frac{[n_2n_3(1-\cos\theta)+n_1\sin\theta+c\cdot\Delta y]F}{n_3^2 + (1-n_3^2)\cos\theta + c\cdot\Delta z} \tag{6}$$

$$a_7 = \frac{n_1n_3(1-\cos\theta)-n_2\sin\theta+a\cdot\Delta z}{n_3^2 + (1-n_3^2)\cos\theta + c\cdot\Delta z} \qquad a_8 = \frac{n_2n_3(1-\cos\theta)+n_1\sin\theta+b\cdot\Delta z}{n_3^2 + (1-n_3^2)\cos\theta + c\cdot\Delta z}$$

where for simplicity we have set F = 1.

It was shown in [1][3] using Lie Group Theory that given two perspective views at t_1 and t_2, the eight pure parameters a_1, a_2, ... , a_8 are unique, and they can be estimated by solving a system of linear equations.

III. COMPUTING ACTUAL MOTION PARAMETERS FROM PURE PARAMETERS

By using the rigidity constraint and the fact that a plane in 3-space can be oriented in at most two possible directions in order to cut a circle in an ellipsoid, we shall prove that the number of possible solutions for the motion parameters can never exceed two aside from a scale factor for the translation parameters. The number of solutions depends upon the multiplicity of the singular values of the following matrix A consisting of the eight pure parameters a_i's:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix} \quad (7)$$

The SVD of A is given by

$$A = U \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} V = U \Lambda V^T \quad (8)$$

where λ_i's are the singular values of A, and U,V are 3 x 3 orthonormal matrices.

Let $k = n_3^2 + (1-n_3^2)\cos\theta + c \cdot \Delta z$; then it can be readily shown that

$$A = k^{-1} \left\{ R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} \right\} \quad (9)$$

or

$$kA = R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}$$

From (3) and (4), it can be seen that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left\{ R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} \right\} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= k\,A \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{10}$$

If we transform the original coordinate system with the orthonormal matrix V in (8) as depicted in Fig. 2, where (x_n, y_n, z_n) is the new coordinate system after transformation, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = V \cdot \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{11}$$

and

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = V \cdot \begin{bmatrix} x'_n \\ y'_n \\ z'_n \end{bmatrix} \tag{12}$$

Substituting (11), (12) into (10) gives

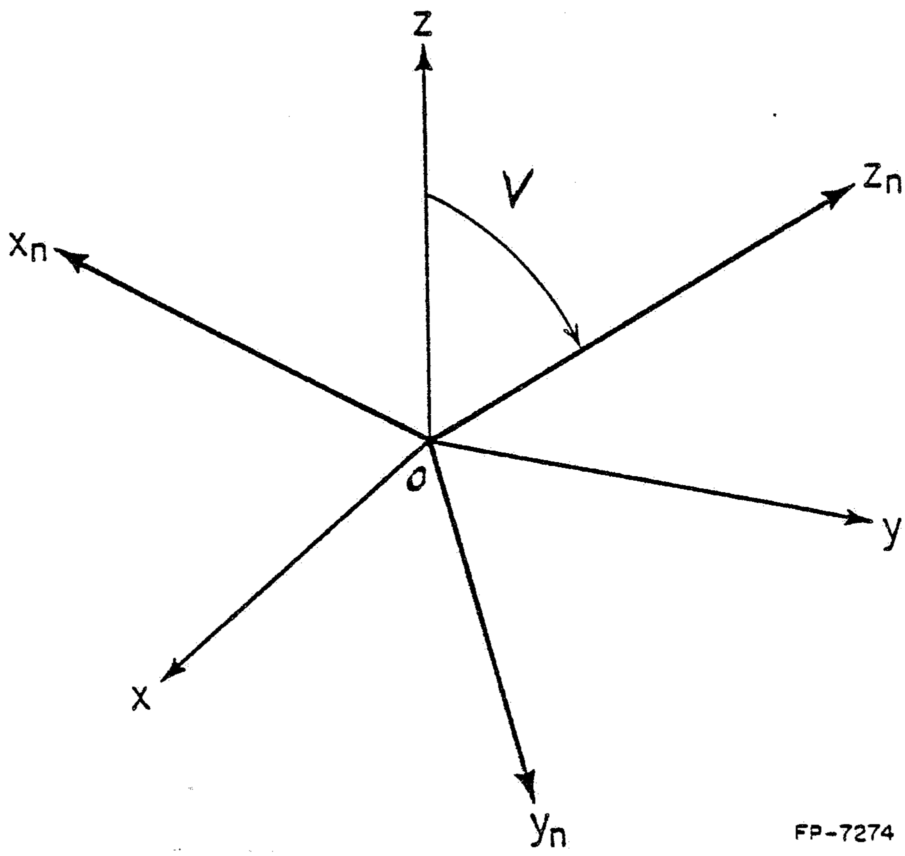

Fig. 2 The relationship between the (x,y,z) and (x_n, y_n, z_n) coordinate systems.

$$V \begin{bmatrix} x_n' \\ y_n' \\ z_n' \end{bmatrix} = k\, A\, V \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{13}$$

By taking the Euclidean norms of the vectors on both sides of (13), we obtain

$$\begin{bmatrix} x_n' & y_n' & z_n' \end{bmatrix} V^T\, V \begin{bmatrix} x_n' \\ y_n' \\ z_n' \end{bmatrix} = k^2 \begin{bmatrix} x_n & y_n & z_n \end{bmatrix} V^T A^T A\, V \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{14}$$

Since V is orthonormal, $V^T \cdot V$ on the left hand side can be replaced by an identity matrix. Substituting (8) into (14) gives

$$x_n'^2 + y_n'^2 + z_n'^2 = k^2 \begin{bmatrix} x_n & y_n & z_n \end{bmatrix} V^T V \Lambda U^T U \Lambda V^T V \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \quad (15)$$

Replacing U^TU in (15) by an identity matrix gives

$$x_n'^2 + y_n'^2 + z_n'^2 = k^2(\lambda_1^2 x_n^2 + \lambda_2^2 y_n^2 + \lambda_3^2 z_n^2) \quad (16)$$

This is the key equation that will lead us to the solution of the uniqueness problem, as will be seen hereafter.

In the following, we state and prove three theorems regarding the uniqueness and computation of the motion parameters given the pure parameters, and the physical characterizations of the motion in the object space for different multiplicities of the singular values of the matrix A.

THEOREM I

If the multiplicity of the singular values of A is two, e.g., $\lambda_1 = \lambda_2 \neq \lambda_3$, then the solution for the motion and geometrical parameters is unique aside from a common scale factor for the translation parameters, and

$$R = \lambda_1^{-1} A - (\frac{\lambda_3}{\lambda_1} - s)\, U_3 V_3^T ,$$

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = w^{-1}(\frac{\lambda_3}{\lambda_1} - s)\, U_3 \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = w\, V_3$$

where $s = \det(U)\det(V)$

w is a scale factor

a,b,c are the parameters in (3) which is the planar equation of the object surface at time t_1

U_3, V_3 are the third columns of U and V respectively.

Furthermore, a necessary and sufficient condition for the multiplicity of the singular values of A to be 2 is that the motion can be realized by rotating the object around the origin and then translating it along the normal direction of the object surface.

[Proof]

The two sides of (16) can be equated to a collection of positive values corresponding to the range of values for x, y, z and x', y', z'. Let d be one such value. Then we have

$$d = x_n'^2 + y_n'^2 + z_n'^2 \tag{17}$$

and

$$d = k^2(\lambda_1^2 x_n^2 + \lambda_2^2 y_n^2 + \lambda_3^2 z_n^2) \tag{18}$$

Clearly, (17) defines a sphere in the (x_n', y_n', z_n') space, while (18) defines an ellipsoid in the (x_n, y_n, z_n) space. Since $\lambda_1 = \lambda_2$, two of the three principal axes of the ellipsoid are equally long. Since the object surface is assumed to be planar, the collection of the points on the object surface that also satisfy (17) must be the circle which lies on the intersection of the sphere and the object surface at time t_2 (see Fig. 3). Because of (16), (17) and (18), all the points on this circle at time t_2 must also satisfy (18) at time t_1, i.e. they must lie on the intersection of the object surface and the ellipsoid. Due to the rigidity constraint, this intersection should also be a circle. But the only possibility for a plane to cut a circle out of an ellipsoid with two of the three principal

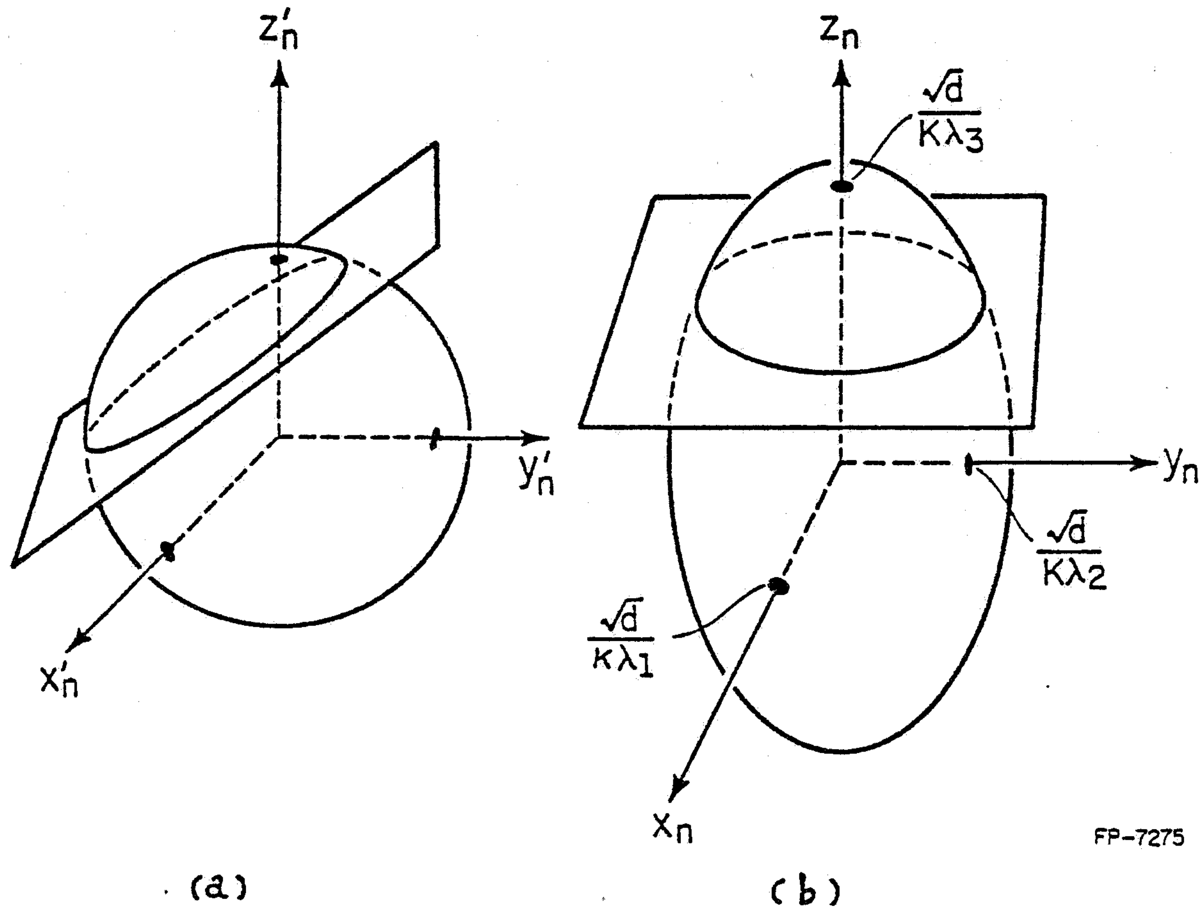

Fig. 3 (a) The object surface always intercepts the sphere defined by (17) at a circular cross-section.

(b) The object surface intercepts the ellipsoid defined by (18) at a circular cross-section only if it is properly oriented.

axes equally long is that the plane be perpendicular to the major axis (the longest one) of the ellipsoid, as depicted in Fig. 4. That is to say, there is only one possible orientation for the object surface before the motion. This is the key step that will lead us to the conclusion that the motion parameters are unique, as will be seen shortly.

Note that λ_1 (and λ_2) can never be zero since were this to be true, the ellipsoid defined by (18) would have degenerated into two parallel planes, and there is no way the object surface can intercept two parallel planes at a circular cross-section.

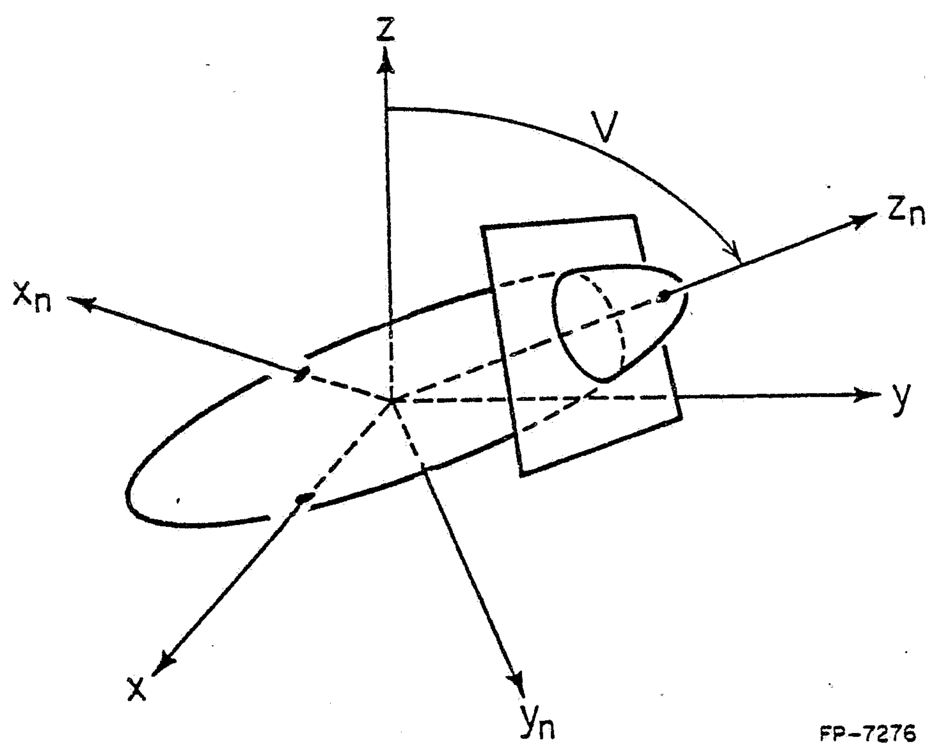

Fig. 4. The object surface must be perpendicular to the z_n axis to intercept the ellipsoid at a circular cross-section when $\lambda_1 = \lambda_2 \neq \lambda_3$.

Since, as depicted in Fig. 4, the object surface must be perpendicular to the z_n axis, and since the z_n axis is obtained by rotating the z axis with the orthonormal matrix V as in Fig. 2, it is seen that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = V \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} = w\, V_3 \tag{19}$$

where a,b,c are the parameters in (3),

V_3 is the third column of V in (8), and

w is an arbitrary constant.

Substituting (19) and (8) into (9) gives

$$R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} 0 & 0 & w \end{bmatrix} \cdot V^T = k\, U \Lambda V^T \tag{20}$$

Premultiply (20) by U^T and postmultiply (20) by V to give

$$U^T R\, V + U^T \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} 0 & 0 & w \end{bmatrix} V^T V = k\Lambda \tag{21}$$

or

$$R' + \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} \begin{bmatrix} 0 & 0 & w \end{bmatrix} = k \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \tag{22}$$

where

$$R' = U^T R\, V \tag{23}$$

$$\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = U^T \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{24}$$

(22) gives

$$R' = \begin{bmatrix} k\lambda_1 & 0 & -w\cdot\Delta x' \\ 0 & k\lambda_1 & -w\cdot\Delta y' \\ 0 & 0 & k\lambda_3 - w\cdot\Delta z' \end{bmatrix} \tag{25}$$

It will be shown now that $\Delta x'$ and $\Delta y'$ in (25) are zero, therefore R' is diagonal.

Since U, V and R in (23) are all orthonormal, R' is also. Taking the inner product of the 2nd and 3rd columns of R', and equating it to zero gives

$$k\lambda_1 w \cdot \Delta y' = 0 \tag{26}$$

λ_1 and k cannot be zero since were λ_1 or k to be zero, the 1st and 2nd columns of (25) would be zero, which contradicts the fact that R' is orthonormal. Obviously, w cannot vanish either, otherwise, a, b and c would vanish, which contradicts (3). Therefore, (26) implies that $\Delta y' = 0$.

Similarly, one can show that $\Delta x' = 0$. Thus, (25) is diagonal, which, when combined with the fact that R' is orthonormal, gives the following:

$$k\lambda_1 = +1 \text{ or } -1 \tag{27}$$

$$k\lambda_3 - w\Delta z' = +1 \text{ or } -1 \tag{28}$$

We show that k has to be positive:

From (10), we have

$$z' = k(a_7 x + a_8 y + z) \tag{29}$$

For x = 0 and y = 0, z' = kz. Since the object must be in front of the camera, z and z' are both positive, which implies that k is positive.

Since λ_1 is nonnegative by definition, the right hand side of (27) can not be -1. Therefore,

$$k = \frac{1}{\lambda_1}$$

and

$$R' = \begin{bmatrix} 1 & & \\ & 1 & \\ & & s \end{bmatrix} \tag{30}$$

where s is either +1 or -1.

Since $R' = URV^T$, we have det(R') = det(U)det(R)det(V) = det(U)det(V). Thus s = det(U)det(V). (28) gives the following:

$$\Delta z' = w^{-1}\left(\frac{\lambda_3}{\lambda_1} - s\right) \tag{31}$$

From (24), (31) and the fact that $\Delta x' = \Delta y' = 0$, we have

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = U \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = U \cdot \begin{bmatrix} 0 \\ 0 \\ w^{-1}\left(\frac{\lambda_3}{\lambda_1} - s\right) \end{bmatrix}$$

$$= w^{-1}\left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 \tag{32}$$

(19), (20) and (32) imply that

$$R = \lambda_1^{-1}A - \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}$$

$$= \lambda_1^{-1} A - \left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 V_3^T \tag{33}$$

In the following, it will be shown that the solutions for the rotation matrix R in (33) is unique, and that aside from a scaling factor, the translation parameters Δx, Δy, Δz in (32) and the geometrical parameters a, b, c in (19) are also unique.

The first thing to show is that, once A is given, U_3 is fixed except for the sign. From (8),

$$A\, A^T\, U_3 = \lambda_3^2\, U_3 \tag{34}$$

Let Q be any orthonormal eigenvector matrix of $A \cdot A^T$. Then

$$A\, A^T = Q^T \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix} Q \tag{35}$$

From (34) and (35), we have

$$\left\{ Q^T \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix} Q - \lambda_3^2\, I \right\} U_3 = 0$$

or

$$P\, U_3 = 0$$

where

$$P \triangleq AA^T - \lambda_3^2 I = Q^T \begin{bmatrix} \lambda_1^2-\lambda_3^2 & & \\ & \lambda_1^2-\lambda_3^2 & \\ & & 0 \end{bmatrix} Q \tag{36}$$

P has rank 2 since $\lambda_1^2 - \lambda_3^2$ on the diagonal of the diagonal matrix in

(36) is nonzero. Also, P is fixed once A is given since $P = AA^T - \lambda_3^2 I$. Therefore, U_3 is fixed except for the sign.

The next thing to prove is that (33) is unique, i.e.

$$\left(\frac{\lambda_3}{\lambda_1} - s\right) U_3 V_3^T = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}$$

is unique once A is given. Two cases are to be discussed. The first is when $\lambda_3 \neq 0$. In this case, s = sgn(det(A)) since $A = U \Lambda V^T$ and thus $s = \det(U)\det(V) = (\lambda_1\lambda_2\lambda_3)^{-1}\det(A) = \mathrm{sgn}(\det(A))$. Therefore, given A, s is fixed. The next thing to prove is that $U_3V_3^T$ in (33) is unique.

Since U_3 and V_3 are fixed except for the sign, all one has to show is that when V_3 changes its sign, U_3 must also.

From (8) we have

$$A\,V = U\,\Lambda = \begin{bmatrix} \lambda_1 U_1 & \lambda_2 U_2 & \lambda_3 U_3 \end{bmatrix}$$

thus

$$A\,V_3 = \lambda_3 U_3$$

Since A and λ_3 are fixed given two perspective views, we see that when V_3 changes its sign, U_3 must also. Therefore, $U_3V_3^T$ has fixed sign. We have thus proved that the product

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}$$

and therefore R in (33) are all unique.

For the second case for which $\lambda_3 = 0$, we have from (33) that

$$R = \lambda_1^{-1}A + s\ U_3V_3^T$$
$$= \lambda_1^{-1}A + \det(U) \cdot U_3 \cdot \det(V) \cdot V_3^T \qquad (37)$$

If U_3 changes its sign, det(U) will also. Thus the sign of det(U) U_3 in (37) remains unchanged. Similarly, the sign of det(V) V_3^T also is fixed when V_3 changes its sign. Therefore, the uniqueness of (37) is not shaken by the ambiguity of the signs of U_3 and V_3.

Since A^TA have double eigenvalues, the eigenvectors V_1 and V_2 that correspond to the multiple eigenvalues λ_1 ($= \lambda_2$) are orthonormal to each other but may be anywhere in a certain fixed plane perpendicular to V_3 (Note that we are now interpreting eigenvectors geometrically as some vectors in 3-space.) If the order of V_1 and V_2 on the plane are interchanged while keeping V_3 fixed, the sign of det(V) will change. We are now to prove that when this does happen, the sign of det(U) will also change, thereby keeping (37) fixed. It is obvious from (8) that

$$(\lambda_1^{-1}A)\ V_1 = U_1$$
$$(\lambda_1^{-1}A)\ V_2 = U_2$$

Since λ_1 and A are fixed, when V_1 and V_2 are interchanged, U_1 and U_2 will also. Therefore, when det(V) changes sign, det(U) will also. Thus for the case when $\lambda_3 = 0$, the product

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}$$

as well as R in (33) are unique.

We are now to prove that a necessary condition for $\lambda_1 = \lambda_2 \neq \lambda_3$ is that the translation vector $\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$ is parallel to the normal direction of the object surface at time t_2.

Since before the motion, i.e. at t_1, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is normal to the object surface, this vector is rotated by R at time t_2 and becomes $R \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.
It is only necessary to prove that there exists a scalar q such that

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = q\, R \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{38}$$

From (19), (32), (33) and (38), we have

$$w^{-1}(\frac{\lambda_3}{\lambda_1} - s)\, U_3 = q[\lambda_1^{-1}A - (\frac{\lambda_3}{\lambda_1} - s)\, U_3\, V_3^T]\, w\, V_3$$

$$= q\, w\lambda_1^{-1}A\, V_3 - q\, w\, (\frac{\lambda_3}{\lambda_1} - s)\, U_3\, V_3^T\, V_3 \tag{39}$$

But

$A = \lambda_1\, U_1\, V_1^T + \lambda_2\, U_2\, V_2^T + \lambda_3\, U_3\, V_3^T$, thus $A\, V_3$ in (39) becomes

$$A\, V_3 = \lambda_1 U_1\, V_1^T\, V_3 + \lambda_2 U_2\, V_2^T\, V_3 + \lambda_3 U_3\, V_3^T\, V_3$$

$$= 0 + 0 + \lambda_3 U_3 \tag{40}$$

Substituting (40) into (39) gives

$$w^{-1}(\frac{\lambda_3}{\lambda_1} - s)\, U_3 = q\, w\, (\frac{\lambda_3}{\lambda_1} - \frac{\lambda_3}{\lambda_1} + s)\, U_3 = q\, w\, s\, U_3 .$$

Therefore, if we take q as $w^{-2}s^{-1}(\frac{\lambda_3}{\lambda_1} - s)$, then (38) will be satisfied. We have thus proved that the necessary condition for $\lambda_1 = \lambda_2 \neq \lambda_3$ is that the motion can be realized by first rotating the object around an axis passing through the origin, and then translating it along the normal direction of the object surface at time t_2.

In the proof of Theorem II, it will be shown that if the translation is along the normal direction of the object surface at t_2, then the singular values of A can not be all distinct. This fact, together with Theorem III, provide the sufficiency part. Q.E.D.

THEOREM II

If the singular values of A are all distinct, e.g., $\lambda_1 > \lambda_2 > \lambda_3$, then there are exactly two solutions for the motion and geometrical parameters aside from a scale factor for the translation and geometrical parameters, and that

$$R = U \begin{bmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ -s\beta & 0 & s\alpha \end{bmatrix} V^T$$

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = w^{-1}[-\beta U_1 + (\frac{\lambda_3}{\lambda_2} - s\alpha)\, U_3]$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = w\,[\delta V_1 + V_3]$$

where $\delta = \pm\left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_2^2 - \lambda_3^2}\right)^{\frac{1}{2}}$

$$\alpha = \frac{\lambda_1 + s\lambda_3\delta^2}{\lambda_2(1 + \delta^2)}$$

$$\beta = \pm\sqrt{1 - \alpha^2}$$

$$s = \det(U)\det(V)$$

(in each of the two solutions, $\mathrm{sgn}(\beta) = -\mathrm{sgn}(\delta)$.)

Furthermore, a necessary and sufficient condition for distinct singular values is that the motion can be decomposed into rotation around an axis through the origin followed by translation along a direction different from the normal direction of the object surface at time t_2.

[Proof]:

Since the three singular values are distinct, the three principal axes of the ellipsoid defined by (18) are of different lengths. By using the same argument as in Theorem I, the object surface at t_1 must be oriented in such a way that it cuts a circle out of this ellipsoid. It is easy to verify using basic analytical geometry that a plane can be oriented in only two possible directions (see Fig. 5) in order to cut a circle out of an ellipsoid whose three principal axes are of different lengths. Since $\lambda_1 > \lambda_2 > \lambda_3$, the longest principal axis is aligned with z_n axis, and the vector normal to the object surface is

$$w\begin{bmatrix}\delta \\ 0 \\ 1\end{bmatrix}, \quad \text{where } \delta = \pm\left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_2^2 - \lambda_3^2}\right)^{\frac{1}{2}}$$

in the (x_n, y_n, z_n) coordinate system, and w is some constant.

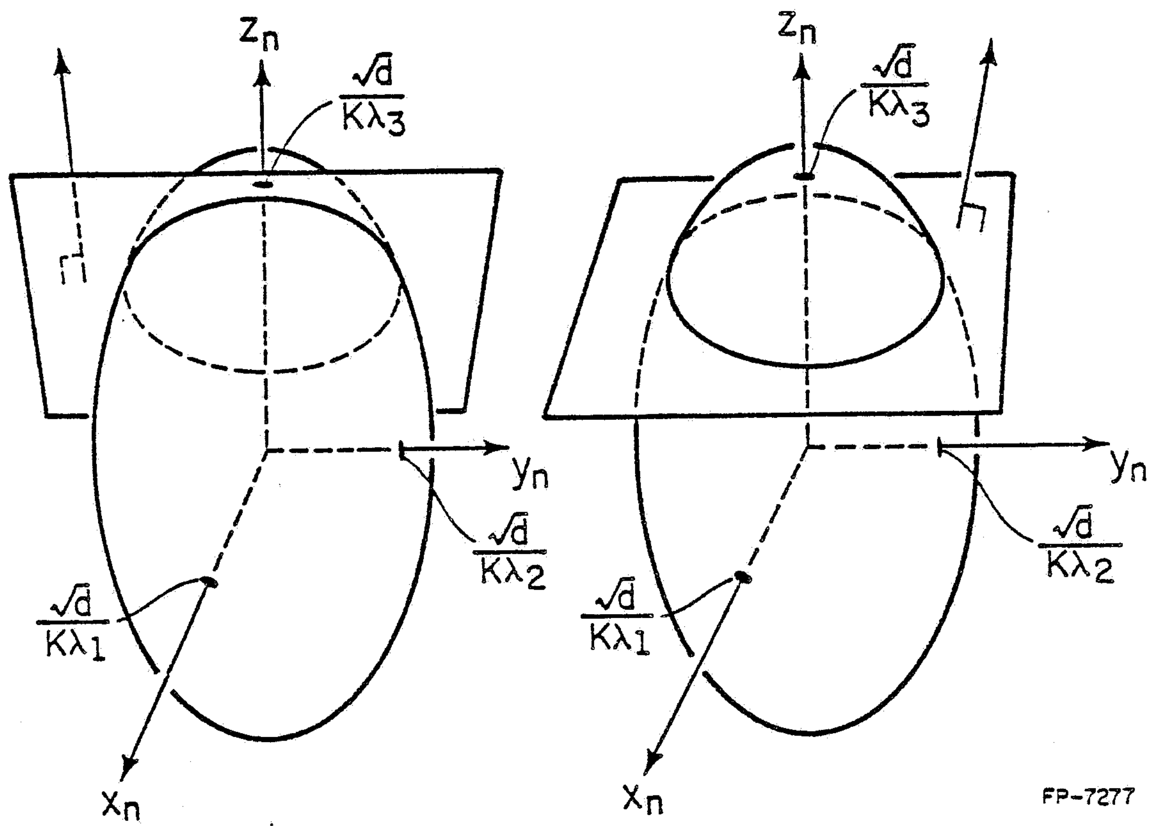

Fig. 5 The object surface can be oriented in exactly two possibl directions in order to intercept an ellipsoid defined by at a circular cross-section when $\lambda_1 > \lambda_2 > \lambda_3$.

Since, as shown in Fig. 2, the (x_n, y_n, z_n) axes are obtained by rotating the (x, y, z) axes with the orthonormal matrix V, we see that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = w\, V \begin{bmatrix} \delta \\ 0 \\ 1 \end{bmatrix} \tag{41}$$

where w is some constant. Substituting (41) into (9) gives

$$k\, A = k\, U\, \Lambda\, V^T = R + w \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} \delta & 0 & 1 \end{bmatrix} V^T \tag{42}$$

Premultiplying (42) by U^T and postmultiplying (42) by V give

$$k\Lambda = U^T R\, V + w\, U^T \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \begin{bmatrix} \delta & 0 & 1 \end{bmatrix} V^T V$$

Thus

$$R' = k\, \Lambda - \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} \begin{bmatrix} \delta & 0 & 1 \end{bmatrix} \tag{43}$$

where $R' \triangleq U^T R\, V$ (44)

$$\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} \triangleq w\, U^T \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{45}$$

From (43),

$$R' = \begin{bmatrix} k\lambda_1 - \delta\cdot\Delta x' & 0 & -\Delta x' \\ -\delta\cdot\Delta y' & k\lambda_2 & -\Delta y' \\ -\delta\cdot\Delta z' & 0 & k\lambda_3 - \Delta z' \end{bmatrix} \tag{46}$$

Since U, R and V are orthonormal, (46) implies that R' is also orthonormal. Taking the inner product of columns 2 and 3 and equating the result to zero gives

$$k\lambda_2 \cdot \Delta y' = 0 \tag{47}$$

Since $\lambda_1 > \lambda_2 > \lambda_3 \geq 0$, we have $\lambda_2 > 0$. Therefore, (47) implies that $\Delta y' = 0$. Thus (46) becomes

$$R' = \begin{bmatrix} k\lambda_1 - \delta\cdot\Delta x' & 0 & -\Delta x' \\ 0 & k\lambda_2 & 0 \\ -\delta\cdot\Delta z' & 0 & k\lambda_3 - \Delta z' \end{bmatrix} \tag{48}$$

The normality of column 2 implies that $k\lambda_2 = \pm 1$. But since $k > 0$ and $\lambda_2 > 0$, we have $k\lambda_2 = 1$, or $k = 1/\lambda_2$. Furthermore, from the fact that columns 1 and 3, as well as rows 1 and 3 of R' are mutually orthogonal, and that the norms of the rows and columns of R' are unity, it can be shown that

$$R' = \begin{bmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ -s\beta & 0 & s\alpha \end{bmatrix} \tag{49}$$

where $$\alpha = \frac{\lambda_1}{\lambda_2} - \delta\cdot\Delta x' = s(k\lambda_3 - \Delta z') \tag{50}$$

$$\beta = -\Delta x' = s\delta\cdot\Delta z' = \pm\sqrt{1 - \alpha^2} \tag{51}$$

$$s = \det(U)\det(V) \tag{52}$$

Since U and V are orthonormal, from (52), s is either +1 or -1. It is to be shown that although det(U) and det(V) may be +1 or -1 for a particular A, s is unique once A is given.

Recall that U_1, U_2 and U_3 are the eigenvectors of AA^T corresponding to eigenvalues λ_1^2, λ_2^2, and λ_3^2 respectively. Since λ_1, λ_2 and λ_3 are distinct, U_1, U_2, U_3, V_1, V_2 and V_3 are all fixed except for the signs. However, as was seen in the proof of Theorem I, we have

$$A\,V_1 = \lambda_1 U_1$$

$$A\,V_2 = \lambda_2 U_2$$

$$A\,V_3 = \lambda_3 U_3$$

Therefore, when U_i changes its sign, V_i will also, where i = 1, 2, 3. Hence the sign of det(U)det(V) remains fixed. Thus, s is unique.

From (50) and (51), we have

$$\alpha - \frac{\lambda_1}{\lambda_2} = \beta \cdot \delta \tag{53}$$

$$\delta\left(\alpha + s\frac{\lambda_3}{\lambda_2}\right) = \beta \tag{54}$$

Cancelling β in (53) and (54) gives

$$\alpha = \frac{\lambda_1 + s\lambda_3\delta^2}{\lambda_2(1+\delta^2)}$$

where $$\delta = \pm\left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_2^2 - \lambda_3^2}\right)^{\frac{1}{2}}$$

From (50) and (51), and the fact that $\Delta y' = 0$, we have

$$\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} -\beta \\ 0 \\ \frac{\lambda_3}{\lambda_2} - s\alpha \end{bmatrix} \tag{55}$$

From (45), (47) and (55),

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = w^{-1} U \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = w^{-1} U \cdot \begin{bmatrix} -\beta \\ 0 \\ \frac{\lambda_3}{\lambda_2} - s\alpha \end{bmatrix}$$

$$= w^{-1}[-\beta\ U_1 + (\frac{\lambda_3}{\lambda_2} - s\alpha)\ U_3] \tag{56}$$

From (44) and (49),

$$R = U \begin{bmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ -s\beta & 0 & s\alpha \end{bmatrix} V^T \tag{57}$$

From (41), (56), (57), and the fact that s is fixed, we see that there are exactly two solutions aside from a scaling factor for the translation and geometrical parameters.

It is to be shown that a necessary and sufficient condition for the singular values to be distinct is that the translation vector is not aligned with the normal direction of the object surface after rotation (or at time t_2). The sufficiency part was proved in Theorem I. The necessity part is proved by contradiction. We shall show that if the translation vector is along the normal direction of the object surface at t_2, then the singular values cannot be distinct.

It was indicated in the proof of Theorem I that the normal direction of the object surface at t_2 is aligned with $R\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Suppose $\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$ is parallel to $R\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = h\, R \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{58}$$

for some constant h. With (41), (56), (57) and (58), we have

$$w^{-1} U \begin{bmatrix} -\beta \\ 0 \\ \frac{\lambda_3}{\lambda_2} - s\alpha \end{bmatrix} = h\, U \begin{bmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ -s\beta & 0 & s\alpha \end{bmatrix} V^T V \begin{bmatrix} \delta \\ 0 \\ 1 \end{bmatrix}$$

or

$$\begin{bmatrix} -\beta \\ 0 \\ \frac{\lambda_3}{\lambda_2} - s\alpha \end{bmatrix} = w\, h \begin{bmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ -s\beta & 0 & s\alpha \end{bmatrix} \begin{bmatrix} \delta \\ 0 \\ 1 \end{bmatrix}$$

which implies that

$$-\beta = w\, h\, (\alpha \cdot \delta + \beta) \tag{59}$$

and

$$\frac{\lambda_3}{\lambda_2} - s\alpha = w\, h\, (-s\beta \cdot \delta + s\alpha) \tag{60}$$

Substituting (50) and (51) into (59) and (60) gives

$$\Delta x' = \frac{wh\delta}{1 + wh(1 + \delta^2)} \frac{\lambda_1}{\lambda_2} \tag{61}$$

and

$$\Delta z' = \frac{wh}{1 + wh(1 + \delta^2)} \frac{\lambda_3}{\lambda_2} \tag{62}$$

But from (51),

$$-\Delta x' = s \, \delta \, \Delta \, z' \tag{63}$$

Substituting (61) and (62) into (63) gives

$$\frac{wh\delta}{1 + wh(1 + \delta^2)} \frac{\lambda_3}{\lambda_2} = \frac{wh\delta}{1 + wh(1 + \delta^2} \frac{-s\lambda_3}{\lambda_2}$$

which implies that $\lambda_1 = -s\lambda_3$. Since λ_1 and λ_3 are nonnegative by definition, we have $\lambda_1 = \lambda_3$. But this contradicts the assumption that $\lambda_1 \neq \lambda_3$. Therefore, the necessity part is proved. Q.E.D.

<u>Theorem III</u>

The necessary and sufficient condition for the multiplicity for the singular values of A to be three, i.e., $\lambda_1 = \lambda_2 = \lambda_3$, is that the motion consist of rotation around an axis through the origin only, i.e. $\Delta x = \Delta y = \Delta z = 0$. Also, the rotation matrix is unique, and $R = \lambda_1^{-1}A$. The object surface can be anywhere.

Proof

If $\lambda_1 = \lambda_2 = \lambda_3$, then (16) gives

$$x_n'^2 + y_n'^2 + z_n'^2 = k^2\lambda_1^2(x_n^2 + y_n^2 + z_n^2) \tag{64}$$

Since any 3-D rigid body motion can be decomposed into rotation followed by translation, we first rotate the object such that (x_n, y_n, z_n) becomes (x''_n, y''_n, z''_n). Then we carry out the translation which changes (x''_n, y''_n, z''_n)

into (x_n', y_n', z_n'). That is,

$$\begin{bmatrix} x_n'' \\ y_n'' \\ z_n'' \end{bmatrix} = R' \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{65}$$

and

$$\begin{bmatrix} x_n' \\ y_n' \\ z_n' \end{bmatrix} = \begin{bmatrix} x_n'' \\ y_n'' \\ z_n'' \end{bmatrix} + \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} \tag{66}$$

where R', $\Delta x'$, $\Delta y'$ and $\Delta z'$ are the motion parameters in the (x_n, y_n, z_n) space as defined by (23) and (24). (65) gives

$$x_n''^2 + y_n''^2 + z_n''^2 = \begin{bmatrix} x_n & y_n & z_n \end{bmatrix} R'^T R' \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$$

$$= x_n^2 + y_n^2 + z_n^2$$

This, when combined with (66), gives

$$x_n^2 + y_n^2 + z_n^2 = (x_n' - \Delta x')^2 + (y_n' - \Delta y')^2 + (z_n' - \Delta z')^2 \tag{67}$$

From (64) and (67), we have

$$(k^2\lambda_1^2-1)x_n'^2 + (k^2\lambda_1^2-1)y_n'^2 + (k^2\lambda_1^2-1)z_n'^2 - [2\Delta x'\cdot x_n' + 2\Delta y'\cdot y_n'$$

$$+ 2\Delta z'\cdot z_n' - \Delta x'^2 - \Delta y'^2 - \Delta z'^2]\, k^2\lambda_1^2 = 0 \tag{68}$$

Since (68) is true for all x_n', y_n' and z_n', by equating the coefficients of all powers of x_n', y_n' and z_n' to zero, we have

$$\Delta x' = \Delta y' = \Delta z' = 0$$

and

$$k\lambda_1 = 1 \text{ or } k = \frac{1}{\lambda_1}$$

Therefore, from (24),

$$\Delta x = \Delta y = \Delta z = 0.$$

Then, (9) gives

$$R + 0 = k\,A, \text{ or } R = \lambda_1^{-1} A$$

Therefore, we ahve proved that if $\lambda_1 = \lambda_2 = \lambda_3$, then the motion consists of rotation around an axis through the origin only, and the solution for the rotation matrix is unique. The object surface can be anywhere. This proves the necessity part.

We now proceed to prove the sufficiency part. If the motion consists of rotation around an axis through the origin only, i.e., $\Delta x = \Delta y = \Delta x = 0$, then from (9),

$$A = k^{-1} R \tag{69}$$

Let $U_A = R$, $V_A = I$ and $\Lambda_A = k^{-1} I$. Then (69) becomes

$$A = U_A \Lambda_A V_A^T$$

$$= U_A \begin{bmatrix} k^{-1} & & \\ & k^{-1} & \\ & & k^{-1} \end{bmatrix} V_A^T \tag{70}$$

Since U_A and V_A are orthonormal, (70) gives the SVD of A, with singular values k^{-1}, k^{-1} and k^{-1}. Then from the fact that the singular values of any matrix are unique, we see that A has three identical singular values. This proves the sufficiency part. Q.E.D.

IV. CONCLUSIONS

Three theorems have been stated and proved regarding the uniqueness and the computation of the motion parameters, and the physical descriptions and classifications of the actual three-dimensional motion for a rigid planar patch. The motion parameters are unique aside from a scale factor for the translation parameters if the singular values of the 3 x 3 matrix consisting of the 8 pure parameters are not all distinct; otherwise, the number of solutions is two. The distinction between the cases of multiplicity 1 and 2 lies in whether or not the translation vector coincides with the normal direction of the object surface at t_2. If there is no translation at all, then the singular values are all identical. In any case, once the eight pure parameters are estimated, which can be done by solving a system of linear equations, computing the singular value decomposition of a 3 x 3 matrix is all that it takes to obtain the 3-D motion parameters and the directional cosines of the normal direction of the planar patch.

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Estimating Three-Dimensional Motion Parameters

of a Rigid Planar Patch, III: Finite Point Correspondences

and the Three-View Problem

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ABSTRACT

Two results are presented in this paper. First, it is shown that in estimating three-dimensional motion of a rigid planar patch, the eight pure parameters used in [1] and [2] are uniquely determined from the image correspondences of four points, no three colinear, and can be estimated by solving a set of linear equations. The second result concerns the three-view problem. It is proved that given four image point correspondences in three perspective views of a planar patch undergoing general three-dimensional rigid body motion, the number of solutions for the motion parameters is one, as opposed to two [2] when only two perspective views are given.

I. Introduction

The interest in motion estimation using image sequences has been growing rapidly in many fields of research in the past few years. The efforts in the 70's were primarily focused upon two-dimensional motion estimation [11-18]. Recently, attention has been gradually shifted toward three-dimensional motion estimation [1-10, 19-21]. The difference between 2-D and 3-D motion estimation is not just the degree of difficulty or complexity in solving the motion equations. The issues of uniqueness, the minimum information required to ensure uniqueness and the 3-D structure interpretation, which are not present in the study of 2-D motion estimation, make the study of 3-D motion estimation more challenging and interesting. Furthermore, due to the nonlinearity and the increase of the number of unknowns of the motion equations for 3-D motion estimation, the development of more clever and efficient ways of solving the motion equations becomes also extremely important.

For the case of estimating 3-D motion of rigid curved surfaces, [3] presented an efficient algorithm for determining the motion parameters exactly without having to solve nonlinear equations, and was the first to analyze the problem of how many image point correspondences are required to ensure the uniqueness of motion parameters.

For the case of 3-D motion estimation of a rigid planar patch, a brief introduction is given in Sec. II. In this paper, it is proved that the eight pure parameters [1,2] in the two-view problem are unique given the image correspondences of four points no three colinear, and can be estimated by solving a set of linear equations. For the three-view problem, it is proved that given four image point correspondences in three (distinct) perspective views, the solutions for the motion parameters are unique.

II. The Eight Pure Parameters and the A Matrix

The basic geometry of the problem considered in [1] and [2] is repeated here in Figure 1. Consider two frames at time t_1 and t_2 ($t_1 < t_2$). For a particular point P on the object, let

(x,y,z) = object-space coordinates of a point P at time t_1.

(x',y',z') = object-space coordinates of P at time t_2.

(X,Y) = image-space coordinates of P at t_1.

(X',Y') = image-space coordinates of P at t_2.

It is obvious from Figure 1 that

$$X = \frac{x}{z} \qquad Y = \frac{y}{z}$$

$$X' = \frac{x'}{z'} \qquad Y' = \frac{y'}{z'} \tag{0}$$

where the focal length is normalized to one for convenience. It was shown in [1] and [2] that for a rigid planar patch undergoing 3-D motion (a rotation with an angle θ around an axis through the origin with directional cosines n_1, n_2, n_3, followed by a translation with translation vector (Δx, Δy,Δz))

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{1}$$

where R is a 3 x 3 orthonormal matrix of the first kind

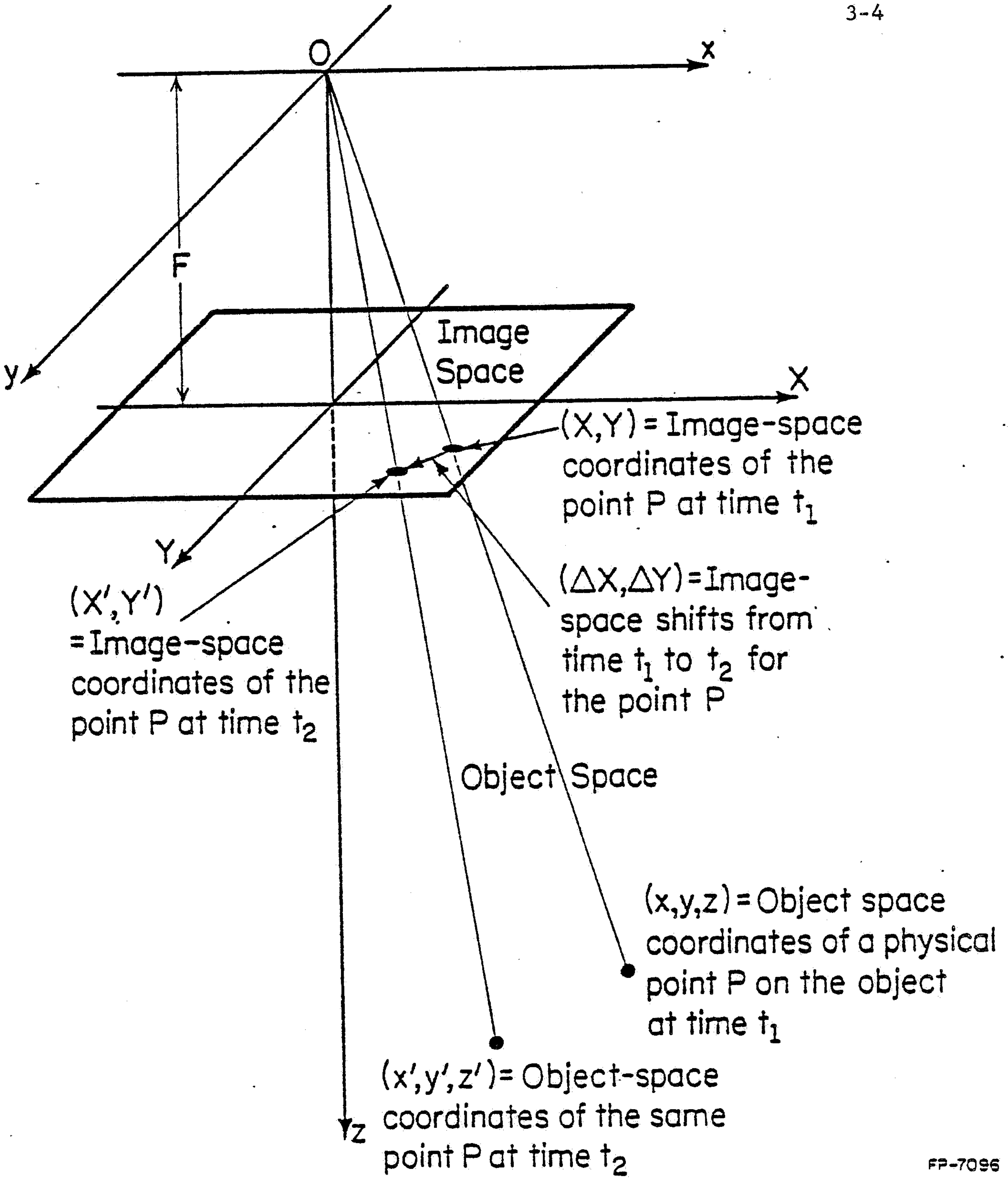

Fig. 1 Basic geometry for three-dimensional motion estimation.

$$R = \begin{bmatrix} n_1^2+(1-n_1^2)\cos\theta & n_1n_2(1-\cos\theta)-n_3\sin\theta & n_1n_3(1-\cos\theta)+n_2\sin\theta \\ n_1n_2(1-\cos\theta)+n_3\sin\theta & n_2^2+(1-n_2^2)\cos\theta & n_2n_3(1-\cos\theta)-n_1\sin\theta \\ n_1n_3(1-\cos\theta)-n_2\sin\theta & n_2n_3(1-\cos\theta)+n_1\sin\theta & n_3^2+(1-n_3^2)\cos\theta \end{bmatrix} \quad ($$

the image-space coordinates before and after the motion are related by

$$X' = \frac{a_1X + a_2Y + a_3}{a_7X + a_8Y + 1}$$

$$Y' = \frac{a_4X + a_5Y + a_6}{a_7X + a_8Y + 1} \qquad (3)$$

where the a_i's are such that if we define R to be such that

$$k = n_3^2 + (1-n_3^2)\cos\theta + c.\Delta Z$$

and let $ax + by + cz = 1$ be the equation describing the object surface before motion (at t_1), then

$$A \triangleq \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix} = k^{-1} \left\{ R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \quad b \quad c] \right\} \qquad (4)$$

The eight a_i's in A are called the pure parameters [1]. Given A, the actual motion parameters can be obtained by simply computing the SVD of the matrix A and the number of solutions for the motion parameters is either one or two depending on the multiplicity of singular values of A [2]. As for the uniqueness of the pure parameters given the image motion, it was shown either using Lie Group Theory [1] or elementary algebra [20] that given the image point correspondences of the whole plane, the pure parameters are unique. It is the purpose of Section III to show that it takes only four image point correspondences, no three colinear to determine the pure parameters uniquely. The algebraic proof in Section III.1 is direct in the sense that the coefficient matrix of the set of linear equations that the eight pure parameters must satisfy given four image point correspondences is proved to be nonsingular, which directly leads to the conclusion on the uniqueness of the pure parameters. The geometrical proof in Section III.2 is indirect in the sense that the pure parameters are not directly shown to be unique, but rather the image point correspondences of the whole image plane are proved to be fixed given four image point correspondences. In order to ultimately prove that the eight pure parameters are unique, two more results are needed. [1] and [20] proved that given the image point correspondences of the whole plane, the eight pure parameters are unique if the 3 x 3 A matrix is nonsingular. This is the first result needed. The second one is Lemma II for the algebraic proof in Section III.1. With these two facts and the geometrical proof in Section III.2, one can conclude that the eight pure parameters are unique given four image point correspondences, noncolinear <u>both</u> before and after the motion.

III.1 Algebraic Proof for the Uniqueness of the Eight Pure Parameters Given Four Image Point Correspondences.

From (3), we have

$$M \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix} = \begin{bmatrix} X_1' \\ Y_1' \\ X_2' \\ Y_2' \\ \vdots \\ X_4' \\ Y_4' \end{bmatrix} \tag{5}$$

where

$$M \triangleq \begin{bmatrix} X_1 & Y_1 & 1 & 0 & 0 & 0 & -X_1X_1' & -Y_1X_1' \\ 0 & 0 & 0 & X_1 & Y_1 & 1 & -X_1Y_1' & -Y_1Y_1' \\ X_2 & Y_2 & 1 & 0 & 0 & 0 & -X_2X_2' & -Y_2X_2' \\ 0 & 0 & 0 & X_2 & Y_2 & 1 & -X_2Y_2' & -Y_2Y_2' \\ X_3 & Y_3 & 1 & 0 & 0 & 0 & -X_3X_3' & -Y_3X_3' \\ 0 & 0 & 0 & X_3 & Y_3 & 1 & -X_3Y_3' & -Y_3Y_3' \\ X_4 & Y_4 & 1 & 0 & 0 & 0 & -X_4X_4' & -Y_4X_4' \\ 0 & 0 & 0 & X_4 & Y_4 & 1 & -X_4Y_4' & -Y_4Y_4' \end{bmatrix} \tag{6}$$

In order to prove that given four image point correspondences, the pure parameters are unique, we take the following approach. Let (X_i', Y_i') be transformed from (X_i, Y_i), $i = 0, \ldots, 3$, with some reference pure parameters $a_i^{(o)}$s' such that

$$X_i' = \frac{a_1^{(o)} X_i + a_2^{(o)} Y_i + a_3^{(o)}}{a_7^{(o)} X_i + a_8^{(o)} Y_i + 1}$$

$$Y_i' = \frac{a_4^{(o)} X_i + a_5^{(o)} Y_i + a_6^{(o)}}{a_7^{(o)} X_i + a_8^{(o)} Y_i + 1} \qquad (7)$$

Then the elements of the matrix M in (6) contain the image coordinates before motion (i.e., X_is' and Y_is') and the reference pure paramters $a_i^{(o)}$'s. Let A_0 be defined as

$$Ao = \begin{bmatrix} a_1^{(o)} & a_2^{(o)} & a_3^{(o)} \\ a_4^{(o)} & a_5^{(o)} & a_6^{(o)} \\ a_7^{(o)} & a_8^{(o)} & 1 \end{bmatrix} \qquad (8)$$

It is to be shown that if Ao containing the reference pure parameters is nonsingular, then give four image point correspondences with no three co-linear, (5) yields only one set of solutions for the a_i's, namely the reference pure parameters $a_i^{(o)}$'s.

In order to simplify the analysis, let the origin of the image coordinate system at t_1 be located at one of the image points, say (X_0, Y_0), at t_1. It will be shown at the end of this section why this simplification does not result in any loss of generality.

By setting X_0 and Y_0 in (3) and (7) to zero, we have

$$X_0' = a_3 = a_3^{(o)}$$

$$Y_0' = a_6 = a_6^{(o)} \qquad (9)$$

With (9), the number of unknowns now becomes six. Let G be defined as

$$G \triangleq \begin{bmatrix} e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ d_1 & d_2 & d_3 \end{bmatrix} \triangleq A\ H \qquad (10)$$

where

$$H \triangleq \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{bmatrix} \qquad (11)$$

Substituting (9), (10) and (11) into (5) gives

$$D \begin{bmatrix} a_1 \\ a_2 \\ a_4 \\ a_5 \\ a_7 \\ a_8 \end{bmatrix} = B \qquad (12)$$

where

$$
D \triangleq \begin{bmatrix}
d_1X_1 & d_1Y_1 & 0 & 0 & e_1X_1 & e_1Y_1 \\
d_2X_2 & d_2Y_2 & 0 & 0 & e_2X_2 & e_2Y_2 \\
d_3X_3 & d_3Y_3 & 0 & 0 & e_3X_3 & e_3Y_3 \\
0 & 0 & d_1X_1 & d_1X_1 & f_1X_1 & f_1Y_1 \\
0 & 0 & d_2X_2 & d_2Y_2 & f_2X_2 & f_2Y_2 \\
0 & 0 & d_3X_3 & d_3Y_3 & f_3X_3 & f_3Y_3
\end{bmatrix} \qquad (13)
$$

$$
B \triangleq \begin{bmatrix}
\begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{bmatrix} & \begin{bmatrix} a_3^{(o)}a_7^{(o)} - a_1^{(o)} \\ a_3^{(o)}a_8^{(o)} - a_2^{(o)} \end{bmatrix} \\
\begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{bmatrix} & \begin{bmatrix} a_6^{(o)}a_7^{(o)} - a_6^{(o)} \\ a_6^{(o)}a_8^{(o)} - a_5^{(o)} \end{bmatrix}
\end{bmatrix} \qquad (14)
$$

Note that B does not contain the unknowns a_i's, $i = 1,2,4,5,7,8$. Therefore, if D is nonsingular, the solution for the unknowns, $a_1, a_2, a_4, a_5, a_7, a_8$, is unique. Lemma I, below, gives the exact conditions for D to be singular.

Lemma I:

Let D be given by (13). Then

$$\det(D) = d_1 d_2 d_3 \det(Ao) \det(H) (X_1Y_2-X_2Y_1)(X_2Y_3-X_3Y_2)(X_3Y_1-X_1Y_3)$$

where

Ao is defined in (8)

H is defined in (11)

d_1, d_2, d_3 are given in (10)

[Proof] The cofactor of the (11)th element of (13) is given by

$$\mathrm{cof}(d_1X_1) = \begin{vmatrix} d_2Y_2 & 0 & 0 & e_2X_2 & e_2Y_2 \\ d_3Y_3 & 0 & 0 & e_3X_3 & e_3Y_3 \\ 0 & d_1X_1 & d_1Y_1 & f_1X_1 & f_1Y_1 \\ 0 & d_2X_2 & d_2Y_2 & f_2X_2 & f_2Y_2 \\ 0 & d_3X_3 & d_3Y_3 & f_3X_3 & f_3Y_3 \end{vmatrix}$$

After some straightforward derivations, the above becomes

$$\mathrm{cof}(d_1X_1) = d_2d_3(X_2Y_3-X_3Y_2)[e_2(d_3f_1-d_1f_3)Y_3-e_3(d_2f_1-d_1f_2)Y_2]$$

Similarly,

$$\mathrm{cof}(d_2X_2) = d_1d_3(X_1Y_3-X_3Y_1)[e_1(d_3f_2-d_2f_3)Y_3-e_3(d_1f_2-d_2f_1)Y_1]$$

and $$\mathrm{cof}(d_3X_3) = d_2d_1(X_2Y_1-X_1Y_2)[e_2(d_1f_3-d_3f_1)Y_1-e_1(d_2f_3-d_3f_2)Y_2]$$

Therefore,

$$\det(D) = d_1X_1\mathrm{cof}(d_1X_1)-d_2X_2\mathrm{cof}(d_2X_2)+d_3X_3\mathrm{cof}(d_3X_3)$$

$$= d_1d_2d_3[e_3(d_1f_2-d_2f_1)(X_2Y_3-X_3Y_2)(X_3Y_1-X_1Y_3)(X_1Y_2-X_2Y_1)$$

$$- e_2(d_1f_3-d_3f_1)(X_1Y_2-X_2Y_1)(X_2Y_3-X_3Y_2)(X_3Y_1-X_1Y_3)$$

$$+ e_1(d_2f_3-d_3f_2)(X_1Y_2-X_2Y_1)(X_2Y_3-X_3Y_2)(X_3Y_1-X_1Y_3)$$

$$= d_1d_2d_3[e_1(d_2f_3-d_3f_2)-e_2(d_1f_3-d_3f_1)+e_3(d_1f_2-d_2f_1)]$$

$$(X_1Y_2-X_2Y_1)(X_2Y_3-X_3Y_2)(X_3Y_1-X_1Y_3)$$

$$= d_1d_2d_3.\det(A_o)(X_1Y_2-X_2Y_1)(X_2Y_3-X_3Y_2)(X_3Y_1-X_1Y_3)$$

O.E.D.

It is obvious from Lemma I that if d_1, d_2 and d_3 are never zero (to be shown later), the D is singular and only if

(i) $\det(A_o) = 0$ (i.e., A_o is singular)

(ii) $\det(H) = 0$ (i.e., point 1, 2 and 3 are colinear)

(iii) $X_1Y_2-X_2Y_1 = 0$ (i.e., point 0, 1 and 2 are colinear)

(iv) $X_2Y_3-X_3Y_2 = 0$ (i.e., point 0, 2 and 3 are colinear)

(v) $X_3Y_1-X_1Y_3 = 0$ (i.e., point 0, 1 and 3 are colinear)

Note that (ii), (iii), (iv) and (v) exhaust all the possibilities for any three among the four points to be colinear. We now show that d_1, d_2 and d_3 are strictly positive.

It was shown in [2] that

$$\begin{bmatrix} x_i' \\ y_i' \\ z_i' \end{bmatrix} = k_o A_o \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} , \quad i = 1, 2, 3 \tag{15}$$

From (15),

$$z_i' = k_o(a_7^{(o)} x_i + a_8^{(o)} y_i + z_i)$$

Substituting (o) into the above gives

$$z_i' = k_o(a_7^{(o)} z_i X_i + a_8^{(o)} z_i Y_i + z_i)$$

or

$$a_7^{(o)} X_i + a_8^{(o)} Y_i + 1 = \frac{z_i'}{z_i} k_o \; , \; i = 1, 2, 3 \tag{16}$$

Since the object points must be in front of the camera, $z_i' > 1$ (the normalized focal length) and $z_i > 1$. [2] shows that $k_o < 0$ corresponds to the case that the object points move to the back of the camera. k_o obviously cannot be zero, otherwise (15) would imply that all the object points move to the origin. Therefore, from (16),

$$a_7^{(o)} X_i + a_8^{(o)} Y_i + 1 = \frac{z_i'}{z_i} \; k_o > 0 \; , \; i = 1, 2, 3.$$

From (10),

$$d_i = a_7^{(o)} X_i + a_8^{(o)} Y_i + 1 \; , \; i = 1, 2, 3.$$

Thus

$d_i > 0$ for $i = 1, 2, 3.$

We have proved that D in (12) is nonsingular if, and only if, A_o is nonsingular and no three points among the four are colinear. It is shown in the following Lemma that the restriction on A_o being nonsingular need not be imposed if none of the three points are colinear both before and after the motion.

Lemma II:

Given the fact that the image points before motion (at t_1) are not colinear (or equivalently, at least three points are noncolinear), then the following three statements are equivalent:

(i) A_o is sungular

(ii) The object surface passes through the origin at t_2.

(iii) All the image points after motion (at t_2) are colinear.

[Proof] We prove that (i) iff (ii) and (ii) iff (iii).

[(i) $\Longrightarrow$ (ii)]

Let the SVD of A_o be given by

$$A_o = U \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} V^T \tag{17}$$

where λ_1, λ_2 and λ_3 are the singular values of A_o. If A is singular, then one of the singular values must be zero, since from (17) we have

$$\det(A_o) = \det(U)\, \lambda_1\lambda_2\lambda_3\, \det(V)$$

and therefore, if $\det(A_o) = 0$, then one of the λ_is' must be zero. Substituting (17) into (15) gives

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = k_o U \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} V^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = k_o \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} V^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= k_o \begin{bmatrix} \lambda_1\ V_1^T J \\ \lambda_2\ V_2^T J \\ \lambda_3\ V_3^T J \end{bmatrix} \tag{18}$$

where

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \triangleq U^T \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$J \triangleq \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{19}$$

$V_j \stackrel{\Delta}{=}$ the ith column of V,j = 1,2,3. Since U^T is orthonormal, (19) can be regarded as a rotation of the object-space coordinate system around an axis through the origin at time t_2. From the fact established earlier that one of the λ_is' must be zero if A_o is singular, we see from (18) that by rotating the coordinate system around an axis through the origin, the object surface coincides either with the x"y" plane or x"z" plane or y"z" plane. This implies that before rotating the coordinate system using (19), the object surface must be passing through the origin.

[(ii) $\Longrightarrow$ (i)]

Since the object surface passes through the origin at t_2, by assumption, (18) becomes

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} \lambda_1 & V_1^T J \\ \lambda_2 & V_2^T J \\ \lambda_3 & V_3^T J \end{bmatrix} \qquad (20)$$

at the origin. Were A_o to be nonsingular, λ_1, λ_2 and λ_3 would be nonzero. Then (20) would give

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V_1^T \\ V_2^T \\ V_3^T \end{bmatrix} J = V^T J \qquad \text{or}$$

$$V^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = J \qquad \text{or} \qquad J = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that the object surface at t_1 passes through the origin. The there exists $\alpha, \beta \in R$ such that

$$z = \alpha x + \beta y$$

From Equation (0) and the above,

$$z = \alpha X z + \beta Y z$$

Thus

$$\alpha X + \beta Y = 1$$

which implies that all the image points are colinear at t_1, contradicting the premise of the lemma. Thus A_o has to be singular.

[(iii) $\Longrightarrow$ (ii)]

From (iii), there exists $\alpha, \beta \rightarrow R$ such that

$$Y' = \alpha X' + \beta$$

From Equation (0) and the above,

$$\frac{y'}{z'} = \alpha \frac{x'}{z'} + \beta$$

or $\alpha x' - y' + \beta z' = 0$, which implies (ii)

[(ii) $\Longrightarrow$ (iii)]

From (ii), there exists $\alpha, \beta \in R$ such that

$$z = \alpha x + \beta y = \alpha X z + \beta Y z$$

or

$$\alpha X + \beta Y = 1$$

which implies (iii).

* End of Proof for Lemma II *

Up to this point, we have proved that given four image point correspondences no three points colinear both before and after the motion, the pure parameters are unique, if the origin of the image coordinate system coincides with one of the image points at t_1. It is shown now that the latter assumption does not cause any loss of generality.

First, we shall show that given four image point correspondences in two frames, one can derive from this the image correspondences of the same four object points in two frames taken by the camera in the same position, but oriented differently such that one of the object points coincides with the optical axis, i.e., the z axis. Since it has been shown earlier that the pure parameters are unique given four image point correspondences with one of the image point at frame 1 at the origin, we see that the 3 x 3 matrix containing the pure parameters for this new configuration designated as A_n, is unique. Next, we shall show that the A matrix for the original configuration is similar to A_n, and can be determined uniquely from A_n. The proof would then be completed. Now we furnish the details.

Since rotating the camera is equivalent to rotating the object points, we now look for a rotation matrix R_o, which can rotate the point (x_o, y_o, z_o) to the z axis, i.e.,

$$\begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} = R_o^T \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}$$

or

$$R_o \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} \tag{21}$$

where α is some constant. Let R_{oi} be the ith column of R_o, $i = 1,2,3$. Then, from (21)

$$\alpha R_{o3} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = z_o \begin{bmatrix} x_o/z_o \\ y_o/z_o \\ 1 \end{bmatrix} = z_o \begin{bmatrix} X_0 \\ Y_0 \\ 1 \end{bmatrix}$$

Since R_{o3} is normalized, we have

$$R_{o3} = \frac{1}{\sqrt{x_o^2 + y_o^2 + 1}} \begin{bmatrix} X_0 \\ Y_0 \\ 1 \end{bmatrix}$$

and $\alpha = z_o \sqrt{x_o^2 + y_o^2 + 1}$

Thus

$$R_o = \begin{bmatrix} R_{o1} & R_{o2} & \begin{matrix} sX_o \\ sY_o \\ s \end{matrix} \end{bmatrix} \tag{22}$$

where $s = (X_0^2 + Y_0^2 + 1)^{-\frac{1}{2}}$ and R_{o1}, R_{o2} are two arbitrary column vectors such that R_o is orthonormal. Note that although R_o is not unique, any arbitrary choice of R_o will lead to the desired conclusion, as to be seen later.

It is to be shown that the image coordinates of the four points at t_1 and t_2 for the new configuration (all the points are rotated by R_o) can be derived from the image coordinates for the original configuration.

Let (x_{ni}, y_{ni}, z_{ni}) be the object coordinates of the ith point after being rotated with R_o and (x_{ni}, y_{ni}) be its image corrdinates. Then

$$\begin{bmatrix} x_{ni} \\ y_{ni} \\ z_{ni} \end{bmatrix} = R_o^T \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} & R_{o1}^T & \\ & R_{o2}^T & \\ sX_o & sY_o & s \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

$$= \begin{bmatrix} R_{o1}^T \; J \\ R_{o2}^T \; J \\ s[X_0 Y_0 1] \; J \end{bmatrix} \quad \text{where } J \triangleq \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

Thus

$$X_{ni} = \frac{x_{ni}}{z_{ni}} = \frac{R_{o1}^T J}{s[X_0 Y_0 1] J}$$

$$= \frac{R_{o1}^T J z_i^{-1}}{s[x_o y_o 1] J z_i^{-1}} = \frac{R_{o1}^T J'}{s[X_0 Y_0 1] J'} \qquad (23)$$

where

$$J' \triangleq \begin{bmatrix} x_i/z_i \\ y_i/z_i \\ 1 \end{bmatrix} = \begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix}$$

Similarly,

$$Y_{ni} = \frac{R_{o2}^{T} J'}{s[X_o Y_o 1]\ J'} \tag{24}$$

and

$$X_{ni}' = \frac{R_{o1}^{T} J''}{s[X_o' Y_o' 1]\ J''} \tag{25}$$

$$Y_{ni}' = \frac{R_{o2}^{T} J''}{s[X_o' Y_o' 1]\ J''} \tag{26}$$

where

$$J'' \triangleq \begin{bmatrix} X_i' \\ Y_i' \\ 1 \end{bmatrix}$$

From (23) - (26), we see that X_{ni}, Y_{ni}, X_{ni}', Y_{ni}' are functions of X_i, Y_i, X_i', Y_i' only. Therefore, the image point correspondences for the new configuration can be determined directly from those for the original configuration.

From [2],

$$\begin{bmatrix} x_n' \\ y_n' \\ z_n' \end{bmatrix} = k_n A_n \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \quad (27)$$

and

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = k A \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (28)$$

Since

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_o \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$$

and

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_o \begin{bmatrix} x_n' \\ y_n' \\ z_n' \end{bmatrix}$$

(28) becomes

$$R_o \begin{bmatrix} x_n' \\ y_n' \\ z_n' \end{bmatrix} = k A R_o \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$$

$$\begin{bmatrix} x_n' \\ y_n' \\ z_n' \end{bmatrix} = k\, R_o^T A R_o \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{29}$$

Since it was proved earlier in this section that the pure parameters are unique given four image point correspondences with one of the point at the origin, A_n is unique. Therefore, comparing (27) and (29),

$$A_n = R_o^T A R_o$$

or

$$A = R_o A_n R_o^T \tag{30}$$

Although R_o in (22) is not unique, A is still unique since for any arbitrary choice of R_o in (22) (Note that different R_o would result in a different A_n), (30) is the necessary condition for all possible A's. Therefore, the pure parameters for the original configuration are unique.

We have proved that given four image point correspondences, no three image points colinear both before and after the motion, the pure parameters are unique.

III.2 Geometrical Proof for the Uniqueness of the Point Correspondences of the Whole Image Plane Given Four Point Correspondences

It is worth noting that the geometrical proof presented in this section does not lead to the conclusion on the uniqueness of the pure parameters directly. As explained at the end of Section II, it takes Lemma II in Section III.1 and the results in [1][20] to complete the proof.

It is to be proved that given the correspondences of four image points in two perspective views no three colinear, the image correspondence of any other point can be determined uniquely. In particular, let A, B, C, D and E be five arbitrary points in frame 1, such that no three are colinear, and let A', B', C', D' be the given corresponding points of A, B, C, D in frame 2, as depicted in Figure 2. We would like to show that the corresponding point E' of E in frame 2 is uniquely determined.

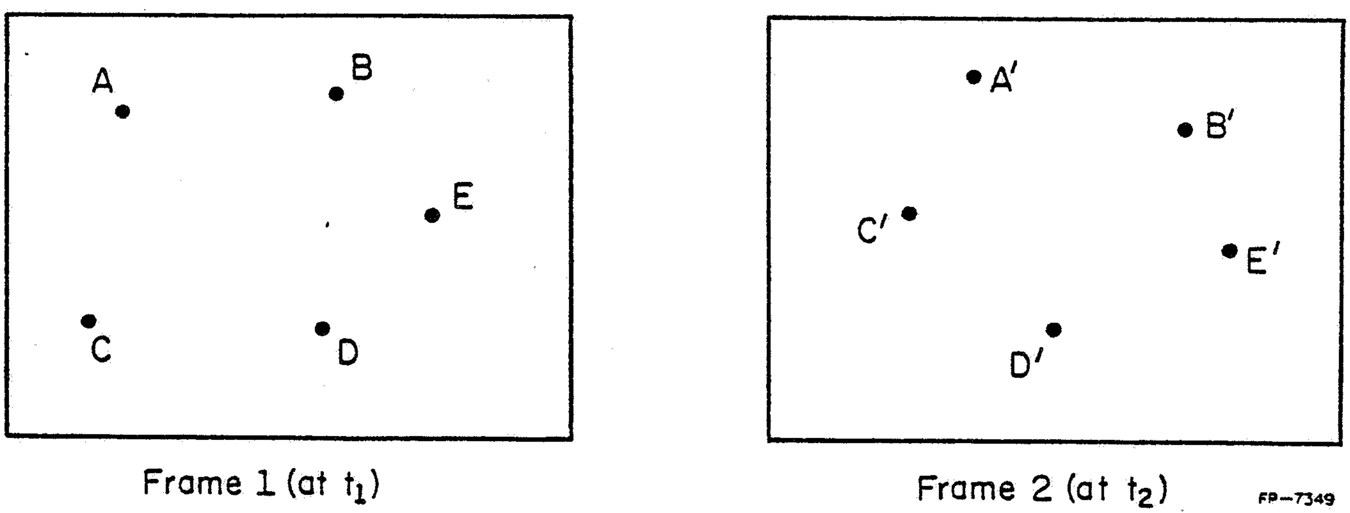

Figure 2 Five point correspondences in two perspective views.

It is easy to see that if a set of points in the image space are colinear, the corresponding points on the planar patch in the object space must also be colinear and vise versa. In fact, let (X_i, Y_i), $i = 1, \ldots,$ n, be the image coordinates of n colinear points on the image plane and (x_i, y_i, z_i), $i = 1, \ldots, n$, be the corresponding points on the planar patch. Then there exist $a, b \in R$ such that

$$X_i = aY_i + b$$

Substituting (o) into the above gives

$$x_i = ay_i + bz_i \tag{31}$$

which indicates that the object points are on a plane passing through the origin. Since A, B, C, D are not colinear, from Lemma II of Section III.1, the object surface cannot pass through the origin. Therefore, the object surface must lie on the intersection of the object surface and the plane described by (31). Thus, the object points must be colinear. The converse is obviously true since we can regard the object surface as the image plane and vise versa and then repeat the above argument.

Next, it is to be shown that given the correspondences of three colinear points, the correspondences of all the other points on the line passing through these three points are determined.

Consider an arbitrary 4th point on the line passing through the given three points in frame 1. Since it was shown earlier in this section that the points in the object space corresponding to a set of colinear points in the image plane must also be colinear, we can see that the two sets of four points, one set on the image plane, the other set on the planar patch, are in perspective correspondence by definition [22]. Therefore, the cross ratio [22,23] of the four points in the image plane is the same as the cross ratio of the four points on the planar patch at t_1, which remains unchanged from t_1 to t_2 since the object undergoes rigid body motion. Similarly, the two sets of points, one on the image plane and the other on the planar patch at t_2, also have the same cross ratios. Therefore, the cross ratios for the two sets of four points, one at t_1 and the other at t_2, are the same. Then by definition [22], these two sets of image points are in projective correspondence. It is well known in projective geometry (p. 83 [23]) that the projective correspondence between two lines is fully established when three pairs of corresponding points are given. Therefore, we can see that given the correspondences of three colinear points, the correspondence of any other point on the line is determined.

Since there always exist two straight lines not parallel to each other such that one line passes through two points among the given four points A, B, C, D and the second line passes through the other two points, we can assume without losing generality that the line passing through points A,D, denoted by $\overleftrightarrow{AD}$, is not parallel to $\overleftrightarrow{BC}$. Obviously, $\overleftrightarrow{A'D'}$ is also not parallel to $\overleftrightarrow{B'C'}$ in this case. Since none of the three among A,B,C,D are colinear, the point lying on the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$, denoted by G, does not coincide with any one among A, B, C and D. Similarly, the point lying on the intersection of $\overleftrightarrow{A'D'}$ and $\overleftrightarrow{B'D'}$, denoted by G', does not coincide with any one among A', B', C' and D'. If E lies on either one of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$, say $\overleftrightarrow{AD}$, the corresponding point E' of E is fixed, since the correspondences of the three points A, G, C, which are colinear with E, are fixed. On the other hand, if E does not lie on either $\overleftrightarrow{AD}$ or $\overleftrightarrow{BC}$, let

L_1 and L_2 be two lines in frame 1 not parallel to any of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ as depicted in Figure 3, and let the points at the intersections of L_1, L_2 and $\overleftrightarrow{BC}$ be denoted by H and J, respectively, and the points at the intersections of L_1, L_2 and $\overleftrightarrow{AD}$ be denoted by I and K, respectively.

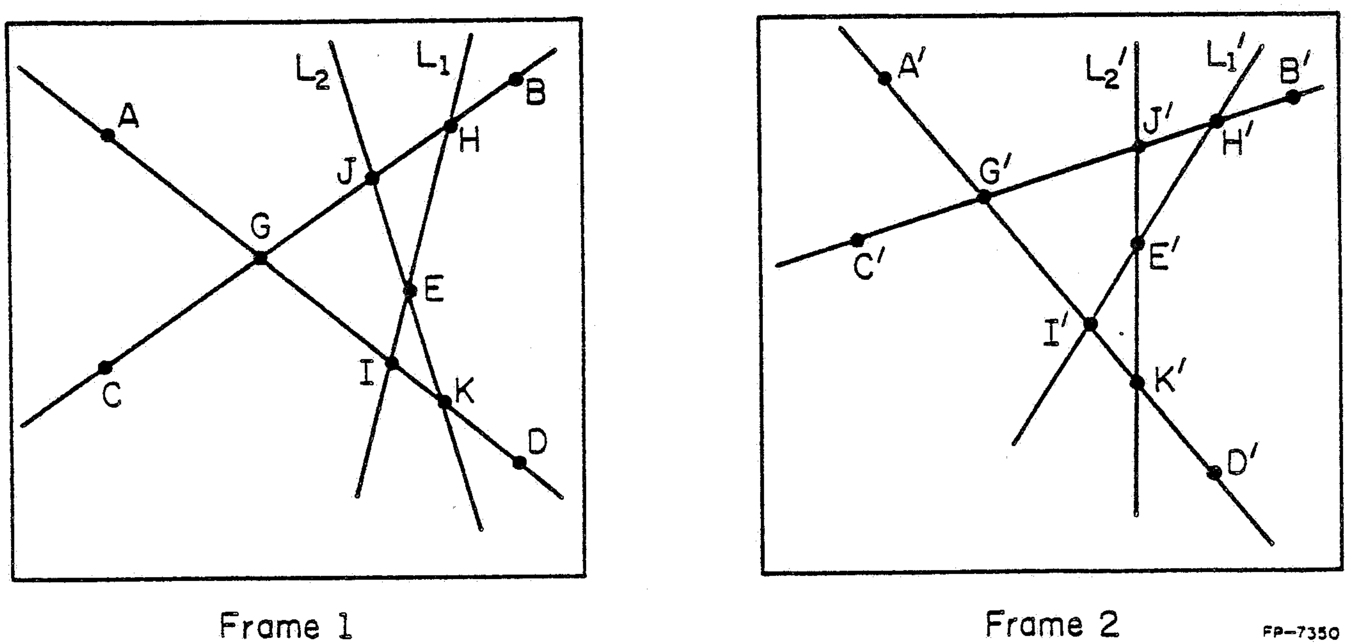

Figure 3. The point correspondence of E can be determined from the point correspondences of A, B, C and D.

Since the correspondences of B, G and C are fixed, the correspondences of all the points on the line $\overleftrightarrow{BC}$ can be uniquely determined. Therefore, the correspondences of H and J, denoted by H' and J' respectively, are fixed. Similarly, the correspondences of I and K, denoted by I' and K' respectively, are also fixed. Therefore, the corresponding point of E in frame 2, denoted by E', which lies on the intersection of $\overleftrightarrow{I'H'}$ and $\overleftrightarrow{J'K'}$, is fixed. We have thus proved that given the correspondences of four image points with none of the three colinear, the correspondence of any other point in the image plane can be uniquely determined. Therefore, the mapping $(x,y) \longrightarrow (x',y')$ is fixed for all $(x,y) \in R^2$. Since the image points are not colinear, according the Lemma II in Section III.1, the matrix A_o is nonsingular. Then, from [1][20], the pure parameters are unique.

IV. Uniqueness of the Motion Parameters Given Four Point Correspondences in Three Image Frames

Consider three distinct image frames, taken at three time instances t_1, t_2 and t_3 ($t_1 < t_2 < t_3$), of a rigid planar patch undergoing three-dimensional motion. It was proved in Section III that given four image point correspondences in two image frames, the pure parameters are unique, and from [2], given the pure parameters, the number of solutions for the real motion parameters is two in general, unless the A matrix in (4) has multiple singular values. In this section, it is proved that with four point correspondences in three image frames, the solution for the motion parameters is unique.

Let A_{ij} be the 3 × 3 matrix containing the eight pure parameters for the motion from t_i to t_j, where i = 1, 2, 3 and j = 1, 2, 3 and let k_{ij} be the associated constant k as used in (4). Consider a particular point P on an object. Let

(x,y,z) = object-space coordinates of P at t_2.

(x',y',z') = object-space coordinates of P at t_1.

(x'',y'',z'') = object-space coordinates of P at t_3.

(X,Y) = image space coordinates of P at t_2.

(X',Y') = image space coordinates of P at t_1.

(X'',Y'') = image space coordinates of P at t_3.

It can be shown that $A_{ij} = A_{ji}^{-1}$ and $k_{ij} = k_{ji}^{-1}$. In fact, from (15),

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = k_{21}A_{21} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{32}$$

and

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_{12}A_{12} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{33}$$

Since four image point correspondences with no three colinear are given, according to Lemma II in Section III.1, A_{ij} is nonsingular. Therefore, (32) gives

$$k_{21}^{-1}A_{21}^{-1} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{34}$$

Comparing (33) and (34) shows that

$$A_{12} = A_{21}^{-1}, \quad k_{12} = k_{21}^{-1} \tag{35}$$

is one possibility. Since it was proved in Section III.1 that given four image point correspondences, the matrix A is unique, this must be the only possibility. We are now to verify the following composition rules for A_{ij}'s and k_{ij}'s:

$$A_{ij} = A_{nj} A_{in} \tag{36}$$

$$k_{ij} = k_{nj} k_{in} \tag{37}$$

where $i = 1, 2, 3,\ \ j = 1, 2, 3,\ \ n = 1, 2, 3$ and $n \neq i,\ n \neq j$.

From (15), we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_{12} A_{12} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{38}$$

and

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = k_{23} A_{23} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{39}$$

Substituting (38) into (39) gives

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = k_{23}k_{12}A_{23}A_{12} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{40}$$

But by definition,

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = k_{13}A_{13} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{41}$$

From (40) and (41), we see that

$$A_{13} = A_{23}A_{12}$$

$$k_{13} = k_{23}k_{12}$$

is one possiblity. Since A_{13} is unique given four image point correspondences, this must be the only possiblity. As for other values of i, j, n in (36) and (37), the proof is the same except for the change of indices.

Since moving the object is equivalent to moving the camera so far as the image point correspondences are concerned, the situation can be depicted in Figure 4 where 0_1, 0_2 and 0_3 are the three focal points for the three image frames when the planar patch is considered to be stationery while the camera is moving, for the purpose of showing the relationships between A_{ij}'s and the three image frames.

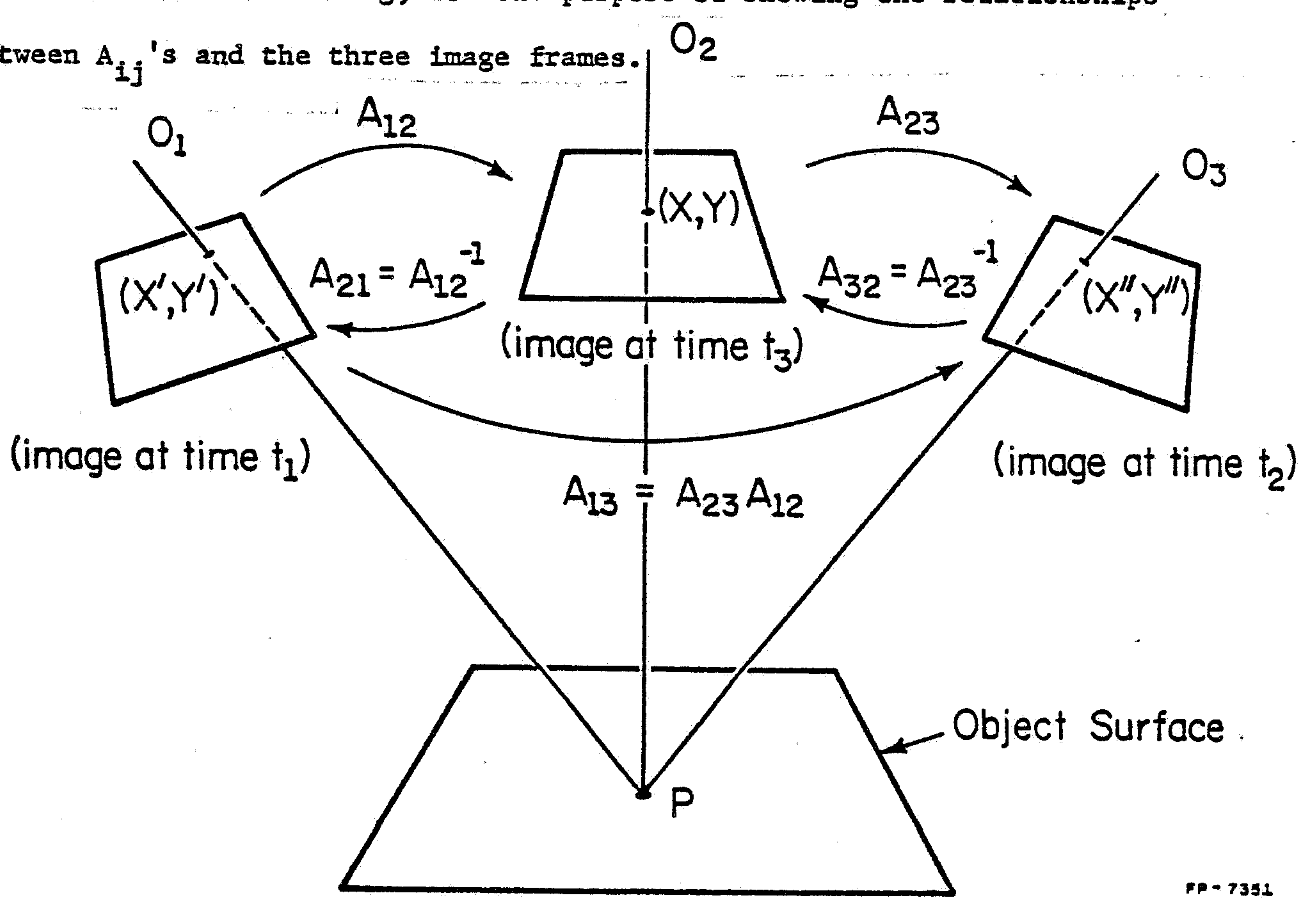

Figure 4. The A_{ij}'s and the three image frames of a rigid planar patch.

Since four image point correspondences are given, A_{12} and A_{23} are fixed. Therefore, all A_{ij}'s, for i = 1, 2, 3 and j = 1, 2, 3 are fixed. Then, from [2], for the motion from t_2 to t_1, there are two sets of solutions for the motion parameters given A_{21} and for the motion from t_2 to t_3, there are also two sets of solutions for the motion parameters given A_{23}. Since these two motions, one from t_2 to t_1 and the other from t_2 to t_3, can be completely independent in general, the only possibility for the solution of the motion parameters to be unique is that not both of the two solutions for the orientations (i.e., the directional cosines of the normal directions of the object surface) of the planar patch corresponding to the two solutions of the motion parameters for the motion from t_2 to t_1 coincide with those for the motion from t_2 to t_3. This is to be proved by contradiction. Assume that there are indeed two solutions. Let the SVD of A_{ij} be

$$A_{ij} = U_{ij} \Lambda_{ij} V_{ij}^T \tag{42}$$

where

$$\Lambda_{ij} = \begin{bmatrix} \lambda_1^{(i,j)} & & \\ & \lambda_2^{(i,j)} & \\ & & \lambda_3^{(i,j)} \end{bmatrix} \tag{43}$$

The approach we shall take is outlined below:

(i) Prove that

$$V_{21} = V_{23} \begin{bmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{bmatrix}$$

(ii) Prove that $\Lambda_{21} = g\,\Lambda_{23}$ for some constant g.

(iii) Prove that $\Lambda_{13} = g^{-1}\,I$. Then the three singular values for the motion from t_1 to t_3 are identical, which implies from [2] that the solution of the motion parameters for the motion from t_1 to t_3 is unique, contradicting to the assumption that there are two solutions.

The details for the above three steps are now given:

(i) Since the order of $\lambda_1^{(i,j)}$, $\lambda_2^{(i,j)}$, $\lambda_3^{(i,j)}$ can be rearranged by parmutating the columns of U_{ij} and V_{ij} in (42), we can always assume that

$$\lambda_1^{(2,1)} \geq \lambda_2^{(2,1)} \geq \lambda_3^{(2,1)} \tag{44}$$

and

$$\lambda_1^{(2,3)} \geq \lambda_2^{(2,3)} \geq \lambda_3^{(2,3)} \tag{45}$$

If any of the equality signs in (44) holds, i.e., A_{21} has multiple singular values, then for the motion from t_2 to t_1, the solution for the motion parameters and the orientation of the object surface at t_2 are unique according to Theorem I in [2]. Then, for the motion from t_2 to t_3, the solution for the motion parameters must also be unique, since were this false, the A_{23} must have distinct singular values and there would be two solutions for the orientation of the object surface at t_2. Similarly, if any of the equality signs in (45) holds, the solutions of the motion parameters are unique for both the motions from t_2 to t_1 and t_2 to t_3. If only the inequality signs in (34) and (35) hold, then from Theorem II in [2], the two solutions for the directional cosines of the planar patch for the motion from t_2 to t_1 are

$$\vec{d_1} \triangleq \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \triangleq w\, V_{21} \begin{bmatrix} \delta \\ 0 \\ 1 \end{bmatrix} \tag{46}$$

and

$$\vec{d_2} \triangleq \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \triangleq w\, V_{21} \begin{bmatrix} -\delta \\ 0 \\ 1 \end{bmatrix} \tag{47}$$

and for the motion from t_2 to t_3,

$$\vec{d_1'} \triangleq \begin{bmatrix} a_1' \\ b_1' \\ c_1' \end{bmatrix} \triangleq w\, V_{23} \begin{bmatrix} \delta' \\ 0 \\ 1 \end{bmatrix} \tag{48}$$

and

$$\overrightarrow{d_2'} \triangleq \begin{bmatrix} a_2' \\ b_2' \\ c_2' \end{bmatrix} \triangleq w' \, V_{23} \begin{bmatrix} -\delta' \\ 0 \\ 1 \end{bmatrix} \tag{49}$$

where w and w' are some normalizing constants

$$\delta = \left\{ \frac{\left[\lambda_1^{(2,1)}\right]^2 - \left[\lambda_2^{(2,1)}\right]^2}{\left[\lambda_2^{(2,1)}\right]^2 - \left[\lambda_3^{(2,1)}\right]^2} \right\}^{1/2}$$

$$\delta' = \left\{ \frac{\left[\lambda_1^{(2,3)}\right]^2 - \left[\lambda_2^{(2,3)}\right]^2}{\left[\lambda_2^{(2,3)}\right]^2 - \left[\lambda_3^{(2,3)}\right]^2} \right\}^{1/2}$$

Since the two solutions for the directional cosines of the object surface at t_2 for the motion from t_2 to t_1 are assumed to be the same as those for the motion from t_2 to t_3, either

$$\overrightarrow{d_1} = \overrightarrow{d_1'} \ , \quad \overrightarrow{d_2} = \overrightarrow{d_2'} \tag{50}$$

or

$$\overrightarrow{d_1} = \overrightarrow{d_2'} \ , \quad \overrightarrow{d_2} = \overrightarrow{d_1'} \tag{51}$$

Let $V_n^{(i,j)}$ be defined as the nth column of V_{ij} for n = 1, 2, 3. Then, from (46) and (47),

$$\vec{d_1} + \vec{d_2} = w\, V_{21} \begin{bmatrix} \delta \\ 0 \\ 1 \end{bmatrix} + w\, V_{21} \begin{bmatrix} -\delta \\ 0 \\ 1 \end{bmatrix}$$

$$= w\, V_{21} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = w\, V_3^{(2,1)}$$

Since the norm of $V_3^{(2,1)}$ is unity, we have from the above,

$$\frac{\vec{d_1} + \vec{d_2}}{\| \vec{d_1} + \vec{d_2} \|} = \pm\, V_3^{(2,1)} \tag{52}$$

We now show that $V_2^{(2,1)}$ is given by the normalized outer product of $\vec{d_1}$ and $\vec{d_2}$.

$$d_1 \times d_2 = w\, V_{21} \begin{bmatrix} \delta \\ 0 \\ 1 \end{bmatrix} \times w\, V_{21} \begin{bmatrix} -\delta \\ 0 \\ 1 \end{bmatrix}$$

$$= w^2 \, [\delta\, V_1^{(2,1)} + V_3^{(2,1)}] \times [-\delta\, V_1^{(2,1)} + V_3^{(2,1)}]$$

$$= w^2 \, [-\delta^2\, V_1^{(2,1)} \times V_1^{(2,1)} + \delta\, V_1^{(2,1)} \times V_3^{(2,1)}$$

$$- \delta\, V_3^{(2,1)} \times V_1^{(2,1)} + V_3^{(2,1)} \times V_3^{(2,1)}] \tag{53}$$

where " X " stands for vector outer product.

Since $V_1^{(2,1)} \times V_1^{(2,1)} = 0$, $V_3^{(2,1)} \times V_3^{(2,1)} = 0$, and $V_3^{(2,1)} \times V_1^{(2,1)} = -V_1^{(2,1)} \times V_3^{(2,1)}$, we have from (53),

$$\vec{d_1} \times \vec{d_2} = 2w^2 \delta V_1^{(2,1)} \times V_3^{(2,1)} \tag{54}$$

Since $V^{(2,1)}$ is an orthonormal matrix, we have

$$V_1^{(2,1)} \times V_3^{(2,1)} = \pm V_2^{(2,1)}$$

Substituting the above into (54) gives

$$\vec{d_1} \times \vec{d_2} = \pm 2w^2 \delta V_2^{(2,1)}$$

Since $V_2^{(2,1)}$ is normalized, we have from the above

$$\frac{\vec{d_1} \times \vec{d_2}}{\| \vec{d_1} \times \vec{d_2} \|} = \pm V_2^{(2,1)} \tag{55}$$

Since V_{21} is orthonormal, we have

$$V_1^{(2,1)} = \pm V_2^{(2,1)} \times V_3^{(2,1)}$$

Substituting (52) and (55) into the above gives

$$V_1^{(2,1)} = \pm \left[\frac{\vec{d_1} \times \vec{d_2}}{\| \vec{d_1} \times \vec{d_2} \|} \times \frac{\vec{d_1} + \vec{d_2}}{\| \vec{d_1} + \vec{d_2} \|} \right] \tag{56}$$

Similarly, for the motion from t_2 to t_3, one can show using exactly the procedure as above that columns of V_{23} can be expressed as functions of $\vec{d_i'}$ and $\vec{d_2'}$ in (48) and (49) as follows:

$$V_3^{(2,3)} = \pm \frac{\overrightarrow{d_1'} + \overrightarrow{d_2'}}{\| \overrightarrow{d_1'} + \overrightarrow{d_2'} \|} \tag{57}$$

$$V_2^{(2,3)} = \pm \frac{\overrightarrow{d_1'} \times \overrightarrow{d_2'}}{\| \overrightarrow{d_1'} \times \overrightarrow{d_2'} \|} \tag{58}$$

$$V_1^{(2,3)} = \pm \frac{\overrightarrow{d_1'} + \overrightarrow{d_2'}}{\| \overrightarrow{d_1'} + \overrightarrow{d_2'} \|} \times \frac{\overrightarrow{d_1'} \times \overrightarrow{d_2'}}{\| \overrightarrow{d_1'} \times \overrightarrow{d_2'} \|} \tag{59}$$

With either (50) or (51), we have from (52) and (57), $V_3^{(2,1)} = \pm V_3^{(2,3)}$ and from (55) and (58), $V_2^{(2,1)} = \pm V_2^{(2,3)}$ and from (56) and (59), $V_1^{(2,1)} = \pm V_1^{(2,3)}$. Thus, we have proved that

$$V_{21} = V_{23} \begin{bmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{bmatrix} \tag{60}$$

(ii) Let (x_n, y_n, z_n), (x_n', y_n', z_n') and (x_n'', y_n'', z_n'') be the new coordinate systems obtained by rotating the coordinate systems (x,y,z) at t_1, (x',y',z') at t_2 and (x'',y'',z'') at t_3, respectively, using the orthonormal matrix V_{21} as follows:

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \triangleq V_{21}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{61}$$

$$\begin{bmatrix} x_n' \\ y_n' \\ z_n' \end{bmatrix} \triangleq V_{21}^T \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{62}$$

$$\begin{bmatrix} x_n'' \\ y_n'' \\ z_n'' \end{bmatrix} \triangleq V_{21}^T \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \tag{63}$$

From (60) and (61), we have

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \left\{ V_{23} \begin{bmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{bmatrix} \right\}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{bmatrix} V_{23}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or

$$\begin{bmatrix} s_1 x_n \\ s_2 y_n \\ s_3 z_n \end{bmatrix} = V_{23}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (64)$$

where $s_i = \pm 1$, $i = 1, 2, 3$. Similarly, from (60) and (63), we have

$$\begin{bmatrix} s_1 x_n'' \\ s_2 y_n'' \\ s_3 z_n'' \end{bmatrix} = V_{23}^T \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \qquad (65)$$

(64) and (65) indicate that the new coordinate systems (s_n, y_n, z_n) in (61) and (x_n'', y_n'', z_n'') in (63) can also be obtained, except for the signs by rotating the old coordinate systems using V_{23} instead of V_{21}. Note that in (64) and (65), (x, y, z) is the coordinate system at t_2 and (x'', y'', z'') is the coordinate system at t_3, while V_{23} is the matrix in (42) containing the singular vectors for the matrix A_{23} which characterizes the motion from t_2 to t_3. Similarly, in (61) and (62), (x, y, z) is the coordinate system at t_2 and (x', y', z') is the coordinate system at t_1, while V_{21} is the matrix in (42) containing the singular vectors for the matrix A_{21} which characterizes the motion from t_2 to t_1. According to [2], if the original coordinate systems in the object space are transformed as in (61) and (62) using V_{21} for the motion from t_2 to t_1, there is a rigid circle lying on the intersection of the object surface and the ellipsoid

$$d_1 = k_{21}^2 \{ [\lambda_1^{(2,1)}]^2 x_n^2 + [\lambda_2^{(2,1)}]^2 y_n^2 + [\lambda_3^{(2,1)}]^2 z_n^2 \} \qquad (66)$$

at t_2, while at t_1, this rigid circle must lie on the intersection of the object surface and the sphere

$$d_1 = x_n'^2 + y_n'^2 + z_n'^2 \tag{67}$$

where d_1 is some constant. We would like to show that this rigid circle must be concentric, on the object surface, with another rigid circle that lies on the intersection of the object surface and the ellipsoid

$$d_2 = k_{23}^2 \{[\lambda_1^{(2,3)}]^2 x_n^2 + [\lambda_2^{(2,3)}]^2 y_n^2 + [\lambda_3^{(2,3)}]^2 z_n^2\} \tag{68}$$

at t_2 and on the intersection of the object surface and the sphere

$$d_2 = x_n''^2 + y_n''^2 + z_n''^2 \tag{69}$$

at t_3 for some d_2.

Because of (60), (61) and (64), we can see that the principle axes of the two ellipsoids in (66) and (68) are the same. From (46), (47) and (61), the solutions for the directional cosines of the planar patch at t_2 in the new coordinate system for the motion from t_2 to t_1 are given by

$$V_{21}^T w V_{21} \begin{bmatrix} \pm\delta \\ 0 \\ 1 \end{bmatrix} = w' \begin{bmatrix} \pm\delta \\ 0 \\ 1 \end{bmatrix}$$

Similarly, from (48), (49) and (64), the solutions of the directional cosines of the planar patch at t_2 in the new coordinate system for the motion from t_2 to t_3 are given by

$$V_{23}^T \cdot w'V_{23} \begin{bmatrix} \pm\delta' \\ 0 \\ 1 \end{bmatrix} = w' \begin{bmatrix} \pm\delta' \\ 0 \\ 1 \end{bmatrix}$$

From the above two equations, we can see that in the new coordinate system, the normal directions of the object surface at t_2 must be perpendicular to the y_n axis, both for the motion from t_2 to t_1 and for the motion from t_2 to t_3. Since the principle axes of the two ellipsoids in (66) and (68) coincide with the x_n, y_n and z_n axes and since the normal direction of the planar patch is perpendicular to the y_n axis (or equivalently, the planar patch is parallel to the y_n axis) we see that the centers of the rigid circles lying on the intersections of the planar patch and the ellipsoids either in (66) or (68) must be on the x_n z_n plane. Obviously, for a particular planar patch, as d_1 increases, the dimension of the ellipsoid in (66) also increases and, consequently, the center of the rigid circle that lies on the intersection of the planar patch and the ellipsoid becomes closer to the z_n axis. In the limit, as d_1 goes to infinity, the center is on the z_n axis. Similarly, as d_2 becomes large, the center of the rigid circle on the intersection of the planar patch and the ellispoid in (68) approaches the z_n axis. On the other hand, as d_1 decreases, the rigid circle lying on the intersection of the planar patch and the ellipsoid in (66) gradually shrinks to a point. For a particular planar patch, let the distance between this limiting point and the z_n axis be P_1. Similarly, as d_2 decreases, the rigid circle lying on the intersection of the planar patch and the ellipsoid in (68) also shrinks to a point. Let the distance between this limiting point and the z_n axis be P_2. Then it is seen that the distances between the z_n axis and the centers of the collection of circles lying on the planar patch and the ellipsoids in (66) for some range of d_1 vary between 0 and p_1, while the distances between the z_n axis and the centers of the collection of circles lying on the planar patch and the ellipsoids in (68) for some range of d_2 vary between 0 and p_2. Let p_3 be such that

$$0 < p_3 < \min(p_1, p_2).$$

From the above, it is obvious that there exist at least two rigid circles lying on the intersections of the planar patch and the ellipsoids in (66) and (68), respectively, such that the centers coincide with each other and the distances between the center and the z_n axis is p_3.

Let the equation describing the object surface at t_2 be expressed as

$$\alpha x + \beta y + \gamma z = 1 \qquad (70)$$

for some $\alpha, \beta, \gamma \in R$. Note that at this point we do not know whether α, β and γ in (70) are unique or not. However, any choice of α, β and γ will lead us to the conclusion we are expecting for step (ii) as to be seen in the following. Substituting (70) into (66) gives

$$\left\{ \left[\lambda_1^{(2,1)}\right]^2 + \left[\frac{\lambda_3^{(2,1)}}{\gamma}\alpha\right]^2 \right\} x_n^2 - \frac{2\,d}{\gamma^2}\left[\lambda_3^{(2,1)}\right]^2 x_n + \left[\lambda_2^{(2,1)}\right]^2 y_n^2 + \left[\frac{\lambda_3^{(2,1)}}{\gamma}\right]^2 = d_1 \qquad (71)$$

Similarly, substituting (70) into (68) gives

$$\left\{[\lambda_1^{(2,3)}]^2 + \left[\frac{\lambda_3^{(2,3)}}{\gamma}\alpha\right]^2\right\}x_n^2 - 2\alpha\left[\frac{\lambda_3^{(2,3)}}{\gamma}\right]^2 x_n + [\lambda_2^{(2,3)}]^2 y_n^2$$

$$+ \left[\frac{\lambda_3^{(2,3)}}{\gamma}\right]^2 = d_2 \tag{72}$$

Since z_n no longer appears in (71) and (72), the two curves in the $x_n\ y_n$ plane described by (71) and (72) are the vertical projections of the two concentric circles in the (s_n, y_n, z_n) space into the $x_n\ y_n$ plane and, therefore, must be two ellipses that are identical up to a scale factor. By equating the coefficients in (71) and (72) up to a positive proportionality constant h, we have

$$\left[\lambda_1^{(2,1)}\right]^2 + \left[\frac{\lambda_3^{(2,1)}}{\gamma}\alpha\right]^2 = \frac{hd_1}{d_2}\left\{\left[\lambda_1^{(2,3)}\right]^2 + \left[\frac{\lambda_3^{(2,3)}}{\gamma}\alpha\right]^2\right\} \tag{73}$$

$$-2\frac{\alpha}{d_1}\left[\frac{\lambda_3^{(2,1)}}{\gamma}\right]^2 = \frac{h}{d_2}\left\{-2\alpha\left[\frac{\lambda_3^{(2,3)}}{\gamma}\right]^2\right\} \tag{74}$$

$$\frac{\left[\lambda_2^{(2,1)}\right]^2}{d_1} = \frac{h}{d_2}\left[\lambda_2^{(2,3)}\right]^2 \tag{75}$$

$$\frac{1}{d_1}\left[\frac{\lambda_3^{(2,1)}}{\gamma}\right]^2 = \frac{h}{d_2}\left[\frac{\lambda_3^{(2,3)}}{\gamma}\right]^2 \tag{76}$$

Let $g = \left(\frac{d_1}{d_2}h\right)^{\frac{1}{2}}$. (Note that d_1, d_2 and h are all positive). Then (74) becomes

$$\lambda_3^{(2,1)} = \pm g\,\lambda_3^{(2,3)}$$

But since the singular values are nonnegative by definition, we have

$$\lambda_3^{(2,1)} = g \cdot \lambda_3^{(2,3)} \tag{77}$$

Similarly, (75) gives

$$\lambda_2^{(2,1)} = g\, \lambda_2^{(2,3)} \tag{78}$$

Substituting (78) into (73) gives

$$[\lambda_1^{(2,1)}]^2 + [\frac{g\alpha}{\gamma}\lambda_3^{(2,3)}]^2 = [g\lambda_1^{(2,3)}]^2 + [\frac{g\alpha}{\gamma}\lambda_3^{(2,3)}]^2$$

or $$[\lambda_1^{(2,1)}]^2 = [g\, \lambda_1^{(2,3)}]^2$$

or $$\lambda_1^{(2,1)} = g\, \lambda_1^{(2,3)} \tag{79}$$

From (77), (78) and (79), we have

$$\Lambda_{21} = g\, \Lambda_{23} \tag{80}$$

(iii) From (36), we have

$$A_{13} = A_{23}\, A_{12} \tag{81}$$

(35) and (81) give

$$\begin{aligned} A_{13} &= A_{23}\, A_{21}^{-1} \\ &= (U_{23}\, \Lambda_{23}\, V_{23}^T)(U_{21}\, \Lambda_{21}\, V_{21}^T)^{-1} \\ &= (U_{23}\, \Lambda_{23}\, V_{23}^T)(V_{21}\, \Lambda_{21}^{-1}\, U_{21}^T) \end{aligned} \tag{82}$$

Substituting (60) and (80) into (82) gives

$$A_{13} = U_{23}\,\Lambda_{23}\,V_{23}^T \cdot V_{23} \begin{bmatrix} \pm1 & & \\ & \pm1 & \\ & & \pm1 \end{bmatrix} g^{-1}\,\Lambda_{23}^{-1}\,U_{21}^T$$

$$= g^{-1}\,U_{23}\,\Lambda_{23} \cdot I \cdot \begin{bmatrix} \pm1 & & \\ & \pm1 & \\ & & \pm1 \end{bmatrix} \Lambda_{23}^{-1}\,U_{23}^T$$

$$= U_{23} \begin{bmatrix} \pm1 & & \\ & \pm1 & \\ & & \pm1 \end{bmatrix} g^{-1}\,\Lambda_{23}\,\Lambda_{23}^{-1}\,U_{23}^T$$

$$= U_{23} \begin{bmatrix} \pm1 & & \\ & \pm1 & \\ & & \pm1 \end{bmatrix} (g^{-1}\,I)\,U_{23}^T$$

$$= U\,\Lambda\,V^T \tag{83}$$

where $\Lambda \triangleq g^{-1}\,I$ (84)

$$U \triangleq U_{23} \begin{bmatrix} \pm1 & & \\ & \pm1 & \\ & & \pm1 \end{bmatrix} \tag{85}$$

$$V \triangleq U_{23} \tag{86}$$

Since U_{23} is orthonormal, it is obvious from (85) and (86) that U and V are orthonormal. Also, since $g \geq 0$, Λ in (84) is a diagonal matrix with nonnegative diagonal elements. Therefore, from definition, (83) is the singular value decomposition of A_{13}. But since the singular values are unique given A_{13}, we have

$$\Lambda_{13} = \Lambda = g^{-1} \cdot I \tag{87}$$

Therefore, $\lambda_1^{(1,3)} = \lambda_2^{(1,3)} = \lambda_3^{(1,3)} = g^{-1}$.

Since the three singular values of A_{13} for the motion from t_1 to t_3 are identical, from Theorem III in [2], the motion parameters are unique, contradicting the assumption that there are two solutions. We have thus proved that given four image point correspondences in three image frames, the solution for the motion parameters is unique.

V. Conclusions

We have shown that in estimating three-dimensional motion parameters of a rigid planar patch the eight pure parameters used in [1] and [2] are unique, and can be determined by solving a set of eight nonsingular linear equations given the image correspondences of four points with no three colinear both before and after the motion. In [2] it was shown that given the eight pure parameters, there are two possible solutions to the motion parameters. It is proved in this paper that given four image point correspondences in three (distinct) perspective views, the motion parameters are uniquely determined.

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