CONTROL SYSTEMS LABORATORY

Report R-102

Binary Quantization of Signal Amplitudes:
Effect for Radar Angular Accuracy

Prepared by: Duane H. Cooper

May 1959

Contract DA-36-039-SC-56695
D/A Sub-Task 3-99-06-111

UNIVERSITY OF ILLINOIS · URBANA · ILLINOIS

The research reported in this document was made possible by support extended to the University of Illinois, Control Systems Laboratory, jointly by the Department of the Army (Signal Corps and Ordnance Corps), Department of the Navy (Office of Naval Research), and the Department of the Air Force (Office of Scientific Research), Air Research and Development Command) under Signal Corps Contract DA-36-039-SC-56695, D/A Sub-Task 3-99-06-111.

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Project D/A Sub-task 3-99-06-111
Numbered Pages: 33
Distribution List: 8

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Abstract

An investigation, made by Peter Swerling (Proc. I.R.E. Vol. 44, No. 9, Sept. 1956, p. 1146), determining the limits that receiver noise imposes upon the accuracy with which the angular position of a target may be estimated by means of a pulsed search radar, is extended to determine additional accuracy limitations imposed by the quantization of signal strengths prior to angle estimation. The analysis follows that of Swerling in form. The results are presented in graphs. For many cases of practical interest, the additional limitation in accuracy may be neglected.

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I. Introduction

Binary quantization is intended to mean a process in which a continuous signal is converted into one with a rectangular waveform, which has a fixed magnitude during times for which the original signal exceeds some preassigned value, and a zero magnitude otherwise.* There are at least two instances in which it is desirable to know the effects of imposing binary quantization as an early step in the process of information extraction. These instances will be mentioned in connection with the quantization of the signal from a pulsed search radar.

For the first instance, binary quantization may be taken as an exteme example of simple amplitude saturation, such as may occur in video amplifiers, or in cathode-ray display devices. As has been shown by J. I. Marcum, if binary quantization does occur, then the imposition of any other monotonic-increasing non-linearity, prior to quantization, causes no further loss of information in the quantized signal.[1] It is in this sense that binary quantization may be taken as an extreme example of such a non-linearity.

For the second instance, one may consider the mechanization of the radar observation process. This usually involves the storage of a representation of the radar signal, in order that certain computations, as entailed in the observation process, may be performed. The required storage capacity may be greatly reduced if it should prove that a binary quantized

* Quantization into more than two levels, or that quantization which preserves only the polarity information of an oscillatory signal, are beyond the scope of the present consideration.

version of the radar signal is an adequate representation for the purpose at hand.*

The consequences of binary quantization for detection have already been investigated. J. V. Harrington, in an approximate calculation of the enhancement type, investigated signal-to-noise ratios in signals generated as sums of repeated pulse in noise, where the pulses summed were quantized, on the one hand, and unquantized, on the other.[2] The present author, in some unpublished work, and also Peter Swerling, have made numerical analyses, which yielded detection sensitivities in terms of the input signal-to-noise ratios required for given probabilities of detection, for fixed false alarm rates, also given.[3] These examples of such investigations have indicated that the loss in detectability is approximately one decibel, the calculations of the enhancement type indicating a somewhat larger loss than the more direct calculations of detection sensitivities.

The work here reported extends the investigation of the losses resulting from binary quantization to a consideration of the degradation of maximum angular accuracy, for the pulsed search radar, as entailed in binary quantization. The effect will be seen to be small.

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* One could expect that the adequacy of the binary representation might sharply depend on the purposes at hand, i.e. on exactly what information is to be extracted from the signal.

II. Derivation of the Formula for σ_{min}

Peter Swerling has calculated the maximum angular accuracy for a pulse search radar in the case where no quantization is imposed.[4] He was able to treat the case in which there were no fluctuations in target reflectivity, or, with suitable interpretation, the case in which such fluctuations were very slow; he was also able to treat the case in which such fluctuations were very rapid.

The present analysis follows the procedures established by Swerling, except that a treatment of the rapid fluctuation case is omitted. After introducing some notation, a certain conditional probability, called L, is defined, in terms of which a theorem from the theory of statistical estimation may be quoted. From this theorem, a formula for the maximum angular accuracy is derived, with many of the details being developed in the appendix.

One designates the azimuth angle for the target as θ_T, the quantity to be estimated, and further designates the azimuth angle for the i-th radar pulse as θ_i. In the time interval between pulses, the radar antenna turns through an angular interval of length $\Delta\theta$. The signal-to-noise ratio for the i-th pulse will be a function of the difference between θ_i and θ_T. This function will be taken to be deterministic, and the noise contribution will be taken to be independent from pulse-to-pulse. One may write this function in terms of the axial value, x_o, of the signal-to-noise ratio and a symmetrical function, $f(u)$, representing the antenna radiation pattern, thus:

$$x_i = x_o f(u_i), \tag{1}$$

in which

$$u_i = (\theta_i - \theta_T)/\beta, \tag{2}$$

with β being a parameter designating the width of the radiation pattern.*

One then constructs the function

$$L\{v_1, v_2, \ldots, v_k | \theta_T\}, \qquad (3)$$

which is the joint conditional probability that one will observe voltages $v_1, v_2, \ldots, v_k$, as the antenna scans through the target sector, if θ_T is the target location.** In the quantized case, each v_i may be either 0 or 1. This function is

$$L = (1/\Delta\theta) \int_{\Delta\theta} \prod_{i=i}^{k} p\{v_i | x_i\} \, d\theta. \qquad (4)$$

The process of averaging over the interval $\Delta\theta$, indicated here, is intended to take account of the fact that the generation of the radar pulses may not be synchronized with the turning of the radar antenna, so that the location of the angle θ_i will show a scan-to-scan jitter, uniformly over the angular interval $\Delta\theta$.

This averaging process may be now dispensed with, by introducing the assumption that x_i shows sufficiently small variations with θ_i over the interval $\Delta\theta$. Thus one writes

$$L\{v_1, v_2, \ldots, v_k | \theta_T\} = \prod_{i=i}^{k} p\{v_i | x_i\} = \prod_{i=i}^{i} p_i, \qquad (5)$$

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* A more complicated dependence upon θ_i and θ_T might be more realistic if non-rotating structure, or those not having the correct rotational symmetry, helped determine the radiation pattern. Such complications are beyond the scope of this work. Similar remarks apply to the assumption of symmetry for $f(u)$.

** Since the question of target existence may be regarded as having to do with a separate problem, that of detection, the target is assumed to exist within a certain angular sector, which may be as wide as 360 degrees. A "scan" is defined to be one complete angular search through that sector, with returned radar pulses being possible for the angles $\theta_1, \theta_2, \ldots, \theta_k$, covering that sector.

in which p_i is the conditional probability of observing v_i for the i-th radar pulse, if the signal-to-noise ratio is x_i. The functional dependence of p_i upon v_i is as follows:

$$p_i = \begin{cases} q(x_i), \text{ if } v_i = 1 \\ 1-q(x_i), \text{ if } v_i = 0. \end{cases} \quad (6)$$

This specification of p_i in terms of q_i achieves the introduction of a process of binary quantization.

If the natural logarithm of L is

$$\lambda = \ln L = \sum_i \ln p_i, \quad (7)$$

then the smallest possible root mean square error in estimating θ_T, which shall be called σ_{min}, is given by

$$(1/\sigma_{min})^2 = E\{(\partial\lambda/\partial\theta_T)^2\}(1-\rho^2), \quad (8)$$

in which ρ is a term indicating the extent to which errors in estimating x_o affect the errors in angular estimation.* If x_o is known <u>a priori</u>, then one puts $\rho = 0$. However, even in the present case, one has $\rho \approx 0$, as will be shown in the appendix.** This leads to the simpler formula

$$(1/\sigma_{min})^2 = E\{(\partial\lambda/\partial\theta_T)^2\}, \quad (9)$$

in which $E\{\}$ means "the expectation value of", sometimes called an ensemble average, and this expectation is to be taken with respect to the joint distribution of $v_1, v_2, \ldots, v_k$.

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* This is the content of the major theorem quoted by Swerling from Cramer. The reader is referred to Swerling's work cited in Reference 4 for more details.

** This result indicates that it should be possible to devise a θ_T estimator not requiring a knowledge of x_o.

The required expectation value is readily computed, and the details are given in the appendix. The result is

$$(1/\sigma_{min})^2 = \sum_i (\partial q_i/\partial \theta_T)^2/(q_i - q_i^2). \tag{10}$$

Recalling that

$$q_i = q(x_i), \tag{11}$$

one may write, with the aid of equations (1) and (2),

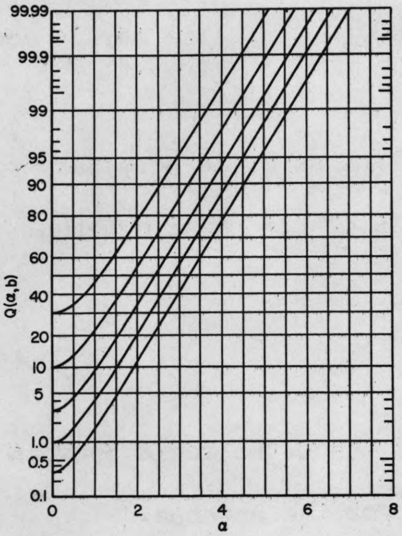
$$(1/\sigma_{min})^2 = x_o^2 \sum_i (\partial q_i/\partial x_i)^2 (f_i'/\beta)^2/(q_i - q_i^2), \tag{12}$$

which is a form in which the beam pattern function f makes explicit appearance, since the notation is intended to indicate that $f' = df/du$. Also in this form, it is evident that there is no dependence on θ_T, and it may be taken to have some convenient value, such as zero.

The function q(x) is the probability that the quantized signal will have the value "1" instead of "0". Its structure depends on the manner in which the quantized signal is derived. Actually, the formalism thus far developed is valid for a variety of such quantization procedures, but at this point it is convenient to assume that the envelope values of the radar IF signal are compared with a threshold bias, b, so as to produce a unit voltage during those intervals in which b is exceeded, and zero voltage otherwise. In representing this quantization procedure, the signal voltages will be measured in units of rms noise voltage, as will the bias.

The assumption of this quantization procedure provides a description of q(x) in terms of the Q-functions described by J. I. Marcum[1], as follows:

$$q(\alpha^2/2) = Q(\alpha, b) = \int_b^\infty e^{-(v^2+\alpha^2)/2} I_o(\alpha v) v \, dv, \tag{13}$$



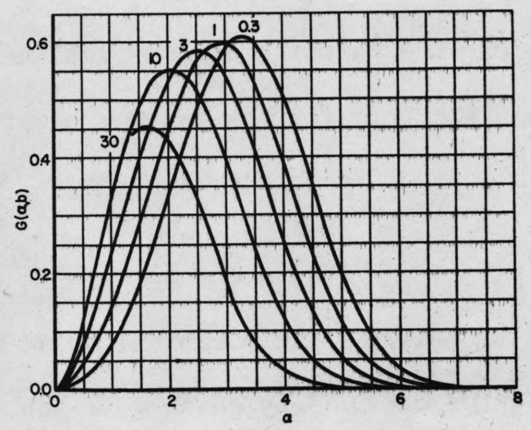


Figure 1. The function $Q(\alpha,b) = \int_b^\infty e^{-(\alpha^2+v^2)/2} I_0(\alpha v) v \, dv$ plotted versus α for certain values of $Q(0,b)$. The scale of ordinates is a normal probability scale in units of percent.

Figure 2. The function $G(\alpha,b) = (\partial Q/\partial\alpha)^2/(Q-Q^2)$ for which $Q = Q(\alpha,b)$ as plotted in Figure 1. The parameter label is the single-pulse noise quantization probability, $Q(0,b)$.

in which we have written the signal-to-noise ratio in amplitude units as α, instead of using power units.* In terms of the amplitude signal-to-noise ratio, the same quantity in power units is

$$x = \alpha^2/2. \qquad (14)$$

The function $I_n()$ is the modified Bessel function of the first kind, order n.

A plot of $Q(\alpha,b)$ is given in Figure 1. The vertical scale is a normal probability scale in units of percent. The individual curves are labeled by the value of $Q(0,b)$, which implicitly gives the value of b via

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* S. O. Rice has given a prior description of the Q-function as defined here. See Reference 6. The function is one of the incomplete Toronto functions, and has been called the non-central chi-squared distribution.

$$q_o = Q(0,b) = e^{-b^2/2}. \tag{15}$$

In Figure 2 are displayed plots of $(\partial Q/\partial \alpha)/(Q-Q^2)$. The curves are labeled in a manner similar to that in Figure 1.

For the purposes of numerical calculations it is necessary to choose a specific form for the two-way power gain of the radar antenna. Following Swerling, the choice is

$$f(u) = e^{-u^2}. \tag{16}$$

With this choice, 2β turns out to be 0.85 times the angle between 3 db points in the one-way power gain pattern for the antenna.

Finally, it is convenient to replace the sum over radar pulses by an integration over azimuth angle, again invoking the assumption that variations over the angle $\Delta\theta$ are sufficiently small. This conversion to integral form is achieved by multiplying the summand by $d\theta/\Delta\theta$, where $\Delta\theta$ is the angle turned between radar pulses, and by replacing the operation of summation by one of integration, where $d\theta$ represents the differential of the variable of integration. In this, $\Delta\theta$ is given by

$$\Delta\theta = 2\beta/N, \tag{17}$$

in which N is the number of pulses produced as the antenna turns through an angle 2β. Further, it is a convenient and good approximation to infinitely extend the limits of integration.

All of this manipulation leads ultimately to the formula

$$(1/\sigma_{min})^2 = (Nx_o/\beta^2)\int_{-\infty}^{\infty} G(\alpha,b)e^{-u^2}u^2du, \tag{18}$$

in which

$$G(\alpha,b) = (\partial Q/\partial \alpha)^2/(Q-Q^2), \tag{19}$$

with

$$Q = Q(\alpha, b), \tag{20}$$

as hitherto defined, and

$$\alpha = \sqrt{2x_o}\, e^{-u^2/2}. \tag{21}$$

The variable of integration in this case is $u = \theta/\beta$. A choice of α may be made for the variable of integration, leading to the formula

$$(1/\sigma_{min})^2 = (N/\beta^2)\int_o^{\sqrt{2x_o}} G(\alpha, b)\sqrt{\ln(2x_o/\alpha^2)}\ \alpha\, d\alpha, \tag{22}$$

a form which is convenient for the purposes of numerical quadrature.

III. Results of the Calculation

The numerical quadratures have been performed, and the results may be seen plotted in Figure 3, in which $(x_o \sqrt{N}/\beta)\sigma_{min}$ is plotted versus x_o, the axial value of signal-to-noise ratio shown in decibels. The bias b is indicated near each of the curves in terms of the noise probability q_o given in equation (15). Also plotted is Swerling's result for the analog (i.e., non-quantized) case. This is the curve marked A.

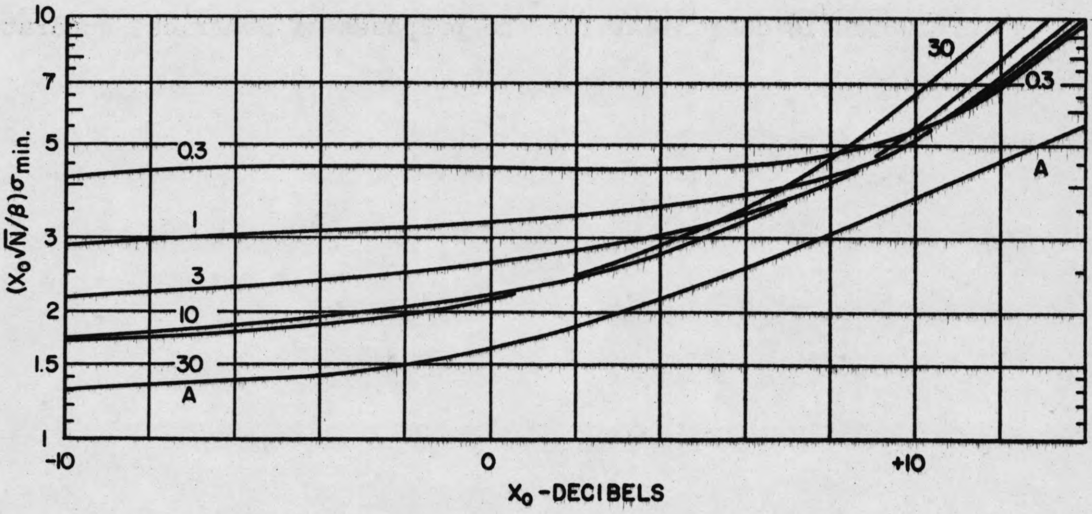

Figure 3. The minimum rms angular error, σ_{min}, multiplied by $x_o \sqrt{N}/\beta$, and plotted versus x_o, the axial signal-to-noise ratio. The scale of abscissae give the values of x_o in units of decibels. The scale of ordinates in logarithmic. The curve marked A is for the analog case as computed by Swerling (Reference 5). The others are for the quantized case, the parameter label being the single-pulse noise quantization probability in units of percent.

A power series expansion may be made, suitable for small x_o, for which the first two terms are given in the following formula:

$$A + Bx_o = (x_o \sqrt{N}/\beta)\sigma_{min}, \qquad (23)$$

in which

$$A^2 = 8(1-q_o)\sqrt{(2/x)}/(q_o b^4), \tag{24}$$

and

$$B^2 = (A^2/54)[4-b^2 q_o/(1-q_o)]^2. \tag{25}$$

For the values of q_o indicated in Figure 3, and for x_o less than 0.1 (-10db), this formula may be used with an accuracy of better than two percent.

For large signal-to-noise ratios, an asymptotic formula, discussed in the appendix, may be used. It is

$$\sqrt{\pi/8}\,[K \ln(x_o/k)]^{-1/4} \backsim (\sqrt{N}/\beta)\sigma_{min}, \tag{26}$$

in which K and k may be regarded as functions of either b or q_o. These functions are plotted, along with $b^2/2$, in Figure 4. There, the

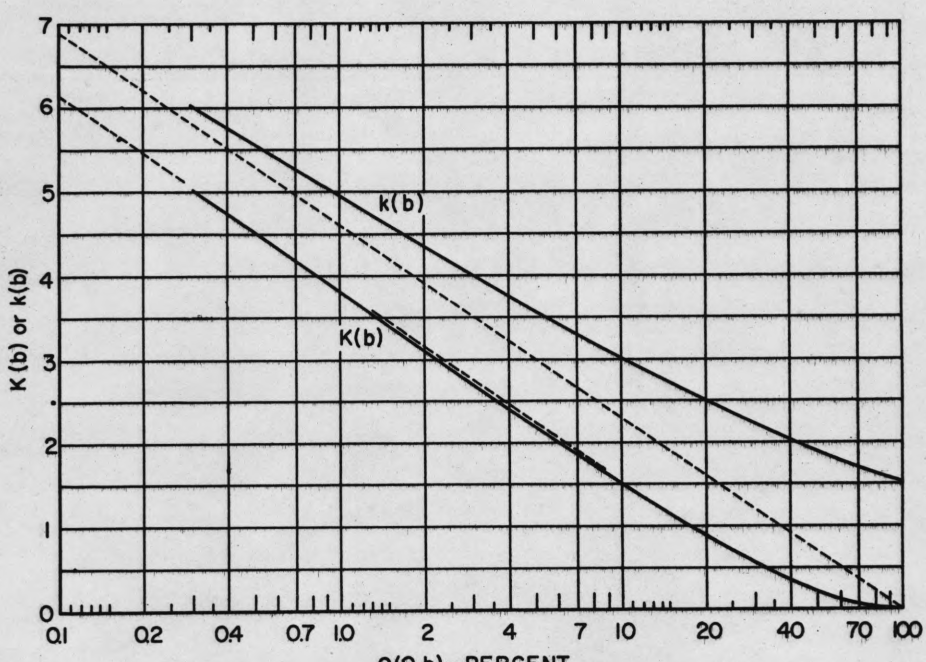

Figure 4. The functions K(b) and k(b) for use in the asymptotic formula, (26), good for $x_o > 4k$. The dependence on b is shown via Q(0,b), since $\ln Q(0,b) = -b^2/2$. Asymptotes to the curves are $b^2/2$ and $(b^2/2)-3/4$, and are shown as dashed lines. The scale of abscissae is logarithmic.

independent variable is taken to be q_o, shown on a logarithmic scale. For $x_o > 4k$, the formula yields results with errors less than about two percent.

For large values of bias, an asymptotic formula may be derived for those signal levels equal to the bias, i.e. for the signal-to-noise ratio, $x_o = b^2/2$. This formula is

$$1.093x_o^{-1/8}(1-0.24x_o^{-1/2}) \backsim (\sqrt{N}/\beta)\sigma_{min}, \tag{27}$$

in which the coefficients have been derived numerically in a manner indicated in the appendix. For x_o greater than about five, the formula is good to within about one percent.

IV. Comparison of the Quantized and Unquantized Cases

The most pertinent comparison of the quantized and the analog case, for angular accuracy, is, however, not that suggested by Figure 3, where a comparison at corresponding values of x_o is readily made. Rather, what would be more pertinent, in an operational sense, since observations of x_o are made only with great difficulty, would be a comparison at corresponding values of detection probability, as Swerling has indicated[4].

Accordingly, calculations of detection probability were made for two hypothetical radars each operating at a pulse frequency of 360 per second, with the antenna turning at a speed of 6 revolutions per minute. The radars were chosen to have beamwidths, measured between half power points of the one-way pattern, of 3 degrees and 1 degree, respectively. The corresponding number of pulses generated as the antenna turned through an angle equal to the beamwidth were to be 30 and 10, respectively. The value of β for the two cases are 1.275 degrees and 0.425 degree, respectively, and the values of N corresponding to 2β are 25.5 and 8.5, respectively.

The detection probabilities were calculated according to the procedures of Reference 1, using the graphs of the incomplete Toronto function presented there, for the analog case. For the quantized case, essentially the same procedures were used, except that the Q-functions, used in conjunction with the binomial distribution functions, replaced the incomplete Toronto functions. In the latter case also, optimal values for the "second threshold" were chosen from Reference 3. The false alarm probability was chosen to be about 5×10^{-5} per revolution (of the antenna) per range element, in each case. Finally the standard 1.6 db beam-shape loss factor was taken into account for each case.[4]

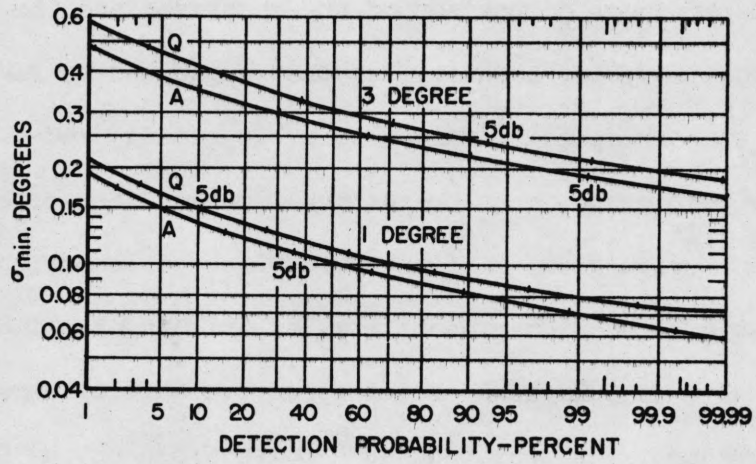

Figure 5. A comparison of minimum rms angular errors for quantized (Q) and analog (A) cases for 3 degree and 1 degree beamwidth radars. The radars are assumed to pulse at a rate of 360 pps and to turn at 6 rpm. The false alarm rate is set at the very low value of one in 20,000 revolutions per range element. Marks along the curve are spaced at intervals of one decibel of the axial signal-to-noise ratio. The scale of abscissae is a normal probability scale; the scale of ordinates is logarithmic.

It should be noted that the detection probabilities as calculated are so-called "single-scan detection probabilities", in that no target reflectivity fluctuations are assumed. This calculation is applicable to the case where such fluctuations are so slowly occuring that the reflectivity may be regarded as essentially constant during the few tens of milliseconds that the radar beam is "on-target". In this latter case, the probabilities should be interpreted as conditional, the condition being the attainment of the particular value of x_o. No treatment of rapid fluctuations (changing reflectivity by significant amounts in times less than a few tens of milliseconds) is given.*

* Also, no account is given of the effect of "human factors". The probabilities here computed are applicable to the case of an optimal mechanization of the observation process.

With the aid of detection curves, so computed, and the curves of Figure 3, the variable x_o may be eliminated, and one may plot σ_{min}, in degrees, versus the detection probability, in percent, as displayed in Figure 5. It is evident from this latter figure that the loss in accuracy as a result of quantization is, indeed, almost inconsiderable. The angular error is from 10 to 15 percent greater in the quantized case.* One might well have expected that the loss in sensitivity resulting from quantization, by requiring less "noisy" data, would almost entirely make up for the displacement of the curves of Figure 3 from curve A, and, in fact, this certainly appears to be the case.

In order to fully treat the case for slow fluctuations of target reflectivity, one ought to multiply the detection probability, as calculated in the above manner, by the probability of obtaining the value of x_o, and use this product to weight σ_{min} in an average over x_o. Similarly, one obtains an average detection probability. This would yield the results corresponding to Figure 5 for the slow fluctuations. This has not been done, but there already is abundant evidence in Figure 5 itself that the main conclusion would still stand: the effect of binary quantization for radar angular accuracy may be neglected in many cases of interest.

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* The absolute size of the angular errors is atypically small, for most cases of interest. This is a consequence of the very small false alarm rates chosen, which, in turn, require larger signal-to-noise ratios for detection than usual, by perhaps 3 decibels. These low false alarm rates do, however, put the quantized case in a poorer light, relative to the analog, than would the more "normal" false alarm rates.

V. Acknowledgements

Thanks are due to Miss J. Timko and Mrs. S. Goldberg for performing much of the numerical work.

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APPENDIX

I. Evaluation of ρ.

The quantity ρ is determined by the equation[4]

$$\rho^2 E\{(\partial\lambda/\partial\theta_T)^2\}E\{(\partial\lambda/\partial x_o)^2\} = E^2\{(\partial\lambda/\partial\theta_T)(\partial\lambda/\partial x_o)\}, \quad (1A)$$

and, since it is evident, from what follows, that

$$E\{(\partial\lambda/\partial\theta_T)^2\} \neq 0, \quad (2A)$$

and

$$E\{(\partial\lambda/\partial x_o)^2\} \neq 0, \quad (3A)$$

then if ρ is to be zero, it is sufficient for one to establish the relation

$$E\{(\partial\lambda/\partial\theta_T)(\partial\lambda/\partial x_o)\} = 0. \quad (4A)$$

By definition, one has

$$(\partial\lambda/\partial\theta_T)(\partial\lambda/\partial x_o) = [\Sigma_i(\partial p_i/\partial\theta_T)/p_i] \times [\Sigma_j(\partial p_j/\partial x_o)/p_j], \quad (5A)$$

which may be written

$$(\partial\lambda/\partial\theta_T)(\partial\lambda/\partial x_o) = \Sigma_i(\partial p_i/\partial\theta_T)(\partial p_i/\partial x_o)/p_i^2$$
$$+ \sum_{i\neq j}(\partial p_i/\partial\theta_T)(\partial p_j/\partial x_o)/(p_i p_j). \quad (6A)$$

The first sum on the right may be written as

$$(x_o/\beta)\,\Sigma_i(\partial p_i/\partial x_i)^2 f_i' f_i/p_i^2, \quad (7A)$$

in which pulse-to-pulse independence on the part of v_i allows expectations to be taken term by term. A single term is

$$(x_o f_i f_i'/\beta)E\{(\partial p_i/\partial x_i)^2/p_i^2\}. \quad (8A)$$

Determining the expectation yields

$$E\{(\partial p_i/\partial x_i)^2/p_i^2\} = (\partial p_i/\partial x_i)^2/q_i + [\partial(1-q_i)/\partial x_i]^2/(1-q_i), \quad (9A)$$

which is

$$E\{(\partial p_i/\partial x_i)^2/p_i^2\} = (\partial q_i/\partial x_i)^2/(q_i-q_i^2). \tag{10A}$$

Thus, the desired expectation is, for the first summation,

$$E\{\textstyle\sum_i(\partial p_i/\partial\theta_T)(\partial p_i/\partial x_o)/p_i^2\} = x_o\textstyle\sum_i(f_i'f_i/\beta)(\partial q_i/\partial x_i)^2/(q_i-q_i^2). \tag{11A}$$

In the order of approximation with which this work is being done, this summation would be replaced by integration. Then, since f is an angle dependent variable, and is even, also since q is a single valued function of $x = x_o f$, one observes that

$$(\partial q/\partial x)^2 f/(q-q^2) \tag{12A}$$

is an even function of angle. However, f' is odd. Thus, in the approximation in which the summation is replaceable by integration, this summation vanishes.

For the second summation in equation (6A), each term may be written, because of the pulse-to-pulse independence of v_i,

$$E\{(\partial p_i/\partial\theta_T)/p_i\}E\{(\partial p_j/\partial x_o)p_j\}, \tag{13A}$$

in which

$$E\{(\partial p_i/\partial\theta_T)/p_i\} = (\partial q_i/\partial\theta_T) + \partial(1-q_i)/\partial\theta_T, \tag{14A}$$

which clearly vanishes.

Thus, to the extent that the integral approximation is valid, one has

$$\rho \approx 0, \tag{15A}$$

provided that $E\{(\partial\lambda/\partial x_o)^2\}$ does not vanish. This expectation has essentially the same formal structure as $E\{(\partial\lambda/\partial\theta_T)^2\}$ evaluated below, so that its non-vanishing character may be deduced from what follows.

II. Evaluation of $E\{(\partial\lambda/\partial\theta_T)^2\}$

By definition one has

$$(\partial\lambda/\partial\theta_T)^2 = [\Sigma_i(\partial p_i/\partial\theta_T)/p_i] \times [\Sigma_j(\partial p_j/\partial\theta_T)/p_j], \qquad (16A)$$

which may be written

$$(\partial\lambda/\partial\theta_T)^2 = \Sigma_i(\partial p_i/\partial\theta_T)^2/p_i^2 + \sum_{i\neq j}(\partial p_i/\partial\theta_T)(\partial p_j/\partial\theta_T)/(p_i p_j). \qquad (17A)$$

Taking term-by-term expectations, one deduces for the first summation that

$$E\{(\partial p_i/\partial\theta_T)^2/p_i^2\} = (\partial q_i/\partial\theta_T)^2/(q_i-q_i^2) \qquad (18A)$$

and similarly, that, for the second summation,

$$E\{(\partial p_i/\partial\theta_T)/p_i\}\, E\{(\partial p_j/\partial\theta_T)/p_j\}, \qquad (19A)$$

each term vanishes in the same manner as (14A). Thus, one has

$$E\{(\partial\lambda/\partial\theta_T)^2\} = \Sigma_i(\partial q_i/\partial\theta_T)^2/(q_i-q_i^2), \qquad (20A)$$

as was to be shown.

III. Properties of the function $Q(\alpha,b)$

In this part of the Appendix are set forth some properties of the function $Q(\alpha,b)$ which are particularly pertinent to numerical work with this function.

The definition of this function is taken to be[1,6]

$$Q(\alpha,b) = \int_b^\infty e^{-(v^2+\alpha^2)/2} I_o(\alpha v) v \, dv, \tag{13}$$

in which the notation $I_n()$ is intended to designate the modified Bessel function of the first kind, order n.

By means of a partial differentiation with respect to α, followed by an integration by parts, one may deduce that

$$\partial Q/\partial\alpha = b e^{-(\alpha^2+b^2)/2} I_1(\alpha b), \tag{21A}$$

and from equation (15)

$$Q(0,b) = e^{-b^2/2}, \tag{15}$$

a second integral definition of $Q(\alpha,b)$ is obtained:

$$Q(\alpha,b) = e^{-b^2/2} + \int_o^\alpha e^{-(v^2+b^2)/2} I_1(vb) b \, dv. \tag{22A}$$

This second definition is most useful if one wishes to tabulate both $\partial Q/\partial\alpha$ and Q itself, the latter being readily derived from the former by a rather economical procedure of numerical quadrature.

From the second integral definition, another integration by parts will yield a third integral definition, which is

$$Q(\alpha,b) = e^{-(\alpha^2+b^2)/2} I_o(\alpha b) + \int_o^\alpha e^{-(v^2+b^2)/2} I_o(vb) v \, dv. \tag{23A}$$

This definition is useful, since from it and the first, one may deduce, using

$$Q(\alpha,0) = Q(\infty,b) = 1, \tag{24A}$$

that

$$Q(b,b) = [1+e^{-b^2}I_0(b^2)]/2. \tag{25A}$$

This is a form which will provide the first term in a Taylor's series expansion of $Q(\alpha,b)$ for α near b, each term of which may be written in closed form: writing

$$Q(\alpha,b) = \sum_n (1/n!)Q^{(n)}(b,b)(\alpha-b)^n, \tag{26A}$$

in which

$$Q^{(n)}(b,b) = (\partial/\partial\alpha)^n Q(\alpha,b)\Big|_{\alpha=b}, \tag{27A}$$

then the first four coefficients are

$$\left.\begin{aligned} Q(b,b) &= [1+e^{-b^2}I_0(b^2)]/2, \\ Q'(b,b) &= be^{-b^2}I_1(b^2), \\ Q''(b,b) &= b^2[e^{-b^2}I_0(b^2)-(1+2\gamma)e^{-b^2}I_1(b^2)], \\ Q'''(b,b) &= 2b^3[(1+\gamma+4\gamma^2)e^{-b^2}I_1(b^2)-(1+\gamma)e^{-b^2}I_0(b^2)], \end{aligned}\right\} \tag{28A}$$

in which

$$2\gamma = 1/b^2. \tag{29A}$$

The principal use of this expansion has been in checking and correcting quadrature errors for Q near $Q(b,b) \approx 1/2$, i.e. for values of Q when α is near b.

Another expansion, derivable from the first integral definition of

$Q(\alpha,b)$, by successive integration by parts, is the infinite series[1,6]

$$Q(\alpha,b) = e^{-(\alpha^2+b^2)/2}\sum_n (\alpha/b)^n I_n(\alpha b). \tag{30A}$$

With the aid of the power series expansion for the modified Bessel functions of the first kind, one is led to the following double series:

$$Q(\alpha,b) = e^{-(\alpha^2+b^2)/2}\sum_{k=0}^{\infty}\left[(\alpha^2/2)^k/k!\sum_{r=0}^{k}(b^2/2)^r/r!\right], \tag{31A}$$

which is useful if α is small. This will lead to an expression involving series for $(\partial Q/\partial\alpha)^2/(Q-Q^2)$, which is

$$G(\alpha,b) = b^2[I_1(\alpha b)]^2/(S_1S_2), \tag{32A}$$

in which

$$S_1 = \sum_{k=0}^{\infty}\left[(\alpha^2/2)^k/k!\sum_{r=0}^{k}(b^2/2)^r/r!\right], \tag{33A}$$

and

$$S_2 = \sum_{k=0}^{\infty}\left[(\alpha^2/2)^k/k!\sum_{r=k+1}^{\infty}(b^2/2)^r/r!\right]. \tag{34A}$$

This development was used to derive the first two terms of the series (23) in powers of x_o for $(x_o\sqrt{N}/\beta)\sigma_{min}$. Another use was the derivation of the series expansions (45A) and (46A) in powers of $b^2/2$ for the functions $k(b)$ and $K(b)$, as appearing in the asymptotic formula (26).

Asymptotic representations of $Q(\alpha,b)$ may be obtained using asymptotic expansions of the Bessel function.[1,6]

IV. Asymptotic Formulae for σ_{min}

The asymptotic formula (26) may be expressed as follows:

$$N(\beta/\sigma_{min})^2 = \int_0^{\alpha_0} G(\alpha,b)\sqrt{2\ln(\alpha_0/\alpha)}\ \alpha\, d\alpha \sim H_0\sqrt{2\ln(\alpha_0/h)}, \quad (35A)$$

holding for $\alpha_0 \to \infty$. The quantities H_0 and h are functions of b to be determined.*

To establish (35A), each side of the relation may be divided by $\sqrt{2\ln\alpha_0}$, leaving

$$\int_0^{\alpha_0} G(\alpha,b)\sqrt{1-\xi(\alpha)}\ \alpha d\alpha \sim H_0\sqrt{1-\xi(h)}, \quad (36A)$$

in which

$$\xi(h) = (\ln h)/(\ln\alpha_0). \quad (37A)$$

Then, the radical in each side of the asymptotic relation (36A) is expanded in a series in powers of ξ. Since $G(\alpha,b)$ vanishes with at least exponential speed for $\alpha \gg b$, the integration on the left may be done in the series on a term-by-term basis, for $\alpha_0 \gg b$, and further, the upper limit of the integration may be extended to infinity.

These things being done, the result is an asymptotic relation between two series, each in powers of

$$1/(\ln\alpha_0), \quad (38A)$$

in which the correspondence of the coefficients is

$$\int_0^{\infty} G(\alpha,b)(\ln\alpha)^n \alpha d\alpha = H_n(b) \leftrightarrow H_0(b)(\ln h)^n. \quad (39A)$$

---

* In this formula, α_0 is written instead of $\sqrt{2x_0}$.

If this correspondence is replaced by an equality for the first (n=0) terms, one has a definition for H_o, and the asymptotic formula (36A) is established, independently of the value of h, save that lnh be finite. The quantity h may then be determined in such a way as to maximize the accuracy of the formula for finite values of α_o. Clearly this is achieved by replacing the correspondence by an equality for the second (n=1) term, as well. Thus, the equation

$$\ln h(b) = H_1/H_o \tag{40A}$$

defines this function.

Normalizations have been made, so that the roles of h and H_o are filled by the functions k and K, causing the asymptotic formula to appear as presented in the text:

$$\sqrt{\pi/8}\,[K \ln(x_o/k)]^{-1/4} \sim (\sqrt{N}/\beta)\sigma_{min}. \tag{26}$$

The normalizations are

$$k(b) = h^2/2, \tag{41A}$$

$$K(b) = (H_o\pi/8)^2, \tag{42A}$$

and were chosen so that, for large values of b, the function k would be asymptotic to $b^2/2$, and K would be asymptotic to $(b^2/2)-3/4$. In the case of K, the exact choice depends upon the value of the constant

$$\int_o^\infty e^{-2u^2}[1-\mathrm{Erf}^2(u)]^{-1}du, \tag{43A}$$

in which

$$\mathrm{Erf}(u) = \sqrt{4/\pi}\int_o^u e^{-v^2}dv. \tag{44A}$$

By means of numerical quadrature, the value of this constant has been determined to be unity with an accuracy of better than one part in 5000.

For small values of b, the behaviors of k and K are characterized by

$$k = e^{c}[1-(b^2/2)(2-\ln 2)/4+\ldots], \tag{45A}$$

$$K = (\pi/4)^2(b^2/2)^2[1-(b^2/2)/2+\ldots], \tag{46A}$$

in which c is the complement of the Euler-Mascheroni constant,

$$c = 1 - \lim_{m\to\infty}\left\{ \sum_{n=1}^{m} (1/n) - \ln m \right\} = 0.42278\ 4335\ldots\ . \tag{47A}$$

For finite values of b, the values of k and K have been computed by numerical quadrature. These functions are plotted in Figure 4.

Formula (27), an asymptotic expression giving σ_{min} for large values of b, but with $x_o = b^2/2$, was derived using an asymptotic representation of $\alpha G(\alpha,b)$. This representation involves the integrand of (43A), and, in consequence, the numerical coefficients of (27) involve moments of that integrand. These moments, which are of half-integer order, were evaluated by numerical quadrature.

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